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PSEUDO ALMOST AUTOMORPHIC MILD SOLUTIONS FOR TWO-TERM FRACTIONAL ORDER DIFFERENTIAL EQUATIONS VIA MEASURE THEORY

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Abstract. In this article, by measure theory, we introduce and study the concept of μ-Stepanov like functions. We present new result on composition theorem for the space of such functions. As applications, we establish some sufficient criteria for the existence and uniqueness of pseudo almost automorphic mild solutions to two-term fractional differential equations, under the assumption that the forcing terms of the equation are μ-Stepanov like pseudo almost automorphic function. The working tools are an operator theoretical approach based on certain families of strongly continuous operators Definition 2.1, Banach fixed point theorem and Leray-Shauder alternative theorem. An example is given for illustration.

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1. Introduction

The concept of almost automorphic functions was first introduced in the literature by Bochner [13] in relation to some aspects of differential geometry, it is a natural generalization of almost periodicity [12], [13], for more details about this topic we refer the reader to the book [29] where the author gave an important overview about the theory of almost automorphic functions and their applications to differential equations.

The generalizations of almost automorphy follow a historical developpment (see Table 1 and the references cited therein for more details). The relationship between the various almost automorphy is depicted in Fig. 1, where $MPAAS^p$ is the class of functions to be explored in this paper.

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In the last two decade, many authors have produced extensive literature on the theory of almost automorphy, pseudo almost automorphy and weighted pseudo almost automorphy due to its significance and applications in physics, mechanics, control theoy and mathematical biology. The existence of almost automorphic solutions and its generalizations is among the most attractive topics in qualitative theory of differential equations, one can see [[10], [15], [44], [37]] for more details.

The concept of Stepanov-like pseudo almost automorphic functions was first studied in [16], it generalizes the Bochner almost automorphy in a natural way. N'Guérékata and Pankov [30] introduced the concept of Stepanov-like pseudo almost automorphy and applied this notion to study the existence and uniqueness of an almost automorphic solution to the autonomous semilinear equation. Mophou [41] studied the existence and uniqueness of a weighted pseudo almost automorphic mild solution to a semilinear differential equation. Moreover, Chang, Zhang and N'Guérékata [46] established some properties, new composition theorems of Stepanov-like weighted pseudo almost automorphic functions. Recently, Blot et al. [8] introduced the concept of μ-pseudo almost automorphic functions which is more general than the class of weighted pseudo almost automorphic functions. N'Guérékata et all [17] studied the existence and uniqueness of μ-pseudo almost automorphic mild solutions to nonautonomous evolution equation. Baroun et al [6], studied the existence and uniqueness of μ-pseudo almost automorphic mild solutions to semilinear parabolic evolution equation.

Fractional differential equations have gained considerable importance due to their applications in various fields of the science, such as physics, mechanics, chemistry engineering, etc. We refer to the monographs of Kibas and al. [35,36], Diethelm [25] and the papers of Agarwal and al. [2], Benchohra and al [1,7], Mophou and al [40], N'guerekata [31]. In particular, the study of abstract semilinear fractional differential equation is one of the great interests. Many works have been done to prove existence, uniqueness of the mild solutions with a prescribed qualitative property. Almost automorphy of semilinear fractional differential equations was initiated by Araya and Lizama [4]. Zhang, Chang and N'Guérékata [18] studied weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations. Therefore, many authors have made important contributions on these topics [3,19,34,41,43]. However, the study of μ-pseudo almost automorphic mild solutions for two-term fractional differential equations with the forcing terms of the equation are μ-Stepanov like pseudo almost automorphic function remains open.

The objective of this paper is to present new result on composition theorem of $\mu-$Stepanov like pseudo almost automorphic functions Theorem 3.2. We establish sufficient conditions ensuring the existence and uniqueness of $\mu-$pseudo almost automorphic mild solutions for two-term fractional differential equations in the form:

$$D_t^{\alpha}u'(t)+\gamma D_t^{\beta}u(t)=Au(t)+D_t^{\alpha}f(t,u(t)),\ t\in\mathbb{R},\ 0<\alpha\leqslant\beta<1,\ \gamma\geqslant 0, \tag{1}$$

where A is a sectorial operator of angle $\beta\ \frac{\pi}{2}$, the fractional derivative is understood in the Weyl sense and the forcing term f of the equation is μ-Stepanov-like pseudo almost automorphic. $\mathbb{X}$ is a Banach space and $BC(\mathbb{R},\mathbb{X})$ denotes the Banach space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$.

TABLE 1. Historical development of almost periodicity and almost automorphy

Function	Original reference
Almost periodic (AP)	Bochner (1927) [11]
Asymptotic almost periodic (AAP)	Fréchet (1941) [27]
Pseudo almost periodic (PAP)	Zhang (1992) [45]
Pseudo almost periodic with measure $(MPAP)$	Blot and al. (2011) [9]
Stepanov-like almost periodic (APS^p)	Stepanov (1925) [42]
Stepanov-like pseudo almost periodic $(PAPS^p)$	Diagana (2007) [20]
Almost automorphic (AA)	Bochner (1955) [14]
Asymptotic almost automorphic (AAA)	N'Guérékata (1981) [28]
Pseudo almost automorphic (PAA)	Xiao and al. (2008) [44]
Stepanov-like almost automorphic (AAS^p)	Casarino (2000) [16]
Stepanov-like pseudo almost automorphic $(PAAS^p)$	Diagana (2009) [23]
Stepanov-like pseudo almost automorphic with measure $(MPAAS^p)$	K.Ezzinbi (2016) [26]

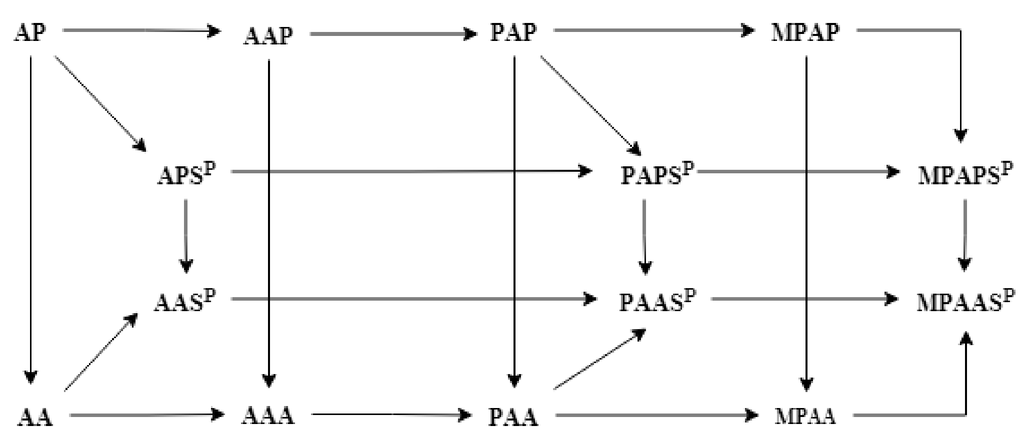

FIGURE 1. Relationship between almost periodic or automorphic functions and their extensions, where " $\longrightarrow$ " denotes subset relation " $\subset$ ".

The organization of this work is as follows: In Section 2, we recall some definitions, theorems on fractional derivative, almost automorphic functions, μ-pseudo almost automorphic functions and

μ-Stepanov-like pseudo almost automorphic functions. In Section 3, we give our main results, we prove the composition result of Stepanov μ-pseudo almost automorphic functions with few assumptions and we prove the existence and uniqueness of a μ-pseudo almost automorphic mild solution to the two-term fractional order differential equation (1) when the forcing term f is μ-Stepanov-like pseudo almost automorphic and Lipschitz continuous Theorem 3.6 and 3.7 and not Lipschitz continuous Theorem 3.8 and 3.9 using nonlinear Leray-Schauder alternative theorem. An application is given in Section 4, to illustrate the results obtained. An historical development of almost periodicity and almost automorphy is given in the end of the article.

2. Preliminaries

In what follows we recall some definitions, notations and some basic results of almost automorphy and its extensions that we need in the sequel. First, we introduce some classical notations.

Let $(\mathbb{X}, \|.\|)$, $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ be two Banach spaces, $C(\mathbb{R}, \mathbb{X})$ (resp. $C(\mathbb{R}\times\mathbb{X}, \mathbb{Y})$) stands of the set of continuous functions from $\mathbb{R}$ to $\mathbb{X}$ (resp. from $\mathbb{R}\times\mathbb{X}$ to $\mathbb{Y}$), $BC(\mathbb{R}, \mathbb{X})$ (resp. $BC(\mathbb{R}\times\mathbb{X}, \mathbb{Y})$) denotes the Banach space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$ (resp. from $\mathbb{R}\times\mathbb{X}$ to $\mathbb{Y}$), equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$.

2.1. Fractional derivative.

Let $\alpha > 0$ be given, we denote $g_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$, where Γ is the Gamma function. Given a vector-valued function $u : \mathbb{R} \to \mathbb{X}$, the Weyl fractional integral of order $\alpha > 0$ is defined by

$$D_t^{-\alpha} u(t) = \int_{-\infty}^{t} g_\alpha(t-s)u(s)ds,\ t \in \mathbb{R},$$

when the integral is convergent.

The Weyl fractional derivative $D_t^\alpha u$ of order $\alpha > 0$ is defined by

$$D_t^\alpha u(t) = \frac{d^n}{dt^n} D_t^{-(n-\alpha)} u(t),\ t\in\mathbb{R},\ \text{where } n = [\alpha] + 1.$$

It is known that $D_t^\alpha D_t^{-\alpha} u(t) = u(t)$ for any $\alpha > 0$. One can see Miller and Ross [39] for more information and further details.

2.2. Sectorial operators and their properties.

In this section, we introduce definitions, notations, lemmas and preliminary facts which are used throughout this work.

First, let us recall that a closed and densely defined linear operator A is said to be ω-sectorial if there exist $\theta \in [0, \frac{\pi}{2}[$, $M > 0$ and $\omega \in \mathbb{R}$ such that its resolvent exists in the sector

$$\omega + S_\theta := \{\omega + \lambda \ :\ \lambda \in \mathbb{C},\ |arg(\lambda)| < \frac{\pi}{2} + \theta\}\backslash\{\omega\},$$

and

$$\|(\lambda - A)^{-1}\| \leqslant \frac{M}{|\lambda - \omega|},\ \lambda \in \omega + S_\theta.$$

Let $\mathscr{L}(\mathbb{X})$ be the space of bounded linear maps from the Banach space $\mathbb{X}$ into itself.

Definition 2.1. [33] *Let $\gamma \geqslant 0$ and $0 \leqslant \alpha, \beta \leqslant 1$ be given. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space $\mathbb{X}$. We call A the generator of an $(\alpha,\beta)_\gamma$-regularized family if there exist $\omega \geqslant 0$ and a strongly continuous function $S_{\alpha,\beta} : \mathbb{R}^+ \to \mathscr{L}(\mathbb{X})$ such that $\{\lambda^{\alpha+1} + \gamma\lambda^\beta : \mathcal{R}e\lambda > \omega\} \subset \rho(A)$ and*

$$H(\lambda)x := \lambda^\alpha(\lambda^{\alpha+1} + \gamma\lambda^\beta - A)^{-1}x = \int_0^\infty \exp(-\lambda t)S_{\alpha,\beta}(t)xdt,\ \mathcal{R}e\lambda > \omega,\ x \in \mathbb{X}.$$

Lemma 2.2. [33] *Let $\alpha \geqslant 0, \beta \leqslant 1, \gamma \geqslant 0$. There exists Laplace transformable functions $a_{\alpha,\beta} \in C^1(\mathbb{R}^+)$ and $K_{\alpha,\beta} \in C^1(\mathbb{R}^+)$ such that*

$$\hat{a}_{\alpha,\beta}(\lambda) = \frac{1}{\lambda^{\alpha+1} + \gamma\lambda^\beta} \text{ and } \widehat{K}_{\alpha,\beta}(\lambda) = \frac{\lambda^\alpha}{\lambda^{\alpha+1} + \gamma\lambda^\beta}.$$

From the above lemma, note that $a_{\alpha,\beta} = g_\alpha * K_{\alpha,\beta}$. Moreover $K_{\alpha,\beta}(0) = 1$ and $K_{\alpha,\beta}$ is differentiable, for an explicit representation see [33].

