

# ADAPTIVE KRYLOV SUBSPACE DIMENSION SELECTION FOR IMAGE DEBLURRING USING GLOBAL GMRES AND TIKHONOV REGULARIZATION

BRAHIM REZKI<sup>1,\*</sup>, KHALED MELKEMI<sup>2</sup>

<sup>1</sup>Laboratory of Applied Mathematics, University of Biskra, Biskra, Algeria

<sup>2</sup>EDPA Laboratory, Department of Mathematics, University of Batna 2, Batna, Algeria

\*Corresponding author: brahim.rezki@univ-biskra.dz

Received Jul. 11, 2026

**ABSTRACT.** In this paper, we propose an adaptive Krylov subspace method for solving large-scale discrete ill-posed problems in image deblurring. By integrating Tikhonov regularization with the global GMRES algorithm, our approach automatically selects the regularization parameter via the generalized cross-validation (GCV) criterion. Unlike standard approaches using a fixed Krylov subspace dimension, the proposed algorithm dynamically identifies an optimal dimension by monitoring the peak signal-to-noise ratio (PSNR) across iterations. The process terminates when successive PSNR improvements become negligible, ensuring an efficient balance between reconstruction quality and computational overhead. Experimental results demonstrate that this adaptive strategy outperforms fixed-dimension methods in both restoration quality and computational efficiency.

2020 Mathematics Subject Classification. 47A52; 68T05; 74P10; 65J20.

Key words and phrases. Ill-posed problem; Tikhonov regularization; Sylvester equation; Arnoldi method.

## 1. INTRODUCTION

Consider the linear least-squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2 \quad (1.1)$$

where  $A \in \mathbb{R}^{m \times n}$ ,  $m \geq n$ , is large-scale and severely ill-conditioned, and the right-hand side  $b = \hat{b} + \eta \in \mathbb{R}^m$  is contaminated by additive white Gaussian noise  $\eta$ . Problems of this form typically arise from the discretization of ill-posed Fredholm integral equations of the first kind with compact kernel [1,3,4], and are particularly prevalent in image deblurring, where  $A$  represents the blurring operator,  $x$  the unknown original image, and  $b$  the observed degraded image [3–5,7]. Because  $A$  is ill-conditioned

and  $b$  is noisy, a naive solution of (1.1) is dominated by noise amplification, this simultaneous ill-conditioning and contamination calls for regularization, while the large scale of  $A$  rules out direct factorization methods in favor of iterative Krylov subspace techniques, in which (1.1) is projected onto a low-dimensional subspace generated by an Arnoldi-type process. Representative methods of this class include Global-GMRES [6]. Among regularization strategies, Tikhonov regularization is one of the most widely used. It replaces (1.1) by the penalized problem

$$\min_{x \in \mathbb{R}^n} (\|Ax - b\|_2^2 + \lambda^2 \|Lx\|_2^2), \quad (1.2)$$

where  $L \in \mathbb{R}^{p \times n}$  is a regularization operator commonly the identity or a discrete first or second-order derivative operator and  $\lambda > 0$  is a regularization parameter that controls the trade-off between fidelity to the data and smoothness (or magnitude) of the solution. Popular strategies for choosing  $\lambda$  include Generalized Cross-Validation (GCV) [15] and the L-curve criterion [16, 17]. In two-dimensional image deblurring, problem (1.1) is more naturally expressed in matrix form, since the unknown  $x = \text{vec}(X)$  actually represents an image  $X \in \mathbb{R}^{n_1 \times n_2}$  stored column-wise, and the operator  $A$  itself has separable (Kronecker) structure,  $A = A_2 \otimes A_1$ , induced by the horizontal and vertical blurring operators  $A_1$  and  $A_2$ . Using the Kronecker product identity

$$\text{vec}(A_1 X A_2^T) = (A_2 \otimes A_1) \text{vec}(X), \quad (1.3)$$

the corresponding degradation model is given by:

$$B = A_1 X A_2^T + N \quad (1.4)$$

which can be equivalently represented in vector form by setting

$$A = A_2 \otimes A_1, \quad x = \text{vec}(X), \quad b = \text{vec}(B), \quad n = \text{vec}(N)$$

where  $X$  denotes the unknown true image,  $B$  is the observed blurred and noisy image and  $N$  represents the additive noise. Recovering  $X$  is then equivalent to solving the matrix least-squares problem

$$\min_X \|A_1 X A_2^T - B\|_F^2, \quad (1.5)$$

Its Tikhonov-regularized formulation is,

$$\min_X (\|A_1 X A_2^T - B\|_F^2 + \lambda^2 \|LX\|_F^2). \quad (1.6)$$

As is common practice, the regularization operator  $L$  in (1.6) is taken as the identity,  $L = I$ . Setting the gradient of (1.6) with respect to  $X$  to zero yields the generalized Sylvester equation

$$A_2^T A_2 X A_1^T A_1 + \lambda^2 X = A_2^T B A_1, \quad (1.7)$$

which is generally too large to be solved directly. This motivates the use of Krylov subspace projection methods, which have proven effective for computing accurate approximate solutions of (1.7) at low

computational cost [2,11]. Building on this framework, we propose an **adaptive hybrid Global-GMRES method** for image deblurring, in which the Krylov subspace dimension  $m$  is selected automatically. Specifically:

1. equation (1.7) is projected, via the Global-Arnoldi process, onto a Krylov subspace of increasing dimension  $m$
2. at each step, the regularization parameter  $\lambda$  is chosen by minimizing the GCV function on the projected problem, and the corresponding approximate solution  $X_k$  is computed
3. the reconstruction quality is monitored through the Peak Signal-to-Noise Ratio (PSNR), and  $m$  is increased until the change in PSNR between two consecutive values falls below a prescribed tolerance.

