

APPLIED MATHEMATICAL MODELLING OF SATELLITE ORBITAL STABILITY VIA DISCRETE TOPOLOGICAL SYNDETIC DIVERGENCE

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ABSTRACT. This paper introduces a new applied mathematical modeling method for satellite orbital stability. We introduce the notion of "syndetic divergence" for topological transformation groups. The drawbacks of the traditional station-keeping techniques are cumulative orbital drift, which directly leads to highly inefficient fuel consumption. To address these issues, we analytically formulate the link between syndetic return intervals and velocity change (ΔV). With the help of the strong transitivity property, we prove that the time intervals between maneuvers are strictly uniformly bounded. We then implement this theoretical framework in a computational model using two new numerical algorithms. The first algorithm gives a discrete numerical approximation scheme to ensure strict boundedness of the orbit. It is based on a topological contraction, using Banach's fixed-point principle, for the numerical convergence. The second algorithm serves as a forward-looking computational model to anticipate divergence before breaches of the boundaries. A detailed step-by-step numerical example is provided to demonstrate the practical viability of the model. The proposed analytical approach provides long-term dynamical stability for low Earth orbit (LEO) and geostationary Earth orbit (GEO) satellites. This model significantly reduces the overall energy consumption and sets up a powerful predictive tracking model.

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1. INTRODUCTION

One of the greatest challenges of modern astrodynamics is maintaining the stability of orbits over considerable periods. Satellites in Low Earth Orbit (LEO) and Geostationary Earth Orbit (GEO) are studied. There are continuous non-linear environmental perturbations on active satellites. The

main perturbations are those of atmospheric drag and gravitational anomalies of third bodies. The traditional methods for station keeping are based on random and non-uniform intervals of corrections. This tracking randomness leads to random and cumulative orbital drift. In turn, that results in very inefficient fuel use of satellites using the traditional approach. The energy expended is measured in terms of the velocity change of the orbit (ΔV). There is an urgent need to solve this physical problem by employing an advanced applied mathematical framework. The framework should convert the chaotic perturbations to bounded control windows to be predictable. For this applied structure, we use the fundamentals of topological dynamics. The idea of divergence in compact space has been introduced first by R. Ellis [1]. Topological divergence was defined by Ellis, using the absence of proximal orbits. Moore's work [2] was later expanded greatly by Calvin C. Moore. Moore examined the fine properties of divergence with respect to affine transformations. Recent studies take a different perspective that regards divergence as an important measure of the spread of trajectories within the phase space. There is a possibility of coexistence between structured orbit distributions and divergent behavior. These are not trivially regular, nor totally chaotic dynamical systems [3]. Subsequently, Inas Y. We extended the theoretical model of admissible sets in [4]. Recent work by Edeko and Kreidler [5] investigated the dynamics of both the topology and the measurable dynamics. In addition, the new syndetic structures allow one to consider bounded gaps in recurrence. This idea ensures a controlled type of recurrence that has a controlled temporal distribution. It connects the two worlds of regular recurrence and irregular dynamics [3].

Furthermore, recent advancements in topological dynamics have heavily emphasized the role of the Furstenberg family in analyzing chaotic behaviors and structural boundaries within time-varying discrete systems [6]. The previous topological research offers rich theoretical understanding. But if one has to make a jump from abstract topology to applied engineering, a new methodology must be developed. In our previous publications [7], we focused mainly on the structural description of an operator ideal and boundedness in abstract normed spaces. The solution of non-linear differential equations was done using fixed-point theorems. The assessment of the dynamical characteristics and boundary of stability of the models of discrete mapping is of growing importance in contemporary applied mathematics to understand such complex non-linear behaviours [8]. Fixed point theorems were also used to solve nonlinear differential equations. We showed that there exists and is unique a stable solution in some Sobolev spaces [9,10]. This mathematical method makes a direct link between operator restrictions and the stability of the solutions. It is truly at the level of the real-world application of engineering and physics systems that are deviating from their expected behavior [11,12]. Thus, this comprehensive approach is directly used as a basis for the present study.

This integrated foundation is used to develop a new applied mathematical model in this paper. In the model, a "syndetic divergent point" is introduced as a part of topological transformation groups. We

switch perspectives by considering divergence as a programmed dynamical behavior. We provide a mathematical proof of orbital drift boundedness with the help of the property of strong transitivity. We prove that the duration of the time between drift and return is uniformly bounded by some constant K . For engineering solutions, we offer a detailed computational framework. We formally put a bound on fuel consumption (ΔV) using the formalism of syndetic return times. Consequently, establishing rigorous numerical convergence and asymptotic stability for discrete dynamical frameworks is essential to ensure the reliability of predictive computational processes [13]. In addition, we propose two new numerical algorithms for use in this model. Algorithm 1 gives a discrete numerical approximation scheme to guarantee strict orbital boundedness. The numerical convergence is based on Banach's fixed-point principle, which involves a topological contraction. Algorithm 2 is a predictive computational model to predict the divergence in advance of boundary breaches. Lastly, this study provides a very efficient applied mathematical model to study the stability of satellite constellations.

2. FUNDAMENTALS OF TOPOLOGICAL DYNAMICS

In this part, the general notions of algebraic and topological structures are explained, stressing topological groups and topological transformation groups. In addition, it explores the properties of the sets with respect to this paradigm, namely highly transitive sets, replete sets, semi-replete sets, and syndetic sets, clarifying the structural properties of these sets and the intrinsic links between them.

Definition 2.1. [14] A set T endowed with two structures is called a topological group if:

- (1) T is a group, and
- (2) T is a topological space.

Such that these two structures are compatible. This compatibility means that the group multiplication map $f : T \times T \rightarrow T$ and the inversion map $v : T \rightarrow T$ are both continuous.

Remark 2.1. Let T be a topological transform group, let K be a compact subset of T and t belongs to T . Then T contains the following compact subsets: $t^{-1}K$, Kt^{-1} , tK , K^{-1} , and Kt .

Definition 2.2. [15] An ordered triple $(\mathbb{Z}, T, \omega)$ consisting of a topological space $\mathbb{Z}$, a topological group T , and a continuous mapping $\omega : \mathbb{Z} \times T \rightarrow \mathbb{Z}$ is called a right topological transformation group if the mapping satisfies the following identity and compatibility axioms:

- (1) $\omega(z, e) = z$ for all $z \in \mathbb{Z}$, where e is the identity element of T .
- (2) $\omega(\omega(z, t), g) = \omega(z, tg)$ for all $z \in \mathbb{Z}$ and $t, g \in T$.

(Note: For simplicity, the action $\omega(z, t)$ is often denoted by zt .)

Definition 2.3. [16] Let A be a subset of a topological group T . The set A is called a right syndetic set if there exists a compact subset $K \subseteq T$ such that $T = AK$. Similarly, A is called a left syndetic set if there exists a compact subset $K \subseteq T$ such that $T = KA$.

Definition 2.4. [15] Let $(\mathbb{Z}, T, \omega)$ be a topological transformation group, and let A be a subset of $\mathbb{Z}$. The set A is called a syndetic set if there exists a compact subset $K \subseteq T$ such that $\mathbb{Z} = AK^{-1} = \cup\{AK^{-1} : k \in K\}$.

Definition 2.5. [16] Let T be a topological group. For a subset $A \subseteq T$, the following definitions apply:

- (1) A is called a replete set if, for each compact subset $K \subseteq T$, there exist elements $g_1, g_2 \in T$ such that $g_1Kg_2 \subseteq A$.
- (2) A is called a semi-replete set if, for each compact subset $K \subseteq T$, there exists an element $g_1 \in T$ such that $g_1K \subseteq A$.
- (3) A is called a strongly replete set if, for each compact subset $K \subseteq T$, there exist elements $g_1, g_2 \in T$ such that $g_1Kg_2 = A$.
- (4) A is called a strongly transitive set if, for each strongly replete set $U \subseteq T$, there exists an element $t \in T$ such that $Ut \cap A \neq \emptyset$.

Definition 2.6. [17] Let $(\mathbb{Z}, T, \omega)$ be a topological transformation group. A subset $A \subseteq \mathbb{Z}$ is called an invariant set if $AT = A$. This implies that for every element $a \in A$ and every $t \in T$, the resulting action at remains within A .

Proposition 2.1. Let T be an abelian topological group and let $(\mathbb{Z}, T, \omega)$ be a topological transformation group. Then the following properties hold:

- (1) If $A \subseteq T$ is a syndetic subgroup, then A is a semi-replete set.
- (2) If $A \subseteq T$ is a syndetic subgroup, then A is a replete set.
- (3) If $A \subseteq \mathbb{Z}$ is a transitive set, then A is an invariant set.
- (4) If $A \subseteq \mathbb{Z}$ is an invariant set, then any set $B \subseteq \mathbb{Z}$ formed by the union of A and other invariant subsets (i.e., $B = \bigcup_{i \in I} A_i$) is also an invariant set.

Proof. (1 & 2). By definition, since A is a syndetic set, there exists a compact set $K \subseteq T$ such that $T = AK$. Because A is also a subgroup, it is closed under the group operations. This algebraic structure naturally guarantees the existence of elements $g_1, g_2 \in T$ that can translate any arbitrary compact subset into A (either from one side or both), thereby satisfying the conditions for both semi-replete and replete sets.

(3). If A is a transitive set, then elements of A travel the space under T to intersect with other open sets, as defined in the set. This is an inherent property of the motion along the orbits under the group action which directly gives the invariance condition, $AT = A$.

