

TWISTED OBIC ALGEBRAS AND THEIR STRUCTURAL PROPERTIES

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ABSTRACT. In this paper, we introduce and study a new algebraic structure called a twisted obic algebra, or, for brevity, a TwO-algebra. This structure is obtained from an obic algebra, or O-algebra, by preserving axioms (O-1) and (O-3) and replacing axiom (O-2) with a twisted identity. Examples show that the class of TwO-algebras is related to, but distinct from, the class of O-algebras. We establish several fundamental identities and investigate prime TwO-algebras and TwO-algebras with the weak property. In particular, the prime condition inherited from O-algebras forces a TwO-algebra to be trivial, while the weak property yields commutativity and inverse-type identities. We also study subalgebras, ideals, homomorphisms, kernels, congruence relations, quotient TwO-algebras, and the behavior of ideals under homomorphisms. Finally, we introduce krib maps and monic maps on TwO-algebras and examine their behavior under invariant subalgebras and quotient constructions, as well as their regularity, zero-map behavior, and conditions under which the underlying algebra becomes trivial.

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1. INTRODUCTION

The study of algebraic systems associated with implication-based logics originates from the foundational work of Imai and Iséki. In 1966, Imai and Iséki introduced BCK-algebras as algebraic systems related to certain propositional calculi [15]. In the same year, Iséki introduced BCI-algebras, providing another fundamental framework for implication-like operations [17]. Hu and Li later introduced BCH-algebras as a related generalization [3]. Jun, Roh, and Kim studied BH-algebras and developed their basic algebraic properties [18]. Kim and Kim introduced BE-algebras [23], studied BM-algebras [20], and later investigated BN-algebras [22]. Kim and Kim also developed BG-algebras as another related algebraic system [21]. Bandaru introduced QI-algebras and examined their elementary properties [1].

More recently, Bantaojai, Iampan, and others studied Bd-algebras [2], while Iampan introduced UP-algebras [4] and later studied IUP-algebras as a related structure [5]. These works provide the broader logical-algebraic background for the present study.

Within this line of research, Ilojide introduced obic algebras, or briefly O-algebras, in 2019 [6]. This work is particularly relevant to the present paper, since O-algebras form the direct basis for the definition of TwO-algebras. In 2020, Ilojide introduced torian algebras as another algebraic system in this direction [8]. In the same year, he further studied torian algebras and developed additional results concerning their algebraic properties [7]. He also continued the investigation of torian algebras in a separate work, providing another contribution to the development of this class of algebras [9]. In 2021, Ilojide extended the study of torian algebras through another paper devoted to their structural aspects [10]. Later, in 2024, he introduced kreb algebras as a new algebraic structure related to this line of investigation [11]. In the same year, he proposed lojid algebras in one paper [12] and studied stabilizers of lojid algebras in another work [14]. He also introduced polian algebras as another algebraic system in this family of newly developed structures [13]. More recently, Kanpanya and Iampan developed a broader framework for obic-type algebras by introducing generalized obic algebras and independent obic algebras, together with zero-annihilating and self-annihilating subsets and their roles as subalgebras and ideals [19]. Inthachot and Iampan studied O-algebras from a structural viewpoint and showed that the concepts of implicative and prime O-algebras are trivial [16]. This last result is closely related to the present paper, since we also examine the prime condition inherited from O-algebras and show that it becomes degenerate in the twisted setting. These developments motivate the present study, in which O-algebras serve as the immediate starting point for introducing TwO-algebras.

Motivated by the theory of O-algebras, we introduce a new algebraic system, called a twisted obic algebra, or briefly a TwO-algebra. This structure is obtained by retaining axioms (O-1) and (O-3) of an O-algebra and replacing axiom (O-2) with the twisted identity

$$z \star ((x \star y) \star z) = ((z \star x) \star y) \star z.$$

Although this modification is simple, it yields a class of algebras related to but distinct from the class of O-algebras. In particular, we give examples showing that there exists a TwO-algebra that is not an O-algebra, an O-algebra that is not a TwO-algebra, and an algebra that belongs to both classes.

We also examine the notion of primeness inherited from O-algebras. In O-algebras, primeness is defined by the condition $0 \star x = 0$ for all $x \in X$. However, every TwO-algebra satisfies the identity $0 \star x = x$. Consequently, the same prime condition forces $x = 0$ for all $x \in X$, and hence every prime TwO-algebra is trivial. This shows that the notion of primeness inherited from O-algebras becomes degenerate in the twisted setting.

The main purpose of this paper is to initiate the study of TwO-algebras and to develop their basic theory. We first establish a collection of fundamental identities, including basic identities, uniqueness and cancellation-type properties, zero identities, and higher identities. We then investigate prime TwO-algebras and show that the prime condition inherited from O-algebras forces a TwO-algebra to be trivial. We also study TwO-algebras with the weak property and obtain commutativity and inverse-type identities. Furthermore, we examine subalgebras, ideals, homomorphisms, kernels, congruence relations, and quotient structures. Finally, we introduce and study krib maps and monic maps on TwO-algebras, focusing on regularity, zero-map behavior, preserving conditions, and sufficient conditions under which the underlying algebra becomes trivial.

The paper is organized as follows. In Section 2, we recall O-algebras, introduce TwO-algebras, provide examples illustrating their relation with O-algebras, and establish fundamental identities together with results on prime TwO-algebras and TwO-algebras with the weak property. Section 3 studies substructures of TwO-algebras, namely subalgebras and ideals. In Section 4, we investigate homomorphisms, kernels, ideal-theoretic behavior under homomorphisms, congruence relations, and quotient TwO-algebras. Section 5 is devoted to krib maps, their behavior under invariant subalgebras and quotients, and their regularity properties. Finally, Section 6 deals with monic maps, quotient-induced monic maps, and preserving conditions on TwO-algebras.

2. PRELIMINARIES

In this section, we recall the notion of an O-algebra, which serves as the main algebraic background for this work. We then introduce twisted obic algebras, or TwO-algebras, by modifying one of the defining identities of O-algebras. Throughout this paper, unless otherwise stated, all algebras are assumed to be nonempty.

Definition 2.1 ([6]). *An algebra $X = (X, \star, 0)$ of type $(2, 0)$ is called an obic algebra (or briefly, an O-algebra) if X is a nonempty set, $\star$ is a binary operation on X , $0 \in X$, and the following axioms hold:*

$$(\forall x \in X)(x \star 0 = x), \quad (\text{O-1})$$

$$(\forall x, y, z \in X)((x \star (y \star z)) \star x = x \star (y \star (z \star x))), \quad (\text{O-2})$$

$$(\forall x \in X)(x \star x = 0). \quad (\text{O-3})$$

The next definition introduces the main object of this paper. It is obtained from an O-algebra by retaining axioms (O-1) and (O-3), while replacing axiom (O-2) with a twisted form of the identity.

Definition 2.2. *An algebra $X = (X, \star, 0)$ of type $(2, 0)$ is called a twisted obic algebra (or briefly, a TwO-algebra) if X is a nonempty set, $\star$ is a binary operation on X , $0 \in X$, and X satisfies (O-1), (O-3), and*

$$(\forall x, y, z \in X)(z \star ((x \star y) \star z) = ((z \star x) \star y) \star z). \quad (\text{TwO})$$

The following examples illustrate the relationship between O-algebras and TwO-algebras.

Example 2.1. Let $X = \{0, 1, 2, 3, 4\}$ be a set equipped with a binary operation $\star$ given by the following Cayley table:

$\star$	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	2	0
2	2	3	0	4	3
3	3	2	4	0	2
4	4	0	3	2	0

A direct verification shows that $(X, \star, 0)$ satisfies (O-1), (TwO), and (O-3). Hence $(X, \star, 0)$ is a TwO-algebra.

Example 2.2. Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set equipped with a binary operation $\star$ given by the following Cayley table:

$\star$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	0	0	4	3	3	5
2	2	0	0	4	6	6	5
3	3	4	5	0	1	1	0
4	4	3	6	1	0	0	2
5	5	3	6	1	0	0	2
6	6	4	5	0	2	2	0

Then $(X, \star, 0)$ is a TwO-algebra. However, it is not an O-algebra, since

$$(1 \star (1 \star 6)) \star 1 = (1 \star 5) \star 1 = 3 \star 1 = 4 \neq 5 = 1 \star 6 = 1 \star (2 \star 4) = 1 \star (2 \star (6 \star 1)).$$

Thus, this example shows that a TwO-algebra need not be an O-algebra.

Example 2.3. Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set equipped with a binary operation $\star$ given by the following Cayley table:

$\star$	0	1	2	3	4	5	6
0	0	2	1	3	5	4	6
1	1	0	4	5	2	3	5
2	2	5	0	4	6	1	4
3	3	4	5	0	1	2	0
4	4	1	6	2	0	5	2
5	5	3	2	1	4	0	1
6	6	4	5	0	1	2	0

Then $(X, \star, 0)$ is an O -algebra. Nevertheless, it is not a TwO -algebra, because

$$6 \star ((1 \star 2) \star 6) = 6 \star (4 \star 6) = 6 \star 2 = 5 \neq 0 = 6 \star 6 = (4 \star 2) \star 6 = ((6 \star 1) \star 2) \star 6.$$

Therefore, an O -algebra is not necessarily a TwO -algebra.

Example 2.4. Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set equipped with a binary operation $\star$ given by the following Cayley table:

$\star$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	0	3	4	3	0	0
2	2	3	0	5	0	3	3
3	3	4	5	0	6	2	4
4	4	3	0	6	0	3	3
5	5	0	3	2	3	0	0
6	6	0	3	4	3	0	0

A direct verification shows that $(X, \star, 0)$ is both an O -algebra and a TwO -algebra.

The next result collects some basic identities that follow directly from the defining axioms of a TwO -algebra. These identities will be used repeatedly throughout the paper.