This adaptive stopping criterion removes the need to fix  $m$  in advance, and provides an effective compromise between restoration quality and computational cost. The remainder of the paper is organized as follows. Section 2 recalls preliminaries on Tikhonov regularization and the Global-Arnoldi process. Section 3 formulates the image deblurring problem within the Global-GMRES framework. Section 4 presents the proposed adaptive algorithm. Section 5 reports numerical experiments, and Section 6 concludes the paper.

## 2. PRELIMINARIES AND DEFINITIONS

### 2.1. Kronecker tensor product of matrices.

**Definition 1.** Let  $M$  and  $N$  be two real matrices defined by:

$$\begin{cases} M = (m_{ij}), & \text{Where } 1 \leq i \leq p, \text{ and } 1 \leq j \leq q \\ \text{and} \\ N = (n_{ij}), & \text{Where } 1 \leq i \leq r, \text{ and } 1 \leq j \leq s \end{cases}$$

The kronecker tensor product of the matrices  $M$  and  $N$  is defined as follows:

$$M \otimes N = K \quad \text{where } K \text{ is of size } pr \times qs \text{ and } K = (m_{ij}N) \quad \begin{matrix} 1 \leq i \leq p \\ 1 \leq j \leq q \end{matrix}$$

**2.2. Kronecker tensor product properties.** Let  $M, N, P$  and  $Q$  be a matrices, we have the following properties

1.  $(M \otimes N)(P \otimes Q) = (MP) \otimes (NQ)$
2.  $(M \otimes N)^{-1} = (M)^{-1} \otimes (N)^{-1}$  (if  $M$  and  $N$  are square and invertible matrices)
3.  $(M \otimes N)^T = (M)^T \otimes (N)^T$
4.  $\text{Vect}(MXN) = (N^T \otimes M)\text{Vect}(X)$ , where  $X$  is a matrix of size  $q \times r$

5. if  $M, N$  and  $P$  are matrices of sizes  $p \times q, q \times r$  and  $r \times s$  respectively, then

$$\text{tr}(MNP) = (\text{Vect}(M^T))^T (I_q \otimes N) \text{Vect}(P)$$

Where Vect is an operator which transforms the matrix to a vector by concatenating the columns of the matrix.

**2.3. Frobeinus inner product.** The F-inner product of the matrices  $A$  and  $B$  is given by the folowing formula

$$\langle A, B \rangle_F = \text{tr}(A^T B), \text{ where } \text{tr}(C) \text{ denotes the trace of the square matrix } C$$

The Frobeinus norm denoted by  $\|\cdot\|_F$  is given by:  $\|A\|_F = (\langle A, A \rangle_F)^{\frac{1}{2}}$ .

The set of matrices of order  $n \times p$  is said to be F-orthogonal if it is orthogonal with respect to the Frobeinus inner product  $\langle \cdot, \cdot \rangle_F$

In the context of image restoration process, we can factored the blurring matrix  $A$  defined in (1,1) as a kronecker product  $A = A_1 \otimes A_2$  of a suitable sizes.

**2.4. Tikhonov regularization method.** As shown in [8,9], the Tikhonov regularization technique relies on the equivalence between problem (1.1) and the penalized problem (1.2). The following proposition illustrates how to transform the penalized problem (1.2) into a generalized Sylvester matrix equation.

**Proposition 1.** *Solving the Tikhonov regularization problem (1.2) is equivalent to solving a generalized Sylvester matrix equation of the form:*

$$MXN + SXT = E$$

*Proof.*

$$\min_x (\|Ax - b\|_2^2 + \lambda^2 \|Lx\|_2^2) \iff \hat{x} = \underset{x}{\text{argmin}} \left\| \begin{bmatrix} A \\ \lambda L \end{bmatrix} x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|_2^2 \quad (2.1)$$

The minimizer solution of (2.1) is calculated as follows:

$$\begin{aligned} \nabla \left\langle \begin{pmatrix} A \\ \lambda L \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix}, \begin{pmatrix} A \\ \lambda L \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\rangle &= 0 \\ \implies (A^T A + \lambda^2 L^T L)x &= A^T b \\ \iff A_\lambda x &= A^T b, \text{ where } A_\lambda = A^T A + \lambda^2 L^T L \end{aligned} \quad (2.2)$$

Now we consider the Kronecker product factorization of the matrices  $A$  and  $L$  as:

$A = A_1 \otimes A_2$  and  $L = L_1 \otimes L_2$ , where  $A_1$  and  $L_1$  are two squares matrices of size  $n \times n$ ,  $A_2$  and  $L_2$  are two squares matrices of sizes  $p \times p$  then the problem (2.2) becomes

$$((A_1 \otimes A_2)^T (A_1 \otimes A_2) + \lambda^2 (L_1 \otimes L_2)^T (L_1 \otimes L_2)) X = (A_1 \otimes A_2)^T b$$

using the Kronecker property, we obtain

$$\begin{aligned} (A_1^T \otimes A_2^T)(A_1 \otimes A_2)x + \lambda^2(L_1^T \otimes L_2^T)(L_1 \otimes L_2)x &= (A_1^T \otimes A_2^T)b \\ \iff (A_1^T A_1 \otimes A_2^T A_2)x + \lambda^2(L_1^T L_1 \otimes L_2^T L_2)x &= (A_1^T \otimes A_2^T)b \\ \iff (A_2^T A_2 X A_1^T A_1) + \lambda^2(L_2^T L_2 X L_1^T L_1) &= (A_2^T B A_1) \end{aligned} \quad (2.3)$$

Where  $\text{vec}(B) = b$  and  $\text{vec}(X) = x$ .