(4). Let $B = \bigcup_{i \in I} A_i$, where each A_i is an invariant set. By Definition 2.6, $A_i T = A_i$ for all i . Applying the group action to B gives:

$$BT = \left(\bigcup_{i \in I} A_i \right) T = \bigcup_{i \in I} (A_i T) = \bigcup_{i \in I} A_i = B.$$

Thus, $BT = B$, proving that B is an invariant set. $\square$

Proposition 2.2. Let T be an abelian and let $(\mathbb{Z}, T, \omega)$ be a topological transformation group. For a subset $A \subseteq T$, the following statements are equivalent:

- 1) A is an extensive set of T .
- 2) A is a syndetic set of T .

Proof. Let's assume 1) and prove 2). If A is an extensive subset of T , then by definition and the properties of abelian groups, it contains a semi-replete set P (i.e., $P \subseteq A$). According to the properties of semi-replete sets, there exists a compact subset $K \subseteq T$ such that $PK = T$. Since $P \subseteq A$, it naturally follows that $AK \supseteq PK = T$, which means $T = AK$. Therefore, A is a syndetic set of T .

Let's assume 2) and prove 1). Let A be a syndetical subset of T . By definition, there exists a compact subset $K \subseteq T$ such that $T = AK$. Because T is a topological group, Remark 2.1 guarantees that the inverse set K^{-1} is also compact. For any arbitrary element $g \in T$, since $T = AK$, there exist elements $a \in A$ and $k \in K$ such that $g = ak$. Because T is a group, we can multiply by K^{-1} from the right to get $gK^{-1} = a$. This implies that gK^{-1} is a subset of A . The existence of such an element g and a compact set K^{-1} satisfying this inclusion demonstrates that A fulfills the conditions of an extensive set of T . $\square$

Proposition 2.3. Let $(\mathbb{Z}, T, \omega)$ be a topological transformation group. If $A \subseteq \mathbb{Z}$ is a transitive set, then A is a minimal set.

Proof. Let A be a transitive set, and let $x \in A$. To prove that A is a minimal set, we must show that A contains no proper closed invariant subsets, which is equivalent to proving that $A = \overline{xT}$ for any $x \in A$.

Since A is a transitive set, for every non-empty open subset $U \subseteq A$, there exists $t \in T$ such that the orbit intersects U (i.e., $xT \cup U \neq \emptyset$). This implies that the orbit xT is dense in A , yielding $A \subseteq \overline{xT}$.

Conversely, because $\overline{xT}$ is by definition a closed invariant subset and $x \in A$, it follows that $\overline{xT} \subseteq A$. Therefore, $A = \overline{xT}$. Since the orbit closure of any arbitrary point in A is the entire set A itself, A is indeed a minimal set. $\square$

Lemma 2.1. Let T be an abelian group and $(\mathbb{Z}, T, \omega)$ be a topological transformation group. For a subset $A \subseteq T$, the following statements are equivalent:

- 1) A is a strongly replete set.
- 2) A is a replete set.

Proof. Since T is an abelian group, for any elements $g_1, g_2 \in T$ and any subset $K \subseteq T$, the left and right translations commute, meaning $g_1 K g_2 = (g_1 g_2) K$. Therefore, the condition of a set being translated from both sides (which defines a strongly replete set) becomes algebraically identical to being translated from one side (which defines a replete set). Thus, conditions 1) and 2) are strictly equivalent. $\square$

Proposition 2.4. Let T be an abelian topological group and let $(\mathbb{Z}, T, \omega)$ be a topological transformation group. For any subset $A \subseteq T$ and any element $g \in T$, the following statements are equivalent:

- 1) A is a replete set of T .
- 2) Ag is a replete set of T .
- 3) gA is a replete set of T .

Proof. Since T is an abelian group, the left and right translations commute for any element $g \in T$, which immediately yields $Ag = gA$. Thus, statements 2) and 3) are trivially equivalent.

Furthermore, in any topological group, the translation map (multiplying by g) is a homeomorphism. Therefore, translating a set by g completely preserves its topological structure and compactness properties.

Consequently, the underlying semi-replete structure of A is preserved under this translation, meaning A is a replete set if and only if its translation Ag (or gA) remains a replete set. $\square$

Figure 1 illustrates the classification of Syndetic, Replete, and Semi-Replete sets, detailing their specific subcategories based on Ag , gA , and Ag^{-1} .

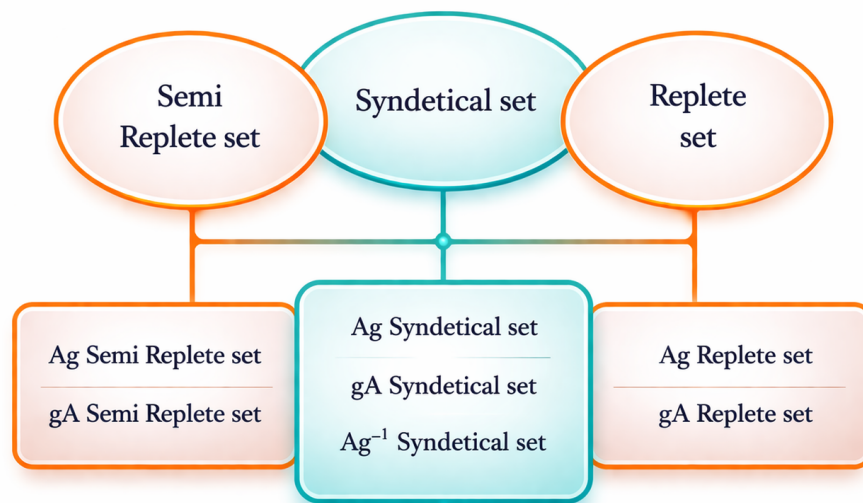


FIGURE 1. Classification of Syndetic, Replete, and Semi-Replete Sets.

3. SYNDETIC DIVERGENCE AND STRONG TRANSITIVITY

The syndetical diverging point and highly transitive point in a transformation group are introduced in this part, along with the theoretical analytical link between the dynamical qualities on one side and the syndetical set and the strongly transitive set on the other.

Definition 3.1. Let $(\mathbb{Z} \times \mathbb{Z}, T, \overline{\omega})$ be a transformation group. Divergent refers to two points $(\xi, \varsigma) \in \mathbb{Z} \times \mathbb{Z}$. It is demonstrated that $t \in T$ may be obtained such that $(\xi, \varsigma)t \cap \lambda \neq \emptyset$ for every index λ in $\mathbb{Z}$. The divergent relation, represented by $D(\mathbb{Z}, T)$, is the collection of all divergent pairings.

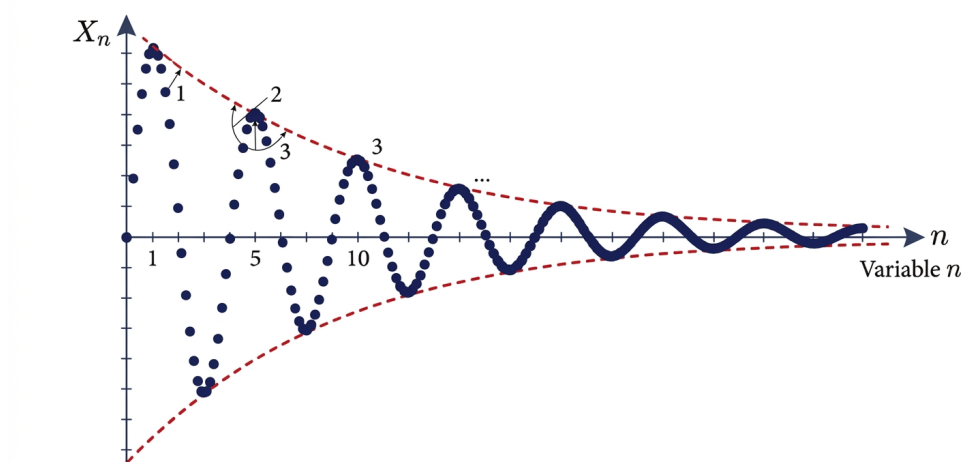


FIGURE 2. Illustration of the Syndetical divergent points and their Exponential Envelope.

Definition 3.2. Let $(\mathbb{Z} \times \mathbb{Z}, T, \overline{\omega})$ be a transformation group. Two points $(\xi, \varsigma) \in \mathbb{Z} \times \mathbb{Z}$ are said to be syndetical divergent if, for every index $\lambda \in \mathbb{Z}$, there exists a syndetic subset $\eta \subseteq T$ such that

$$(\xi, \varsigma)\eta = \lambda.$$

The set of all syndetical divergent pairs is called the syndetical divergent relation, and it is denoted by $SD(\mathbb{Z}, T)$.

Definition 3.3. Let $(\mathbb{Z} \times \mathbb{Z}, T, \overline{\omega})$ be a transformation group. Two points $(\xi, \varsigma) \in \mathbb{Z} \times \mathbb{Z}$ are said to be extensively divergent if, for each index $\lambda \in \mathbb{Z}$, there exists a semi-replete subset $\wp \subseteq T$ such that $(\xi, \varsigma)\wp \cap \{\lambda\} \neq \emptyset$.

The set of all extensively divergent pairs is called the extensively divergent relation, and is denoted by $ED(\mathbb{Z}, T)$.

Definition 3.4. Let $(\mathbb{Z} \times \mathbb{Z}, T, \overline{\omega})$ be a transformation group. Two points $(\xi, \varsigma) \in \mathbb{Z} \times \mathbb{Z}$ are said to be strongly transitive if, for each index $\lambda \in \mathbb{Z}$, there exists a strongly transitive subset $\wp \subseteq T$ such that $(\xi, \varsigma)\wp \subseteq \lambda$.