Proposition 2.3. [33] *Let $0 < \alpha, \beta \leqslant 1$ and $\gamma \geqslant 0$. Let $S_{\alpha,\beta}(t)$ be an $(\alpha,\beta)_\gamma$-regularized family on $\mathbb{X}$ with generator A. Then the following assertions hold true:*

- ***(i)*** *$S_{\alpha,\beta}(t)$ is strongly continuous and $S_{\alpha,\beta}(0) = I$.*
- ***(ii)*** *For all $x \in D(A)$ and $t \geqslant 0$, we have $S_{\alpha,\beta}(t)x \in D(A)$ and $AS_{\alpha,\beta}(t)x = S_{\alpha,\beta}(t)Ax$.*
- ***(iii)*** *Let $x \in \mathbb{X}$ and $t \geqslant 0$. Then $\int_0^t a_{\alpha,\beta}(t-s)S_{\alpha,\beta}(s)xds \in D(A)$ and*

$$S_{\alpha,\beta}(t)x = K_{\alpha,\beta}(t)x + A\int_0^t (g_\alpha * K_{\alpha,\beta})(t-s)S_{\alpha,\beta}(s)xds.$$

- ***(iv)*** *For all $x \in D(A)$ we have $S_{\alpha,\beta}(.)x \in C^1(\mathbb{R}^+;\mathbb{X})$.*

Theorem 2.4. [33] *Let $0 < \alpha \leqslant \beta \leqslant 1,\ \gamma > 0$ and $\omega < 0$. Assume that A is an ω-sectorial operator of angle $\beta\frac{\pi}{2}$, then A generates an $(\alpha,\beta)_\gamma$-regularized family $S_{\alpha,\beta}(t)$ satisfying the estimate*

$$\|S_{\alpha,\beta}(t)\| \leqslant \frac{C}{1 + |\omega|(t^{\alpha+1} + \gamma t^\beta)},\ t \geqslant 0,$$

for some constant $C > 0$ depending only on α, β.

Lemma 2.5. [18] *Let $\{S(t)\}_{t\geqslant 0} \subset B(\mathbb{X})$ be a strongly continuous family of bounded and linear operators such that $\|S(t)\| \leqslant \varpi(t),\ t \in \mathbb{R}^+$, where $\varpi \in L^1(\mathbb{R}^+)$ is nonincreasing. Then, for each $f \in AAS^p(\mathbb{R},\mathbb{X})$,*

$$\int_{-\infty}^t S(t-s)f(s)ds \in AA(\mathbb{R},\mathbb{X}).$$

2.3. **Almost automorphic functions.**

Definition 2.6. [30](*Bochner*). *A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that:*

$$g(t) = \lim_{n\to\infty} f(t+s_n) \text{ is well defined for each } t \in \mathbb{R} \text{ and}$$

$$\lim_{n\to\infty} g(t-s_n) = f(t) \text{ for each } t \in \mathbb{R}.$$

The collection of all such functions will be denoted by $AA(\mathbb{R},\mathbb{X})$.

Remark 2.7. *Note that in the above limits the function g is just measurable. If the convergence in both limits is uniform in $t \in \mathbb{R}$, then f is almost periodic. The concept of almost automorphy is then larger than almost periodicity. If f is almost automorphic, then its range is relatively compact, thus bounded in norm.*

Example 2.8. *Let $K : \mathbb{R} \to \mathbb{R}$ be such that*

$$K(t) = sin\left(\frac{1}{2+cost+cos\sqrt{2}t}\right) \text{ for } t \in \mathbb{R}.$$

Then K is almost automorphic, but it is not uniformly continuous on $\mathbb{R}$. Therefore, it is not almost periodic.

Proposition 2.9. [29] *Let $f_1, f_2, f \in AA(\mathbb{R},\mathbb{X})$ and $\lambda \in \mathbb{R}$. Then, the following are true:*

- ***(i)*** $\lambda f_1 + f_2 \in AA(\mathbb{R},\mathbb{X})$.
- ***(ii)*** *The set $\{f(t) : t \in \mathbb{R}\}$ is relatively compact in $\mathbb{X}$.*
- ***(iii)*** *The space $AA(\mathbb{R},\mathbb{X})$ is translation invariant, i.e., for all $\tau \in \mathbb{R}$, $f \in AA(\mathbb{R},\mathbb{X})$ implies $f(.+\tau) \in AA(\mathbb{R},\mathbb{X})$.*
- ***(iv)*** *The space $AA(\mathbb{R},\mathbb{X})$ equipped with the supnorm is a Banach space.*
- ***(v)*** *f is bounded i.e., $\sup_{t\in\mathbb{R}} \|f(t)\| < \infty$.*

Definition 2.10. [22] *A continuous function $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ is called almost automorphic in $t \in \mathbb{R}$ uniformly in $x \in B$ where $B \subset \mathbb{X}$ is any bounded subset, if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that*

$$g(t,x) = \lim_{n\to\infty} f(t+s_n,x) \text{ is well defined for each } t \in \mathbb{R} \text{ and}$$

$$\lim_{n\to\infty} g(t-s_n,x) = f(t,x) \text{ for each } t \in \mathbb{R} \text{ uniformly in } x \in B.$$

The collection of those almost automorphic functions is denoted by $AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$.

Theorem 2.11. [22] *Let $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ be an almost automorphic function. Suppose that f is Lipschitzian in $t \in \mathbb{R}$, that is, there exists $L > 0$ such that*

$$\|f(t,x) - f(t,y)\| \leqslant L\|x-y\|, \text{ for all } (x,y) \in \mathbb{X}\times\mathbb{X} \text{ and } t \in \mathbb{R}.$$

If $\varphi \in AA(\mathbb{R},\mathbb{X})$, then $F : \mathbb{R} \to \mathbb{X}$ defined by $F(.) = f(.,\varphi(.))$ belongs to $AA(\mathbb{R},\mathbb{X})$.

2.4. μ-pseudo almost automorphic functions.

Definition 2.12. [38]

(i) *A bounded continuous function with vanishing mean value can be defined as*

$$AA_0(\mathbb{R},\mathbb{X}) = \left\{ f \in BC(\mathbb{R},\mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|f(\sigma)\| d\sigma = 0 \right\}.$$

(ii) *We also define $AA_0(\mathbb{R}\times\mathbb{X},\mathbb{X})$ to be the collection of all functions* $f : t \longrightarrow f(t,x) \in BC(\mathbb{R}\times\mathbb{X},\mathbb{X})$ *satisfying*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|f(\sigma,x)\| d\sigma = 0,$$

uniformly for any bounded subset of $\mathbb{X}$.

Definition 2.13. [38] *A continuous function $f : \mathbb{R} \to \mathbb{X}$ ($\mathbb{R}\times\mathbb{X} \longrightarrow \mathbb{X}$) is said to be pseudo-almost automorphic if it can be decomposed as $f = g + \varphi$, where $g \in AA(\mathbb{R},\mathbb{X})$ ($AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$) and $\varphi \in AA_0(\mathbb{R},\mathbb{X})$ ($\varphi \in AA_0(\mathbb{R}\times\mathbb{X},\mathbb{X})$).*

Denote by $PAA(\mathbb{R},\mathbb{X})$ ($PAA(\mathbb{R}\times\mathbb{X},\mathbb{X})$) the set of all such functions.
The space $PAA(\mathbb{R},\mathbb{X})$ ($PAA(\mathbb{R}\times\mathbb{X},\mathbb{X})$) is a Banach space.

Theorem 2.14. [37] *Let $f = g + \varphi \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ with $g(t,x) \in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $\varphi(t,x) \in AA_0(\mathbb{R}\times\mathbb{X},\mathbb{X})$. Assume:*

(i) *$g(t,x)$ is uniformly continuous on any bounded subset uniformly for $t \in \mathbb{R}$.*

(ii) *There exists a nonnegative function $\mu \in L^p(\mathbb{R})$, $(1 \leqslant p \leqslant \infty)$ such that*

$$\|f(t,x) - f(t,y)\| \leqslant \mu(t)\|x-y\|, \text{ for all } x,y \in \mathbb{X} \text{ and } t \in \mathbb{R}.$$

If $x(t) \in PAA(\mathbb{R},\mathbb{X})$, then $f(.,x(.)) \in PAA(\mathbb{R},\mathbb{X})$.

In the following we denote by $\mathbb{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathbb{M}$ the set of all positive measures μ on $\mathbb{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty,$ for all $a,b \in \mathbb{R}$ $(a \leqslant b)$.

Definition 2.15. [8] *Let $\mu \in \mathbb{M}$. A bounded continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-ergodic if*

$$\lim_{r\to+\infty} \frac{1}{(\mu[-r,r])}\int_{-r}^{r} \|f(t)\|\, d\mu(t) = 0.$$

We denote the space of all such functions by $MAA_0(\mathbb{R},\mathbb{X},\mu)$.

$$MAA_0(\mathbb{R},\mathbb{X},\mu) = \left\{ f \in BC(\mathbb{R},\mathbb{X}) : \lim_{r\to+\infty} \frac{1}{\mu([-r,r])}\int_{-r}^{r} \|f(t)\| d\mu(t) = 0 \right\}.$$

Definition 2.16. [8] *Let $\mu \in \mathbb{M}$. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form $f = g + \varphi$, where $g \in AA(\mathbb{R},\mathbb{X})$ and $\varphi \in MAA_0(\mathbb{R},\mathbb{X},\mu)$.*

We denote the space of all such functions by $MPAA(\mathbb{R},\mathbb{X},\mu)$. Thus we have

$$AA(\mathbb{R},\mathbb{X}) \subset PAA(\mathbb{R},\mathbb{X}) \subset MPAA(\mathbb{R},\mathbb{X},\mu) \subset BC(\mathbb{R},\mathbb{X}).$$

Definition 2.17. [8] *A continuous function $f : \mathbb{R}\times\mathbb{X} \to \mathbb{Y}$ is said to be almost automorphic in t uniformly with respect to x in $\mathbb{X}$ if the following two conditions are hold*

(*i*) *For all $x \in \mathbb{X}$, $f(.,x) \in AA(\mathbb{R},\mathbb{Y})$.*

(*ii*) *f is uniformly continuous on each compact set K in $\mathbb{X}$ with respect to the second variable x, namely, for each compact set K in $\mathbb{X}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2 \in K$, one has*

$$\| x_1 - x_2 \| \leqslant \delta \Rightarrow \sup_{t\in\mathbb{R}} \|f(t,x_1) - f(t,x_2)\| \leqslant \varepsilon.$$

Denote by $AAU(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ the set of all such functions.

Definition 2.18. [8] *Let $\mu \in \mathbb{M}$. A continuous function $f : \mathbb{R}\times\mathbb{X} \to \mathbb{Y}$ is said to be μ-ergodic in t uniformly with respect to x in $\mathbb{X}$ if the following two conditions are true:*

(*i*) *For all $x \in \mathbb{X}$, $f(.,x) \in MAA_0(\mathbb{R},\mathbb{Y},\mu)$.*

(*ii*) *f is uniformly continuous on each compact set K in $\mathbb{X}$ with respect to the second variable x.*

Denote by $MAA_0U(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$ the set of all such functions.