(2.3) is the linear matrix problem, which represents the generalized Sylvester matrix equation.

also, we rewritten it as:

$$MXN + SXT = E \quad (2.4)$$

Where  $M = A_2^T A_2$ ,  $N = A_1^T A_1$ ,  $S = \lambda^2 L_2^T L_2$ ,  $T = L_1^T L_1$ , and  $E = A_2^T B A_1$   $\square$

The next theorem ensures the existence and uniqueness of a solution to the generalized Sylvester matrix equation (2.4)

**Theorem 1.** Assume  $(M - \lambda S)$  and  $(N - \lambda T)$  are two regular matrix pencils of order  $n$  and  $p$ , respectively. We denote by  $\mathcal{L}(M, S)$  and  $\mathcal{L}(N, T)$  the sets of generalized eigenvalues corresponding to the pairs  $(M, S)$  and  $(N, T)$ , respectively. If  $\mathcal{L}(M, S) \cap \mathcal{L}(N, T) = \emptyset$ , then equation (2.4) has a unique solution. For all details concerning the proof of this theorem, see [14].

**2.5. Arnoldi Krylov subspace.** Constructing a Krylov subspace is a crucial step in building an approximate solution for the Sylvester matrix equation problem (2.4). In this subsection, we first present the definition of the Krylov subspace and in the second part, we introduce an algorithm for constructing a basis that generates the Krylov subspace.

**Definition 2.** Let  $\mathcal{A} : \mathbb{R}^{n \times p} \rightarrow \mathbb{R}^{n \times p}$  be an operator, and let  $b \in \mathbb{R}^{n \times p}$  be a matrix. The Krylov subspace generated by the pair  $(\mathcal{A}, b)$  is defined by:

$$\mathcal{K}_k(\mathcal{A}, b) = \text{span} \{b, \mathcal{A}b, \dots, \mathcal{A}^{k-1}b\} \subset \mathbb{R}^{n \times p} \quad (2.5)$$

Where  $K_k = [b, \mathcal{A}b, \dots, \mathcal{A}^{k-1}b]$  is called Krylov matrix

To determine its basis, we use the Arnoldi process [1], shown below, as a tool for constructing an orthonormal basis  $\{\omega_1, \omega_2, \dots, \omega_k\}$

$$\text{i.e. } \langle \omega_i, \omega_j \rangle = \delta_{ij} \text{ (}\delta_{ij} \text{ represents the Kronecker symbol).}$$

**Algorithm 1** Modified Global Arnoldi Algorithm

---

```

1: Inputs: An operator  $\mathcal{A}$ ,  $b$  a matrix of size  $n \times p$  and an integer  $k$ .
2: Step 0 :  $\omega_1 = \frac{b}{\|b\|_F}$ ;
3: Step 1 :
4: for  $j=1,\dots,k$  do
5:    $\bar{\omega} = \mathcal{A}\omega_j$ ;
6:   for  $i=1,\dots,j$  do
7:      $h_{i,j} = \langle \omega_i, \bar{\omega} \rangle_F$ ;
8:      $\bar{\omega} = \bar{\omega} - h_{i,j}\omega_i$ ;
9:   end for
10:   $h_{j+1,j} = \|\bar{\omega}\|_F$ ;
11:   $\omega_{j+1} = \frac{\bar{\omega}}{h_{j+1,j}}$ ;
12: end for

```

---

**Remark 1.** The outputs of the previous Arnoldi Algorithm are the following matrices:

$$W_k = [\omega_1, \omega_2, \dots, \omega_k] \text{ and } \omega_{k+1} \in \mathbb{R}^{n \times p}$$

$$\tilde{H}_k = \begin{bmatrix} H_k \\ h_{k+1,k} e_k^T \end{bmatrix}$$

$$\text{Where } H_k = \begin{bmatrix} h_{11} & \cdots & h_{1k} \\ \vdots & \ddots & \vdots \\ h_{k1} & \cdots & h_{kk} \end{bmatrix}, e_k \text{ is the } k\text{-th unit vector of the } \mathbb{R}^n \text{ canonical basis and the matrices}$$

$W_k, \tilde{H}_k$  verify:

$$[\mathcal{A}(\omega_1), \mathcal{A}(\omega_2), \dots, \mathcal{A}(\omega_k)] = W_{k+1} \left( \tilde{H}_k \otimes I_p \right) = W_k (H_k \otimes I_p) + \omega_k e_k^T \quad (2.6)$$

Where:  $W_{k+1} = [\omega_1, \omega_2, \dots, \omega_{k+1}]$

### 3. DEBLURRING PROCESS USING GLOBAL-GMRES METHOD

The Global Generalized Minimum Residual method (Global-GMRES) for solving the Sylvester matrix equation plays a fundamental role in image deblurring. In this section, we outline the various steps involved in this method, and some necessary theoretical tools.

**3.1. Global-GMRES method for solving generalized Sylvester matrix equation.** The generalized Sylvester matrix equation is in the form (2.4) i.e.

$$MXN + SXT = E$$

Where:  $M, S \in \mathbb{R}^{n \times n}$ ,  $N, T \in \mathbb{R}^{p \times p}$ , and  $X \in \mathbb{R}^{n \times p}$ . This kind of equations occur in different numerical approaches [12, 13, 19] such as: boundary value problems, reduction methods for large scale dynamical systems and large eigenvalue problems. Among the best-known methods for solving this type of equation is the Global-GMRES method [6, 18]. It is founded on the following steps:

1. We construct an F-orthonormal basis  $\{\omega_1, \omega_2, \dots, \omega_k\}$  of the Krylov subspace associated to the pair  $(\mathcal{A}, \omega)$ ,  $K_k(\mathcal{A}, \omega) = \text{span} [\omega, \mathcal{A}(\omega), \dots, \mathcal{A}^{k-1}(\omega)]$  using modified global Arnoldi algorithm such that:  $\mathcal{A}(X) = MXN + SXT$
2. Starting from an initial vector  $X_0 \in \mathbb{R}^{n \times p}$ , the corresponding residual  $R_0 = E - \mathcal{A}(X_0)$ .
3. At iteration  $k$ , the approximate solution of the generalized Sylvester matrix equation (2.4) is defined by:

$$X_k = X_0 + F_k, \quad \text{where } F_k \in K_k(\mathcal{A}, R_0) \quad (3.1)$$

and

$$R_k = E - \mathcal{A}(X_k) \perp_F K_k(\mathcal{A}, \mathcal{A}(R_0)) \quad (3.2)$$

The key to the G-GMRES method resides in reducing the problem's size. The next theorem illustrates how to achieve this reduction in dimensionality.