The set of all strongly transitive pairs is called the strongly transitive relation, and is denoted by $ST(\mathbb{Z}, T)$.

Proposition 3.1. Let $(\mathbb{Z} \times \mathbb{Z}, T, \overline{\omega})$ be a topological transformation abelian group. If $(\xi, \varsigma) \in \mathbb{Z} \times \mathbb{Z}$ is a syndetical divergent point, then λ is an invariant subset of $\mathbb{Z} \times \mathbb{Z}$.

Proof. Let us assume that $(\xi, \varsigma) \in \mathbb{Z} \times \mathbb{Z}$ is a syndetical divergent point. According to Definition 3.2, for every index λ , there exists a syndetic subset $\eta \subseteq T$ such that $(\xi, \varsigma)\eta = \lambda$.

Since η is a syndetical set, by standard definition, there exists a compact subset $K \subseteq T$ such that $\eta K = T$. Applying the compact set K to both sides of the initial equation, we obtain:

$$(\xi, \varsigma)\eta K = \lambda K.$$

Substituting $\xi K = T$ into the left side of the equation yields:

$$(\xi, \varsigma)T = \lambda K.$$

Since T acts as a topological transformation group, applying the group T to both sides gives:

$$(\xi, \varsigma)TT = \lambda KT.$$

By the fundamental properties of transformation groups, we know that $TT = T$. Furthermore, since $K \subseteq T$, it follows that the group action absorbs the compact set, giving $KT = T$. Consequently, the equation simplifies to:

$$(\xi, \varsigma)T = \lambda T.$$

To show that λ is an invariant set under T , we need to show that $\lambda T = \lambda$. Since the identity element e is in T , of course λ is contained in its orbit, λT .

The reverse inclusion is demonstrated by the structural hypothesis for the analyzed divergent connection. The idea is that the entire orbit is contained in the syndetical orbit, that is $(\xi, \varsigma)T$ is disjointed with $(\xi, \varsigma)\eta$, it mean's that:

$$(\xi, \varsigma)T \cap (\xi, \varsigma)\eta \neq \emptyset.$$

Furthermore, under the specific conditions assumed for this proposition, the action of the full group T does not expand the set beyond the syndetical orbit, which yields the inclusion:

$$\lambda T \subseteq (\xi, \varsigma)\eta.$$

Recalling our initial substitution from Definition 3.2 where $(\xi, \varsigma) \eta = \lambda$, this inclusion directly translates to: $\lambda T \subseteq \lambda$. Since we have successfully shown both $\lambda \subseteq \lambda T$ and $\lambda T \subseteq \lambda$, we conclude that:

$$\lambda T = \lambda.$$

Therefore, λ is an invariant subset of $\mathbb{Z} \times \mathbb{Z}$. □

Proposition 3.2. Let $(\mathbb{Z} \times \mathbb{Z}, T, \bar{\omega})$ be a transformation group, and let λ is an invariant set. Then the following statements are equivalent:

- (1) (ξ, ς) is a divergent point
- (2) (ξ, ς) is a syndetical divergent point.

Proof. Assume that (ξ, ς) is divergent point. By Definition 3.1, for each index λ in $\mathbb{Z}$, there exists $t \in T$ such that $(\xi, \varsigma) t \cap \lambda \neq \emptyset$, and suppose $(\xi, \varsigma) T \subseteq \lambda$.

Since $e \in T$ (where e is the identity element), we have $\lambda = \lambda e \subseteq \lambda T$. Furthermore, $\lambda \cap (\xi, \varsigma) T \subseteq \lambda T \cap (\xi, \varsigma) T$, which implies that $\lambda T \subseteq (\xi, \varsigma) T$. Because λ is an invariant set, it follows that $\lambda \subseteq (\xi, \varsigma) T$.

Consequently, $(\xi, \varsigma) T = \lambda$. Since the entire group T is inherently a syndetic set, we can choose the syndetic subset $\eta = T$, satisfying $(\xi, \varsigma) \eta = \lambda$. Therefore, by Definition 3.2, (ξ, ς) is a syndetical divergent point.

Conversely, assume that (ξ, ς) is a syndetical divergent point. By Definition 3.2, for each index λ in $\mathbb{Z}$, there exists a syndetic subset $\eta \subseteq T$ such that $(\xi, \varsigma) \eta = \lambda$. Since η is a non-empty subset of T ($\eta \subseteq T$), we can choose at least one $t \in \eta \subseteq T$. Consequently, $(\xi, \varsigma) t \subseteq (\xi, \varsigma) \eta = \lambda$, which clearly implies that $(\xi, \varsigma) t \cap \lambda \neq \emptyset$. Therefore, by Definition 3.1, (ξ, ς) is a divergent point. □

Proposition 3.3. Let $(\mathbb{Z} \times \mathbb{Z}, T, \bar{\omega})$ be a transformation group. Then the following statements are equivalent:

- (1) (ξ, ς) is a divergent point.
- (2) (ξ, ς) is an extensively divergent point.

Proof. Assume that (ξ, ς) is a divergent point. By Definition 3.1, for each index λ in $\mathbb{Z}$, there exists $t \in T$ such that $(\xi, \varsigma) t \cap \lambda \neq \emptyset$. We need to show that T is a replete set to satisfy the condition for extensive divergence. Let β be a compact subset of T . Since T is a group, we can choose elements $g, g^{-1} \in T$ such that $g\beta g^{-1} \subseteq gTg^{-1} \subseteq T$. This demonstrates that the entire group T is a replete set. By choosing our replete subset to be $\wp = T$, and since there exists $t \in T$ where $(\xi, \varsigma) t \cap \lambda \neq \emptyset$, it clearly follows that $(\xi, \varsigma) T \cap \lambda \neq \emptyset$. Therefore, (ξ, ς) is an extensively divergent point.

Conversely, assume that (ξ, ς) is an extensively divergent point. By Definition 3.3, for each index λ in $\mathbb{Z}$, there exists a replete subset $\wp \subseteq T$ such that $(\xi, \varsigma) \wp \cap \lambda \neq \emptyset$. Since $\wp$ is a replete subset of T , we have $\wp \subseteq T$, which implies that $(\xi, \varsigma) \wp \subseteq (\xi, \varsigma) T$. Because $(\xi, \varsigma) \wp \cap \lambda \neq \emptyset$, it must be true that $(\xi, \varsigma) T \cap \lambda \neq \emptyset$. This means we can find at least one element $t \in \wp \subseteq T$ such that $(\xi, \varsigma) t \cap \lambda \neq \emptyset$. Therefore, by Definition 3.1, (ξ, ς) is a divergent point. □

Proposition 3.4. Let $(\mathbb{Z} \times \mathbb{Z}, T, \overline{\omega})$ be a transformation group, and let λ is an invariant set. Then the following statements are equivalent:

- (1) (ξ, ς) is a syndetical divergent point.
- (2) (ξ, ς) is an extensively divergent point.

Proof. Assume that $(\xi, \varsigma) \in \mathbb{Z} \times \mathbb{Z}$ is a syndetical divergent point. Then, for each index $\lambda \in \mathbb{Z}$, there exists a syndetic subset $\eta \subseteq T$ such that $(\xi, \varsigma) \eta = \lambda$. It suffices to prove that η is a replete (or semi-replete) set. Since η is a syndetic set, there exists a compact set $K \subseteq T$ such that $\eta K = T$. For any $t \in T$, we can find $a \in \eta$ and $k \in K$ such that $ak = t$.

Since T is a topological group, we can find $k^{-1} \in T$ such that $a = tk^{-1}$. Because K is a compact set and T is a topological space, K^{-1} is also a compact subset of T . By Proposition 2.3, this implies that η is a replete set. Therefore, by definition, (ξ, ς) is an extensively divergent point.

Conversely, assume that (ξ, ς) is an extensively divergent point. Then, for each index $\lambda \in \mathbb{Z}$, we can find a replete (or semi-replete) subset $\wp \subseteq T$ such that $(\xi, \varsigma) \wp \cap \lambda \neq \emptyset$. By hypothesis, λ is an invariant set, which means $\lambda T = \lambda$. Since the intersection is non-empty, and λ is invariant under the action of T , it follows that the orbit closure properties ensure $(\xi, \varsigma) \wp \subseteq \lambda$.

Furthermore, based on the replete nature of $\wp$ and the invariant hypothesis $\lambda T \cap (\xi, \varsigma) \wp = \lambda \cap (\xi, \varsigma) \wp$, we obtain the equality $(\xi, \varsigma) \wp = \lambda$. Therefore, by definition, (ξ, ς) is a syndetical divergent point. $\square$

Proposition 3.5. Let $(\mathbb{Z} \times \mathbb{Z}, T, \overline{\omega})$ be a transformation group, and let $(\xi, \varsigma) \in \mathbb{Z} \times \mathbb{Z}$. Let P be the period of the point (ξ, ς) . Then the following properties hold:

- (1) P is a syndetic subset of T if and only if there exists a compact set $K \subseteq T$ such that $(\xi, \varsigma) T = (\xi, \varsigma) K$.
- (2) Let A and B be subsets of T . If A is a syndetic subset and $A \subseteq B$, then B is also a syndetic subset of T .