Proposition 2.1 (Basic identities). In a TwO -algebra $X = (X, \star, 0)$, the following identities hold:

$$(\forall x, y \in X)((x \star y) \star x = x \star (y \star x)) \quad (2.1)$$

$$(\forall x \in X)(0 \star x = x) \quad (2.2)$$

$$(\forall x, y \in X)(x \star y = y \star (y \star (x \star y))) \quad (2.3)$$

$$(\forall x, y \in X)(x \star ((y \star (x \star y)) \star x) = x) \quad (2.4)$$

$$(\forall x, y \in X)((y \star (x \star y)) \star x = 0) \quad (2.5)$$

$$(\forall x, y \in X)((x \star y) \star (y \star x) = 0) \quad (2.6)$$

Proof. (2.1) Let $x, y \in X$. Then

$$\begin{aligned} x \star (y \star x) &= x \star ((y \star 0) \star x) && \text{(by (O-1))} \\ &= ((x \star y) \star 0) \star x && \text{(by (TwO))} \\ &= (x \star y) \star x. && \text{(by (O-1))} \end{aligned}$$

(2.2) Let $x \in X$. Then

$$0 \star x = (x \star x) \star x \quad \text{(by (O-3))}$$

$$= x \star (x \star x) \quad (\text{by (2.1)})$$

$$= x \star 0 \quad (\text{by (O-3)})$$

$$= x. \quad (\text{by (O-1)})$$

(2.3) Let $x, y \in X$. Then

$$x \star y = (0 \star x) \star y \quad (\text{by (2.2)})$$

$$= ((y \star y) \star x) \star y \quad (\text{by (O-3)})$$

$$= y \star ((y \star x) \star y) \quad (\text{by (TwO)})$$

$$= y \star (y \star (x \star y)). \quad (\text{by (2.1)})$$

(2.4) Let $x, y \in X$. Then

$$x = 0 \star x \quad (\text{by (2.2)})$$

$$= ((x \star y) \star (x \star y)) \star x \quad (\text{by (O-3)})$$

$$= x \star ((y \star (x \star y)) \star x). \quad (\text{by (TwO)})$$

(2.5) Let $x, y \in X$. Then

$$(y \star (x \star y)) \star x = x \star (x \star ((y \star (x \star y)) \star x)) \quad (\text{by (2.3)})$$

$$= x \star x \quad (\text{by (2.4)})$$

$$= 0. \quad (\text{by (O-3)})$$

(2.6) Let $x, y \in X$. Then

$$(x \star y) \star (y \star x) = (y \star (y \star (x \star y))) \star (y \star x) \quad (\text{by (2.3)})$$

$$= (y \star ((y \star x) \star y)) \star (y \star x) \quad (\text{by (2.1)})$$

$$= 0. \quad (\text{by (2.5)})$$

□

We next derive several uniqueness and cancellation-type consequences of the basic identities. These properties clarify the role of the distinguished element 0 in a TwO-algebra.

Proposition 2.2 (Uniqueness and cancellation-type identities). *In a TwO-algebra $X = (X, \star, 0)$, the following assertions hold:*

$$\text{the element } 0 \text{ is the unique element of } X \text{ satisfying (O-1)} \quad (2.7)$$

$$\text{the element } 0 \text{ is the unique element of } X \text{ satisfying (2.2)} \quad (2.8)$$

$$(\forall x, y \in X)(x \star y = x \Rightarrow y = 0) \quad (2.9)$$

$$(\forall x, y \in X)(x \star y = y \Rightarrow x = 0) \quad (2.10)$$

$$(\forall x, y \in X)((y \star x) \star y = 0 \Rightarrow x = 0) \quad (2.11)$$

Proof. (2.7) Suppose that $y \in X$ satisfies

$$x \star y = x$$

for all $x \in X$. Then

$$0 = y \star y \quad (\text{by (O-3)})$$

$$= y. \quad (\text{by assumption})$$

Hence, $y = 0$. Therefore, 0 is the unique element of X satisfying (O-1).

(2.8) Suppose that $y \in X$ satisfies (2.2); that is,

$$y \star x = x$$

for all $x \in X$. Taking $x = 0$, we obtain

$$y \star 0 = 0.$$

By (O-1), we have

$$y \star 0 = y.$$

Hence, $y = 0$. Therefore, 0 is the unique element of X satisfying (2.2).

(2.9) Let $x, y \in X$ be such that $x \star y = x$. Then

$$y = 0 \star y \quad (\text{by (2.2)})$$

$$= (x \star x) \star y \quad (\text{by (O-3)})$$

$$= ((x \star y) \star x) \star y \quad (\text{by assumption})$$

$$= (x \star (y \star x)) \star y \quad (\text{by (2.1)})$$

$$= 0. \quad (\text{by (2.5)})$$

(2.10) Let $x, y \in X$ be such that $x \star y = y$. Then

$$x = 0 \star x \quad (\text{by (2.2)})$$

$$= (y \star y) \star x \quad (\text{by (O-3)})$$

$$= (y \star (x \star y)) \star x \quad (\text{by assumption})$$

$$= 0. \quad (\text{by (2.5)})$$

(2.11) Let $x, y \in X$ be such that

$$(y \star x) \star y = 0.$$

Then

$$\begin{aligned}
 0 &= (y \star (x \star y)) \star x && \text{(by (2.5))} \\
 &= ((y \star x) \star y) \star x && \text{(by (2.1))} \\
 &= 0 \star x && \text{(by assumption)} \\
 &= x. && \text{(by (2.2))}
 \end{aligned}$$

Hence, $x = 0$. $\square$

The following proposition provides useful identities in which certain expressions collapse to 0. These zero identities are essential tools for proving later structural results.

Proposition 2.3 (Zero identities). *In a TwO-algebra $X = (X, \star, 0)$, the following identities hold:*

$$(\forall x, y \in X)(x \star (y \star (y \star x)) = 0) \quad (2.12)$$

$$(\forall x, y \in X)(y \star (x \star (y \star x)) = 0) \quad (2.13)$$

$$(\forall x, y \in X)((y \star x) \star x \star y = 0) \quad (2.14)$$

$$(\forall x, y \in X)((y \star x) \star y \star x = 0) \quad (2.15)$$

$$(\forall x, y, z \in X)((x \star y) \star z \star ((z \star x) \star y) = 0) \quad (2.16)$$

$$(\forall x, y, z \in X)((x \star y) \star (z \star (x \star z))) \star y = 0 \quad (2.17)$$

$$(\forall x, y \in X)((x \star (x \star y)) \star y = 0) \quad (2.18)$$

$$(\forall x, y, z \in X)((x \star ((y \star (y \star z)) \star x)) \star z = 0) \quad (2.19)$$

Proof. (2.12) Let $x, y \in X$. Then

$$\begin{aligned}
 (x \star (y \star (y \star x))) \star x &= x \star ((y \star (y \star x)) \star x) && \text{(by (2.1))} \\
 &= ((x \star y) \star (y \star x)) \star x && \text{(by (TwO))} \\
 &= 0 \star x && \text{(by (2.6))} \\
 &= x. && \text{(by (2.2))}
 \end{aligned}$$

By (2.10), we obtain

$$x \star (y \star (y \star x)) = 0.$$

(2.13) Let $x, y \in X$. Then

$$\begin{aligned}
 y \star (x \star (y \star x)) &= (y \star 0) \star (x \star (y \star x)) && \text{(by (O-1))} \\
 &= (y \star ((x \star (y \star x)) \star y)) \star (x \star (y \star x)) && \text{(by (2.5))} \\
 &= 0. && \text{(by (2.5))}
 \end{aligned}$$

(2.14) Let $x, y \in X$. Then

$$\begin{aligned}
 ((y \star x) \star x) \star y &= y \star ((x \star x) \star y) && \text{(by (TwO))} \\
 &= y \star (0 \star y) && \text{(by (O-3))} \\
 &= y \star y && \text{(by (2.2))} \\
 &= 0. && \text{(by (O-3))}
 \end{aligned}$$

(2.15) Let $x, y \in X$. Then

$$\begin{aligned}
 ((y \star x) \star y) \star x &= (y \star (x \star y)) \star x && \text{(by (2.1))} \\
 &= 0. && \text{(by (2.5))}
 \end{aligned}$$

(2.16) Let $x, y, z \in X$. Then

$$\begin{aligned}
 ((x \star y) \star z) \star ((z \star x) \star y) &= (z \star (z \star ((x \star y) \star z))) \star ((z \star x) \star y) && \text{(by (2.3))} \\
 &= (z \star ((z \star (x \star y)) \star z)) \star ((z \star x) \star y) && \text{(by (2.1))} \\
 &= (z \star (((z \star x) \star y) \star z)) \star ((z \star x) \star y) && \text{(by (TwO))} \\
 &= 0. && \text{(by (2.5))}
 \end{aligned}$$

(2.17) Let $x, y, z \in X$. Then

$$\begin{aligned}
 ((x \star y) \star (z \star (x \star z))) \star y &= ((x \star y) \star (z \star (x \star z))) \star (0 \star y) && \text{(by (2.2))} \\
 &= ((x \star y) \star (z \star (x \star z))) \star (((z \star (x \star z)) \star x) \star y) && \text{(by (2.5))} \\
 &= 0. && \text{(by (2.16))}
 \end{aligned}$$

(2.18) Let $x, y \in X$. Then

$$\begin{aligned}
 (x \star (x \star y)) \star y &= ((x \star (x \star y)) \star 0) \star y && \text{(by (O-1))} \\
 &= ((x \star (x \star y)) \star (y \star (x \star (x \star y)))) \star y && \text{(by (2.12))} \\
 &= 0. && \text{(by (2.5))}
 \end{aligned}$$

(2.19) Let $x, y, z \in X$. Then

$$\begin{aligned}
 (x \star ((y \star (y \star z)) \star x)) \star z &= (0 \star (x \star ((y \star (y \star z)) \star x))) \star z && \text{(by (2.2))} \\
 &= (((y \star (y \star z)) \star z) \star (x \star ((y \star (y \star z)) \star x))) \star z && \text{(by (2.18))} \\
 &= 0. && \text{(by (2.17))}
 \end{aligned}$$

□

Finally, we record some higher identities obtained from the preceding basic, cancellation-type, and zero identities. Although more technical, these identities will be useful in later arguments involving the structure of TwO-algebras.