**Theorem 2.** *At the iteration number  $k$ , the approximate solution to the problem (2.4) is given by:*

$$X_k = X_0 + W_k(y_k \otimes I_p) \quad (3.3)$$

*Such that:  $y_k$  represents the solution of the following reduced problem*

$$\min_{y \in \mathbb{R}^k} \left\| \|\mathbb{R}_0\|_F e_1 - \tilde{H}_k y \right\|_2 \quad (3.4)$$

Where  $e_1$  is the first vector of the canonical basis for  $\mathbb{R}^{k+1}$ .

*Proof.* Our proof consists of two steps

Step 01: (The proof of (3.4))

According to the Global-GMRES method we have:

$$\begin{aligned} X_k &= X_0 + F_k / F_k \in K_k(\mathcal{A}, R_0) \\ &= X_0 + \sum_{i=1}^k y_i^k \omega_i \end{aligned}$$

Where  $\{\omega_1, \omega_2, \dots, \omega_k\}$  is the obtained orthonormal basis for  $K_k(\mathcal{A}, R_0)$  using Arnoldi algorithm

$$\begin{aligned} \Rightarrow X_k &= X_0 + [\omega_1, \omega_2, \dots, \omega_k] (y_k \otimes I_p) \\ &= X_0 + W_k (y_k \otimes I_p) \end{aligned}$$

$$\text{st: } y_k = (y_1^k, y_2^k, \dots, y_k^k)^T$$

Step 02: (The proof of (3.5))

We have  $R_k = E - \mathcal{A}(X_k)$  such that:  $X_k = X_0 + F_k/F_k \in K_k(\mathcal{A}, R_0)$  then:

$$\begin{aligned} E - \mathcal{A}(X_k) &= E - \mathcal{A}(X_0) - \sum_{i=1}^k y_i^k \omega_i \\ &= R_0 - \mathcal{A}(W_k) (y_k \otimes I_p) \\ &= \|R_0\|_F \omega_1 - \mathcal{A}(W_k) (y_k \otimes I_p) \\ &= \|R_0\|_F W_{k+1} (e_1 \otimes I_p) - W_{k+1} (\tilde{H}_k \otimes I_p) (y_k \otimes I_p) \\ &= \|R_0\|_F W_{k+1} (e_1 \otimes I_p) - W_{k+1} (\tilde{H}_k y_k \otimes I_p) \\ &= W_{k+1} (\|R_0\|_F (e_1 \otimes I_p) - (\tilde{H}_k y_k \otimes I_p)) \\ &= W_{k+1} (\|R_0\|_F e_1 - \tilde{H}_k y_k) \otimes I_p \end{aligned}$$

$$\begin{aligned} \Rightarrow \|E - \mathcal{A}(X_k)\|_F &= \left\| W_{k+1} (\|R_0\|_F e_1 - \tilde{H}_k y_k) \otimes I_p \right\|_F \\ &= \|W_{k+1}\|_F \left\| (\|R_0\|_F e_1 - \tilde{H}_k y_k) \otimes I_p \right\|_F \\ &= \left\| (\|R_0\|_F e_1 - \tilde{H}_k y_k) \otimes I_p \right\|_F, \text{ (Because } W_{k+1} \text{ is a unitary matrix)} \end{aligned}$$

On the other hand, we have: for any vector  $V$ ;  $\|V \otimes I_p\|_F = \sqrt{p} \|V\|_2$ , where  $I_p$  is the  $p \times p$  identity matrix.

$$\begin{aligned} \Rightarrow \|E - \mathcal{A}(X_k)\|_F &= \sqrt{p} \left\| \|R_0\|_F e_1 - \tilde{H}_k y_k \right\|_2 \\ \Rightarrow \min_{X \in X_0 + K_k(\mathcal{A}, R_0)} \|E - \mathcal{A}(X_k)\|_F &= \min_{y \in \mathbb{R}^k} \left\| \|R_0\|_F e_1 - \tilde{H}_k y \right\|_2 \end{aligned}$$

□

**Remark 2.** The problem (3.4) is equivalent to the following problem

$$\widetilde{R}'_k y = \|R_0\|_F \widetilde{Q}'_k^T e_1 \quad (3.5)$$

Where  $\tilde{H}_k = \tilde{Q}_k \tilde{R}_k$  (represents the QR factorization of  $\tilde{H}_k$ .  $\tilde{R}'_k$  is the square matrix obtained from  $\tilde{R}_k$  by removing its last row, while  $\tilde{Q}'_k$  is similarly obtained from  $\tilde{Q}_k$  by removing its last row). For more details, see [1].

**Theorem 3.** Let  $R_k$  be a residual obtained by the global-GMRES method at step  $k$  for solving generalized Sylvester matrix equation (2.4), then  $R_k$  satisfies the following properties:

$$\begin{cases} R_k = \alpha_{k+1} W_{k+1} (\tilde{Q}_k e_{k+1} \otimes I_p) \\ \text{and} \\ \|R_k\|_F = \sqrt{p} |\alpha_{k+1}| \end{cases} \quad (3.6)$$

where  $\alpha_{k+1}$  is the last component of the vector  $\|R_0\|_F \tilde{Q}_k^T e_1$ , and  $e_{k+1}^T = (0, 0, \dots, 0, 1) \in \mathbb{R}^{k+1}$ .