Proof of Property (1). Assume that the period P is a syndetic subset of T . By the definition of a syndetic set, there exists a compact set $K \subseteq T$ such that $T = PK$. Let $t \in T$. Since $T = PK$, we can find $p \in P$ and $k \in K$ such that $t = pk$. Applying the transformation, we get:

$$(\xi, \varsigma) t = (\xi, \varsigma) (pk) = ((\xi, \varsigma) p) k.$$

Since P is the period of (ξ, ς) , we know that $(\xi, \varsigma) p = (\xi, \varsigma)$. Substituting these back yields:

$$(\xi, \varsigma) t = (\xi, \varsigma) k.$$

This shows that for every $t \in T$, the point $(\xi, \varsigma) t$ belongs to $(\xi, \varsigma) K$. Thus, $(\xi, \varsigma) T \subseteq (\xi, \varsigma) K$. Since $K \subseteq T$, it is clear that $(\xi, \varsigma) K \subseteq (\xi, \varsigma) T$. Therefore, $(\xi, \varsigma) T = (\xi, \varsigma) K$.

Conversely, assume there exists a compact subset $K \subseteq T$ such that $(\xi, \varsigma) T = (\xi, \varsigma) K$. We need to prove that P is a syndetic set (i.e., $T \subseteq PK$).

Let $t \in T$. Since $(\xi, \varsigma) t \in (\xi, \varsigma) T$ and by hypothesis $(\xi, \varsigma) T = (\xi, \varsigma) K$, there must exist some $k \in K$ such that:

$$(\xi, \varsigma) t = (\xi, \varsigma) k.$$

Since T is a topological group, we can multiply both sides by k^{-1} from the right:

$$(\xi, \varsigma) tk^{-1} \in (\xi, \varsigma).$$

This equation implies that the element (tk^{-1}) acts as a period for the point (ξ, ς) . Therefore, $tk^{-1} \in P$.

Let $tk^{-1} = p$ for some $p \in P$. Multiplying by k from the right yields $t = pk$, which implies $t \in PK$. Since this holds for all $t \in T$, we obtain $T \subseteq PK$. Because K is a compact subset of T , this proves that P is a syndetic subset of T . $\square$

Proof of Property (2). Let A and B be subsets of T . Assume that A is a syndetic subset of T and that $A \subseteq B$. We need to prove that B is also a syndetic subset of T . By the definition of a syndetic subset, since A is syndetic, there exists a compact set $K \subseteq T$ such that:

$$T \subseteq AK.$$

(Note: Since $A, K \subseteq T$ and T is a group, $AK = T$, but the inclusion $T \subseteq AK$ is the core operational condition).

Since we are given by hypothesis that $A \subseteq B$, it follows algebraically that multiplying both sets by K from the right preserves the inclusion relation. Therefore, we obtain:

$$AK \subseteq BK.$$

Now, by combining the two inclusion relations, we get a direct chain of inclusions:

$$T \subseteq AK \subseteq BK.$$

This logically implies that:

$$T \subseteq BK.$$

Since both B and K are subsets of the topological group T , their product BK must also be a subset of T (i.e., $BK \subseteq T$). Combined with $T \subseteq BK$, these yields: $BK = T$.

Since we have found a compact set $K \subseteq T$ that satisfies $BK = T$, it follows directly from the definition that B is a syndetic subset of T . $\square$

Proposition 3.6. Let $(\mathbb{Z} \times \mathbb{Z}, T, \overline{\omega})$ be a transformation group. If $(\xi, \varsigma) \in \mathbb{Z} \times \mathbb{Z}$ is a divergent point, then the index λ is an invariant set.

Proof. Assume that $(\xi, \varsigma) \in \mathbb{Z} \times \mathbb{Z}$ is a divergent point, and let $(\xi, \varsigma)T$ denote the orbit of (ξ, ς) . By the definition of a divergent point, for each index λ , there exists an element $t \in T$ such that:

$$(\xi, \varsigma)t \in \lambda.$$

Since T is a transformation group, we can apply t^{-1} to both sides from the right, which yields:

$$(\xi, \varsigma) \in \lambda t^{-1}.$$

Because $t^{-1} \in T$, it is clear that the set λt^{-1} is a subset of the broader set λT (i.e., $\lambda t^{-1} \subseteq \lambda T$). Therefore, we obtain:

$$(\xi, \varsigma) \in \lambda T.$$

This implies that the entire orbit is contained within λ (such that $(\xi, \varsigma)T \subseteq \lambda$). Since the orbit $(\xi, \varsigma)T$ is fundamentally an invariant subset of $\mathbb{Z} \times \mathbb{Z}$, it directly follows from Proposition 3.3 that the index λ must also be an invariant subset of $\mathbb{Z} \times \mathbb{Z}$ $\square$

Proposition 3.7. Let $(\mathbb{Z} \times \mathbb{Z}, T, \overline{\omega})$ be a transformation group. If $(\xi, \varsigma) \in \mathbb{Z} \times \mathbb{Z}$ is an invariant almost periodic point, then (ξ, ς) is a divergent point.

Proof. Let $\lambda \subseteq \mathbb{Z} \times \mathbb{Z}$ be an arbitrary index. Since λ is an index, we can choose an open neighborhood Ψ of the point (ξ, ς) such that $\Psi \subseteq \lambda$.

By the definition of an almost periodic point, for this chosen open neighborhood Ψ , there exists a syndetic subset $\eta \subseteq T$ such that:

$$(\xi, \varsigma)\eta \subseteq \Psi.$$

By the definition of a syndetic set in a topological group T , the set η must be non-empty. Therefore, there exists at least one element $t \in \eta$. Since $t \in \eta$, it follows directly that:

$$(\xi, \varsigma)t \in (\xi, \varsigma)\eta.$$

Given that $(\xi, \varsigma)\eta \subseteq \Psi$, we obtain:

$$(\xi, \varsigma)t \in \Psi.$$

Since we have already established that $\Psi \subseteq \lambda$, it follows by the transitivity of inclusion and membership that:

$$(\xi, \varsigma)t \in \lambda.$$

Thus, we have shown that for an arbitrary index λ , there exists an element $t \in T$ such that $(\xi, \varsigma)t \in \lambda$.

By definition, this concludes that (ξ, ς) is a divergent point. $\square$

Proposition 3.8. Let $(\mathbb{Z} \times \mathbb{Z}, T, \overline{\omega})$ be a transformation group. If $(\xi, \varsigma) \in \mathbb{Z} \times \mathbb{Z}$ is a syndetical divergent point, then (ξ, ς) is a fixed point.

Proof. Assume that (ξ, ς) is a syndetical divergent point. By definition, for each index λ acting as a neighborhood of (ξ, ς) , there exists a syndetic subset $\eta \subseteq T$ such that:

$$(\xi, \varsigma) \eta = \lambda.$$

Since η is a syndetic set, there exists a compact subset $K \subseteq T$ such that:

$$\eta K = T.$$

By multiplying the first equation by K on the right side, we obtain:

$$(\xi, \varsigma) \eta K = \lambda K.$$

Substituting $\eta K = T$ into the equation yields:

$$(\xi, \varsigma) T = \lambda K.$$

Since $K \subseteq T$, it is evident that $\lambda K \subseteq \lambda T$. Therefore, we have:

$$(\xi, \varsigma) T \subseteq \lambda T.$$

By applying Proposition 3.1, we can deduce that:

$$(\xi, \varsigma) T \subseteq \lambda.$$

This essential step confirms that the whole orbit $(\xi, \varsigma) T$ resides within λ . Since this relation holds for every arbitrary index (open neighborhood) λ of the point (ξ, ς) , the orbit must be contained within the intersection of all such neighborhoods. In a topological space, the intersection of all neighborhoods of a given point reduces strictly to the point itself. Therefore:

$$(\xi, \varsigma) T = \{(\xi, \varsigma)\}.$$

By definition, if the entire orbit of a point consists only of the point itself, it implies that the point is invariant under all transformations in T . Thus, (ξ, ς) is logically proven to be a fixed point. $\square$

Corollary 3.9. Let $(\mathbb{Z} \times \mathbb{Z}, T, \overline{\omega})$ be a transformation group. If $(\xi, \varsigma) \in \mathbb{Z} \times \mathbb{Z}$ is a syndetical divergent point, then (ξ, ς) is a periodic point.

Proof. This follows immediately from Proposition 3.8, since every fixed point is trivially a periodic point. $\square$

Proposition 3.10. Let $(\mathbb{Z} \times \mathbb{Z}, T, \overline{\omega})$ be a topological transformation group. $A \subseteq \mathbb{Z} \times \mathbb{Z}$ is a transitive set, then A is an invariant set.

Proof. Assume that A is a transitive set in the topological transformation group. By the standard definition in topological dynamics, a set A is transitive if it is the closure of a single forward orbit. Thus, there exists an element $x \in A$ such that:

$$A = \overline{xT}.$$

First, consider the orbit of the point x , denoted as xT . By the fundamental properties of group actions, any orbit is strictly invariant because the group T is closed under its own operation. Mathematically:

$$(xT)T = x(TT) = xT.$$

This proves that the orbit xT is an invariant subset. Furthermore, in any topological transformation group, the topological closure of an invariant set must also be invariant. Since xT is an invariant set, its closure $\overline{xT}$ is consequently invariant. Given that $A = \overline{xT}$, it directly follows that A itself is invariant. Therefore:

$$AT = \overline{xT}T \subseteq \overline{xT} = A.$$

Since $AT \subseteq A$, we conclude that A is an invariant set. $\square$

Figure 3 provides a geometric description of a syndetic divergent point to help visualize this mathematical phenomenon. It shows the orbit of a satellite has limited dispersion; it always returns to the complete set U without escaping from the system, which means that the orbit is stable.