Definition 2.19. [8] *Let $\mu \in \mathbb{M}$. A continuous function $f : \mathbb{R}\times\mathbb{X} \to \mathbb{Y}$ is said to be μ-pseudo almost automorphic in t uniformly with respect to x in $\mathbb{X}$ if f is written in the form: $f = g+\varphi$, where $g \in AAU(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ and $\varphi \in MAA_0U(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$.*

$MPAAU(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$ denotes the set of all such functions.
We have the following inclusion:

$$AAU(\mathbb{R}\times\mathbb{X},\mathbb{Y}) \subset MPAAU(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu).$$

As a consequence of definitions 2.17, 2.18 and 2.19 , we deduce the following result.

Proposition 2.20. [8] *Let $\mu \in \mathbb{M}$ and $f : \mathbb{R}\times\mathbb{X} \to \mathbb{Y}$ be μ-pseudo almost automorphic in t uniformly with respect to x in $\mathbb{X}$. Then*

(*i*) *For all $x \in \mathbb{X}$, $f(.,x) \in MPAA(\mathbb{R},\mathbb{Y},\mu)$.*

(*ii*) *f is uniformly continuous on each compact set K in $\mathbb{X}$ with respect to the second variable x.*

Remark 2.21. [8]

(*i*) *Without assumption on the measure μ, decomposition of the corresponding μ-pseudo almost automorphic function is not unique.*

(*ii*) *The ρ-weighted pseudo almost automorphic function is μ-pseudo almost automorphic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is ρ: $\dfrac{d\mu(t)}{dt} = \rho(t)$.*

For $\mu \in \mathbb{M}$ and $\tau \in \mathbb{R}$, we denote μ_τ the positive measure on $(\mathbb{R}, \mathbb{B})$ defined by

$$\mu_\tau(A) = \mu(\{a + \tau : a \in A\}) \text{ for } A \in \mathbb{B}.$$

From $\mu \in \mathbb{M}$, we formulate the following hypothesis.

(C) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu_\tau(A) \leqslant \beta\mu(A), \text{ when } A \in \mathbb{B} \text{ satisfies } A \cap I = \emptyset.$$

Theorem 2.22. [8] *Let $\mu \in \mathbb{M}$ satisfies **(C)**. Then $MAA_0(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, therefore $MPAA(\mathbb{R}, \mathbb{X}, \mu)$ is also translation invariant.*

Denote by $L^1(\mathbb{R}, \mathscr{L}(\mathbb{X}))$ the Lebesgue space with respect to the Lebesgue measure on $\mathbb{R}$.

Theorem 2.23. [8] *Let $\mu \in \mathbb{M}$ satisfies **(C)**. If $f \in MPAA(\mathbb{R}, \mathbb{X}, \mu)$ and $G \in L^1(\mathbb{R}, \mathscr{L}(\mathbb{X}))$, then the convolution product $f * G$ defined by*

$$(f * G)(t) = \int_{-\infty}^{\infty} G(s)f(t-s)ds \text{ for } t \in \mathbb{R},$$

is also μ-pseudo almost automorphic. In fact if $f \in AA(\mathbb{R}, \mathbb{X})$, then $f * G \in AA(\mathbb{R}, \mathbb{X})$ and if $f \in MAA_0(\mathbb{R}, \mathbb{X}, \mu)$, then $f * G \in MAA_0(\mathbb{R}, \mathbb{X}, \mu)$.

Theorem 2.24. [8] *Let $\mu \in \mathbb{M}$, $f \in MPAAU(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$ and $x \in MPAA(\mathbb{R}, \mathbb{X}, \mu)$. Assume that the following hypothesis holds.*

(***H***) *For all bounded subset B of $\mathbb{X}$, f is bounded on $\mathbb{R} \times B$.*

Then $[t \longmapsto f(t, x(t))] \in MPAA(\mathbb{R}, \mathbb{Y}, \mu)$.

Corollary 2.25. [8] *Let $\mu \in \mathbb{M}$ and $\psi \in C(\mathbb{X}, \mathbb{Y})$ be bounded namely ψ takes bounded sets in bounded sets. If $x \in MPAA(\mathbb{R}, \mathbb{X}, \mu)$, then $[t \longmapsto \psi(x(t))] \in MPAA(\mathbb{R}, \mathbb{Y}, \mu)$.*

2.5. **Stepanov-like μ-pseudo-almost automorphic functions.** Let $L^p(\mathbb{R}, \mathbb{X})$ denote the space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}$) of measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $\|f\| \in L^p(\mathbb{R}, \mathbb{R})$. Let $L^p_{loc}(\mathbb{R}, \mathbb{X})$ stand for the space of all classes of equivalence of measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that the restriction of f to every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R}, \mathbb{X})$.

Definition 2.26. [24](*Bochner transform*) *The Bochner transform $f^b(t, s)$ with $t \in \mathbb{R}$, $s \in [0, 1]$ of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by $f^b(t, s) = f(t + s)$.*

Remark 2.27. [24]

- **(i)** *A function $\psi(t,s),\ t\in\mathbb{R},\ s\in[0,1]$, is the Bochner transform of a certain function f, $\psi(t,s)=f^b(t,s)$, if and only if $\psi(t+\tau,s-\tau)=\psi(s,t)$ for all $t\in\mathbb{R}$, $s\in[0,1]$ and $\tau\in[s-1,s]$.*
- **(ii)** *Note that if $f=h+\varphi$, then $f^b=h^b+\varphi^b$. Moreover, $(\lambda f)^b=\lambda f^b$ for each scalar λ.*

Definition 2.28. [24] *The Bochner transform $f^b(t,s,u)$ with $t\in\mathbb{R},\ s\in[0,1],\ u\in\mathbb{X}$ of a function $f:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ is defined by $f^b(t,s,u)=f(t+s,u)$ for each $u\in\mathbb{X}$.*

Definition 2.29. [24] *Let $p\in[1,\infty)$. The space $BS^p(\mathbb{R},\mathbb{X})$ of all Stepanov bounded functions, with exponent p, consists of all measurable functions $f:\mathbb{R}\to\mathbb{X}$ such that $f^b\in L^\infty(\mathbb{R},L^p(0,1;\mathbb{X}))$. This is a Banach space with the norm*

$$\|f\|_{S^p}=\|f^b\|_{L^\infty(\mathbb{R},L^p)}=\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|f(\tau)\|^p d\tau\right)^{\frac{1}{p}}.$$

Remark 2.30. *Let $1\leqslant p<\infty$. A function $f\in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be bounded in the sense of Stepanov if $\|f\|_{S^p}$ is finite, where*

$$\|f\|_{S^p}:=\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}=\sup_{t\in\mathbb{R}}\left(\int_0^1\|f(t+s)\|^p ds\right)^{\frac{1}{p}}=\sup_{t\in\mathbb{R}}\|f(t+.)\|_{L^p([0,1],\mathbb{X})}.$$

The following inclusions hold: $BC(\mathbb{R},\mathbb{X})\subset BS^p(\mathbb{R},\mathbb{X})\subset L^p_{loc}(\mathbb{R},\mathbb{X})$.

Remark 2.31. [26] *Functions of $BS^p(\mathbb{R},\mathbb{X})$ may not be bounded. For example, let $p\geqslant 1$ and f be the function defined by*

$$f(t)=\begin{cases} k & k\leqslant t\leqslant k+\dfrac{1}{k^p} \text{ with } k\in\mathbb{N}^*,\\ 0 & \text{otherwise.}\end{cases}$$

Then, f is not bounded ; however, $f\in BS^p(\mathbb{R},\mathbb{X})$, in fact

$$\begin{aligned}\int_t^{t+1}\|f(s)\|^p ds &\leqslant \int_{[t]}^{[t]+2}\|f(s)\|^p ds\\ &= \sum_{k=[t]}^{[t]+1}\int_k^{k+\frac{1}{k^p}}\|f(s)\|^p ds=2.\end{aligned}$$

Definition 2.32. [30] *The space $AAS^p(\mathbb{R},\mathbb{X})$ of Stepanov-like almost automorphic (or S^p-almost automorphic) functions consists of all $f\in BS^p(\mathbb{R},\mathbb{X})$ such that $f^b\in AA(\mathbb{R},L^p(0,1;\mathbb{X}))$.*

In other words, a function $f\in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b:\mathbb{R}\to L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $g\in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty}\left(\int_0^1\|f(t+s_n+s)-g(t+s)\|^p ds\right)^{\frac{1}{p}}=0 \text{ and}$$

$$\lim_{n\to\infty}\left(\int_0^1\|g(t-s_n+s)-f(t+s)\|^p ds\right)^{\frac{1}{p}}=0,$$

pointwise on $\mathbb{R}$.

Remark 2.33. [30] *It is clear that if* $1 \leqslant p < q < \infty$ *and* $f \in L^q_{loc}(\mathbb{R}, \mathbb{X})$ *is* S^q*-almost automorphic, then* f *is* S^p*-almost automorphic. Also if* $f \in AA(\mathbb{R}, \mathbb{X})$*, then* f *is* S^p*-almost automorphic for any* $1 \leqslant p < \infty$*.*

Theorem 2.34. [30] *For all* $1 \leqslant p < \infty$, $(AAS^p(\mathbb{R}, \mathbb{X}), \|.\|_{S^p})$ *is a Banach space.*

Definition 2.35. [21] *A function* $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ *given by* $(t, u) \mapsto f(t, u)$ *with* $f(., u) \in L^p(\mathbb{R}, \mathbb{X})$ *for each* $u \in \mathbb{X}$*, is said to be* S^p*-almost automorphic in* $t \in \mathbb{R}$ *uniformly in* $u \in \mathbb{X}$ *if* $t \mapsto f(t, u)$ *is* S^p*-almost automorphic for each* $u \in \mathbb{X}$*. That means, for every sequence of real numbers* $(s'_n)_{n\in\mathbb{N}}$ *there exists a subsequence* $(s_n)_{n\in\mathbb{N}}$ *and a function* $g(., u) \in L^p(\mathbb{R}, \mathbb{X})$ *such that*

$$\lim_{n\to\infty} \left(\int_0^1 \|f(t + s_n + s, u) - g(t + s, u)\|^p ds\right)^{\frac{1}{p}} = 0 \text{ and}$$

$$\lim_{n\to\infty} \left(\int_0^1 \|g(t - s_n + s, u) - f(t + s, u)\|^p ds\right)^{\frac{1}{p}} = 0,$$

pointwise on $\mathbb{R}$ *and for each* $u \in \mathbb{X}$*.*

We denote by $AAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ the set of all such functions.

Lemma 2.36. [6] *Let* $1 \leqslant p < +\infty$ *and* $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{Y}$ *be a function such that* $f(., x) \in L^p_{loc}(\mathbb{R}, \mathbb{Y})$ *for each* $x \in \mathbb{X}$*. Then,* $f \in AAS^pU(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ *if and only if the following hold:*

- **(*i*)** *For each* $x \in \mathbb{X}$, $f(., x) \in AAS^p(\mathbb{R}, \mathbb{Y})$*.*
- **(*ii*)** f *is* S^p*-uniformly continuous with respect to the second argument on each compact subset* K *in* $\mathbb{X}$*, namely: for all* $\varepsilon > 0$ *there exists* $\delta_{K,\varepsilon}$ *such that for all* $x_1, x_2 \in K$*, we have*

$$\|x_1 - x_2\| \leqslant \delta_{K,\varepsilon} \Rightarrow \left(\int_t^{t+1} \|f(s, x_1) - f(s, x_2)\|^p_{\mathbb{Y}} ds\right)^{\frac{1}{p}} \leqslant \varepsilon \text{ for all } t \in \mathbb{R}.$$

Theorem 2.37. [6]
Let $1 \leqslant p < +\infty$ *and* $f \in AAS^pU(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$*. Assume that* $u \in AA(\mathbb{R}, \mathbb{X})$*. Then,* $f(., u(.)) \in AAS^p(\mathbb{R}, \mathbb{Y})$*.*

Now, we give the definitions and important properties of μ-S^p-pseudo almost automorphic functions.