*Proof.*

1. We have:  $R_k = E - \mathcal{A}(X_k) / X_k = X_0 + W_k (y_k \otimes I_p)$

$$\implies R_k = E - \mathcal{A}(X_0) - \mathcal{A}W_k (y_k \otimes I_p)$$

According to Remark 1, we have:  $\mathcal{A}W_k = W_{k+1} (\tilde{H}_k \otimes I_p)$

$$\begin{aligned} \implies R_k &= \|R_0\|_F W_{k+1} (e_1 \otimes I_p) - W_{k+1} (\tilde{H}_k \otimes I_p) (y_k \otimes I_p) \\ &= W_{k+1} \left[ (\|R_0\|_F e_1 - \tilde{H}_k y_k) \otimes I_p \right] \end{aligned}$$

Using  $QR$  factorization of the matrix  $\tilde{H}_k$ , we obtain:

$$R_k = W_{k+1} (\tilde{Q}_k \otimes I_p) \left[ (\|R_0\|_F \tilde{Q}_k^T e_1 - \tilde{R}_k y_k) \otimes I_p \right]$$

Let  $V_k = \|R_0\|_F \tilde{Q}_k^T e_1 = (\alpha_1, \alpha_2, \dots, \alpha_{k+1})^T$

Such that:  $(\alpha_1, \alpha_2, \dots, \alpha_{k+1})^T$  is the first column of  $\|R_0\|_F \tilde{Q}_k^T$ . On the other hand we have,  $y_k$  is the solution of the system:  $\tilde{R}_k y_k = \|R_0\|_F \tilde{Q}_k^T e_1 = (\alpha_1, \alpha_2, \dots, \alpha_k)^T$ .

So,  $\tilde{R}_k y_k = (\alpha_1, \alpha_2, \dots, \alpha_k, 0)^T$

$$\begin{aligned} \implies R_k &= W_{k+1} (\tilde{Q}_k \otimes I_p) \left[ ((\alpha_1, \alpha_2, \dots, \alpha_{k+1})^T - (\alpha_1, \alpha_2, \dots, \alpha_k, 0)^T) \otimes I_p \right] \\ &= W_{k+1} (\tilde{Q}_k \otimes I_p) [\alpha_{k+1} e_{k+1} \otimes I_p] / e_{k+1} = (0, 0, \dots, 1)^T \in \mathbb{R}^{k+1} \end{aligned}$$

finally,

$$R_k = \alpha_{k+1} W_{k+1} (\tilde{Q}_k e_{k+1} \otimes I_p)$$

2. we have  $R_k = \alpha_{k+1} W_{k+1} (\tilde{Q}_k e_{k+1} \otimes I_p)$  then:

$$\begin{aligned} \|R_k\|_F^2 &= \text{Tr} \left( \alpha_{k+1} W_{k+1} (\tilde{Q}_k e_{k+1} \otimes I_p) \right)^T \left( \alpha_{k+1} W_{k+1} (\tilde{Q}_k e_{k+1} \otimes I_p) \right) \\ &= \alpha_{k+1}^2 \text{Tr} \left( e_{k+1}^T \tilde{Q}_k^T \tilde{Q}_k e_{k+1} \otimes I_p \right) \end{aligned}$$

on the other hand we have:  $\tilde{Q}_k^T \tilde{Q}_k = I_{k+1}$  ( $\tilde{Q}_k$  is a unitary matrix) and  $(e_{k+1}^T e_{k+1} = 1)$

$$\implies \|R_k\|_F^2 = \alpha_{k+1}^2 \text{Tr} (I_p) = \alpha_{k+1}^2 p$$

$$\implies \|R_k\|_F = \sqrt{p} |\alpha_{k+1}|$$

□

---

**Algorithm 2** Global-GMRES for generalized Sylvester matrix equation
 

---

- 1: Choose  $X_0$ , a tolerance  $\varepsilon$ ,  $m$  is the Krylov subspace dimension and set  $k = 0$ .
  - 2: Compute  $R_0 = E - (MX_0N + SX_0T)$ , set  $\beta = \|R_0\|_F$  and  $\omega_1 = R_0/\beta$ .
  - 3: **if**  $\beta < \varepsilon$  **then**
  - 4:     **Stop**
  - 5: **else**
  - 6:     Set  $X_0 = X_k$
  - 7:      $k = k + 1$
  - 8:     Compute  $W_k^{(m+1)}, W_k^{(m)}$  and  $\tilde{H}_k^m$  applying Algorithm 1 to the pair  $(\mathcal{A}, \omega_1)$ .
  - 9:     Perform the QR factorization:  $\tilde{H}_k^m = \tilde{Q}_k^{(m)} \tilde{R}_k^{(m)}$
  - 10:    Determine  $y_k$  as the solution of (3.5), and compute  $X_k = X_0 + W_k^{(m)}(y_k \otimes I_p)$
  - 11:    Using (3.6), compute  $R_k$  and  $\beta = \|R_k\|_F$
  - 12:    Set  $\omega_1 = R_k/\beta$
  - 13:    Go to 3.
  - 14: **end if**
- 

#### 4. PROPOSED METHOD

The deblurring problem involves estimating  $\hat{x}$ , which is the solution of the following minimization problem:

$$\min_{x \in \mathbb{R}^{np}} \|Ax - b\|$$

Here,  $A$  represents the blurring matrix,  $b = \text{Vect}(B)$  such that:  $B$  is the blurred image corrupted by a Gaussian noise and  $x = \text{Vect}(X)$  where  $X$  is the deblurred image. Applying Tikhonov regularization technique we obtain the penalized problem in the following form:

$$\min_{x \in \mathbb{R}^{np}} (\|Ax - b\|_2^2 + \lambda^2 \|Lx\|_2^2)$$

In our case, the regularization matrix  $L$  is chosen to be the identity matrix  $I_{np}$ . Based on Proposition 1 in Section 2, we obtain an equivalent formulation of the penalized problem in the form of a Generalized Sylvester matrix equation which is defined as follows:

$$A_2^T A_2 X A_1^T A_1 + \lambda^2 X = A_2^T B A_1 / A = A_1 \otimes A_2 \quad (4.1)$$