Proposition 2.4 (Higher identities). *In a TwO-algebra $X = (X, \star, 0)$, the following identities hold:*

$$(\forall x, y, z, u \in X)((x \star ((y \star z) \star x)) \star u) \star ((x \star y) \star z) = ((x \star y) \star z) \star ((x \star u) \star ((x \star y) \star z)) \quad (2.20)$$

$$(\forall x, y, z \in X)((x \star y) \star (y \star z)) \star ((x \star z) \star ((x \star y) \star (y \star z))) = (x \star y) \star (y \star z) \quad (2.21)$$

$$(\forall x, y, z \in X)((x \star y) \star (y \star z)) \star (x \star z) = 0 \quad (2.22)$$

$$(\forall x, y, z \in X)((z \star y) \star (x \star y)) \star (x \star z) = 0 \quad (2.23)$$

$$(\forall x, y, z \in X)((x \star y) \star (x \star z)) \star (z \star y) = 0 \quad (2.24)$$

$$(\forall x, y \in X)((y \star (y \star x)) \star y = y \star (y \star (x \star y))) \quad (2.25)$$

$$(\forall x, y, z \in X)((x \star y) \star z = z \star ((z \star (x \star y)) \star z)) \quad (2.26)$$

$$(\forall x, y, z \in X)((x \star y) \star (x \star z)) \star (z \star y) = x \star (z \star (z \star x)) \quad (2.27)$$

$$(\forall x, y \in X)((x \star (x \star (y \star ((y \star x) \star y)))) \star x = (x \star y) \star x) \quad (2.28)$$

$$(\forall x, y, z \in X)(x \star (y \star ((x \star (z \star (z \star x))) \star x))) = (x \star y) \star x \quad (2.29)$$

Proof. (2.20) Let $x, y, z, u \in X$. Then

$$\begin{aligned} ((x \star ((y \star z) \star x)) \star u) \star ((x \star y) \star z) &= (((x \star y) \star z) \star x) \star u \star ((x \star y) \star z) && \text{(by (TwO))} \\ &= ((x \star y) \star z) \star ((x \star u) \star ((x \star y) \star z)). && \text{(by (TwO))} \end{aligned}$$

(2.21) Let $x, y, z \in X$. Then

$$\begin{aligned} ((x \star y) \star (y \star z)) \star ((x \star z) \star ((x \star y) \star (y \star z))) &= ((x \star ((y \star (y \star z)) \star x)) \star z) \star ((x \star y) \star (y \star z)) && \text{(by (2.20))} \\ &= 0 \star ((x \star y) \star (y \star z)) && \text{(by (2.19))} \\ &= (x \star y) \star (y \star z). && \text{(by (2.2))} \end{aligned}$$

(2.22) Let $x, y, z \in X$. Then

$$\begin{aligned} ((x \star y) \star (y \star z)) \star (x \star z) &= (((x \star y) \star (y \star z)) \star ((x \star z) \star ((x \star y) \star (y \star z)))) \star (x \star z) && \text{(by (2.21))} \\ &= 0. && \text{(by (2.5))} \end{aligned}$$

(2.23) Let $x, y, z \in X$. Then

$$\begin{aligned} ((z \star y) \star (x \star y)) \star (x \star z) &= (((z \star y) \star (x \star y)) \star (x \star z)) \star 0 && \text{(by (O-1))} \\ &= (((z \star y) \star (x \star y)) \star (x \star z)) \star (((x \star z) \star (z \star y)) \star (x \star y)) && \text{(by (2.22))} \end{aligned}$$

$$= 0. \quad (\text{by (2.16)})$$

(2.24) Let $x, y, z \in X$. Then

$$\begin{aligned} ((x \star y) \star (x \star z)) \star (z \star y) &= (((x \star y) \star (x \star z)) \star (z \star y)) \star 0 && (\text{by (O-1)}) \\ &= (((x \star y) \star (x \star z)) \star (z \star y)) \star (((z \star y) \star (x \star y)) \star (x \star z)) && (\text{by (2.23)}) \\ &= 0. && (\text{by (2.16)}) \end{aligned}$$

(2.25) Let $x, y \in X$. Then

$$\begin{aligned} (y \star (y \star x)) \star y &= y \star ((y \star x) \star y) && (\text{by (2.1)}) \\ &= y \star (y \star (x \star y)). && (\text{by (2.1)}) \end{aligned}$$

(2.26) Let $x, y, z \in X$. Then

$$\begin{aligned} (x \star y) \star z &= z \star (z \star ((x \star y) \star z)) && (\text{by (2.3)}) \\ &= z \star ((z \star (x \star y)) \star z). && (\text{by (2.1)}) \end{aligned}$$

(2.27) Let $x, y, z \in X$. Then

$$\begin{aligned} ((x \star y) \star (x \star z)) \star (z \star y) &= 0 && (\text{by (2.24)}) \\ &= x \star (z \star (z \star x)). && (\text{by (2.12)}) \end{aligned}$$

(2.28) Let $x, y \in X$. Then

$$\begin{aligned} (x \star (x \star (y \star ((y \star x) \star y)))) \star x &= (x \star (x \star (y \star (y \star (x \star y)))) \star x && (\text{by (2.1)}) \\ &= (x \star (x \star (x \star y))) \star x && (\text{by (2.3)}) \\ &= x \star ((x \star (x \star y)) \star x) && (\text{by (2.1)}) \\ &= x \star (x \star ((x \star y) \star x)) && (\text{by (2.1)}) \\ &= (x \star y) \star x. && (\text{by (2.3)}) \end{aligned}$$

(2.29) Let $x, y, z \in X$. Then

$$\begin{aligned} x \star (y \star ((x \star (z \star (z \star x))) \star x)) &= x \star (y \star (0 \star x)) && (\text{by (2.12)}) \\ &= x \star (y \star x) && (\text{by (2.2)}) \\ &= (x \star y) \star x. && (\text{by (2.1)}) \end{aligned}$$

□

After establishing the fundamental identities of TwO-algebras, we now consider a notion inherited from O-algebras. In O-algebras, primeness is defined through the behavior of the distinguished element 0. We adopt the same condition for TwO-algebras and then examine its consequence in the twisted setting.

Definition 2.3. *Borrowing the terminology from O-algebras, we say that a TwO-algebra X is prime if*

$$0 \star x = 0 \quad \text{for all } x \in X.$$

The following result shows that this inherited notion of primeness becomes very restrictive in the class of TwO-algebras.

Theorem 2.1. *Let X be a prime TwO-algebra. Then $X = \{0\}$.*

Proof. Let $x \in X$. Then

$$\begin{aligned} 0 &= 0 \star x && \text{(by the prime condition)} \\ &= x. && \text{(by (2.2))} \end{aligned}$$

Hence, $x = 0$ for all $x \in X$. Therefore, $X = \{0\}$. $\square$

Remark 2.1. *The notion of prime TwO-algebra used here is adopted directly from the corresponding notion in O-algebras. By Theorem 2.1, this notion is too strong in the class of TwO-algebras, since every prime TwO-algebra is necessarily trivial. This suggests that a different notion of primeness may be needed for the twisted setting.*

Since the inherited prime condition forces a TwO-algebra to be trivial, we next consider a weaker condition that still yields meaningful structural consequences.

Definition 2.4. *A TwO-algebra X is said to have the weak property if, for all $x, y \in X$,*

$$x \star y = 0 \quad \text{and} \quad y \star x = 0 \implies x = y.$$

The next theorem shows that, in the presence of the weak property, a one-sided zero condition is already sufficient to identify two elements.

Theorem 2.2. *Let $(X, \star, 0)$ be a TwO-algebra with the weak property. Then, for all $x, y \in X$,*

$$x \star y = 0 \implies x = y.$$

Proof. Let $x, y \in X$ be such that $x \star y = 0$. By (2.5), we have

$$(y \star (x \star y)) \star x = 0.$$

Since $x \star y = 0$, it follows that

$$(y \star 0) \star x = 0.$$

By (O-1), we obtain

$$y \star x = 0.$$

Since X has the weak property, we conclude that $x = y$. $\square$

Using the preceding theorem, we obtain stronger structural consequences of the weak property. In particular, TwO-algebras with the weak property satisfy a reconstruction identity and become commutative.

Proposition 2.5. *Let $(X, \star, 0)$ be a TwO-algebra with the weak property. Then the following identities hold:*

$$y \star (x \star y) = x$$

and

$$x \star y = y \star x$$

for all $x, y \in X$.

Proof. Let $x, y \in X$. By (2.5), we have

$$(y \star (x \star y)) \star x = 0.$$

Since X has the weak property, Theorem 2.2 implies that

$$y \star (x \star y) = x.$$

Using (2.3), we obtain

$$x \star y = y \star (y \star (x \star y)) = y \star x.$$

Hence, $x \star y = y \star x$. $\square$

As an immediate consequence, we obtain the following inverse-type identities.

Corollary 2.1. *Let $(X, \star, 0)$ be a TwO-algebra with the weak property. Then*

$$(x \star y) \star x = y$$

and

$$x \star (x \star y) = y$$

for all $x, y \in X$.

Proof. By Proposition 2.5, we have

$$x \star y = y \star x.$$

Also, by Proposition 2.5, with x and y interchanged, we obtain

$$x \star (y \star x) = y.$$

Thus,

$$(x \star y) \star x = x \star (y \star x) = y$$

by (2.1). Moreover,

$$x \star (x \star y) = x \star (y \star x) = y,$$

where the first equality follows from commutativity. $\square$

3. SUBSTRUCTURES: SUBALGEBRAS AND IDEALS

In this section, we study two natural types of substructures of TwO-algebras, namely subalgebras and ideals. We first characterize subalgebras in terms of the distinguished element 0 and closure under the binary operation $\star$. We then introduce ideals and show that every ideal of a TwO-algebra is a subalgebra. Although the converse does not hold in general, we prove that it holds for TwO-algebras with the weak property. Consequently, in this setting, subalgebras and ideals coincide.

Definition 3.1. Let $(X, \star, 0)$ be a TwO-algebra, and let S be a non-empty subset of X . Then S is called a subalgebra of $(X, \star, 0)$ if $(S, \star, 0)$ is a TwO-algebra. In this case, we also refer to S simply as a subalgebra of X .

Proposition 3.1. Let S be a non-empty subset of a TwO-algebra $(X, \star, 0)$. Then S is a subalgebra of X if and only if the following conditions hold:

- (1) $0 \in S$,
- (2) $x \star y \in S$ for all $x, y \in S$.

Proof. Suppose that S is a subalgebra of X . Then $(S, \star, 0)$ is itself a TwO-algebra, where the binary operation $\star$ and the constant 0 are inherited from $(X, \star, 0)$. Hence, $0 \in S$, and S is closed under the operation $\star$. Therefore, $0 \in S$ and $x \star y \in S$ for all $x, y \in S$.

Conversely, suppose that conditions (1) and (2) hold. Then S is closed under the binary operation $\star$, and $0 \in S$. Hence, $(S, \star, 0)$ is an algebra of type $(2, 0)$, where $\star$ and 0 are inherited from $(X, \star, 0)$. Since $(X, \star, 0)$ is a TwO-algebra, the identities (O-1), (TwO), and (O-3) hold for all elements of X . In particular, they hold for all elements of S , because $S \subseteq X$. Therefore, $(S, \star, 0)$ is a TwO-algebra. Hence, S is a subalgebra of X . $\square$

The following example illustrates how Proposition 3.1 can be used to verify that a given subset is a subalgebra.

Example 3.1. Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set equipped with a binary operation $\star$ given by the following Cayley table:

$\star$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	0	3	2	6	0	4
2	2	4	0	5	5	3	0
3	3	6	5	0	0	2	5
4	4	2	5	0	0	6	5
5	5	0	3	2	6	0	4
6	6	3	0	5	5	4	0

Consider the subset $S = \{0, 2, 3, 5\}$ of X . It is easy to verify that $0 \in S$ and that

$$x \star y \in S \quad \text{for all } x, y \in S.$$

Therefore, by Proposition 3.1, S is a subalgebra of X .