Definition 2.38. [26] *Let* $\mu \in \mathbb{M}$*. A function* $f \in BS^p(\mathbb{R}, \mathbb{X})$ *is said to be* μ*-ergodic in the sense of Stepanov (or* μ*-*S^p*-ergodic) if*

$$\lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \left(\int_t^{t+1} \|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) = 0.$$

The space of all such functions is denoted by $MAA_0S^p(\mathbb{R}, \mathbb{X}, \mu)$.

$$MAA_0S^p(\mathbb{R}, \mathbb{X}, \mu) = \left\{ f \in BS^p(\mathbb{R}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \left(\int_t^{t+1} \|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) = 0 \right\}.$$

Remark 2.39. [5] *Using the above definition, we obtain that,* $f \in MAA_0S^p(\mathbb{R}, \mathbb{X}, \mu)$ *if and only if* $f^b \in MAA_0(\mathbb{R}, L^p(0, 1; \mathbb{X}), \mu)$*.*

Theorem 2.40. [26] *Let μ satisfy (C) and $f \in MAA_0(\mathbb{R},\mathbb{X},\mu)$. Then, $f \in MAA_0S^p(\mathbb{R},\mathbb{X},\mu)$ for all $p \geqslant 1$.*

Definition 2.41. [5] *Let $\mu \in \mathbb{M}$. A function $f : \mathbb{R}\times\mathbb{X} \longrightarrow \mathbb{Y}$ such that $f(.,x) \in BS^p(\mathbb{R},\mathbb{Y})$ for each $x \in \mathbb{X}$ is said to be μ-S^p-ergodic in t with respect to x in $\mathbb{X}$ if the two following hold*

- (i) *For all $x \in \mathbb{X}$, $f(.,x) \in MAA_0S^p(\mathbb{R},\mathbb{Y},\mu)$,*
- (ii) *f is S^p-uniformly continuous with respect to the second argument on each compact subset K in $\mathbb{X}$ in the following sense: for all $\varepsilon > 0$ there exists $\delta_{K,\varepsilon}$ such that for all $x_1, x_2 \in K$ one has*

$$\|x_1 - x_2\| \leqslant \delta_{K,\varepsilon} \Rightarrow \left(\int_t^{t+1} \|f(s,x_1) - f(s,x_2)\|_{\mathbb{Y}}^p ds\right)^{\frac{1}{p}} \leqslant \varepsilon \text{ for all } t \in \mathbb{R}.$$

Denote by $MAA_0S^pU(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ the set of all such functions.

Corollary 2.42. [6] *Let $\mu \in \mathbb{M}$ satisfies (C). Assume that $u \in AA(\mathbb{R},\mathbb{X})$ and $f \in MAA_0S^pU(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$. Then $f(.,u(.)) \in MAA_0S^p(\mathbb{R},\mathbb{Y},\mu)$.*

Definition 2.43. [17] *Let $\mu \in \mathbb{M}$. A function $f \in BS^p(\mathbb{R},\mathbb{X})$ is said to be μ-Stepanov-like pseudo almost automorphic (or μ-S^p-pseudo almost automorphic) if it can be expressed as $f = g + \varphi$, where $g \in AAS^p(\mathbb{R},\mathbb{X})$ and $\varphi^b \in MAA_0(\mathbb{R},L^p(0,1;\mathbb{X}),\mu)$.*

In other words, a function $f \in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be μ-Stepanov-like pseudo almost automorphic relatively to the measure μ, if its Bochner transform $f^b : \mathbb{R} \longrightarrow L^p(0,1;\mathbb{X})$ is μ pseudo almost automorphic in the sense that there exist two functions $g,\varphi : \mathbb{R} \longrightarrow \mathbb{X}$ such that $f = g+\varphi$, where $g \in AAS^p(\mathbb{R},\mathbb{X})$ and $\varphi^b \in MAA_0(\mathbb{R},L^p(0,1;\mathbb{X}),\mu)$, that is $\varphi^b \in BC(\mathbb{R},L^p(0,1;\mathbb{X}),\mu)$ and

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left(\int_t^{t+1}\|\varphi(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) = 0.$$

Denote by $MPAAS^p(\mathbb{R},\mathbb{X},\mu)$ the set of all such functions.

Definition 2.44. [17] *Let $\mu \in \mathbb{M}$. A function $f : \mathbb{R}\times\mathbb{X} \to \mathbb{Y},\ (t,u) \to f(t,u)$ with $f(.,u) \in L^p_{loc}(\mathbb{R},\mathbb{Y})$ for each $u \in \mathbb{X}$, is said to be μ-Stepanov-like pseudo almost automorphic (or μ-S^p-pseudo almost automorphic) if it can be expressed as $f = g + \varphi$, where $g \in AAS^p(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ and $\varphi^b \in MAA_0(\mathbb{R},L^p(0,1;\mathbb{Y}),\mu)$.*

We denote by $MPAAS^P(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$ the set of all such functions.

Theorem 2.45. [17] *Let $\mu \in \mathbb{M}$ satisfies (C). If $f \in MPAA(\mathbb{R},\mathbb{X},\mu)$ then $f \in MPAAS^P(\mathbb{R},\mathbb{X},\mu)$ for each $1 \leqslant p < \infty$. In other words, $MPAA(\mathbb{R},\mathbb{X},\mu) \subset MPAAS^P(\mathbb{R},\mathbb{X},\mu)$.*

Theorem 2.46. [17] *Let $\mu \in \mathbb{M}$ satisfies (C). Then $MAA_0(\mathbb{R},L^p(0,1;\mathbb{X}),\mu)$ is translation invariant, therefore $MPAAS^P(\mathbb{R},\mathbb{X},\mu)$ is also translation invariant.*

Theorem 2.47. [17] *Let $\mu \in \mathbb{M}$. Assume that $MPAAS^P(\mathbb{R},\mathbb{X},\mu)$ is translation invariant. Then $(MPAAS^P(\mathbb{R},\mathbb{X},\mu), \|.\|_{S^p})$ is a Banach space.*

Proposition 2.48. [6] *Let $\mu \in M$ and $f \in MPAAS^pU(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$, for $1 \leqslant p < +\infty$. Then, the following holds:*

- ***(i)*** *For each $x \in \mathbb{X}$, $f(.,x) \in MPAAS^p(\mathbb{R},\mathbb{Y},\mu)$.*
- ***(ii)*** *f is S^p-uniformly continuous with respect to the second argument on each compact subset K in $\mathbb{X}$ in the following sense: for all $\varepsilon > 0$ there exists $\delta_{K,\varepsilon}$ such that for all $x_1, x_2 \in K$, one has:*

$$\|x_1 - x_2\| \leqslant \delta_{K,\varepsilon} \Rightarrow \left(\int_t^{t+1} \|f(s,x_1) - f(s,x_2)\|_{\mathbb{Y}}^p ds\right)^{\frac{1}{p}} \leqslant \varepsilon, \text{ for all } t \in \mathbb{R}.$$

Theorem 2.49. [17] *Let $\mu \in \mathbb{M}$. Suppose that $f = g+\varphi \in MPAAS^P(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ with $g \in AAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $\varphi^b \in MAA_0(\mathbb{X}, L^p(0,1;\mathbb{X}),\mu)$ which satisfies the following assumption: There exists a constant $L > 0$ such that for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$,*

$$\|f(t,u) - f(t,v)\| \leqslant L\|u - v\|.$$

If $u = \alpha + \beta \in MPAAS^p(\mathbb{R},\mathbb{X},\mu)$ with $\alpha \in AAS^p(\mathbb{R},\mathbb{X})$, $\beta^b \in MAA_0(L^p(0,1;\mathbb{X}),\mu)$ and $K_1 = \overline{\{\alpha(t), t \in \mathbb{R}\}}$ is compact. Then $f(.,u(.)) \in MPAAS^p(\mathbb{R},\mathbb{X},\mu)$.

Theorem 2.50. [6] *Let $\mu \in \mathbb{M}$ satisfies **(C)** and $f : \mathbb{R}\times\mathbb{X} \to \mathbb{Y}$. Assume that:*

- ***(i)*** *$f = g + \varphi \in MPAAS^PU(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$ with $g \in AAS^pU(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ and $\varphi \in MAA_0S^pU(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$.*
- ***(ii)*** *$u = u_1 + u_2 \in MPAA(\mathbb{R},\mathbb{X},\mu)$, where $u_1 \in AA(\mathbb{R},\mathbb{X})$ and $u_2 \in MAA_0(\mathbb{R},\mathbb{X},\mu)$.*
- ***(iii)*** *For every bounded subset $B \subset \mathbb{X}$ the set $\Lambda := \{f(.,x), x \in B\}$ is bounded in $BS^p(\mathbb{R},\mathbb{X})$.*

Then $f(.,u(.)) \in MPAAS^P(\mathbb{R},\mathbb{Y},\mu)$.

2.6. **Compactness criterion.** We recall a useful compactness criterion. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(t) \geqslant 1$ for all $t \in \mathbb{R}$ and $\lim_{|t|\to\infty} h(t) = \infty$.

We consider the space $C_h(\mathbb{X}) := \left\{u \in C(\mathbb{R},\mathbb{X}), \lim_{|t|\to\infty} \frac{u(t)}{h(t)} = 0\right\}$. Then, $C_h(\mathbb{X})$ is a Banach space endowed with the norm $\|u\|_h = \sup_{t\in\mathbb{R}} \frac{\|u(t)\|}{h(t)}$.

Lemma 2.51. [32]

A subset $K \subset C_h(\mathbb{X})$ is relatively compact set if K verifies the following conditions:

- ***(i)*** *The set $K(t) = \{u(t) : u \in K\}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$.*
- ***(ii)*** *The set K is equicontinuous.*
- ***(iii)*** *For each $\varepsilon > 0$ there exists $L > 0$ such that $\|u(t)\| \leqslant \varepsilon h(t)$ for all $u \in K$ and all $|t| > L$.*

Theorem 2.52. [32] (*Leray-Schauder Alternative Theorem*). *Let D be a closed convex subset of a Banach space $\mathbb{X}$ such that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \rho F(x),\ 0 < \rho < 1\}$ is unbounded or the map F has a fixed point in D.*

3. Main results

In this section, we prove the existence and uniqueness of μ-S^p- pseudo almost automorphic mild solutions to the two-term fractional order differential equation. Firstly, we establish a new composition result of μ-S^p-pseudo almost automorphic functions.

3.1. **Composition result of μS^p pseudo almost automorphic functions.** Let $f = g + \varphi \in MPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$, where $g \in AAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ and $\varphi \in MAA_0S^p(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$. In the basis of Remark 3.1 below, we introduce the following hypothesis:

(***H0***) For all $p \geqslant 1$, there exists a nonnegative function $L_f(.) \in BS^p(\mathbb{R}, \mathbb{R})$ such that:

$$\|f(t,x) - f(t,y)\|_{\mathbb{Y}} \leqslant L_f(t)\|x - y\| \text{ for all } x, y \in \mathbb{X} \text{ and } t \in \mathbb{R}.$$

Remark 3.1.