**4.1. Parameter selection.** As shown in various research studies on applied mathematics, which are based on the Tikhonov regularization method, the selection of the regularization parameter  $\lambda$  is determined through the minimization of the well-known GCV function [20–22] as follows:

$$GCV(\lambda) = \frac{\|Ax_\lambda - b\|_2^2}{[\text{tr}(I - AA_\lambda^{-1}A^T)]^2} = \frac{\|(I - AA_\lambda^{-1}A^T)b\|_2^2}{[\text{tr}(I - AA_\lambda^{-1}A^T)]^2} \quad (4.2)$$

Where  $A_\lambda = A^T A + \lambda^2 L^T L$  ( $A_\lambda$  is an invertible matrix because it is positive definite) and  $x_\lambda$  is the solution of the system  $A_\lambda x = A^T b$ .

**Theorem 4.** If we assume the Generalized Singular Value Decomposition (GSVD) of the matrix pair  $(\tilde{H}_k, L)$  in the following form:

$$\begin{cases} \tilde{H}_k = U \Sigma_{\tilde{H}_k} Z^{-1} \\ L = V \Sigma_L Z^{-1} \end{cases}$$

Where  $U, V$  are two unitary matrices,  $\Sigma_{\tilde{H}_k} = \begin{bmatrix} D_{\tilde{H}_k} \\ 0_{\mathbb{R}^k}^T \end{bmatrix}$ ,  $\Sigma_L = \begin{bmatrix} D_L \\ 0_{\mathbb{R}^k}^T \end{bmatrix}$  and  $Z \in \mathbb{R}^{k \times k}$  is an invertible matrix. (Note that  $D_{\tilde{H}_k} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_k)$  and  $D_L = \text{diag}(l_1, l_2, \dots, l_k)$ ), then the  $GCV(\lambda)$  function can be expressed by the following formula:

$$GCV(\lambda) = \frac{\sum_{i=1}^k \left( \frac{\tilde{b}_i l_i^2}{\sigma_i^2 + \lambda^2 l_i^2} \right)^2}{\left[ \sum_{i=1}^k \left( \frac{l_i^2}{\sigma_i^2 + \lambda^2 l_i^2} \right) \right]^2} / \quad \tilde{b} = U^T b \quad (4.3)$$

For more details see [?]. In our case  $L = I_{np}$  then the GCV formula can be expressed as follows:

$$GCV(\lambda) = \frac{\sum_{i=1}^k \left( \frac{\tilde{b}_i}{\sigma_i^2 + \lambda^2} \right)^2}{\left[ \sum_{i=1}^k \left( \frac{1}{\sigma_i^2 + \lambda^2} \right) \right]^2} / \quad \tilde{b} = U^T b \quad (4.4)$$

**4.2. Proposed algorithm.** In this subsection, we will detail the steps involved in our proposed deblurring method. In the field of image processing, the quality of the restored image is widely regarded as a crucial indicator of a restoration technique's effectiveness. Consequently, our method is built around a trade-off between selecting the Krylov subspace dimension to reduce computational cost and maintaining the quality of the reconstructed image, as measured by the PSNR which is defined by: The following algorithm outlines the different steps of our approach.

**Algorithm 3** Proposed deblurring method

- 
- 1: Set  $m = 2$  is the Krylov subspace dimension and  $PSNR_{m-1} = P_0$  a chosen value
  - 2: Choose  $X_0$ , a tolerance  $\varepsilon$ ,  $\lambda > 0$  and set  $k = 0$ .
  - 3: Compute  $R_0 = A_2^T B A_1 - A_2^T A_2 X_0 A_1^T A_1$ , set  $\beta = \|R_0\|_F$  and  $\omega_1 = \frac{R_0}{\beta}$
  - 4: **if**  $\beta < \varepsilon$  **then**
  - 5:     Stop
  - 6: **else**
  - 7:      $k = k + 1$
  - 8:     Compute  $W_k^{(m+1)}$ ,  $W_k^{(m)}$  and  $\tilde{H}_k^m$  applying Algorithm 1 to the pair  $(\mathcal{A}, \omega_1)$ .
  - 9:     Perform the QR factorization:  $\tilde{H}_k^m = \tilde{Q}_k^{(m)} \tilde{R}_k^{(m)}$  and the (SVD) of the matrix  $\tilde{H}_k^m$
  - 10:     Find  $\lambda_{opt}$  as a solution minimizing the GCV function in (4.4).
  - 11:     Determine  $y_k$  as the solution of (3.5), and compute  $X_k = X_0 + W_k^{(m)} (y_k \otimes I_p)$
  - 12:     Compute  $R_k = A_2^T B A_1 - A_2^T A_2 X_k A_1^T A_1 - \lambda_{opt}^2 X_k$ , set  $\beta = \|R_k\|_F$  and  $\omega_1 = \frac{R_k}{\beta}$
  - 13:     Go to 4
  - 14: **end if**
  - 15: Compute  $PSNR_m$
  - 16: **if**  $|PSNR_m - PSNR_{m-1}| < \epsilon$  **then**
  - 17:     Stop
  - 18: **else**
  - 19:      $m = m + 1$  and  $X_0 = X_k$
  - 20:     Go to 3
  - 21: **end if**
- 