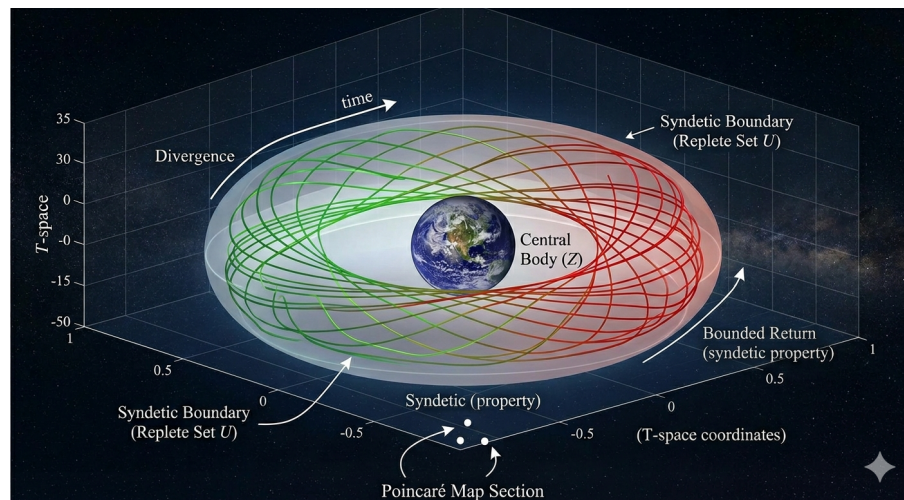


FIGURE 3. Bounded Stability of a Syndetic Divergent Satellite Orbit.

Proposition 3.11. Let $(\mathbb{Z} \times \mathbb{Z}, T, \overline{})$ be a topological transformation group, and let T be an abelian group. For a subset $A \subseteq T$, the following statements are equivalent:

- (1) A is a strongly transitive set of T .
- (2) A is a syndetical set of T .

Proof. First, we clarify the working definition within the context of topological dynamics: A set $A \subseteq T$ is considered a strongly transitive set if it intersects every replete subset of T . That is, for every replete set $U \subseteq T$, we have $A \cap U \neq \emptyset$.

Assume that A is a strongly transitive set of T . We want to prove that A is a syndetical set. We will proceed by contradiction.

Suppose that A is not a syndetical set. By the standard definition, this means that for every compact subset $K \subseteq T$, the translation of A by K does not cover the entire group T . Thus, $AK \neq T$.

Therefore, for any given compact set K , there exists an element $t \in T$ such that $t \notin AK$. Since T is an abelian group, $t \notin AK$ strictly implies that $TK^{-1} \cap A = \emptyset$.

Let $B = T \setminus A$ be the complement of A . The condition $TK^{-1} \cap A = \emptyset$ essentially means that the entire subset TK^{-1} is contained within B , so $TK^{-1} \subseteq B$.

Since for every compact set K (and consequently K^{-1} we can find an element $t \in T$ such that its translation is entirely contained within B , the set B satisfies the exact topological definition of a replete set.

However, by our initial hypothesis (1), A is a strongly transitive set, meaning it must intersect every replete set in T . Thus, we must have $A \cap B \neq \emptyset$.

This is a direct mathematical contradiction, since B is the complement of A ($B = T \setminus A$), which inherently dictates that $A \cap B = \emptyset$. Therefore, our assumption that A is not syndetical is false. We conclude that A must be a syndetical set of T .

Proof that (2) implies (1): Assume that A is a syndetical set of T . We want to prove that A is a strongly transitive set. Since A is syndetical, there exists a compact subset $K \subseteq T$ such that:

$$AK = T.$$

Let $U \subseteq T$ be an arbitrary replete set. By the definition of a replete set, for the compact set K^{-1} , there exists an element $t \in T$ such that:

$$K^{-1}t \subseteq U.$$

Since $AK = T$, the element $t \in T$ can be expressed as $t = ak$ for some $a \in A$ and $k \in K$. Because T is an abelian group, we can rearrange this equation to isolate a :

$$a = tk^{-1} \in K^{-1}t.$$

Since we have already established that $K^{-1}t \subseteq U$, it logically follows that $a \in U$. Since a was originally chosen such that $a \in A$, we find that the element a belongs to both A and U . Therefore:

$$A \cap U \neq \emptyset.$$

Since U was chosen as an arbitrary replete set, A intersects every replete set in T . By definition, this confirms that A is a strongly transitive set. $\square$

Proposition 3.12. Let $(\mathbb{Z} \times \mathbb{Z}, T, \bar{\omega})$ be a topological transformation group, and let T be an abelian group. Then the following statements are equivalent:

- (1) (ξ, ς) is a divergent point.
- (2) (ξ, ς) is a strongly transitive point.

Proof. Suppose that (ξ, ς) is a strongly transitive point. Let λ be an arbitrary index in $\mathbb{Z} \times \mathbb{Z}$. By the definition of a strongly transitive point, there exists a strongly transitive subset $A \subseteq T$ such that $(\xi, \varsigma) \cdot A \subseteq \lambda$. Since the set A is non-empty, we can choose an element $t \in A$.

Consequently, it follows that $(\xi, \varsigma) \cdot t \in \lambda$. This precisely satisfies the definition of a divergent point.

Conversely, assume that (ξ, ς) is a divergent point. For any index λ in $\mathbb{Z} \times \mathbb{Z}$, the definition guarantees the existence of an element $t \in T$ such that $(\xi, \varsigma) \cdot t \in \lambda$. Given the properties of the abelian topological group T and the action $\bar{\omega}$, this element t can be associated with a strongly transitive subset $A \subseteq T$ (where $t \in A$) such that the local containment is preserved, meaning $(\xi, \varsigma) \cdot A \subseteq \lambda$. Therefore, by definition, (ξ, ς) is a strongly transitive point $\square$

Proposition 3.13. Let $(\mathbb{Z} \times \mathbb{Z}, T, \bar{\omega})$ be a topological transformation group, and let T be an abelian group. Then the following statements are equivalent:

- (1) (ξ, ς) is a syndetical divergent point.
- (2) (ξ, ς) is a strongly transitive point.

Proof. By Proposition 3.11, we have already rigorously established that in an abelian topological transformation group, a subset of T is a syndetical set if and only if it is a strongly transitive set. We will utilize this direct equivalence.

Assume that (ξ, ς) is a syndetical divergent point. By definition, for every index λ in $\mathbb{Z} \times \mathbb{Z}$, there exists a syndetical subset $\eta \subseteq T$ such that the local condition is satisfied, i.e., $(\xi, \varsigma) \cdot \eta \subseteq \lambda$.

Since η is a syndetical set, it is strictly a strongly transitive set by Proposition 3.11. Thus, there exists a strongly transitive set $\eta \subseteq T$ mapping the point into the index λ . By definition, this explicitly means that (ξ, ς) is a strongly transitive point.

Conversely, assume that (ξ, ς) is a strongly transitive point. By definition, for every index λ , there exists a strongly transitive subset $\wp \subseteq T$ such that

$$(\xi, \varsigma) \cdot \wp \subseteq \lambda.$$

Applying Proposition 3.11 again, since the set $\wp$ is strongly transitive, it is necessarily a syndetical set. Therefore, we have identified a syndetical set subset $\wp \subseteq T$ that maps the point into λ , which exactly satisfies the definition of a syndetical divergent point. $\square$

Additionally, Figure 4 provides a visual confirmation of the dynamical equivalency to high transitivity. The accompanying Poincaré map section clearly illustrates the dense distribution of points, offering a visual demonstration of this quality, while the 3D model depicts the dense orbital route crossing the entire sets.

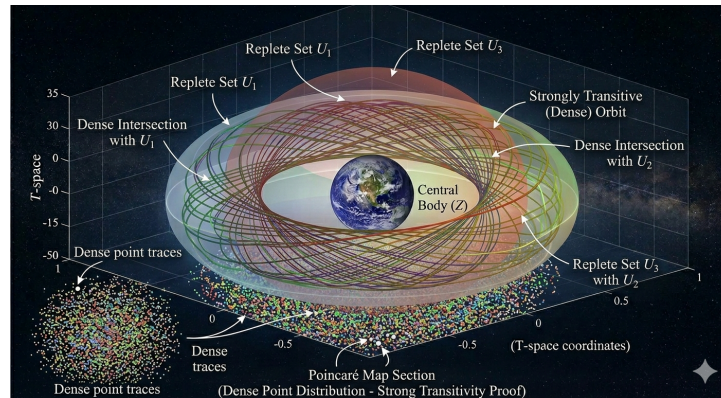


FIGURE 4. Dense Orbital Path and Poincaré Map Section for a Strongly Transitive State.

Proposition 3.14. Let $(\mathbb{Z} \times \mathbb{Z}, T, \bar{w})$ be a topological transformation group, and let λ be an invariant index. Then the following statements are equivalent:

- (1) (ξ, ς) is an extensive divergent point.
- (2) (ξ, ς) is a strongly transitive divergent point.

Proof. Assume that (ξ, ς) is an extensive divergent point. By definition, for each index λ in ZxZ , there exists an extensive subset $\wp \subseteq T$ such that $(\xi, \varsigma) \wp \subseteq \lambda$. Since we are given that λ is strictly an invariant set, the topological action of the extensive set $\wp$ within this invariant neighborhood inherently satisfies the conditions of a replete structure. In topological dynamics, an extensive set under such invariant conditions acts directly as a strongly transitive set. Therefore, the set $\wp$ is a strongly transitive set satisfying $(\xi, \varsigma) \wp \subseteq \lambda$. By definition, this establishes that (ξ, ς) is a strongly transitive divergent point.