(***a***) *Note that under the condition* (***H0***) *the function $f(., x(.)) \in BS^p(\mathbb{R}, \mathbb{Y})$.*

(***b***) *The hypothesis* (***H0***) *implies the statement Lemma 2.36-(ii). Indeed, for all compact subset K of $\mathbb{X}$ and $\epsilon > 0$ there exists $\delta_{k,\epsilon} = \dfrac{\epsilon}{1 + \|L_f\|_{BS^p}}$ such that for all $x_1, x_2 \in K$, one has $\|x_1 - x_2\| \leqslant \delta_{K,\epsilon}$ implies*

$$\begin{aligned}\left(\int_t^{t+1} \|g(s,x_1) - g(s,x_2)\|_{\mathbb{Y}}^p ds\right)^{\frac{1}{p}} &\leqslant \|L_f\|_{S^p}\|x_1 - x_2\| \\ &\leqslant \|L_f\|_{S^p}\delta_{K,\epsilon} \\ &\leqslant \epsilon, \text{ for all } t \in \mathbb{R}.\end{aligned}$$

(***c***) *Assume that* (***H0***) *is satisfied, then the following are true*

(i) *For all function $g : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{Y}$ such that $g(., x) \in L^p_{loc}(\mathbb{R}, \mathbb{Y})$ for each $x \in \mathbb{X}$ which satisfies Lemma 2.36-(i) and by the above Remark (b). Then $g \in AAS^pU(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$.*

(ii) *For all function $\varphi : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{Y}$ such that $\varphi(., x) \in BS^p(\mathbb{R}, \mathbb{Y})$ for each $x \in \mathbb{X}$ which satisfies Definition 2.41-(i) and by the above Remark (b). Then $\varphi \in MAA_0S^pU(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$.*

Now, we present and prove a new composition theorem for μ-S^p-pseudo almost automorphic functions under few conditions.

Theorem 3.2. *Let $\mu \in \mathbb{M}$ satisfies* (***C***)*, $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{Y}$ be a function such that $f = g + \varphi \in MPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$ satisfying:*

(***i***) *For each $x \in \mathbb{X}$, $g(., x) \in AAS^p(\mathbb{R}, \mathbb{Y})$ and $\varphi(., x) \in MAA_0S^p(\mathbb{R}, \mathbb{Y}, \mu)$.*

(***ii***) $u = u_1 + u_2 \in MPAA(\mathbb{R}, \mathbb{X}, \mu)$ *where* $u_1 \in AA(\mathbb{R}, \mathbb{X})$ *and* $u_2 \in MAA_0(\mathbb{R}, \mathbb{X}, \mu)$.

(***iii***) f *satisfies the hypothesis* (***H0***).

Then $f(., u(.)) \in MPAAS^P(\mathbb{R}, \mathbb{Y}, \mu)$.

Proof. Let $f = g + \varphi \in MPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$, where $g \in AAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$, $\varphi \in MAA_0S^p(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$. Then, $g \in AAS^pU(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ (resp. $\varphi \in MAA_0S^pU(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$), by using the above Remark 3.1-(c)-(i) (resp. (c)-(ii)). Then, we have the following decomposition

$$f(t, u(t)) = g(t, u_1(t)) + [f(t, u(t)) - f(t, u_1(t))] + \varphi(t, u_1(t)).$$

Define $G(t) = g(t, u_1(t)),\ F(t) = f(t, u(t)) - f(t, u_1(t)),\ \Lambda(t) = \varphi(t, u_1(t))$.
Since $g \in AAS^pU(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ and $u_1 \in AA(\mathbb{R}, \mathbb{X})$, it follows from Theorem 2.37 that $G(.) \in AAS^p(\mathbb{R}, \mathbb{Y})$. Since $\varphi \in MAA_0S^p(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$ and $u_1 \in AA(\mathbb{R}, \mathbb{X})$, by using Corollary 2.42, we deduce that $\Lambda(.) \in MAA_0S^p(\mathbb{R}, \mathbb{Y}, \mu)$.
To show that $f(., u(.)) \in MPAAS^p(\mathbb{R}, \mathbb{Y}, \mu)$, it is sufficient to prove that $F(t) \in MAA_0S^p(\mathbb{R}, \mathbb{Y}, \mu)$.
Indeed, in view of the hypothesis (**H0**), we claim for $r > 0$ large enough, that

$$\begin{aligned}
&\frac{1}{\mu([-r,r])} \int_{-r}^{r} \left(\int_t^{t+1} \|F(s)\|_{\mathbb{Y}}^p ds \right)^{\frac{1}{p}} d\mu(t) \\
&\quad \leqslant \frac{1}{\mu([-r,r])} \int_{-r}^{r} \left(\int_t^{t+1} [L_f(s)]^p \|u(s) - u_1(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) \\
&\quad = \frac{1}{\mu([-r,r])} \int_{-r}^{r} \left(\int_t^{t+1} [L_f(s)]^p \|u_2(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) \\
&\quad = \frac{1}{\mu([-r,r])} \int_{-r}^{r} \left(\int_0^{1} [L_f(t+s)]^p \|u_2(t+s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) \\
&\quad = \frac{1}{\mu([-r,r])} \int_{-r}^{r} \left| L_f^b(t) \|u_2^b(t)\| \right|_{L^p(0,1;\mathbb{R})} d\mu(t) \\
&\quad \leqslant \frac{\sup_{t\in\mathbb{R}} |L_f^b(t)|_{L^p(0,1;\mathbb{R})}}{\mu([-r,r])} \int_{-r}^{r} \|u_2^b(t)\| d\mu(t) \\
&\quad = \frac{\|L_f\|_{S^p}}{\mu([-r,r])} \int_{-r}^{r} \|u_2^b(t)\| d\mu(t)
\end{aligned}$$

Since $u_2 \in MAA_0(\mathbb{R}, \mathbb{X}, \mu)$ and by using Theorem 2.40, we deduce that $u_2 \in MAA_0S^p(\mathbb{R}, \mathbb{X}, \mu)$.
Therefore, the right hand side in the last inequality goes to 0 as $r \to +\infty$. Consequently,

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \left(\int_t^{t+1} \|F(s)\|_{\mathbb{Y}}^p ds \right)^{\frac{1}{p}} d\mu(t) = 0.$$

□

3.2. **Existence of μ-pseudo-almost automorphic mild solution with Lipschitz condition.** We consider the two-term fractional order linear differential equation

$$D_t^{\alpha+1} u(t) + \gamma D_t^{\beta} u(t) = Au(t) + D_t^{\alpha} f(t),\ t \in \mathbb{R},\ 0 < \alpha \leqslant \beta < 1,\ \gamma \geqslant 0. \tag{2}$$

The following theorem established in [3] guarantees the existence of mild solutions of equation (2).

Theorem 3.3. *Let A be a generator of the $(\alpha,\beta)_\gamma$-regularized family $\{S_{\alpha,\beta}(t)\}_{t\geqslant 0}$. Then, equation (2) admits a mild solution given by*

$$u(t)=\int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s)ds,\ t\in\mathbb{R},$$

provided that $\{S_{\alpha,\beta}(t)\}_{t\geqslant 0}$ exists and integrable.

The following Definition is inspired by the above representation of mild solutions for the problem (2) establish by Alvarez-Pardo and Lizama in [3].

Definition 3.4. [3] *Assume that A generates an integrable $(\alpha,\beta)_\gamma$-regularized family $\{S_{\alpha,\beta}(t)\}_{t\geqslant 0}$. A function $u:\mathbb{R}\to\mathbb{X}$ satisfying the integral equation*

$$u(t):=\int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s,u(s))ds,\ t\in\mathbb{R},$$

is called a mild solution on $\mathbb{R}$ to the problem (1).

We suppose the following:

(H1) A is a sectorial operator of angle $\beta\frac{\pi}{2}$.

(H2) f is Lipschitz in $x\in\mathbb{X}$ uniformly in $t\in\mathbb{R}$, namely, there exists a constant $L>0$ such that:

$$\|f(t,x)-f(t,y)\|\leqslant L\|x-y\|,\ \text{for all } x,y\in\mathbb{X} \text{ and } t\in\mathbb{R}.$$

Lemma 3.5. *Let $\mu\in\mathbb{M}$ satisfies **(C)**. If $f=g+\varphi\in MPAAS^P(\mathbb{R},\mathbb{X},\mu)$ with $g\in AAS^p(\mathbb{R},\mathbb{X})$ and $\varphi^b\in MAA_0(\mathbb{R},L^p(0,1;\mathbb{X}),\mu)$. Then*

$$V(t)=\int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s)ds\in MPAA(\mathbb{R},\mathbb{X},\mu).$$

Proof. Let $f=g+\varphi$ with $g\in AAS^p(\mathbb{R},\mathbb{X})$ and $\varphi^b\in MAA_0(\mathbb{R},L^p(0,1;\mathbb{X}),\mu)$.

$$\begin{aligned} V(t) &= \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s)ds \\ &= \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)[g(s)+\varphi(s)]ds \\ &= \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)g(s)ds+\int_{-\infty}^{t} S_{\alpha,\beta}(t-s)\varphi(s)ds \\ &= G(t)+\Phi(t), \end{aligned}$$

where $G(t)=\displaystyle\int_{-\infty}^{t} S_{\alpha,\beta}(t-s)g(s)ds$ and $\Phi(t)=\displaystyle\int_{-\infty}^{t} S_{\alpha,\beta}(t-s)\varphi(s)ds$.
It is sufficient to show that $G(t)\in AA(\mathbb{R},\mathbb{X})$ and $\Phi(t)\in MAA_0(\mathbb{R},\mathbb{X},\mu)$.

First, we prove that $G(t)\in AA(\mathbb{R},\mathbb{X})$. Since A is an ω- sectorial operator of angle $\beta\frac{\pi}{2}$, then by Theorem 2.4 we have

$$\|S_{\alpha,\beta}(t)\|\leqslant\frac{C}{1+|\omega|(t^{\alpha+1}+\gamma t^{\beta})}\ \text{for } t\geqslant 0,$$

for some constant $C>0$ depending only on α and β.
Since $0<\alpha\leqslant\beta<1$ we deduce that $\varpi(t)=\dfrac{C}{1+|\omega|(t^{\alpha+1}+\gamma t^{\beta})}\in L^1(\mathbb{R}^+)$ and is non-increasing. Indeed, for $0\leqslant t\leqslant y$ we have

$$\begin{aligned}\varpi(t)-\varpi(y) &= \frac{C}{1+|\omega|(t^{\alpha+1}+\gamma t^{\beta})}-\frac{C}{1+|\omega|(y^{\alpha+1}+\gamma y^{\beta})}\\ &= C\frac{1+|\omega|(y^{\alpha+1}+\gamma y^{\beta})-1-|\omega|(t^{\alpha+1}+\gamma t^{\beta})}{(1+|\omega|(y^{\alpha+1}+\gamma y^{\beta}))(1+|\omega|(t^{\alpha+1}+\gamma t^{\beta}))}\\ &= C|\omega|\frac{y^{\alpha+1}-t^{\alpha+1}+\gamma(y^{\beta}-t^{\beta})}{(1+|\omega|(y^{\alpha+1}+\gamma y^{\beta}))(1+|\omega|(t^{\alpha+1}+\gamma t^{\beta}))},\end{aligned}$$

which gives that $\varpi(t)\geqslant\varpi(y)$. Then $\varpi(t)$ is non-increasing and since $g(t)\in AAS^{p}(\mathbb{R},\mathbb{X})$ it follows from Lemma 2.5 that $G(t)\in AA(\mathbb{R},\mathbb{X})$.