## 5. EXPERIMENTAL RESULTS

This section presents some numerical examples concerning the restoration of blurred images. Deblurring images is regarded as a highly significant field in image processing, where the resolution of linear discrete ill-posed problems (1.1) is frequently encountered. In our case, the linear discrete ill-posed problem (1.1) is considered in the following form: We denote by  $X_{\text{org}}$  the original image, and  $x = \text{Vect}(X_{\text{org}})$ , where  $X_{\text{org}}$  is a  $256 \times 256$  matrix of integer values in the range  $[0, 255]$ . The vector  $b = Ax + \eta$  represents the noisy and blurred image, where  $\eta$  denotes the noise vector, whose components distribution follows the normal distribution  $\mathcal{N}(0, \sigma^2)$ , where  $\sigma$  is chosen to be a small positive value.  $A$  is considered a blurring matrix, which is defined in the following manner:  $A = A_1 \otimes A_2 \in \mathbb{R}^{256^2 \times 256^2}$ ,

where  $A_1 = A_2 = (a_{ij})$  is the Toeplitz matrix defined by:

$$a_{ij} = \begin{cases} \frac{1}{\sigma' \sqrt{2\pi}} \exp\left(-\frac{(i-j)^2}{\sigma'^2}\right), & |i-j| \leq r \\ 0 & \text{otherwise} \end{cases} \quad \text{or} \quad a_{ij} = \begin{cases} \frac{1}{2r-1}, & |i-j| \leq r \\ 0 & \text{otherwise} \end{cases}$$

In our case, we use the PSNR to evaluate the quality of the restored image, which is defined by:

$$PSNR = 10 \log_{10} \left( \frac{255^2}{MSE} \right) \quad (5.1)$$

Following the simulation of our algorithm, we obtained the results presented below:

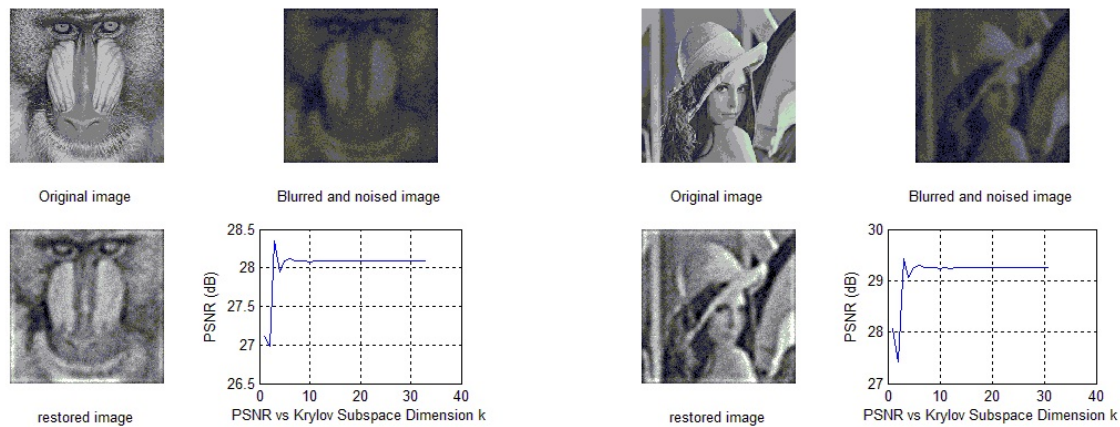


FIGURE 1. Mandril image(left) and Lena image (right)

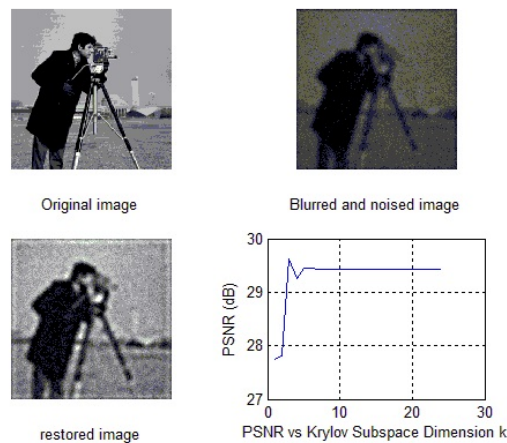


FIGURE 2. cameraman image

| Image                                           | Blurred and Noised image PSNR | Restored image PSNR | Krylov Subspace Dimension |
|-------------------------------------------------|-------------------------------|---------------------|---------------------------|
| Mandril(Gaussian blur $\sigma = 5, r = 35$ )    | 12,3425 dB                    | 28,3472 dB          | k=3                       |
| Lena (uniform blur $r = 35$ )                   | 12,6431 dB                    | 29,4211 dB          | k=3                       |
| Cameraman (Gaussian blur $\sigma = 5, r = 35$ ) | 12,8645 dB                    | 29,6120 dB          | k=3                       |

TABLE 1. Exemple simple de tableau

Table 1 presents the PSNR values for three standard test images (Mandril, Lena, and Cameraman), both before and after the restoration process, along with the dimension of the Krylov subspace used.

- The initial values of PSNR (for blurred and noisy images) are relatively low, around 12-13 dB, reflecting a high degree of image degradation due to blur and noise.
- After restoration, the PSNR increases significantly, reaching values of between 28 and 30 dB, indicating a marked improvement in image quality.
- During our simulations, we observed that a Krylov subspace dimension of  $k=3$  is optimal for all images, indicating that even a reduced dimension can be sufficient to achieve good restoration quality.

## 6. CONCLUSION

In this paper, we propose an improved iterative method for solving large-scale discrete ill-posed linear problems, by adopting an optimal basis selection strategy within the Krylov subspace framework, based on Arnoldi's procedure. The proposed approach has been successfully applied to image deblurring tasks, demonstrating its practical effectiveness. The experimental results confirm the method's ability to improve the visual quality of the restored images while significantly reducing execution time thanks to the optimized reduction of the Krylov subspace basis dimension. These interesting results indicate that the proposed technique could be a valuable tool for other inverse problems in the field of image processing and related areas.