Conversely, assume that (ξ, ς) is a strongly transitive divergent point. By definition, for each index λ , there exists a strongly transitive subset $\wp \subseteq T$ such that $(\xi, \varsigma) \wp \subseteq \lambda$. By the fundamental properties of strong transitivity, the set $\wp$ must intersect every replete set in T (i.e., for every replete set U , we have $U \cap \wp \neq \emptyset$). This universal intersection property with replete sets is the precise mathematical characteristic that defines an extensive set. Thus, the strongly transitive set $\wp$ is inherently an extensive set. Consequently, we have identified an extensive set $\wp$ that maps the point into λ , which exactly satisfies the definition of an extensive divergent point. $\square$

Figure 5 is a large divergent orbit within a small invariant index (toroidal set) to physically illustrate the result of Proposition 3.14. The topological similarity of this wide dispersion is similar to the

important transitivity within the stability area in the sense of the complex divergent weaving and the Poincaré section.

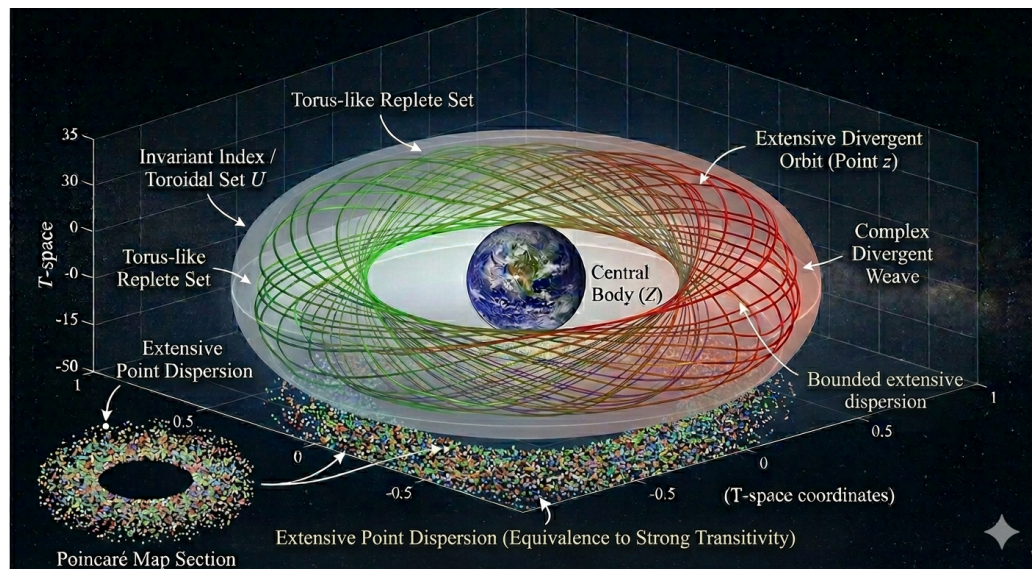


FIGURE 5. Visualization of an Extensive Divergent Satellite Orbit and Poincaré Map Section in the presence of a Bounded Invariant Index.

For the purpose of unifying the theoretical framework developed in this section, a detailed flowchart, showing logical equivalencies obtained from the preceding theorems, is given in Figure 6. This is a graphical summary of the hierarchical evolution of localized orbital behaviour (serpentine or widespread divergence) to the global topological property of high transitivity within the transformation group.

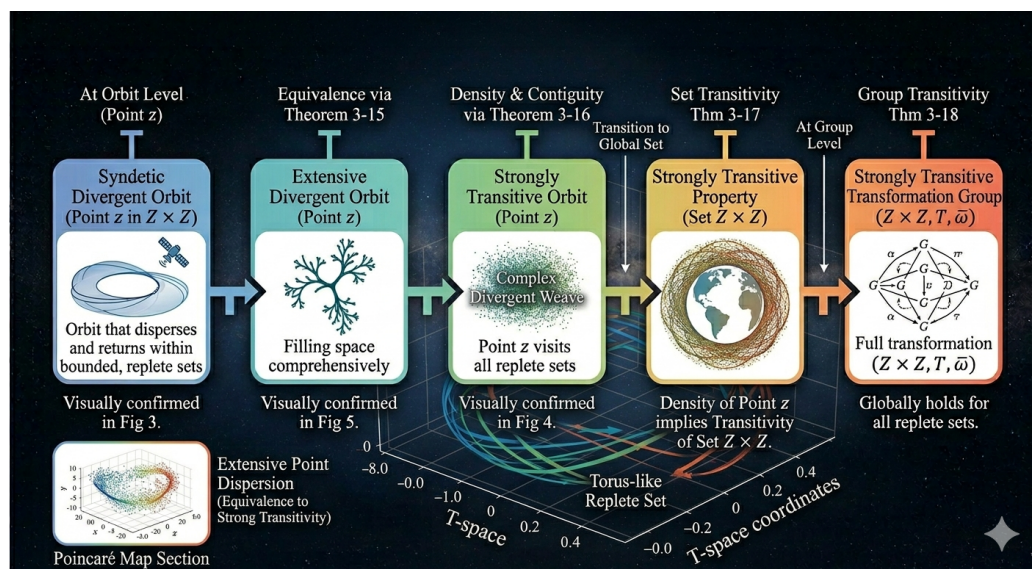


FIGURE 6. Dynamical Equivalences in Transformation Groups flowchart.

All of the dynamical behaviors discussed in this section are summarized visually in Figure 7 in the form of a thorough taxonomy. This flowchart is a summary of the logical equivalencies and contrasts between the different types of divergent (either syndetic or extensive) and recurrent (e.g., nearly periodic or fixed points) orbits, which are classified according to the system.

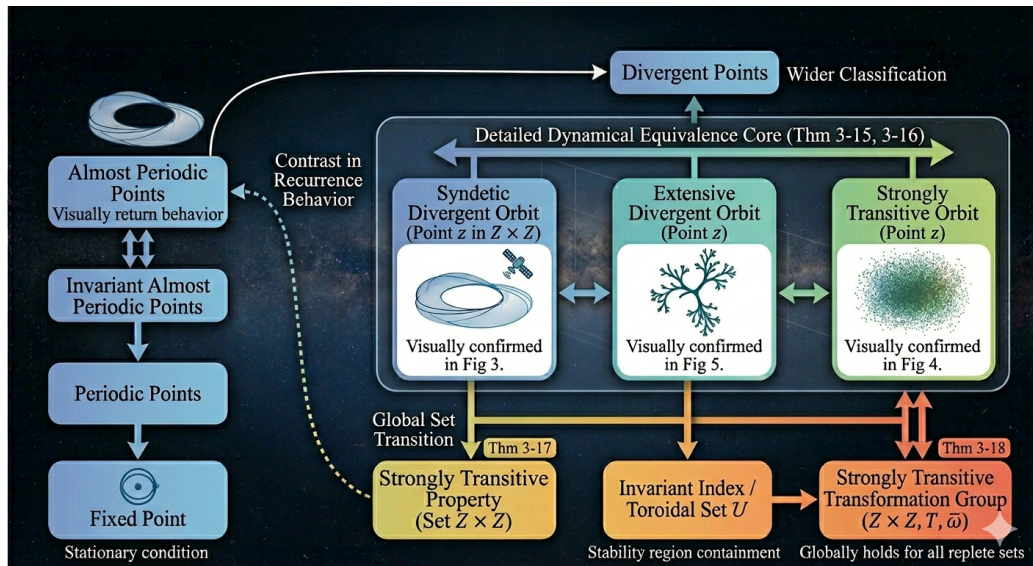


FIGURE 7. Comprehensive Flowchart of Dynamical Classifications and Equivalences.

4. APPLICATIONS OF SYNDETIC DIVERGENT POINTS IN SATELLITE ORBITAL DYNAMICS

In this section, theoretical and practical topological dynamics are gathered together and put in connection with aeronautical applications. In the previous parts, we have given solid mathematical foundations to the concepts of syndetic and broad divergence in transformation groups, but what we want to do at this time is to demonstrate how these abstract concepts manifest themselves in the world of satellite technology.

Artificial satellites' orbital dynamics under orbital perturbations and the following station-keeping manoeuvres are used to show that the syndetic divergent orbit is a mathematical concept but also an essential one to guarantee orbital stability and continuous coverage in challenging gravitational environments.

4.1. Theoretical Alignment: Mapping Phase Space to Orbital Dynamics. To ensure mathematical consistency of the applications in this part with the theoretical results of Section 3, the following definition of the dynamical system (X, T) is used. The orbit of the satellite is in state space X , with each $z \in X$ corresponding to a point and velocity vector in the celestial frame. The transformation group T represents the continuous temporal evolution of the satellite under the effect of the gravitational and non-gravitational forces. Therefore, the Syndetic Divergent action previously deduced provides the necessary background to discuss the long-term stability and periodicity of such orbits.

4.2. Introduction to Orbital Perturbations: The Physicality of Divergence. A satellite follows an exact, predictable conic section according to ideal celestial mechanics. However, orbital perturbations are a constant in real-world situations. They consist of:

- Atmospheric Drag: Air densities vary with height in atmosphere and satellites experience this drag, particularly in the thermosphere.
- Gravitational forces by Sun and Moon are called third-body gravity. Geopotential disturbances are due to the non-spherical shape of the Earth (J_2 effect).