Now, we prove that $\Phi(t)\in MAA_0(\mathbb{R},\mathbb{X},\mu)$. Define

$$\Phi_n(t)=\int_{t-n-1}^{t-n} S_{\alpha,\beta}(t-\sigma)\varphi(\sigma)d\sigma,\ \text{for } n=0,1,2,3...$$

Then,

$$\begin{aligned}\|\Phi_n(t)\| &\leqslant \int_{t-n-1}^{t-n}\|S_{\alpha,\beta}(t-\sigma)\|\|\varphi(\sigma)\|d\sigma\\ &= \int_{n}^{n+1}\|S_{\alpha,\beta}(\sigma)\|\|\varphi(t-\sigma)\|d\sigma\\ &\leqslant \int_{n}^{n+1}\frac{C}{1+|\omega|(\sigma^{\alpha+1}+\gamma\sigma^{\beta})}\|\varphi(t-\sigma)\|d\sigma\\ &\leqslant \int_{n}^{n+1}\frac{C}{1+|\omega|(n^{\alpha+1}+\gamma n^{\beta})}\|\varphi(t-\sigma)\|d\sigma\\ &\leqslant \frac{C}{1+|\omega|(n^{\alpha+1}+\gamma n^{\beta})}\left(\int_{n}^{n+1}\|\varphi(t-\sigma)\|^{p}d\sigma\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore, for $r>0$

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|\Phi_n(t)\|d\mu(t) &\leqslant \frac{C}{1+|\omega|(n^{\alpha+1}+\gamma n^{\beta})}\\ &\times \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left(\int_{n}^{n+1}\|\varphi(t-\sigma)\|^{p}d\sigma\right)^{\frac{1}{p}}d\mu(t).\end{aligned}$$

Using the fact that the space $MAA_0S^{p}(\mathbb{R},L^{p}(0,1;\mathbb{X}),\mu)$ is translation invariant, it follows that $\varphi(t-\sigma)\in MAA_0S^{p}(\mathbb{R},L^{p}(0,1;\mathbb{X}),\mu)$, the above inequality leads to $\Phi_n(t)\in MAA_0(\mathbb{R},\mathbb{X},\mu)$ for each $n=0,1,2,3......$

The above inequality leads also to $\|\Phi_n(t)\|\leqslant\dfrac{C}{1+|\omega|(n^{\alpha+1}+\gamma n^{\beta})}\|\varphi\|_{S^p}$.

Since $0<\alpha\leqslant\beta<1$, then $1<1+\alpha<2$ and therefore

$$I_{\alpha,\beta}=\int_{0}^{\infty}\frac{1}{1+|\omega|(s^{\alpha+1}+\gamma s^{\beta})}ds \tag{3}$$

is convergent on $\mathbb{R}$. Since

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{C}{1+|\omega|(n^{\alpha+1}+\gamma n^{\beta})} &\leqslant C+\sum_{n=1}^{\infty}\int_{n}^{n+1}\frac{C}{1+|\omega|(s^{\alpha+1}+\gamma s^{\beta})}ds\\ &\leqslant C+\int_{0}^{\infty}\frac{C}{1+|\omega|(s^{\alpha+1}+\gamma s^{\beta})}ds\\ &= C(1+I_{\alpha,\beta})<\infty.\end{aligned}$$

It follows from the Weierstrass test that the series $\sum_{n=0}^{\infty} \Phi_n(t)$ is uniformly convergent on $\mathbb{R}$ and $\Phi(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)\varphi(s)ds = \sum_{n=0}^{\infty} \Phi_n(t)$.
Applying $\Phi_n(t) \in MAA_0(\mathbb{R}, \mathbb{X}, \mu)$ and the inequality

$$\begin{aligned} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \|\Phi(t)\| d\mu(t) &\leqslant \frac{1}{\mu([-r,r])} \int_{-r}^{r} \left\| \Phi(t) - \sum_{k=0}^{n} \Phi_k(t) \right\| d\mu(t) \\ &+ \sum_{k=0}^{n} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \|\Phi_k(t)\| \, d\mu(t). \end{aligned}$$

We deduce that the uniformly limit $\Phi(t) = \sum_{n=0}^{\infty} \Phi_n(t) \in MAA_0(\mathbb{R}, \mathbb{X}, \mu)$.
Therefore $V(t) = G(t) + \Phi(t)$ is μ-pseudo almost automorphic. □

Theorem 3.6. *Let $\mu \in \mathbb{M}$ satisfies $(\boldsymbol{C})$. Suppose that $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{X}$ be a function such that $f = g + \varphi \in MPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$ with $g \in AAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and $\varphi^b \in MAA_0(\mathbb{X}, L^p(0,1;\mathbb{X}), \mu)$ and $(\boldsymbol{H1})$, $(\boldsymbol{H2})$ holds. Then, the equation (1) has a unique mild solution in $MPAA(\mathbb{R}, \mathbb{X}, \mu)$ provided that $LC|\omega|^{\frac{-1}{\alpha+1}}\pi < (\alpha+1)\sin(\frac{\pi}{\alpha+1})$.*

Proof. Let the nonlinear operator $F : MPAA(\mathbb{R}, \mathbb{X}, \mu) \to MPAA(\mathbb{R}, \mathbb{X}, \mu)$ defined by

$$(Fu)(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s) f(s, u(s)) ds, \ t \in \mathbb{R}.$$

First, we show that $F(MPAA(\mathbb{R}, \mathbb{X}, \mu)) \subset MPAA(\mathbb{R}, \mathbb{X}, \mu)$.
Let $u \in MPAA(\mathbb{R}, \mathbb{X}, \mu)$. By using the fact that the range of an almost automorphic function is relatively compact, with Theorem 2.45 and Theorem 2.49 one can see that
$f(., u(.)) \in MPAAS^P(\mathbb{R}, \mathbb{X}, \mu)$.
It follows from the preceding Theorem that $(Fu)(.) \in MPAA(\mathbb{R}, \mathbb{X}, \mu)$.
From [19], note that:

$$\int_0^{\infty} \frac{1}{|\omega| t^{\alpha+1}} dt = \frac{|\omega|^{\frac{-1}{\alpha+1}}\pi}{(\alpha+1)\sin(\frac{\pi}{\alpha+1})}, \text{ for } 0 < \alpha < 1.$$

We have:

$$\int_0^{\infty} \frac{1}{1 + |\omega|(s^{\alpha+1} + \gamma s^{\beta})} ds \leqslant \int_0^{\infty} \frac{1}{1 + |\omega| s^{\alpha+1}} ds.$$

Then

$$I_{\alpha,\beta} \leqslant \frac{|\omega|^{\frac{-1}{\alpha+1}}\pi}{(\alpha+1)\sin(\frac{\pi}{\alpha+1})}.$$

Now, let $u, v \in MPAA(\mathbb{R}, \mathbb{X}, \mu)$. Then

$$\begin{aligned}
\|(Fu)(t)-(Fv)(t)\|_\infty &= \left\|\int_{-\infty}^t S_{\alpha,\beta}(t-s)(f(s,u(s))-f(s,v(s)))ds\right\|_\infty \\
&= \sup_{t\in\mathbb{R}}\left\|\int_0^\infty S_{\alpha,\beta}(s)(f(t-s,u(t-s))-f(t-s,v(t-s)))ds\right\| \\
&\leqslant L\sup_{t\in\mathbb{R}}\int_0^\infty \|S_{\alpha,\beta}(s)\|_\infty\|u(t-s)-v(t-s)\|ds \\
&\leqslant L\|u-v\|_\infty C\int_0^\infty \frac{1}{1+|\omega|(s^{\alpha+1}+\gamma s^\beta)}ds \\
&\leqslant LC\frac{|\omega|^{\frac{-1}{\alpha+1}}\pi}{(\alpha+1)\sin(\frac{\pi}{\alpha+1})}\|u-v\|_\infty \\
&\leqslant \|u-v\|_\infty.
\end{aligned}$$

Then, we deduce that F is a contraction, so by the Banach Fixed Point Theorem F has a unique fixed point. □

Theorem 3.7. *Let $\mu \in \mathbb{M}$ satisfies (C). Suppose that $f : \mathbb{R}\times\mathbb{X} \longrightarrow \mathbb{X}$ be a function such that $f = g+\varphi \in MPAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ with $g \in AAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and $\varphi^b \in MAA_0(\mathbb{X}, L^p(0,1;\mathbb{X}),\mu)$ satisfying*

- (i) *For each $x \in \mathbb{X}$, $g(.,x) \in AAS^p(\mathbb{R},\mathbb{Y})$ and $\varphi(.,x) \in MAA_0S^p(\mathbb{R},\mathbb{Y},\mu)$.*
- (ii) *f satisfies the hypothesis **(H0)**, **(H1)**.*

Then the equation (1) has a unique mild solution in $MPAA(\mathbb{R},\mathbb{X},\mu)$ provided that
$$C\|L_f\|_{S^p}\left(1+\frac{|\omega|^{\frac{-1}{\alpha+1}}\pi}{(\alpha+1)\sin(\frac{\pi}{\alpha+1})}\right) < 1.$$

Proof. Let the nonlinear operator $F : MPAA(\mathbb{R},\mathbb{X},\mu) \to MPAA(\mathbb{R},\mathbb{X},\mu)$ defined by

$$(Fu)(t) = \int_{-\infty}^t S_{\alpha,\beta}(t-s)f(s,u(s))ds,\ t\in\mathbb{R}.$$

First, let us prove that $F(MPAA(\mathbb{R},\mathbb{X},\mu)) \subset MPAA(\mathbb{R},\mathbb{X},\mu)$.
Indeed, let $u \in MPAA(\mathbb{R},\mathbb{X},\mu)$, since f satisfies the hypothesis **(i)** and **(ii)**, it follows from the above Theorem 3.2 that $f(.,u(.)) \in MPAAS^P(\mathbb{R},\mathbb{X},\mu)$ and by using Lemma 3.5, we have that $(Fu)(.) \in MPAA(\mathbb{R},\mathbb{X},\mu)$.

Next, we prove that the operator F has a unique fixed point in $MPAA(\mathbb{R},\mathbb{X},\mu)$.
Indeed, for each $t\in\mathbb{R}$, $u, v \in MPAA(\mathbb{R},\mathbb{X},\mu)$, we have

$$\begin{aligned}
\|(Fu)(t)-(Fv)(t)\| &= \left\|\int_{-\infty}^t S_{\alpha,\beta}(t-s)[f(s,u(s))-f(s,v(s))]ds\right\| \\
&\leqslant \int_{-\infty}^t \|S_{\alpha,\beta}(t-s)\|\|f(s,u(s))-f(s,v(s))\|ds \\
&\leqslant \int_{-\infty}^t \frac{C}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^\beta)}L_f(s)\|u(s)-v(s)\|ds
\end{aligned}$$

$$\begin{aligned}
&\leqslant C\|u-v\|_\infty \int_{-\infty}^{t} \frac{1}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})} L_f(s)ds \\
&= C\|u-v\|_\infty \sum_{n=0}^{\infty} \int_{t-n-1}^{t-n} \frac{1}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})} L_f(s)ds \\
&\leqslant C\|u-v\|_\infty \sum_{n=0}^{\infty} \int_{n}^{n+1} \frac{1}{1+|\omega|(s^{\alpha+1}+\gamma s^{\beta})} L_f(t-s)ds \\
&\leqslant C\|u-v\|_\infty \sum_{n=0}^{\infty} \frac{1}{1+|\omega|(n^{\alpha+1}+\gamma n^{\beta})} \int_{n}^{n+1} L_f(t-s)ds \\
&\leqslant C\|u-v\|_\infty \sum_{n=0}^{\infty} \frac{1}{1+|\omega|(n^{\alpha+1}+\gamma n^{\beta})} \left(\int_{t-n-1}^{t-n} \|L_f(s)\|^p ds\right)^{\frac{1}{p}} \\
&\leqslant C\|u-v\|_\infty \|L_f\|_{S^p} \sum_{n=0}^{\infty} \frac{1}{1+|\omega|(n^{\alpha+1}+\gamma n^{\beta})} \\
&\leqslant C\|u-v\|_\infty \|L_f\|_{S^p}(1+I_{\alpha,\beta}).
\end{aligned}$$

Therefore

$$\|(Fu)-(Fv)\|_\infty \leqslant C\|L_f\|_{S^p}\left(1+\frac{|\omega|^{\frac{-1}{\alpha+1}}\pi}{(\alpha+1)\sin(\frac{\pi}{\alpha+1})}\right)\|u-v\|_\infty.$$

This proves that F is a contraction. By the Banach Fixed Point Theorem, we deduce that F has a unique fixed point. □

3.3. **Existence of μ-pseudo-almost automorphic solution without Lipschitz perturbation.** We study the existence of μ-pseudo almost automorphic mild solution of equation (1) when the perturbation f is not Lipschitz continuous.