## AUTHORS' CONTRIBUTIONS

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

## CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## REFERENCES

- [1] A. Bouhamidi, K. Jbilou, Sylvester Tikhonov-Regularization Methods in Image Restoration, *J. Comput. Appl. Math.* 206 (2007), 86–98. <https://doi.org/10.1016/j.cam.2006.05.028>.
- [2] B.N. Datta, M. Heyouni, K. Jbilou, The Global Arnoldi Process for Solving the Sylvester-Observer Equation, *Comput. Appl. Math.* 29 (2010), 527–544. <https://doi.org/10.1590/s1807-03022010000300012>.
- [3] P.C. Hansen, Rank-Deficient and Discrete Ill-Posed Problems: Numerical Aspects of Linear Inversion, SIAM, 1998. <https://doi.org/10.1137/1.9780898719697>.
- [4] N.K. Chada, A.M. Stuart, X.T. Tong, Tikhonov Regularization within Ensemble Kalman Inversion, *SIAM J. Numer. Anal.* 58 (2020), 1263–1294. <https://doi.org/10.1137/19m1242331>.
- [5] A. Bouhamidi, L. Elbouyahyaoui, M. Heyouni, The Constant Solution Method for Solving Large-Scale Differential Sylvester Matrix Equations with Time Invariant Coefficients, *Numer. Algorithms* 96 (2024), 449–488. <https://doi.org/10.1007/s11075-023-01653-3>.
- [6] F. Panjeh Ali Beik, Theoretical Results on the Global GMRES Method for Solving Generalized Sylvester Matrix Equations, *Bull. Iran. Math. Soc.* 40 (2014), 1097–1117.
- [7] A. Buccini, M. Pasha, L. Reichel, Generalized Singular Value Decomposition with Iterated Tikhonov Regularization, *J. Comput. Appl. Math.* 373 (2020), 112276. <https://doi.org/10.1016/j.cam.2019.05.024>.
- [8] H. Xiang, J. Zou, Regularization with Randomized SVD for Large-Scale Discrete Inverse Problems, *Inverse Probl.* 29 (2013), 085008. <https://doi.org/10.1088/0266-5611/29/8/085008>.
- [9] R.I. Boş, E.R. Csetnek, S.C. László, Tikhonov Regularization of a Second Order Dynamical System with Hessian Driven Damping, *Math. Program.* 189 (2021), 151–186. <https://doi.org/10.1007/s10107-020-01528-8>.
- [10] P. Novati, M.R. Russo, A GCV Based Arnoldi-Tikhonov Regularization Method, *BIT Numer. Math.* 54 (2014), 501–521. <https://doi.org/10.1007/s10543-013-0447-z>.
- [11] M. El Guide, A. El Ichi, K. Jbilou, R. Sadaka, Tensor Krylov Subspace Methods via the T-Product for Color Image Processing, *arXiv*, (2020). <https://doi.org/10.48550/arXiv.2006.07133>.
- [12] E. Ringh, G. Mele, J. Karlsson, E. Jarlebring, Sylvester-Based Preconditioning for the Waveguide Eigenvalue Problem, *Linear Algebra Appl.* 542 (2018), 441–463. <https://doi.org/10.1016/j.laa.2017.06.027>.
- [13] M.A. Hamadi, K. Jbilou, A. Ratnani, A Data-Driven Krylov Model Order Reduction for Large-Scale Dynamical Systems, *J. Sci. Comput.* 95 (2023), 2. <https://doi.org/10.1007/s10915-023-02122-8>.
- [14] K.-W.E. Chu, The Solution of the Matrix Equations  $AXB - CXD = E$  and  $(YA - DZ, YC - BZ) = (E, F)$ , *Linear Algebra Appl.* 93 (1987), 93–105. [https://doi.org/10.1016/s0024-3795\(87\)90314-4](https://doi.org/10.1016/s0024-3795(87)90314-4).
- [15] G.H. Golub, M. Heath, G. Wahba, Generalized Cross-Validation as a Method for Choosing a Good Ridge Parameter, *Technometrics* 21 (1979), 215–223. <https://doi.org/10.1080/00401706.1979.10489751>.
- [16] D. Yu, C. Hwang, H. Zhu, S. Ge, The Tikhonov-L-Curve Regularization Method for Determining the Best Geoid Gradients from SWOT Altimetry, *J. Geod.* 97 (2023), 93. <https://doi.org/10.1007/s00190-023-01783-5>.
- [17] P.C. Hansen, The L-Curve and Its Use in the Numerical Treatment of Inverse Problems, in: P. Johnston (Ed.), *Computational Inverse Problems in Electrocardiology*, WIT Press, pp. 119–142, (2001).
- [18] Y. Saad, M.H. Schultz, GMRES: A Generalized Minimal Residual Algorithm for Solving Nonsymmetric Linear Systems, *SIAM J. Sci. Stat. Comput.* 7 (1986), 856–869. <https://doi.org/10.1137/0907058>.
- [19] E. Tohidi, F. Toutounian, Numerical Solution of Time-Dependent Diffusion Equations with Nonlocal Boundary Conditions via a Fast Matrix Approach, *J. Egypt. Math. Soc.* 24 (2016), 86–91. <https://doi.org/10.1016/j.joems.2014.06.018>.

- [20] P. Fika, Approximation of the Tikhonov Regularization Parameter through Aitken's Extrapolation, *Appl. Numer. Math.* 190 (2023), 270–282. <https://doi.org/10.1016/j.apnum.2023.04.008>.
- [21] J.R. Winkler, M. Mitrouli, C. Koukouvinos, The Application of Regularisation to Variable Selection in Statistical Modelling, *J. Comput. Appl. Math.* 404 (2022), 113884. <https://doi.org/10.1016/j.cam.2021.113884>.
- [22] H. Yanagihara, Explicit Solution to the Minimization Problem of Generalized Cross-Validation Criterion for Selecting Ridge Parameters in Generalized Ridge Regression, *Hiroshima Math. J.* 48 (2018), 203–222. <https://doi.org/10.32917/hmj/1533088835>.