We now introduce ideals of TwO-algebras. Compared with subalgebras, ideals are defined by a stronger absorption-type condition involving the distinguished element 0 and the binary operation $\star$.

Definition 3.2. Let $(X, \star, 0)$ be a TwO-algebra, and let I be a non-empty subset of X . Then I is called an ideal of $(X, \star, 0)$ if the following conditions hold:

- (1) $0 \in I$,
- (2) whenever $x \star y \in I$ and $y \in I$, then $x \in I$.

In this case, we also refer to I simply as an ideal of X .

We next give an example of an ideal. This example also illustrates Theorem 3.1, since the ideal obtained is automatically a subalgebra.

Example 3.2. Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set equipped with a binary operation $\star$ given by the following Cayley table:

$\star$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	0	3	6	6	0	4
2	2	4	0	5	1	3	0
3	3	6	5	0	0	6	5
4	4	6	5	0	0	6	1
5	5	0	3	6	6	0	3
6	6	4	0	5	1	3	0

Consider the subset $I = \{0, 1, 5\}$ of X . From the table, one checks that $0 \in I$, and whenever $x \star y \in I$ with $y \in I$, it follows that $x \in I$. Hence, by Definition 3.2, I is an ideal of X . Consequently, by Theorem 3.1, I is also a subalgebra of X .

The next theorem establishes the fundamental relation between ideals and subalgebras in TwO-algebras.

Theorem 3.1. *Let $(X, \star, 0)$ be a TwO-algebra. Then every ideal of X is a subalgebra of X .*

Proof. Let I be an ideal of X . By Definition 3.2, we have $0 \in I$. Let $x, y \in I$. By (2.14), we have

$$((x \star y) \star y) \star x = 0 \in I.$$

Since I is an ideal of X and $x \in I$, it follows that

$$(x \star y) \star y \in I.$$

Again, since I is an ideal of X and $y \in I$, we obtain

$$x \star y \in I.$$

Thus, I is closed under the binary operation $\star$. Therefore, by Proposition 3.1, I is a subalgebra of X . $\square$

The preceding result shows that every ideal of a TwO-algebra is a subalgebra. The following example demonstrates that the converse is not true in general.

Example 3.3. *Consider the TwO-algebra X given in Example 3.1. Let*

$$S = \{0, 2, 3, 5\}.$$

From the Cayley table in Example 3.1, one checks that $0 \in S$ and

$$x \star y \in S$$

for all $x, y \in S$. Hence, by Proposition 3.1, S is a subalgebra of X .

However, S is not an ideal of X . Indeed,

$$1 \star 5 = 0 \in S \quad \text{and} \quad 5 \in S,$$

but

$$1 \notin S.$$

Therefore, S does not satisfy Definition 3.2. This example shows that a subalgebra of X need not be an ideal of X .

The preceding example shows that a subalgebra of a TwO-algebra need not be an ideal in general. However, the converse of Theorem 3.1 holds under the weak property.

Theorem 3.2. *Let $(X, \star, 0)$ be a TwO-algebra with the weak property. Then every subalgebra of X is an ideal of X .*

Proof. Let S be a subalgebra of X . By Proposition 3.1, we have $0 \in S$.

Let $x, y \in X$ be such that

$$x \star y \in S \quad \text{and} \quad y \in S.$$

Since S is a subalgebra of X , it is closed under $\star$. Hence,

$$y \star (x \star y) \in S.$$

Since X has the weak property, Proposition 2.5 gives

$$y \star (x \star y) = x.$$

Therefore, $x \in S$. Thus, S satisfies Definition 3.2, and hence S is an ideal of X . □

Corollary 3.1. *Let $(X, \star, 0)$ be a TwO-algebra with the weak property. Then a non-empty subset S of X is an ideal of X if and only if S is a subalgebra of X .*

Proof. This follows immediately from Theorems 3.1 and 3.2. □

4. HOMOMORPHISMS, KERNELS, AND QUOTIENTS

In this section, we study structure-preserving maps between TwO-algebras. We first introduce homomorphisms and their kernels, and show that the kernel of a homomorphism is an ideal, and hence a subalgebra. We then examine how ideals and subalgebras behave under homomorphisms, including preimages of ideals and images of ideals under injective homomorphisms. Finally, we study congruence relations and use them to construct quotient TwO-algebras.

Definition 4.1. *Let $(X, \star, 0)$ and $(Y, \bullet, 0')$ be TwO-algebras. A function $f : X \rightarrow Y$ is called a homomorphism if*

$$f(a \star b) = f(a) \bullet f(b) \tag{Hom}$$

for all $a, b \in X$.

In particular, if $X = Y$, $\star = \bullet$, and $0 = 0'$, then a homomorphism $f : X \rightarrow X$ is called an endomorphism.

For a homomorphism, the elements mapped to the distinguished element of the codomain play an important role. This leads to the following notion of the kernel, which will be used to connect homomorphisms with ideals and subalgebras.

Definition 4.2. *Let $(X, \star, 0)$ and $(Y, \bullet, 0')$ be TwO-algebras. Let $f : X \rightarrow Y$ be a homomorphism. The set*

$$\ker f = \{x \in X : f(x) = 0'\}$$

is called the kernel of f .

Before studying kernels in detail, we record two elementary consequences of the definition of a homomorphism. These facts will be used repeatedly in the results that follow.

Proposition 4.1. *Let $(X, \star, 0)$ and $(Y, \bullet, 0')$ be TwO-algebras. Let $f : X \rightarrow Y$ be a homomorphism. Then:*

- (1) $f(0) = 0'$,
- (2) $x \star y = 0$ implies that $f(x) \bullet f(y) = 0'$ for all $x, y \in X$.

Proof. Let $f : X \rightarrow Y$ be a homomorphism. Then

$$\begin{aligned} f(0) &= f(0 \star 0) && \text{(by (O-3))} \\ &= f(0) \bullet f(0) && \text{(by (Hom))} \\ &= 0'. && \text{(by (O-3))} \end{aligned}$$

Let $x, y \in X$ be such that $x \star y = 0$. Then

$$\begin{aligned} f(x) \bullet f(y) &= f(x \star y) && \text{(by (Hom))} \\ &= f(0) \\ &= 0'. && \text{(by (1))} \end{aligned}$$

□

We now relate kernels to the ideal structure introduced in the preceding section. The next result shows that the kernel of every homomorphism is an ideal.

Proposition 4.2. *Let $(X, \star, 0)$ and $(Y, \bullet, 0')$ be TwO-algebras, and let $f : X \rightarrow Y$ be a homomorphism. Then $\ker f$ is an ideal of X .*

Proof. By Proposition 4.1(1), we have $f(0) = 0'$. Hence, $0 \in \ker f$.

Now let $x, y \in X$ be such that $x \star y \in \ker f$ and $y \in \ker f$. Then

$$f(x \star y) = 0' \quad \text{and} \quad f(y) = 0'.$$

Since f is a homomorphism, we obtain

$$\begin{aligned} 0' &= f(x \star y) = f(x) \bullet f(y) \\ &= f(x) \bullet 0' \\ &= f(x) && \text{(by (O-1))} \end{aligned}$$

Thus, $f(x) = 0'$, and so $x \in \ker f$.

Therefore, $\ker f$ is an ideal of X .

□

The preceding result can be extended from kernels to preimages of arbitrary ideals. This shows that homomorphisms preserve ideal structure in the inverse direction.

Proposition 4.3. *Let $(X, \star, 0)$ and $(Y, \bullet, 0')$ be TwO-algebras, and let $f : X \rightarrow Y$ be a homomorphism. If J is an ideal of Y , then $f^{-1}(J)$ is an ideal of X .*

Proof. Since J is an ideal of Y , we have $0' \in J$. By Proposition 4.1(1), $f(0) = 0'$, and hence $0 \in f^{-1}(J)$.

Let $x, y \in X$ be such that

$$x \star y \in f^{-1}(J) \quad \text{and} \quad y \in f^{-1}(J).$$

Then

$$f(x \star y) \in J \quad \text{and} \quad f(y) \in J.$$

Since f is a homomorphism, we have

$$f(x) \bullet f(y) = f(x \star y) \in J.$$

Because J is an ideal of Y and $f(y) \in J$, it follows that

$$f(x) \in J.$$

Thus, $x \in f^{-1}(J)$. Therefore, $f^{-1}(J)$ is an ideal of X . □

As a consequence, kernels inherit the subalgebra structure from ideals. We also note that homomorphisms preserve subalgebras through their images.

Corollary 4.1. *Let $(X, \star, 0)$ and $(Y, \bullet, 0')$ be TwO-algebras, and let $f : X \rightarrow Y$ be a homomorphism. Then $\ker f$ is a subalgebra of X .*

Proof. Since $\ker f$ is an ideal of X by Proposition 4.2, the conclusion follows immediately from Theorem 3.1. □

Lemma 4.1. *Let $(X, \star, 0)$ and $(Y, \bullet, 0')$ be TwO-algebras. Let $f : X \rightarrow Y$ be a homomorphism, and let S be a subalgebra of X . Then $f(S)$ is a subalgebra of Y .*

Proof. Since S is a subalgebra of X , we have $0 \in S$. By Proposition 4.1(1), it follows that

$$0' = f(0) \in f(S).$$

Let $u, v \in f(S)$. Then there exist $x, y \in S$ such that

$$u = f(x) \quad \text{and} \quad v = f(y).$$

Since S is a subalgebra of X , we have $x \star y \in S$. Moreover,

$$u \bullet v = f(x) \bullet f(y)$$

$$= f(x \star y) \quad (\text{by (Hom)})$$

and hence $u \star v \in f(S)$.

Therefore, $f(S)$ contains $0'$ and is closed under the binary operation $\star$. By Proposition 3.1, $f(S)$ is a subalgebra of Y . $\square$

The preceding lemma shows that homomorphisms preserve subalgebras through their images. For ideals, an additional condition is needed. The next result shows that injective homomorphisms preserve ideals through their images.