To establish our next result, we consider functions f that satisfy the following boundness conditions.

- **(H3)** For every bounded subset $B \subset \mathbb{X}$ the set $\{f(.,x), x \in B\}$ is bounded in $BS^p(\mathbb{R},\mathbb{X})$.
- **(H4)** There exists a continuous nondecreasing function $W : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(t,x)\| \leqslant W(\|x\|) \text{ for all } t \in \mathbb{R} \text{ and } x \in \mathbb{X}.$$

The following existence result is based upon nonlinear Leray-Shauder alternative theorem.

Theorem 3.8. *Let $\mu \in \mathbb{M}$ satisfies (**C**). Assume that $0 < \alpha \leqslant \beta < 1$, $\gamma > 0$ and $\omega < 0$. Let $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{X}$ be a function such that $f = g + \varphi \in MPAAS^pU(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$ with $g \in AAS^pU(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and $\varphi \in MAA_0S^pU(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$ which satisfies the assumptions (**H1**), (**H3**), (**H4**) and the following additional conditions:*

(***i***) *For each $r \geqslant 0$, $\lim_{|t|\to\infty} \frac{1}{h(t)} \int_{-\infty}^{t} \frac{W(r(h(s)))}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})} ds = 0$*

where h is a function given in lemma 2.51. We set

$$\Omega(r) := C \left\| \int_{-\infty}^{t} \frac{W(r(h(s)))}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})} ds \right\|_h .$$

(*ii*) *For each $\varepsilon > 0$ there is $\delta > 0$ such that for every $u, v \in C_h(\mathbb{X})$, $\|u-v\|_h \leqslant \delta$ implies that*

$$\int_{-\infty}^{t} \frac{\|f(s,u(s)) - f(s,v(s))\|}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})} ds \leqslant \frac{\varepsilon}{C}, \text{ for all } t \in \mathbb{R}.$$

(*iii*) $\liminf_{\xi \to \infty} \frac{\xi}{\Omega(\xi)} > 1.$

(*iv*) *For all $a, b \in \mathbb{R}$, $a < b$ and $r > 0$, the set $\{f(s, h(s)x) : a \leqslant s \leqslant b, x \in C_h(\mathbb{X}), \|x\|_h \leqslant r\}$ is relatively compact in $\mathbb{X}$.*

Then the equation (1) has a μ-pseudo almost automorphic solution.

Proof. We define the nonlinear operator $F : C_h(\mathbb{X}) \to C_h(\mathbb{X})$ by

$$(Fu)(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s) f(s,u(s)) ds, \ t \in \mathbb{R}.$$

We will show that F has a fixed point in $MPAA(\mathbb{R}, \mathbb{X}, \mu)$. For the sake of convenience, we divide the proof into several steps.

Step 1: The nonlinear operator F is well defined. Indeed, for $u \in C_h(\mathbb{X})$, we have that

$$\begin{aligned}
\frac{\|(Fu)(t)\|}{h(t)} &\leqslant \frac{1}{h(t)} \int_{-\infty}^{t} \|S_{\alpha,\beta}(t-s)\| \|f(s,u(s))\| ds \\
&\leqslant \frac{C}{h(t)} \int_{-\infty}^{t} \frac{\|f(s,u(s))\|}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})} ds \\
&\leqslant \frac{C}{h(t)} \int_{-\infty}^{t} \frac{W(\|u(s)\|)}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})} ds \\
&= \frac{C}{h(t)} \int_{-\infty}^{t} \frac{W(\|u\|_h h(s))}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})} ds
\end{aligned}$$

It follows from condition **(i)** that F is well defined.

Step 2: The operator F is continuous. In fact, for any $\varepsilon > 0$, we take $\delta > 0$ involved in condition (ii). If $u, v \in C_h(\mathbb{X})$ and $\|u-v\|_h \leqslant \delta$, then

$$\|(Fu)(t) - (Fv)(t)\| \leqslant C \int_{-\infty}^{t} \frac{\|f(s,u(s)) - f(s,v(s))\|}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})} ds \leqslant \varepsilon \text{ for all } t \in \mathbb{R},$$

which shows the assertion.

Step 3: We will show that the operator F is completely continuous. We set $B_r(Z)$ the closed ball with center at 0 and radius r in the space Z. Let $V = F(B_r(C_h(\mathbb{X})))$ and $v = F(u)$ for $u \in B_r(C_h(\mathbb{X}))$. It is sufficient to prove that the following statements are true.

(i) $V(t)$ is a relatively compact subset in $\mathbb{X}$ for each $t \in \mathbb{R}$.

(ii) V is equicontinuous.

(iii) For each $\varepsilon > 0$ there exists $L > 0$ such that $\|v(t)\| \leqslant \varepsilon h(t)$ for all $v \in V$ and all $|t| > L$.

First, we will prove that $V(t)$ is a relatively compact subset of $\mathbb{X}$ for each $t \in \mathbb{R}$. Since $h(t) \to \infty$ as $t \to \infty$, it follows from condition (i) that for $\varepsilon > 0$, we can choose $a \geqslant 0$ such that

$$C \int_a^\infty \frac{W(rh(t-s))}{1+|\omega|(s^{\alpha+1}+\gamma s^\beta)} ds \leqslant \varepsilon. \tag{4}$$

$$\begin{aligned}\text{Or } v(t) = (Fu)(t) &= \int_{-\infty}^t S_{\alpha,\beta}(t-s) f(s,u(s)) ds \\ &= \int_0^\infty S_{\alpha,\beta}(s) f(t-s,u(t-s)) ds \\ &= \int_0^a S_{\alpha,\beta}(s) f(t-s,u(t-s)) ds + \int_a^\infty S_{\alpha,\beta}(s) f(t-s,u(t-s)) ds.\end{aligned}$$

Since

$$\begin{aligned}\left\| \int_a^\infty S_{\alpha,\beta}(s) f(t-s,u(t-s)) ds \right\| &\leqslant \int_a^\infty \|S_{\alpha,\beta}(s)\| \|f(t-s,u(t-s))\| ds \\ &\leqslant C \int_a^\infty \frac{W(\|u(t-s)\|)}{1+|\omega|((s^{\alpha+1}+\gamma s^\beta)} ds \\ &\leqslant C \int_a^\infty \frac{W(\|u\|_h h(t-s))}{1+|\omega|(s^{\alpha+1}+\gamma s^\beta)} ds \\ &\leqslant C \int_a^\infty \frac{W(rh(t-s))}{1+|\omega|(s^{\alpha+1}+\gamma s^\beta)} ds.\end{aligned}$$

From inequality (4), we conclude that

$$\left\| \int_a^\infty S_{\alpha,\beta}(s) f(t-s,u(t-s)) ds \right\| \leqslant \varepsilon.$$

We get $v(t) \in a\, \overline{C_0(K)} + B_\varepsilon(\mathbb{X})$, where $C_0(K)$ denotes the convex hull of K and
$K = \{S_{\alpha,\beta}(s) f(\xi, h(\xi)x) : 0 \leqslant s \leqslant a, t-a \leqslant \xi \leqslant t, \|x\|_h \leqslant r\}$.
Using the strong continuity of $S_{\alpha,\beta}$ and the property (iv), we deduce that K is a relatively compact set, and $V(t) \subset a\, \overline{C_0(K)} + B_\varepsilon(\mathbb{X})$.

Second, we will show that the set V is equicontinuous. In fact, we can decompose

$$\begin{aligned}v(t+s) - v(t) &= \int_{-\infty}^{t+s} S_{\alpha,\beta}(t+s-y) f(y,u(y)) dy - \int_{-\infty}^t S_{\alpha,\beta}(t-y) f(y,u(y)) dy \\ &= \int_{-\infty}^t [S_{\alpha,\beta}(t+s-y) - S_{\alpha,\beta}(t-y)] f(y,u(y)) dy \\ &+ \int_t^{t+s} S_{\alpha,\beta}(t+s-y) f(y,u(y)) dy \\ &= \int_0^s S_{\alpha,\beta}(y) f(t+s-y, u(t+s-y)) dy \\ &+ \int_0^\infty [S_{\alpha,\beta}(y+s) - S_{\alpha,\beta}(y)] f(t-y,u(t-y)) dy \\ &= \int_0^s S_{\alpha,\beta}(y) f(t+s-y, u(t+s-y)) dy\end{aligned}$$

$$+ \int_0^a [S_{\alpha,\beta}(y+s) - S_{\alpha,\beta}(y)] f(t-y, u(t-y)) dy$$
$$+ \int_a^\infty [S_{\alpha,\beta}(y+s) - S_{\alpha,\beta}(y)] f(t-y, u(t-y)) dy.$$

For each $\varepsilon > 0$, we can choose $a > 0$ and $\delta_1 > 0$ such that

$$\left\| \int_0^s S_{\alpha,\beta}(y) f(t+s-y, u(t+s-y)) dy + \int_a^\infty [S_{\alpha,\beta}(y+s) - S_{\alpha,\beta}(y)] f(t-y, u(t-y)) dy \right\|$$

$$\begin{aligned} &\leqslant C \left(\int_0^s \frac{W(rh(t+s-y))}{1+|\omega|(y^{\alpha+1}+\gamma y^\beta)} dy + 2 \int_a^\infty \frac{W(rh(t-y))}{1+|\omega|(y^{\alpha+1}+\gamma y^\beta)} dy \right) \\ &\leqslant \frac{\varepsilon}{2}, \text{ for } s \leqslant \delta_1 \text{ and small enough.} \end{aligned}$$

Moreover, since $\{f(t-y, u(t-y)) : 0 < y < a, u \in B_r(C_h(\mathbb{X}))\}$ is a relatively compact set and $S_{\alpha,\beta}$ is strongly continuous, we can choose $\delta_2 > 0$ such that

$\|[S_{\alpha,\beta}(y+s) - S_{\alpha,\beta}(y)] f(t-y, u(t-y))\| \leqslant \frac{\varepsilon}{2a}$ for $s \leqslant \delta_2$.

Combining these estimates, we have that $\|v(t+s) - v(t)\| \leqslant \varepsilon$ for s small enough and independent of $u \in C_h(\mathbb{X})$.

Finally,

$$\frac{\|v(t)\|}{h(t)} \leqslant \frac{C}{h(t)} \int_{-\infty}^t \frac{W(\|u\|_h h(s))}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^\beta)} ds.$$

From the condition (i), we conclude that $\dfrac{\|v(t)\|}{h(t)} \to 0$ as $|t| \to \infty$ and this convergence is independent of $u \in B_r(C_h(\mathbb{X}))$.

Hence, by Lemma 2.51 we have that V is relatively compact set in $C_h(\mathbb{X})$.