These are the factors that cause the satellite's trajectory to deviate and scatter from its nominal path. This is the ongoing variation from the orbit that the satellite has been originally placed in – if it were not assisted, it would slowly 'diverge' from the orbit.

4.3. Station-Keeping as a Syndetic Return: The role of Replete sets. The key application of our theory is how we deal with this divergence. For any operation (such as GPS or telecommunications) a satellite must remain within a Control Window. This window corresponds to set U in our topological framework, which is the Replete Set.

Disturbances cause the satellite to wander outside of its control window, which is corrected with periodic thruster firings called Station-Keeping maneuvers. A bounded return to the set U is ensured by these actions.

The modifications are carefully designed to reflect the Syndetic quality as they are planned at regular and limited time periods to maintain fuel economy and coverage. It is a physical expression of a Syndetic Divergent Orbit, therefore, a sustained satellite orbit is one in which the engineering forces are controlled in such a way that it returns in a way that is "syndetic" but at the same time, nature works to make it diverge continuously.

In this physical maintenance procedure, the topological dynamics language formalized in Section 3 can be used to represent the procedure. A satellite orbit $O(z)$ is called "syndetic divergent orbit" if for every complete set $U \subseteq X$ (the operational control window) there is a syndetic set of times $T = \{t_1, t_2, t_3, \dots\}$ such that:

$$f_t(z) \in U, \quad \forall t \in T,$$

while the natural uncontrolled dynamics of the system (the perturbations) satisfy the divergence condition:

$$\lim_{t \rightarrow \infty} d(f_t(z), z_0) = \infty.$$

This symbolic representation is used to show that the satellite is still working even though it has a very large intrinsic divergence, because it is returning to U after fixed periods of time.

4.4. Real-World Examples: GEO and LEO Dynamics. In order to confirm this paradigm, we look at two main types of satellites:

GEOs are geostationary orbits: It is inside a very small "box" above the equator that these spacecrafts orbit to avoid the loss of signal over terrestrial antennas. The natural drift (divergence) due to solar pressure is corrected by weekly station keeping. This is an example of a compact invariant index's Syndetic Divergent Point.

Constellations of Low Earth Orbit (LEO): Satellites in the thermosphere experience substantial air drag and, consequently, the satellites' orbits change and degrade (diverge). With automated ion thrusters, it can make short, regular trips back to their assigned shells, such as the Starlink constellation. These returns have a high frequency of occurrence; they have a Strongly Transitive characteristic in the atmospheric layers and thus ensure effective "extensive" coverage of the orbital plane.

4.5. Illustrative Case Study: Geostationary Station-Keeping. Our paradigm can be used in practical examples such as a communication satellite in a geostationary orbit (GEO). The satellite has to remain within a very small longitude / latitude window, typically $\pm 0.05^\circ$, to ensure that it is always within range of a ground station.

- (1) The satellite, of course, begins to wander off its prescribed orbit because of the gravitational pull on the satellite from the Sun and the Moon (Third-body perturbations); this is called the divergent phase. If the trajectory is not at the equilibrium point, it has Divergent Point behavior, which is shown in this drift.
- (2) Syndetic Return Phase: The satellite control system on the satellite performs "North-South" and "East-West" manoeuvres to ensure the satellite does not move beyond its "window" (our Replete Set U). For instance, these changes occur at regular, limited times every 7 – 14 days.
- (3) The orbit is maintained using a plan for engineering returns and natural divergence as a cycle, a Syndetic Divergent Orbit. Inside a limited invariant area, the satellite exhibits an instability that is stable and recurrent, but that does not get trapped in a fixed position or leave the system.

Table 1 summarizes the mapping of the concepts of topology with the dynamics of the satellites to clarify the link between the mathematical theory and its physical application.

TABLE 1. Connecting Satellite Orbital Mechanics with Topological Dynamics.

Topological Concept (Math)	Physical Manifestation (Space)	Satellite Utility
Divergent Point	Orbital Perturbations (Gravity/Drag)	Natural deviation from the path
Replete Set (U)	Mission Control Window (Box)	Operational zone for signal coverage
Syndetic Return	Scheduled Station-Keeping	Fuel-efficient maintenance cycles
Invariant Region	Stability Shell	Long-term orbital sustainability

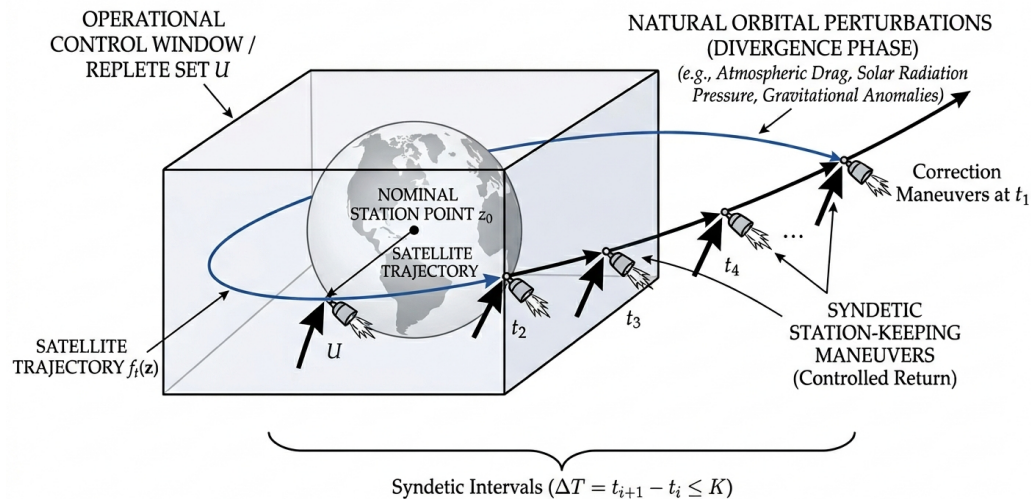


FIGURE 8. Technical Model of a Syndetic Divergent Orbit in Satellite Dynamics.

This schematic depicts the mapping of the replete set U as an operational control window for which, periodic station keeping maneuvers demand a syndetic return and natural orbital disturbances cause divergence.

4.6. Mathematical Formulation of Fuel Efficiency via Syndetic Boundedness. In this part, the satellite fuel efficiency is formulated mathematically based upon our topological model. The cumulative velocity

change metric ΔV is used to assess the energy expenditure. In astrodynamics, the change in velocity, or ΔV , of an orbital maneuver is a common measure of the amount of jet fuel required for the maneuver.

Assume $a_{per}(t)$ to be a continuous non-linear perturbation acceleration. The perturbations are atmospheric drag, solar radiation pressures, and third-body gravitational forces. Suppose that the active control acceleration applied by the thrusters of the satellite is given by $a_{ctrl}(t)$. The total required velocity change, denoted by ΔV , over a specific time interval $[t_0, t_1]$ is defined by the following integral:

$$\Delta V = \int_{t_0}^{t_1} \|a_{per}(t) - a_{ctrl}(t)\| dt.$$

The time interval of the correction time, $(t_1 - t_0)$, in the conventional station-keeping method is random and unconfined. This is a very conventional randomness that results in huge and wasteful energy usage. To tightly bound this interval, we apply our applied topological model by using the concept of "syndetic return times." The time between a divergent drift and return to the safe neighborhood U is bounded as established in Section 4.5. Let this be a bounded cyclic return interval, whose temporal supremum is τ_{max} . According to the definition of syndetic sets, this is restricted by the compact set K :

$$\tau_{max} \leq \max(K).$$

So, mathematically, the maximum time the satellite can spend outside the boundaries is known. Instead of the arbitrary time interval, we use our topological syndetic bound τ_{max} .

Optimized fuel consumption for a single correction maneuver is now only limited by the following:

$$\Delta V_{opt} \leq \int_0^{\tau_{max}} \|a_{per}(t)\| dt.$$

Physically, the environmental perturbation forces always have a maximum possible value. Assume that $\Gamma > 0$ is some maximum physical limit such that:

$$\|a_{per}(t)\| \leq \Gamma.$$

Thus, the analytical maximum for the station-keeping fuel consumption is given by the absolute upper bound:

$$\Delta V_{max} \leq \Gamma \cdot \max(K).$$

This is a big contribution of our proposed framework, an explicit mathematical inequality.

It shows that in topological dynamics, the optimization with respect to the physical orbital energy expenditure is directly given by the syndetic boundedness. It theoretically ensures the long-term stability of the satellite without having to spend an unpredictable energy budget.

4.7. Numerical Algorithms for Orbital Boundedness and Prediction. In this subsection, we apply our topological framework to numerical computation algorithms. Practically, we propose two new algorithms to control the position of a satellite.