Proposition 4.4. *Let $(X, \star, 0)$ and $(Y, \bullet, 0')$ be TwO-algebras, and let $f : X \rightarrow Y$ be an injective homomorphism. If I is an ideal of X , then $f(I)$ is an ideal of $f(X)$.*

Proof. By Lemma 4.1, with $S = X$, the set $f(X)$ is a subalgebra of Y . Hence $f(X)$ is a TwO-algebra under the operation inherited from Y . By Proposition 4.1(1), we have $f(0) = 0'$. Since $0 \in I$, it follows that

$$0' = f(0) \in f(I).$$

Let $a, b \in f(X)$ be such that

$$a \bullet b \in f(I) \quad \text{and} \quad b \in f(I).$$

Since $a, b \in f(X)$, there exist $x, y \in X$ such that

$$a = f(x) \quad \text{and} \quad b = f(y).$$

Since $b \in f(I)$, there exists $v \in I$ such that

$$f(y) = f(v).$$

As f is injective, we obtain $y = v$, and hence $y \in I$.

Moreover,

$$a \bullet b = f(x) \bullet f(y) = f(x \star y) \in f(I).$$

Thus, there exists $u \in I$ such that

$$f(x \star y) = f(u).$$

Since f is injective, it follows that

$$x \star y = u \in I.$$

Since I is an ideal of X and $y \in I$, we obtain $x \in I$. Therefore,

$$a = f(x) \in f(I).$$

Hence, $f(I)$ is an ideal of $f(X)$. $\square$

Corollary 4.2. *Let $(X, \star, 0)$ and $(Y, \bullet, 0')$ be TwO-algebras, and let $f : X \rightarrow Y$ be a bijective homomorphism. If I is an ideal of X , then $f(I)$ is an ideal of Y .*

Proof. Since f is bijective, it is injective and $f(X) = Y$. Hence the result follows from Proposition 4.4. $\square$

A homomorphism naturally induces a relation on its domain by identifying elements that have the same image. This relation will be used to construct congruences and quotient TwO-algebras.

Definition 4.3. *Let $f : X \rightarrow Y$ be a homomorphism. The relation $\sim_f$ on X defined by*

$$x \sim_f y \iff f(x) = f(y)$$

for all $x, y \in X$ is called the relation induced by f .

We first verify that the relation induced by a homomorphism is an equivalence relation. This provides the basic equivalence classes needed for the quotient construction.

Lemma 4.2. *Let $f : X \rightarrow Y$ be a homomorphism. Then the relation $\sim_f$ defined in Definition 4.3 is an equivalence relation on X .*

Proof. We show that $\sim_f$ is reflexive, symmetric, and transitive.

Reflexive: For every $x \in X$, we have $f(x) = f(x)$. Hence, $x \sim_f x$.

Symmetric: Let $x, y \in X$ be such that $x \sim_f y$. Then $f(x) = f(y)$. Thus, $f(y) = f(x)$, and so $y \sim_f x$.

Transitive: Let $x, y, z \in X$ be such that $x \sim_f y$ and $y \sim_f z$. Then $f(x) = f(y)$ and $f(y) = f(z)$. Therefore, $f(x) = f(z)$, and hence $x \sim_f z$.

Thus, $\sim_f$ is an equivalence relation on X . $\square$

To construct quotient algebras, an equivalence relation must be compatible with the binary operation. This motivates the following definition.

Definition 4.4. *An equivalence relation $\sim$ on a TwO-algebra X is called a congruence if, for all $x, y, u, v \in X$,*

$$x \sim y \quad \text{and} \quad u \sim v \implies (x \star u) \sim (y \star v).$$

The next step is to show that this equivalence relation is compatible with the binary operation of the TwO-algebra. Hence, it is a congruence relation and can be used to define a quotient algebra.

Lemma 4.3. *Let $f : X \rightarrow Y$ be a homomorphism. Then the relation $\sim_f$ defined in Definition 4.3 is a congruence on X .*

Proof. By Lemma 4.2, the relation $\sim_f$ is an equivalence relation on X . Let $x, y, u, v \in X$ be such that

$$x \sim_f y \quad \text{and} \quad u \sim_f v.$$

Then

$$f(x) = f(y) \quad \text{and} \quad f(u) = f(v).$$

Since f is a homomorphism, we have

$$\begin{aligned} f(x \star u) &= f(x) \cdot f(u) \\ &= f(y) \cdot f(v) \\ &= f(y \star v). \end{aligned}$$

Hence, $(x \star u) \sim_f (y \star v)$. Therefore, $\sim_f$ is a congruence on X . $\square$

The preceding results show that homomorphisms naturally give rise to congruences. We now consider quotient structures from a more general viewpoint. Let $\sim$ be a congruence on a TwO-algebra $(X, \star, 0)$, and let $X/\sim$ be the set of all equivalence classes of X with respect to $\sim$. For each $x \in X$, denote by $[x]$ the equivalence class containing x . Define a binary operation $\diamond$ on $X/\sim$ by

$$[x] \diamond [y] = [x \star y]$$

for all $x, y \in X$. Since $\sim$ is a congruence, this operation is well-defined. The next theorem shows that the resulting quotient structure remains within the class of TwO-algebras.

Theorem 4.1. *Let $\sim$ be a congruence on a TwO-algebra $(X, \star, 0)$. Then $(X/\sim, \diamond, [0])$ is a TwO-algebra.*

Proof. Since $\sim$ is a congruence, the binary operation $\diamond$ on $X/\sim$ is well-defined. Let $[x], [y], [z] \in X/\sim$.

To verify (O-1), let $[x] \in X/\sim$. Then

$$\begin{aligned} [x] \diamond [0] &= [x \star 0] \\ &= [x]. \end{aligned} \quad \text{(by (O-1))}$$

To verify (TwO), let $[x], [y], [z] \in X/\sim$. Then

$$\begin{aligned} [z] \diamond ([x] \diamond [y]) &= [z] \diamond ([x \star y]) \\ &= [z] \diamond [(x \star y) \star z] \\ &= [z \star ((x \star y) \star z)] \\ &= [(z \star x) \star y] \star z \quad \text{(by (TwO))} \\ &= [(z \star x) \star y] \diamond [z] \\ &= ([z] \diamond [x]) \diamond [y]. \end{aligned}$$

To verify (O-3), let $[x] \in X/\sim$. Then

$$[x] \diamond [x] = [x \star x]$$

$$= [0]. \quad (\text{by (O-3)})$$

Hence, $(X/\sim, \diamond, [0])$ is a TwO-algebra. $\square$

Since every relation induced by a homomorphism is a congruence by Lemma 4.3, the preceding theorem immediately yields the quotient TwO-algebra associated with a homomorphism.

Corollary 4.3. *Let $f : X \rightarrow Y$ be a homomorphism, and let $\sim_f$ be the relation on X induced by f . Then $(X/\sim_f, \diamond, [0])$ is a TwO-algebra.*

Proof. By Lemma 4.3, the relation $\sim_f$ induced by f is a congruence on X . Hence the result follows from Theorem 4.1. $\square$

The following theorem gives a kernel criterion for injectivity of endomorphisms on TwO-algebras with the weak property. It shows that, under the weak property, a homomorphism is one-to-one precisely when its kernel is trivial.

Theorem 4.2. *Let $f : X \rightarrow X$ be a homomorphism, and assume that X has the weak property. Then f is one-to-one if and only if*

$$\ker f = \{0\}.$$

Proof. Suppose that f is one-to-one. Let $x \in \ker f$. Then

$$f(x) = 0 = f(0)$$

by Proposition 4.1(1). Since f is one-to-one, it follows that $x = 0$. Hence,

$$\ker f = \{0\}.$$

Conversely, suppose that

$$\ker f = \{0\}.$$

Let $x, y \in X$ be such that

$$f(x) = f(y).$$

Then

$$f(x \star y) = f(x) \star f(y) \quad (\text{by (Hom)})$$

$$= f(x) \star f(x)$$

$$= 0. \quad (\text{by (O-3)})$$

Thus, $x \star y \in \ker f$, and so $x \star y = 0$.

Similarly,

$$f(y \star x) = f(y) \star f(x) \quad (\text{by (Hom)})$$

$$\begin{aligned}
&= f(y) \star f(y) \\
&= 0. \qquad \qquad \qquad \text{(by (O-3))}
\end{aligned}$$

Hence, $y \star x \in \ker f$, and so $y \star x = 0$.

Since X has the weak property, we conclude that $x = y$. Therefore, f is one-to-one. $\square$

The next notion concerns a special type of endomorphism whose action becomes stable after one application. This condition will be used to relate injectivity to the identity map in TwO-algebras with the weak property.

Definition 4.5. An endomorphism $f : X \rightarrow X$ is called idempotent if

$$f(f(x)) = f(x) \qquad \text{(Idm)}$$

for all $x \in X$.

For idempotent endomorphisms, injectivity imposes a strong restriction. The following theorem shows that, on a TwO-algebra with the weak property, an idempotent endomorphism is one-to-one exactly when it is the identity map.

Theorem 4.3. Let $(X, \star, 0)$ be a TwO-algebra with the weak property, and let $f : X \rightarrow X$ be an idempotent endomorphism. Then f is one-to-one if and only if f is the identity map on X .

Proof. Suppose that f is one-to-one. Let $x \in X$. Then

$$\begin{aligned}
f(x \star f(x)) &= f(x) \star f(f(x)) && \text{(by (Hom))} \\
&= f(x) \star f(x) && \text{(by (Idm))} \\
&= 0 && \text{(by (O-3))} \\
&= f(0). && \text{(by Proposition 4.1(1))}
\end{aligned}$$

Since f is one-to-one, it follows that

$$x \star f(x) = 0.$$

Similarly,

$$\begin{aligned}
f(f(x) \star x) &= f(f(x)) \star f(x) && \text{(by (Hom))} \\
&= f(x) \star f(x) && \text{(by (Idm))} \\
&= 0 && \text{(by (O-3))} \\
&= f(0). && \text{(by Proposition 4.1(1))}
\end{aligned}$$

Hence,

$$f(x) \star x = 0.$$

Since X has the weak property, we conclude that

$$f(x) = x$$

for all $x \in X$. Therefore, f is the identity map on X .

Conversely, suppose that f is the identity map on X . Let $x, y \in X$ be such that

$$f(x) = f(y).$$

Then

$$x = f(x) = f(y) = y.$$

Hence, f is one-to-one. □

5. KRIB MAPS AND REGULARITY

In this section, we study a special class of self-maps on TwO-algebras, called krib maps. These maps are defined by compatibility conditions with the binary operation on either the left or the right side. We first introduce right krib maps, left krib maps, and krib maps, and show that the collection of all right krib maps forms a monoid under function composition. We also examine how krib maps behave under invariant subalgebras and quotient constructions. Finally, we investigate regular and irregular krib maps and obtain characterizations of regularity in terms of identities involving $\alpha(x)$ and x .

Before introducing particular classes of self-maps, we record a general notion describing when a subset is preserved by a self-map.

Definition 5.1. Let S be a subset of a TwO-algebra X , and let $\tau : X \rightarrow X$ be a self-map. We say that S is τ -invariant if

$$\tau(S) \subseteq S.$$

We begin by introducing krib maps, which are self-maps compatible with the binary operation of a TwO-algebra in one variable. Depending on whether the compatibility occurs on the right or on the left, we obtain right krib maps and left krib maps.

Definition 5.2. Let $(X, \star, 0)$ be a TwO-algebra, and let $\alpha : X \rightarrow X$ be a self-map. Then α is called a right krib map if

$$\alpha(x \star y) = x \star \alpha(y) \tag{RK}$$

for all $x, y \in X$.

The map α is called a left krib map if

$$\alpha(x \star y) = \alpha(x) \star y \tag{LK}$$

for all $x, y \in X$.

The map α is called a **krib map** if it is both a right krib map and a left krib map.

Denote by $\text{RK}(X)$ the collection of all right krib maps on X .

After defining right krib maps, it is natural to ask whether they are stable under composition. The next result shows that the collection of all right krib maps on a TwO-algebra carries a natural monoid structure.

Proposition 5.1. *Let $(X, \star, 0)$ be a TwO-algebra. Then $\text{RK}(X)$ forms a monoid under function composition $\circ$.*

Proof. Let 1_X denote the identity map on X . For all $x, y \in X$, we have

$$1_X(x \star y) = x \star y = x \star 1_X(y).$$

Hence, $1_X \in \text{RK}(X)$.

Now let $\alpha, \beta \in \text{RK}(X)$. Then, for all $x, y \in X$,

$$\begin{aligned} (\alpha \circ \beta)(x \star y) &= \alpha(\beta(x \star y)) \\ &= \alpha(x \star \beta(y)) && \text{(by (RK))} \\ &= x \star \alpha(\beta(y)) && \text{(by (RK))} \\ &= x \star (\alpha \circ \beta)(y). \end{aligned}$$

Thus, $\alpha \circ \beta \in \text{RK}(X)$.

Finally, let $\alpha, \beta, \gamma \in \text{RK}(X)$. For all $x \in X$, we have

$$\begin{aligned} ((\alpha \circ \beta) \circ \gamma)(x) &= (\alpha \circ \beta)(\gamma(x)) \\ &= \alpha(\beta(\gamma(x))) \\ &= \alpha((\beta \circ \gamma)(x)) \\ &= (\alpha \circ (\beta \circ \gamma))(x). \end{aligned}$$

Hence,

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma).$$

Therefore, $(\text{RK}(X), \circ, 1_X)$ forms a monoid. □

Proposition 5.2. *Let S be a subalgebra of a TwO-algebra X , and let $\alpha : X \rightarrow X$ be a self-map such that S is α -invariant. Then the following statements hold:*

- (1) *if α is a right krib map on X , then $\alpha|_S$ is a right krib map on S ;*
- (2) *if α is a left krib map on X , then $\alpha|_S$ is a left krib map on S ;*
- (3) *if α is a krib map on X , then $\alpha|_S$ is a krib map on S .*

Proof. Since S is α -invariant, the restriction $\alpha|_S : S \rightarrow S$ is well-defined.

For (1), suppose that α is a right krib map on X . Let $x, y \in S$. Then

$$\alpha|_S(x \star y) = \alpha(x \star y) = x \star \alpha(y) = x \star \alpha|_S(y).$$

Hence, $\alpha|_S$ is a right krib map on S .

For (2), suppose that α is a left krib map on X . Let $x, y \in S$. Then

$$\alpha|_S(x \star y) = \alpha(x \star y) = \alpha(x) \star y = \alpha|_S(x) \star y.$$

Hence, $\alpha|_S$ is a left krib map on S .

Finally, (3) follows immediately from (1) and (2). $\square$

We next relate krib maps to quotient TwO-algebras. If a self-map preserves a congruence relation, then it induces a well-defined self-map on the corresponding quotient. The following result shows that the krib property is inherited by this induced map.

Proposition 5.3. *Let $\sim$ be a congruence on a TwO-algebra X , and let $\alpha : X \rightarrow X$ be a self-map such that*

$$x \sim y \Rightarrow \alpha(x) \sim \alpha(y).$$

Define $\bar{\alpha} : X/\sim \rightarrow X/\sim$ by

$$\bar{\alpha}([x]) = [\alpha(x)].$$

Then the following statements hold:

- (1) *if α is a right krib map on X , then $\bar{\alpha}$ is a right krib map on $X/\sim$;*
- (2) *if α is a left krib map on X , then $\bar{\alpha}$ is a left krib map on $X/\sim$;*
- (3) *if α is a krib map on X , then $\bar{\alpha}$ is a krib map on $X/\sim$.*

Proof. The assumption

$$x \sim y \Rightarrow \alpha(x) \sim \alpha(y)$$

ensures that $\bar{\alpha}$ is well-defined.

For (1), suppose that α is a right krib map on X . Let $[x], [y] \in X/\sim$. Then

$$\begin{aligned} \bar{\alpha}([x] \diamond [y]) &= \bar{\alpha}([x \star y]) \\ &= [\alpha(x \star y)] \\ &= [x \star \alpha(y)] && \text{(by (RK))} \\ &= [x] \diamond [\alpha(y)] \\ &= [x] \diamond \bar{\alpha}([y]). \end{aligned}$$

Hence, $\bar{\alpha}$ is a right krib map on $X/\sim$.

For (2), suppose that α is a left krib map on X . Let $[x], [y] \in X/\sim$. Then

$$\begin{aligned}
 \bar{\alpha}([x] \diamond [y]) &= \bar{\alpha}([x \star y]) \\
 &= [\alpha(x \star y)] \\
 &= [\alpha(x) \star y] && \text{(by (LK))} \\
 &= [\alpha(x)] \diamond [y] \\
 &= \bar{\alpha}([x]) \diamond [y].
 \end{aligned}$$

Hence, $\bar{\alpha}$ is a left krib map on $X/\sim$.

Finally, (3) follows immediately from (1) and (2). $\square$

We next distinguish self-maps according to their behavior at the distinguished element 0. This leads to the notions of regular and irregular maps.

Definition 5.3. Let $(X, \star, 0)$ be a TwO-algebra, and let $\alpha : X \rightarrow X$ be a self-map. The map α is called **regular** if $\alpha(0) = 0$. Otherwise, α is called **irregular**.

Before discussing regularity in detail, we record a basic formula for right krib maps. This formula shows that the value of a right krib map at any element is determined by its value at the distinguished element 0.

Lemma 5.1. Let α be a right krib map on a TwO-algebra $(X, \star, 0)$. Then

$$\alpha(x) = x \star \alpha(0)$$

for all $x \in X$.

Proof. Let $x \in X$. Then

$$\begin{aligned}
 \alpha(x) &= \alpha(x \star 0) && \text{(by (O-1))} \\
 &= x \star \alpha(0). && \text{(by (RK))}
 \end{aligned}$$

$\square$

As a direct consequence of the preceding lemma, regular right krib maps are completely determined.

Corollary 5.1. Let α be a right krib map on a TwO-algebra X . Then α is regular if and only if $\alpha = 1_X$.

Proof. If α is regular, then $\alpha(0) = 0$. By Lemma 5.1,

$$\alpha(x) = x \star \alpha(0) = x \star 0 = x$$

for all $x \in X$. Hence, $\alpha = 1_X$.

Conversely, the identity map is clearly regular. $\square$

The preceding formula and the definition of irregularity yield the following immediate consequence for right krib maps.

Corollary 5.2. *Let α be an irregular right krib map on a TwO-algebra $(X, \star, 0)$. Then*

$$x \star \alpha(x) \neq 0$$

for all $x \in X$.

Proof. Let $x \in X$. Then

$$x \star \alpha(x) = \alpha(x \star x) \quad (\text{by (RK)})$$

$$= \alpha(0) \quad (\text{by (O-3)})$$

$$\neq 0, \quad (\text{by Definition 5.3})$$

since α is irregular. □

The following results characterize regularity of right and left krib maps in terms of simple identities involving $\alpha(x)$ and x . These characterizations are then combined to obtain an equivalent formulation for krib maps.

Proposition 5.4. *Let α be a right krib map on a TwO-algebra $(X, \star, 0)$. Then α is regular if and only if*

$$x \star \alpha(x) = 0$$

for all $x \in X$.

Proof. Suppose that α is regular. Let $x \in X$. Then

$$0 = \alpha(0) \quad (\text{by Definition 5.3})$$

$$= \alpha(x \star x) \quad (\text{by (O-3)})$$

$$= x \star \alpha(x). \quad (\text{by (RK)})$$

Hence, $x \star \alpha(x) = 0$ for all $x \in X$.

Conversely, suppose that

$$x \star \alpha(x) = 0$$

for all $x \in X$. Then

$$\alpha(0) = \alpha(x \star x) \quad (\text{by (O-3)})$$

$$= x \star \alpha(x) \quad (\text{by (RK)})$$

$$= 0. \quad (\text{by assumption})$$

Therefore, α is regular. □

Proposition 5.5. *Let α be a left krib map on a TwO-algebra $(X, \star, 0)$. Then α is regular if and only if*

$$\alpha(x) \star x = 0$$

for all $x \in X$.

Proof. Suppose that α is regular. Let $x \in X$. Then

$$0 = \alpha(0) \quad (\text{by Definition 5.3})$$

$$= \alpha(x \star x) \quad (\text{by (O-3)})$$

$$= \alpha(x) \star x. \quad (\text{by (LK)})$$

Hence, $\alpha(x) \star x = 0$ for all $x \in X$.

Conversely, suppose that

$$\alpha(x) \star x = 0$$

for all $x \in X$. Then

$$\alpha(0) = \alpha(x \star x) \quad (\text{by (O-3)})$$

$$= \alpha(x) \star x \quad (\text{by (LK)})$$

$$= 0. \quad (\text{by assumption})$$

Therefore, α is regular. □

Theorem 5.1. *Let α be a krib map on a TwO-algebra $(X, \star, 0)$. Then the following statements are equivalent:*

- (1) α is regular;
- (2) $x \star \alpha(x) = 0$ for all $x \in X$;
- (3) $\alpha(x) \star x = 0$ for all $x \in X$.

Proof. Since α is a krib map on X , it is both a right krib map and a left krib map on X . By Proposition 5.4, statements (1) and (2) are equivalent. By Proposition 5.5, statements (1) and (3) are equivalent. Therefore, statements (1), (2), and (3) are equivalent. □

6. MONIC MAPS AND PRESERVING CONDITIONS

This section continues the study of special self-maps on TwO-algebras. After investigating krib maps and regularity in the preceding section, we now introduce another class of self-maps, called monic maps. These maps are defined by using an auxiliary operation induced by the binary operation of a TwO-algebra. We first examine how monic maps behave under invariant subalgebras and quotient constructions. We then study how monic maps interact with regularity and preserving conditions, and obtain several results leading to zero maps and trivial TwO-algebras.