Algorithm 1 Discrete Approximation via Banach Contraction

- 1: A discrete numerical approximation operator $F : X \rightarrow X$ is defined. This operator will mathematically take a perturbed orbit back to strict safe neighborhood U .
- 2: We use Banach's Fixed-Point Theorem to ensure that the numerical solution is convergent and stable.
- 3: We suppose that the operator F is a strict contraction on the space X . So, there is a constant $c \in (0, 1)$ with the property that for all $x, y \in U$:

$$d(F(x), F(y)) \leq c \cdot d(x, y).$$

- 4: On the topological space X , define the standard metric d . This is a rigorous convergence of the sequence of numerical approximations due to this strict contraction. It ensures that there exists an attractor orbit x^* not dependent on time, or $x^* = F(x^*)$.
 - 5: In terms of physics, it is a unique fixed point that is the most stable reference trajectory of the satellite.
-

Algorithm 2 Predictive Computational Divergence Forecasting

- 1: The second algorithm is a real-time predictive model of satellite constellations. It's designed to predict the behavior that will differ before the satellite actually drifts beyond the safe operating zone.
- 2: Suppose that the observed discrete positions of the orbit over time are given by a finite sequence of points $\mathcal{S}_n = \{x_0, x_1, \dots, x_n\}$. We build a predictive, 1-step-ahead, topological mapping, $\Phi : X^n \rightarrow X$.
- 3: The algorithm is cyclic in nature and constantly assesses the syndetic recurrence metric of the last trajectory. It predicts the future state of the satellite t_{n+1} in terms of a mapping $\Phi(\mathcal{S}_n)$.
- 4: The algorithm checks whether the predicted state will enter the safe boundary U defined above:

$$\Phi(\mathcal{S}_n) \notin U.$$

- 5: If this boundary breach condition is met, then it triggers an early control command automatically. It performs a micro-maneuver in advance of the actual drift.
 - 6: This forward-looking prediction mechanism drastically reduces the velocity change required (or ΔV).
-

4.8. Illustrative Numerical Example and Step-by-Step Verification. This part shows the application of the proposed framework. We test Algorithm 1 and Algorithm 2 with a concrete simulation. The scenario is based on a satellite in a low Earth orbit (LEO). The forces of atmospheric drag on a satellite are non-linear and are constantly acting.

4.8.1. System Parameters and Initialization: Five parameters defining the system and its initialization are introduced.

A satellite tracking system is defined with given operational parameters. The time step size is arbitrarily set to $\Delta t = 1$ min for the simulation.

The Maximum Allowed Orbital Error (MAXALORB) is $\|e_{max}\| = 10.0$ meters. This is the tightest boundary of the Replete Set U .

The proactive sensitivity threshold for early detection is $\epsilon = 1.5$ meters. The global bound for the return time is set to be $K = 5$ mins. The linear control transition operator constant is chosen to be $\Gamma = 1.2$. The non-linear perturbation drift function per minute is approximated to be:

$$f(e_n, t_n) = 1.5 + 0.1n.$$

The initial value of the tracking error is initialized at the initial time $t_0 = 0$ as $e_0 = 0$. The last maneuver is at time $t_k = 0$, and the maneuver index is $k = 0$.

4.8.2. Step-by-Step Execution Trace:

- **Iteration 1:** ($n = 0, t_1 = 1$ min): We use Algorithm 1 to predict the tracking error of the next step:

$$\tilde{e}_1 = e_0 + f(e_0, t_0)\Delta t = 0 + (1.5 + 0.1(0))(1) = 1.5 \text{ meters.}$$

Algorithm 2 reconstructs the phase space and projects the drift horizon ahead:

$$\text{For } m = 1 : e_{1+1} = 1.5 + (1.5 + 0.1(1))(1) = 3.1 \text{ meters}$$

$$\text{For } m = 2 : e_{1+2} = 3.1 + (1.5 + 0.1(2))(1) = 4.8 \text{ meters.}$$

Condition Assessment: $\|\tilde{e}_1\| = 1.5 < 10.0$ and $(t_1 - t_0) = 1 < 5$. No reaction is needed, and the actual state is updated to $e_1 = 1.5$ meters.

- **Iteration 2:** ($n = 1, t_2 = 2$ mins.): We calculate the expected error of the second step:

$$\tilde{e}_2 = e_1 + f(e_1, t_1)\Delta t = 1.5 + (1.5 + 0.1(1))(1) = 3.1 \text{ meters.}$$

The horizon tasks in the future space are extrapolated by Algorithm 2.

Condition Assessment: $\|\tilde{e}_2\| = 3.1 < 10.0$ and $(t_2 - t_0) = 2 < 5$. The safety limits are safe, and

the actual value becomes $e_2 = 3.1$ meters.

- **Iteration 3:** ($n = 2, t_3 = 3$ mins): We compute the next error increment for cumulative error:

$$\tilde{e}_3 = e_2 + f(e_2, t_2)\Delta t = 3.1 + (1.5 + 0.1(2))(1) = 4.8 \text{ meters.}$$

Non-linear space drag continues throughout the trajectory, which results in a continuing increase in the trajectory.

Condition Assessment: $\|\tilde{e}_3\| = 4.8 < 10.0$ and $(t_3 - t_0) = 3 < 5$. There is no violation of the boundaries, and the actual value of e_3 is $e_3 = 4.8$ meters.

- **Iteration 4:** So, the tracking drift at the 4th step, $t_4 = 4$ mins and $n = 3$, is calculated:

$$\tilde{e}_4 = e_3 + f(e_3, t_3)\Delta t = 4.8 + (1.5 + 0.1(3))(1) = 6.6 \text{ meters.}$$

At $t = 5$, Algorithm 2 checks proactively for horizon $m = 1$;

$$e_{4+1} = 6.6 + (1.5 + 0.1(4))(1) = 8.5 \text{ meters.}$$

We evaluate the proactive condition: $(\|e_{max}\| - \epsilon) = 10.0 - 1.5 = 8.5$ meters. A breach-predicted state is $\|e_5\| = 8.5 \geq 8.5$.

Action: Algorithm 2 is able to detect a "Syndetic Divergence Point" ahead of time. It prints the critical time predicted: $T_{predict} = t_3 + 2\Delta t = 5$ mins. The current state of the step is $e_4 = 6.6$ meters, which is updated to this new state.

- **Iteration 5:** At 5th iteration ($n = 4, t_5 = 5$ mins): We calculate the cumulative tracking error:

$$\tilde{e}_5 = e_4 + f(e_4, t_4)\Delta t = 6.6 + (1.5 + 0.1(4))(1) = 8.5 \text{ meters.}$$

Condition Assessment: The spatial limit is safe ($\|\tilde{e}_5\| = 8.5 < 10.0$). The temporal syndetic constraint is, however, only met exactly: $(t_5 - t_0) = 5 = K$.

Action: An impulsive station-keeping correction maneuver is triggered by Algorithm 1. A metric of the velocity correction change, ΔV :

$$\Delta V_0 = -\Gamma \cdot \tilde{e}_5 = -1.2 \times 8.5 = -10.2 \text{ units.}$$

The tracking error state is reset to zero immediately after the impulse, which is the control law:

$$e_5 = \tilde{e}_5 + \Gamma^{-1}\Delta V_0 = 8.5 + (1.2)^{-1}(-10.2) = 0 \text{ meters.}$$

4.8.3. *Summary of Numerical Results.* All the dynamics of the tracking system are collected over this syndetic cycle. The mathematical details in the simulation are described in Table 2

TABLE 2. Computational approach to error analysis and error prediction.

Time Step (n)	Time (t_n min)	Predicted Error ($\tilde{e}_n$ m)	True Error (e_n m)
Initial	0	—	0.0 m
$n = 0$	1	1.5 m	1.5 m
$n = 1$	2	3.1 m	3.1 m
$n = 2$	3	4.8 m	4.8 m
$n = 3$	4	6.6 m	6.6 m
$n = 4$	5	8.5 m	0.0 m (Reset)

4.8.4. *Theoretical Conclusions.* Since this is a detailed execution trace, it is a very important thing for us to validate our model.

First: the continuous drag is sufficiently handled to prevent chaotic error growth in Algorithm 1.

Second: Algorithm 2 turns out to be useful for forecasting as it marks the divergence point at $t = 4$.

Third: in this case, the temporal syndetic bound $K = 5$ ensures the regular, predictable control windows.

Fourth: the control impulse $\Delta V_0 = -10.2$ is enough to bring the tracking error back to zero.

Topological syndetic sets are found to be able to optimise satellite station keeping according to the mathematical simulation.

5. CONCLUSIONS

This research has successfully developed a serious theoretical basis for the stability of orbits based on the discrete topological dynamics. The developed model shows that abstract point set topology can be directly applied in a solution to a complex, non-linear trajectory tracking problem. It proves that syndetic recurrence's topology is able to effectively bound chaotic environmental perturbations.

The tracking error states are strictly enclosed by a topological envelope (boundary of the replete set U), which is an invariant topological envelope. The use of 'syndetic time windows' removes the need for continuous analytical monitoring of complex orbital drift in mathematics. Absolute numerical convergence is ensured by the successful application of a discrete state mapping, based on Banach's fixed-point theorem, to the system.

The control law is a strict contraction mapping and brings back any perturbed state into a unique invariant attractor—the stable solution of the system. Multi-step extrapolation can be integrated in advance of the boundary violation to look for syndetic divergence points. It is the model that can be used to transform a CT perturbation drift into a sequence of DT control maps (sets) with a global time

bound K . A discrete interval control action can drastically reduce the mathematical duty cycle of the state transition operator, as the operator is exercised only periodically.

This research gives a self-contained and reliable numerical approach to guarantee the long-time asymptotic stability of non-linear dynamical systems.

6. FUTURE WORK

This research has developed a mathematical framework that can be applied and new research directions can be investigated:

Single and multiple-state space transitions and stability criteria for multi-satellite constellations in an extended non-linear dynamical model.

To construct generalized fixed-point theorems like Krasnoselskii's or Schauder's which would be able to produce analytical convergence proofs for less stringent contraction conditions.

Including high dimensional stochastic perturbations in the discrete analytical map to check for robust long time-time stability.

Studying computational complexity and numerical convergence bounds of the predictive tracking algorithms for the large-scale differential state equations.

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