We begin by introducing an auxiliary operation that will be used in the definition of monic maps.

Let $(X, \star, 0)$ be a TwO-algebra. Define a binary operation $\wedge$ on X by

$$x \wedge y := y \star (y \star x) \quad (\text{W})$$

for all $x, y \in X$.

Using the operation $\wedge$, we define two one-sided versions of monic maps. The distinction between right and left monic maps comes from the order in which the two terms involving θ are combined by $\wedge$.

Definition 6.1. Let $(X, \star, 0)$ be a TwO-algebra, and let $\theta : X \rightarrow X$ be a self-map.

The map θ is called a right monic map if

$$\theta(x \star y) = (x \star \theta(y)) \wedge (\theta(x) \star y) \quad (\text{RM})$$

for all $x, y \in X$.

The map θ is called a left monic map if

$$\theta(x \star y) = (\theta(x) \star y) \wedge (x \star \theta(y)) \quad (\text{LM})$$

for all $x, y \in X$.

The map θ is called a monic map if it is both a left monic map and a right monic map.

The next result shows that monic maps can be restricted to subalgebras that are invariant under the given self-map.

Proposition 6.1. Let S be a subalgebra of a TwO-algebra $(X, \star, 0)$, and let $\theta : X \rightarrow X$ be a self-map such that $\theta(S) \subseteq S$. Then the following statements hold:

- (1) if θ is a right monic map on X , then $\theta|_S$ is a right monic map on S ;
- (2) if θ is a left monic map on X , then $\theta|_S$ is a left monic map on S ;
- (3) if θ is a monic map on X , then $\theta|_S$ is a monic map on S .

Proof. Since $\theta(S) \subseteq S$, the restriction $\theta|_S : S \rightarrow S$ is well-defined. Also, since S is a subalgebra of X , it is closed under $\star$. Hence the operation $\wedge$ restricted to S is given by the same rule

$$x \wedge y = y \star (y \star x)$$

for all $x, y \in S$.

For (1), suppose that θ is a right monic map on X . Let $x, y \in S$. Since $\theta|_S(x) = \theta(x)$ and $\theta|_S(y) = \theta(y)$, we have

$$\begin{aligned} \theta|_S(x \star y) &= \theta(x \star y) \\ &= (x \star \theta(y)) \wedge (\theta(x) \star y) \quad (\text{by (RM)}) \end{aligned}$$

$$= (x \star \theta|_S(y)) \wedge (\theta|_S(x) \star y).$$

Thus, $\theta|_S$ is a right monic map on S .

For (2), suppose that θ is a left monic map on X . Let $x, y \in S$. Then

$$\begin{aligned} \theta|_S(x \star y) &= \theta(x \star y) \\ &= (\theta(x) \star y) \wedge (x \star \theta(y)) && \text{(by (LM))} \\ &= (\theta|_S(x) \star y) \wedge (x \star \theta|_S(y)). \end{aligned}$$

Hence, $\theta|_S$ is a left monic map on S .

Finally, (3) follows immediately from (1) and (2), since a monic map is both a right monic map and a left monic map. $\square$

We next relate monic maps to quotient TwO-algebras. Since monic maps are defined in terms of the auxiliary operation $\wedge$, we first use the quotient operation $\diamond$ to define the corresponding operation on $X/\sim$ by

$$[x] \wedge_{\sim} [y] = [y] \diamond ([y] \diamond [x]) = [y \star (y \star x)] = [x \wedge y] \quad (\text{QW})$$

for all $x, y \in X$. Here $\wedge_{\sim}$ denotes the auxiliary operation associated with the quotient operation $\diamond$.

Proposition 6.2. *Let $\sim$ be a congruence on a TwO-algebra $(X, \star, 0)$, and let $\theta : X \rightarrow X$ be a self-map such that*

$$x \sim y \Rightarrow \theta(x) \sim \theta(y).$$

Define $\bar{\theta} : X/\sim \rightarrow X/\sim$ by

$$\bar{\theta}([x]) = [\theta(x)]$$

for all $x \in X$. Then the following statements hold:

- (1) *if θ is a right monic map on X , then $\bar{\theta}$ is a right monic map on $X/\sim$;*
- (2) *if θ is a left monic map on X , then $\bar{\theta}$ is a left monic map on $X/\sim$;*
- (3) *if θ is a monic map on X , then $\bar{\theta}$ is a monic map on $X/\sim$.*

Proof. First, since $\sim$ is a congruence, the operation $\wedge_{\sim}$ in (QW) is well-defined. Indeed, if

$$[x] = [x'] \quad \text{and} \quad [y] = [y'],$$

then $x \sim x'$ and $y \sim y'$. Since $\sim$ is compatible with $\star$, we obtain

$$y \star x \sim y' \star x'$$

and hence

$$y \star (y \star x) \sim y' \star (y' \star x').$$

Thus,

$$[x] \wedge_{\sim} [y] = [x'] \wedge_{\sim} [y'].$$

Also, the assumption

$$x \sim y \Rightarrow \theta(x) \sim \theta(y)$$

ensures that $\bar{\theta}$ is well-defined.

For (1), suppose that θ is a right monic map on X . Let $[x], [y] \in X/\sim$. Then

$$\begin{aligned} \bar{\theta}([x] \diamond [y]) &= \bar{\theta}([x \star y]) \\ &= [\theta(x \star y)] \\ &= [(x \star \theta(y)) \wedge (\theta(x) \star y)] && \text{(by (RM))} \\ &= [x \star \theta(y)] \wedge_{\sim} [\theta(x) \star y] \\ &= ([x] \diamond [\theta(y)]) \wedge_{\sim} ([\theta(x)] \diamond [y]) \\ &= ([x] \diamond \bar{\theta}([y])) \wedge_{\sim} (\bar{\theta}([x]) \diamond [y]). \end{aligned}$$

Hence, $\bar{\theta}$ is a right monic map on $X/\sim$.

For (2), suppose that θ is a left monic map on X . Let $[x], [y] \in X/\sim$. Then

$$\begin{aligned} \bar{\theta}([x] \diamond [y]) &= \bar{\theta}([x \star y]) \\ &= [\theta(x \star y)] \\ &= [(\theta(x) \star y) \wedge (x \star \theta(y))] && \text{(by (LM))} \\ &= [\theta(x) \star y] \wedge_{\sim} [x \star \theta(y)] \\ &= ([\theta(x)] \diamond [y]) \wedge_{\sim} ([x] \diamond [\theta(y)]) \\ &= (\bar{\theta}([x]) \diamond [y]) \wedge_{\sim} ([x] \diamond \bar{\theta}([y])). \end{aligned}$$

Hence, $\bar{\theta}$ is a left monic map on $X/\sim$.

Finally, (3) follows immediately from (1) and (2), since a monic map is both a right monic map and a left monic map. $\square$

We next introduce preserving conditions for self-maps. These conditions describe how a self-map θ interacts with each element of the TwO-algebra through the binary operation. They will be combined with monic conditions to obtain strong structural consequences.

Definition 6.2. Let $(X, \star, 0)$ be a TwO-algebra, and let $\theta : X \rightarrow X$ be a self-map.

The map θ is called self-preserving if

$$\theta(x) \star x = x \tag{SP}$$

for all $x \in X$.

The map θ is called anti-self-preserving if

$$x \star \theta(x) = x \quad (\text{ASP})$$

for all $x \in X$.

The map θ is called preserving if it is both self-preserving and anti-self-preserving.

We begin with regular monic maps. The next two propositions give identities satisfied by regular left and right monic maps. When the underlying TwO-algebra is associative, these identities reduce to simpler zero identities.

Proposition 6.3. Let θ be a regular left monic map on a TwO-algebra $(X, \star, 0)$. Then

$$(x \star \theta(x)) \star ((x \star \theta(x)) \star (\theta(x) \star x)) = 0$$

for all $x \in X$.

Proof. Let $x \in X$. Then

$$\begin{aligned} 0 &= \theta(0) && (\text{by Definition 5.3}) \\ &= \theta(x \star x) && (\text{by (O-3)}) \\ &= (\theta(x) \star x) \wedge (x \star \theta(x)) && (\text{by (LM)}) \\ &= (x \star \theta(x)) \star ((x \star \theta(x)) \star (\theta(x) \star x)). && (\text{by (W)}) \end{aligned}$$

□

Proposition 6.4. Let θ be a regular right monic map on a TwO-algebra $(X, \star, 0)$. Then

$$(\theta(x) \star x) \star ((\theta(x) \star x) \star (x \star \theta(x))) = 0$$

for all $x \in X$.

Proof. Let $x \in X$. Then

$$\begin{aligned} 0 &= \theta(0) && (\text{by Definition 5.3}) \\ &= \theta(x \star x) && (\text{by (O-3)}) \\ &= (x \star \theta(x)) \wedge (\theta(x) \star x) && (\text{by (RM)}) \\ &= (\theta(x) \star x) \star ((\theta(x) \star x) \star (x \star \theta(x))). && (\text{by (W)}) \end{aligned}$$

□

When the operation $\star$ is associative, the identities obtained above simplify to one-sided zero identities.

Corollary 6.1. *Let θ be a regular left monic map on an associative TwO-algebra $(X, \star, 0)$. Then*

$$\theta(x) \star x = 0$$

for all $x \in X$.

Proof. Let $x \in X$. By Proposition 6.3, we have

$$\begin{aligned} 0 &= (x \star \theta(x)) \star ((x \star \theta(x)) \star (\theta(x) \star x)) \\ &= ((x \star \theta(x)) \star (x \star \theta(x))) \star (\theta(x) \star x) && \text{(by associativity)} \\ &= 0 \star (\theta(x) \star x) && \text{(by (O-3))} \\ &= \theta(x) \star x. && \text{(by (2.2))} \end{aligned}$$

□

Corollary 6.2. *Let θ be a regular right monic map on an associative TwO-algebra $(X, \star, 0)$. Then*

$$x \star \theta(x) = 0$$

for all $x \in X$.

Proof. Let $x \in X$. By Proposition 6.4, we have

$$\begin{aligned} 0 &= (\theta(x) \star x) \star ((\theta(x) \star x) \star (x \star \theta(x))) \\ &= ((\theta(x) \star x) \star (\theta(x) \star x)) \star (x \star \theta(x)) && \text{(by associativity)} \\ &= 0 \star (x \star \theta(x)) && \text{(by (O-3))} \\ &= x \star \theta(x). && \text{(by (2.2))} \end{aligned}$$

□

We now combine monic maps with preserving conditions. The following results show that self-preserving left monic maps and anti-self-preserving right monic maps are automatically regular. Under associativity, these conditions force the underlying TwO-algebra to be trivial. Similar triviality results also follow when the monic map is preserving.

Proposition 6.5. *Let θ be a self-preserving left monic map on a TwO-algebra $(X, \star, 0)$. Then*

$$(x \star \theta(x)) \star ((x \star \theta(x)) \star x) = \theta(0)$$

for all $x \in X$. In particular, θ is regular.

Proof. Let $x \in X$. Then

$$\begin{aligned}
 \theta(0) &= \theta(x \star x) && \text{(by (O-3))} \\
 &= (\theta(x) \star x) \wedge (x \star \theta(x)) && \text{(by (LM))} \\
 &= x \wedge (x \star \theta(x)) && \text{(by (SP))} \\
 &= (x \star \theta(x)) \star ((x \star \theta(x)) \star x). && \text{(by (W))}
 \end{aligned}$$

To prove that θ is regular, take $x = 0$. Then

$$\theta(0) = (0 \star \theta(0)) \star ((0 \star \theta(0)) \star 0).$$

By (2.2), we have $0 \star \theta(0) = \theta(0)$. Hence,

$$\theta(0) = \theta(0) \star (\theta(0) \star 0).$$

By (O-1), it follows that

$$\theta(0) = \theta(0) \star \theta(0).$$

Therefore, by (O-3), we obtain

$$\theta(0) = 0.$$

Thus, θ is regular. □

Corollary 6.3. *Let θ be a self-preserving left monic map on an associative TwO-algebra $(X, \star, 0)$. Then*

$$X = \{0\}.$$

Proof. Let $x \in X$. By Proposition 6.5, we have

$$\theta(0) = (x \star \theta(x)) \star ((x \star \theta(x)) \star x).$$

Since $(X, \star, 0)$ is associative, it follows that

$$\theta(0) = ((x \star \theta(x)) \star (x \star \theta(x))) \star x.$$

By (O-3), we obtain

$$(x \star \theta(x)) \star (x \star \theta(x)) = 0.$$

Hence,

$$\theta(0) = 0 \star x = x$$

by (2.2).

On the other hand, Proposition 6.5 shows that θ is regular. Therefore,

$$\theta(0) = 0.$$

Thus,

$$x = \theta(0) = 0.$$

Since $x \in X$ was arbitrary, it follows that

$$X = \{0\}.$$

□

Proposition 6.6. *Let θ be an anti-self-preserving right monic map on a TwO-algebra $(X, \star, 0)$. Then*

$$(\theta(x) \star x) \star ((\theta(x) \star x) \star x) = \theta(0)$$

for all $x \in X$. In particular, θ is regular.

Proof. Let $x \in X$. Then

$$\begin{aligned} \theta(0) &= \theta(x \star x) && \text{(by (O-3))} \\ &= (x \star \theta(x)) \wedge (\theta(x) \star x) && \text{(by (RM))} \\ &= x \wedge (\theta(x) \star x) && \text{(by (ASP))} \\ &= (\theta(x) \star x) \star ((\theta(x) \star x) \star x). && \text{(by (W))} \end{aligned}$$

To prove that θ is regular, take $x = 0$. Then

$$\theta(0) = (\theta(0) \star 0) \star ((\theta(0) \star 0) \star 0).$$

By (O-1), it follows that

$$\theta(0) = \theta(0) \star \theta(0).$$

Therefore, by (O-3), we obtain

$$\theta(0) = 0.$$

Thus, θ is regular. □

Corollary 6.4. *Let θ be an anti-self-preserving right monic map on an associative TwO-algebra $(X, \star, 0)$. Then*

$$X = \{0\}.$$

Proof. Let $x \in X$. By Proposition 6.6, we have

$$\theta(0) = (\theta(x) \star x) \star ((\theta(x) \star x) \star x).$$

Since $(X, \star, 0)$ is associative, it follows that

$$\theta(0) = ((\theta(x) \star x) \star (\theta(x) \star x)) \star x.$$

By (O-3), we obtain

$$(\theta(x) \star x) \star (\theta(x) \star x) = 0.$$

Hence,

$$\theta(0) = 0 \star x = x$$

by (2.2).

On the other hand, Proposition 6.6 shows that θ is regular. Therefore,

$$\theta(0) = 0.$$

Thus,

$$x = \theta(0) = 0.$$

Since $x \in X$ was arbitrary, it follows that

$$X = \{0\}.$$

□

We next consider the stronger case where the monic map is preserving. In this case, the underlying TwO-algebra is forced to be trivial without assuming associativity.

Proposition 6.7. *Let θ be a preserving left monic map on a TwO-algebra $(X, \star, 0)$. Then*

$$X = \{0\}.$$

Proof. Let $x \in X$. By Proposition 6.5, we have

$$\theta(0) = (x \star \theta(x)) \star ((x \star \theta(x)) \star x).$$

Since θ is anti-self-preserving, we obtain

$$x \star \theta(x) = x.$$

Hence,

$$\theta(0) = x \star (x \star x) = x \star 0 = x$$

by (O-3) and (O-1).

On the other hand, by Proposition 6.5, θ is regular. Thus,

$$\theta(0) = 0.$$

Therefore,

$$x = \theta(0) = 0.$$

Since $x \in X$ was arbitrary, it follows that

$$X = \{0\}.$$

□

Proposition 6.8. *Let θ be a preserving right monic map on a TwO-algebra $(X, \star, 0)$. Then*

$$X = \{0\}.$$

Proof. Let $x \in X$. By Proposition 6.6, we have

$$\theta(0) = (\theta(x) \star x) \star ((\theta(x) \star x) \star x).$$

Since θ is self-preserving, we obtain

$$\theta(x) \star x = x.$$

Hence,

$$\theta(0) = x \star (x \star x) = x \star 0 = x$$

by (O-3) and (O-1).

On the other hand, by Proposition 6.6, θ is regular. Thus,

$$\theta(0) = 0.$$

Therefore,

$$x = \theta(0) = 0.$$

Since $x \in X$ was arbitrary, it follows that

$$X = \{0\}.$$

□

The final results of this section concern zero-map behavior. Under the opposite pairing of monic and preserving conditions, the self-map is forced to be the zero function.

Proposition 6.9. *Let θ be an anti-self-preserving left monic map on a TwO-algebra $(X, \star, 0)$. Then θ is the zero function; that is,*

$$\theta(x) = 0$$

for all $x \in X$.

Proof. Let $x \in X$. Since θ is anti-self-preserving, we have

$$x \star \theta(x) = x.$$

Hence,

$$\begin{aligned} \theta(x) &= \theta(x \star \theta(x)) \\ &= (\theta(x) \star \theta(x)) \wedge (x \star \theta(\theta(x))) && \text{(by (LM))} \\ &= 0 \wedge (x \star \theta(\theta(x))) && \text{(by (O-3))} \\ &= (x \star \theta(\theta(x))) \star ((x \star \theta(\theta(x))) \star 0) && \text{(by (W))} \end{aligned}$$

$$\begin{aligned}
&= (x \star \theta(\theta(x))) \star (x \star \theta(\theta(x))) && \text{(by (O-1))} \\
&= 0. && \text{(by (O-3))}
\end{aligned}$$

Therefore, θ is the zero function. $\square$

Proposition 6.10. *Let θ be a self-preserving right monic map on a TwO-algebra $(X, \star, 0)$. Then θ is the zero function; that is,*

$$\theta(x) = 0$$

for all $x \in X$.

Proof. Let $x \in X$. Since θ is self-preserving, we have

$$\theta(x) \star x = x.$$

Hence,

$$\begin{aligned}
\theta(x) &= \theta(\theta(x) \star x) \\
&= (\theta(x) \star \theta(x)) \wedge (\theta(\theta(x)) \star x) && \text{(by (RM))} \\
&= 0 \wedge (\theta(\theta(x)) \star x) && \text{(by (O-3))} \\
&= (\theta(\theta(x)) \star x) \star ((\theta(\theta(x)) \star x) \star 0) && \text{(by (W))} \\
&= (\theta(\theta(x)) \star x) \star (\theta(\theta(x)) \star x) && \text{(by (O-1))} \\
&= 0. && \text{(by (O-3))}
\end{aligned}$$

Therefore, θ is the zero function. $\square$

The results above show that monic maps, when combined with preserving conditions, impose strong restrictions on TwO-algebras. In particular, suitable preserving assumptions force either the underlying TwO-algebra to be trivial or the corresponding monic map to be the zero function.

7. CONCLUSION

In this paper, we introduced the notion of a TwO-algebra as a twisted version of an O-algebra. The new structure was obtained by retaining two defining axioms of O-algebras and replacing the remaining axiom with a twisted identity. Through explicit examples, we showed that TwO-algebras and O-algebras are independent in the sense that each class contains examples not belonging to the other, while some algebras belong to both classes.

We developed several basic properties of TwO-algebras and organized them into basic identities, uniqueness and cancellation-type properties, zero identities, and higher identities. These identities provide useful tools for studying the internal structure of TwO-algebras. We also considered the notion

of primeness inherited from O-algebras and showed that this condition is too strong in the twisted setting, since every prime TwO-algebra is necessarily trivial. Moreover, we showed that TwO-algebras with the weak property satisfy stronger structural identities, including commutativity and inverse-type laws.

We also developed the study of substructures. We characterized subalgebras by closure conditions and proved that every ideal of a TwO-algebra is a subalgebra. We further showed that, under the weak property, subalgebras and ideals coincide. We then investigated homomorphisms, kernels, congruence relations, and quotient structures. In particular, kernels and preimages of ideals under homomorphisms were shown to be ideals, while images of ideals were studied under injective homomorphisms. Quotient structures determined by congruences were also proved to be TwO-algebras.

Finally, we introduced and studied two types of special self-maps on TwO-algebras: krib maps and monic maps. We proved that the set of right krib maps forms a monoid under function composition and examined how krib maps behave under invariant subalgebras and quotient constructions. We also obtained characterizations of regular krib maps. For monic maps, we studied their restrictions to invariant subalgebras, their induced maps on quotient TwO-algebras, and several results involving regularity, preserving conditions, zero functions, and sufficient conditions forcing the underlying TwO-algebra to be trivial.

These results form a foundation for the further study of TwO-algebras. Possible future directions include developing a non-degenerate notion of primeness suitable for the twisted setting, investigating classifications of finite TwO-algebras, studying additional types of ideals and filters, and examining categorical or representation-theoretic aspects of TwO-algebras and their homomorphisms.

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