Step 4: We will show that $\{u^\lambda : u^\lambda = \lambda\, F(u^\lambda),\ 0 < \lambda < 1\}$ is bounded. Indeed, assume that $u^\lambda(.)$ is a solution of the equation $u^\lambda = \lambda\, F(u^\lambda)$ for some $0 < \lambda < 1$. We can estimate

$$\begin{aligned} \|u^\lambda\| &= \lambda \left\| \int_{-\infty}^t S_{\alpha,\beta}(t-s) f(s, u^\lambda(s)) ds \right\| \\ &\leqslant C \int_{-\infty}^t \frac{W(\|u^\lambda\|_h h(s))}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^\beta)} ds \\ &\leqslant h(t)\, \Omega(\|u^\lambda\|_h). \end{aligned}$$

Hence, we get

$$\frac{\|u^\lambda\|_h}{\Omega(\|u^\lambda\|_h)} \leqslant 1$$

and combining with (iii), we conclude that the set $\{u^\lambda : u^\lambda = \lambda\, F(u^\lambda),\ 0 < \lambda < 1\}$ is bounded.

Step 5: It follows from Theorem 2.50 that the function $t \to f(t, u(t))$ belongs to $MPAAS^p(\mathbb{R}, \mathbb{X}, \mu)$. Consider the map $F : MPAA(\mathbb{R}, \mathbb{X}, \mu) \to MPAA(\mathbb{R}, \mathbb{X}, \mu)$. From Lemma 3.5 we infer that $F(MPAA(\mathbb{R}, \mathbb{X}, \mu)) \subset MPAA(\mathbb{R}, \mathbb{X}, \mu)$. Since $MPAA(\mathbb{R}, \mathbb{X}, \mu)$ is a closed subspace of $C_h(\mathbb{X})$, it follows from the previous steps of this proof that F is completely continuous map. Then, by the Leray-Schauder Alternative Theorem we have that F has a fixed point $u \in MPAA(\mathbb{R}, \mathbb{X}, \mu)$. □

In what follows, we will give the existence and uniqueness of nonlinear Leray-Schauder's theorem with fewer assumptions.

Theorem 3.9. *Let $\mu \in \mathbb{M}$ satisfies $(\boldsymbol{C})$. Assume that $0 < \alpha \leqslant \beta < 1$, $\gamma > 0$ and $\omega < 0$. Let $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{X}$ be a function such that $f = g + \varphi \in MPAAS^pU(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$ with $g \in AAS^pU(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and $\varphi \in MAA_0S^pU(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$ which satisfies the assumptions $(\boldsymbol{H1})$, $(\boldsymbol{H3})$ and the following additional conditions:*

- $(\boldsymbol{i})$ *$(S_{\alpha,\beta}(t))_{t\geq 0}$ is compact for $t > 0$.*
- $(\boldsymbol{ii})$ *For each $r > 0$, we set $k := \sup_{t\in\mathbb{R}} \sup_{\|x\|\leqslant r} \|f(t, x)\| < +\infty$.*
- $(\boldsymbol{iii})$ *For each $\varepsilon > 0$ there is $\delta > 0$ such that for every $u, v \in C_h(\mathbb{X})$,*
  $\|u - v\|_h \leqslant \delta$ implies that
$$\int_{-\infty}^{t} \frac{\|f(s, u(s)) - f(s, v(s))\|}{1 + |\omega|((t-s)^{\alpha+1} + \gamma(t-s)^{\beta})} ds \leqslant \frac{\varepsilon}{C}, \text{ for all } t \in \mathbb{R}.$$

Then, the equation (1) has a μ-pseudo almost automorphic solution.

Proof. We define the nonlinear operator $F : C_h(\mathbb{X}) \to C_h(\mathbb{X})$ by
$$(Fu)(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s) f(s, u(s)) ds, \ t \in \mathbb{R}.$$

We will show that F has a fixed point in $MPAA(\mathbb{R}, \mathbb{X}, \mu)$. For the sake of convenience, we divide the proof of several steps.

Step 1: The nonlinear operator F is well defined. Indeed, for $u \in C_h(\mathbb{X})$, we have that
$$\begin{aligned} \|(Fu)(t)\| &\leqslant \int_{-\infty}^{t} \|S_{\alpha,\beta}(t-s)\| \|f(s, u(s))\| ds \\ &\leqslant C \int_{-\infty}^{t} \frac{\|f(s, u(s))\|}{1 + |\omega|((t-s)^{\alpha+1} + \gamma(t-s)^{\beta})} ds. \end{aligned}$$

It follows from condition (ii) that:
$$\frac{\|(Fu)(t)\|}{h(t)} \leqslant \frac{k\, C}{h(t)} \int_{0}^{\infty} \frac{1}{1 + |\omega|(y^{\alpha+1} + \gamma y^{\beta})} dy.$$

Using the fact that $\lim_{|t|\to\infty} h(t) = \infty$ and $\int_{0}^{\infty} \frac{1}{1 + |\omega|(y^{\alpha+1} + \gamma y^{\beta})} dy$ is convergent on $\mathbb{R}$ for $0 < \alpha \leqslant \beta < 1$, it follows that $\lim_{|t|\to\infty} \frac{\|(Fu)(t)\|}{h(t)} = 0$, then F is well defined.

Step 2: We will show that the operator F is continuous. In fact, for any $\varepsilon > 0$, we take $\delta > 0$ involved in condition (iii). If $u, v \in C_h(\mathbb{X})$ and $\|u - v\|_h \leqslant \delta$, then
$$\|(Fu)(t) - (Fv)(t)\| \leqslant C \int_{-\infty}^{t} \frac{\|f(s, u(s)) - f(s, v(s))\|}{1 + |\omega|((t-s)^{\alpha+1} + \gamma(t-s)^{\beta})} ds \leqslant \varepsilon, \ \text{for all } t \in \mathbb{R}.$$

Which shows the assertion.

Step 3: We will show that F is completely continuous. We set $\overline{B}(0, r)$ the closed ball with center at 0 and radius r in the space $C_h(\mathbb{X})$. Let $V = F(\overline{B}(0,r))$ in the space $C_h(\mathbb{X})$ and $v = F(u)$ for $u \in \overline{B}(0,r)$ in the space $C_h(\mathbb{X})$.

First, we will prove that $V(t)$ is a relatively compact subset of $\mathbb{X}$ for each $t \in \mathbb{R}$. We have that

$$\begin{aligned} v(t) = (Fu)(t) &= \int_{-\infty}^{t} S_{\alpha,\beta}(t-s) f(s,u(s))ds \\ &= \int_{0}^{\infty} S_{\alpha,\beta}(s) f(t-s,u(t-s))ds \\ &= \int_{0}^{\varepsilon} S_{\alpha,\beta}(s) f(t-s,u(t-s))ds + \int_{a}^{\infty} S_{\alpha,\beta}(s) f(t-s,u(t-s))ds \\ &+ \int_{\varepsilon}^{a} S_{\alpha,\beta}(s) f(t-s,u(t-s))ds \text{ for } a,\varepsilon > 0. \end{aligned}$$

Here $v(t)$ is the sum of three terms. For the first term

$$\begin{aligned} \left\| \int_{0}^{\varepsilon} S_{\alpha,\beta}(s) f(t-s,u(t-s))ds \right\| &\leqslant \int_{0}^{\varepsilon} \frac{C\ k}{1+|\omega|(s^{\alpha+1}+\gamma s^{\beta})} ds \\ &\leqslant \int_{0}^{\varepsilon} \frac{C\ k}{1+|\omega| s^{\alpha+1}} ds = \delta_1(\varepsilon). \end{aligned}$$

Hence $\int_{0}^{\varepsilon} S_{\alpha,\beta}(s) f(t-s,u(t-s))ds \in \overline{B}(0,\delta_1(\varepsilon))$.

For the second term

$$\begin{aligned} \left\| \int_{a}^{\infty} S_{\alpha,\beta}(s) f(t-s,u(t-s))ds \right\| &\leqslant \int_{a}^{\infty} \|S_{\alpha,\beta}(s)\| \|f(t-s,u(t-s))\| ds \\ &\leqslant \int_{a}^{\infty} \frac{C\ k}{1+|\omega|(s^{\alpha+1}+\gamma s^{\beta})} ds. \end{aligned}$$

Since $\int_{a}^{\infty} \frac{C\ k}{1+|\omega|(s^{\alpha+1}+\gamma s^{\beta})} ds < \int_{a}^{\infty} \frac{C\ k}{s^{\alpha+1}} ds = \delta(a)$, it follows that $\int_{a}^{\infty} S_{\alpha,\beta}(s) f(t-s,u(t-s))ds \in \overline{B}(0,\delta(a))$.

Finally, for the last term, we have that the set $\{S_{\alpha,\beta}(s) f(t-s,u(t-s)), \varepsilon < s < a\}$ is relatively compact in $\mathbb{X}$, since the operator $S_{\alpha,\beta}(t)$ is compact for $t > 0$ and $\sup_{t\in\mathbb{R}} \sup_{\|x\|\leqslant r} \|f(t,x)\| < +\infty$, we deduce the existence of a compact subset K of $\mathbb{X}$ such that

$$V(t) \subset K + \overline{B}(0,\delta(a)) + \overline{B}(0,\delta_1(\varepsilon)). \tag{5}$$

For the sequel, we introduce the Kuratowski measure of non compactness $\alpha(.)$ of bounded subsets B in the Banach space $\mathbb{X}$ defined by

$$\alpha(B) = \inf\{\varepsilon > 0 : \ B \text{ has a finite cover of balls of diameter } < \varepsilon\}$$

The Kuratowski measure of noncompactness verifies the two following properties:

$$\alpha(B_1 + B_2) \leqslant \alpha(B_1) + \alpha(B_2),$$

$$\alpha(B) = 0 \Leftrightarrow B \text{ is relatively compact in } \mathbb{X}.$$

From the inclusion (5), we obtain the inequality

$$\alpha(V(t)) \leqslant \alpha(K) + \alpha(\overline{B}(0,\delta(a))) + \alpha(\overline{B}(0,\delta_1(\varepsilon))).$$

Since $\alpha(\overline{B}(0,\delta(a))) \leqslant 2\delta(a)$, $\alpha(\overline{B}(0,\delta_1(\varepsilon))) \leqslant 2\delta_1(\varepsilon)$ and K is a compact subset of $\mathbb{X}$, it follows that

$$\alpha(V(t)) \leqslant 2(\delta(a) + \delta_1(\varepsilon)).$$

Since $\lim\limits_{a\to\infty} \delta(a) = 0$ and $\lim\limits_{\varepsilon\to 0} \delta_1(\varepsilon) = 0$ then $\alpha(V(t)) = 0$. From properties of the Kuratowski measure of noncompactness, it follows that $V(t)$ is relatively compact in $\mathbb{X}$.

Second, we will show that the set V is equicontinuous. In fact, we can decompose

$$\begin{aligned}
v(t+s) - v(t) &= \int_{-\infty}^{t+s} S_{\alpha,\beta}(t+s-z) f(z,u(z))dz - \int_{-\infty}^{t} S_{\alpha,\beta}(t-z) f(z,u(z))dz \\
&= \int_{-\infty}^{t} [S_{\alpha,\beta}(t+s-z) - S_{\alpha,\beta}(t-z)] f(z,u(z))dz \\
&+ \int_{t}^{t+s} S_{\alpha,\beta}(t+s-z) f(z,u(z))dz \\
&= \int_{0}^{\infty} [S_{\alpha,\beta}(y+s) - S_{\alpha,\beta}(y)] f(t-y,u(t-y))dy \\
&+ \int_{0}^{s} S_{\alpha,\beta}(y) f(t+s-y,u(t+s-y))dy \\
&= \int_{0}^{a} [S_{\alpha,\beta}(y+s) - S_{\alpha,\beta}(y)] f(t-y,u(t-y))dy \\
&+ \int_{a}^{\infty} [S_{\alpha,\beta}(y+s) - S_{\alpha,\beta}(y)] f(t-y,u(t-y))dy \\
&+ \int_{0}^{s} S_{\alpha,\beta}(y) f(t+s-y,u(t+s-y))dy.
\end{aligned}$$

For each $\varepsilon > 0$, we can choose $a > 0$ and $\delta_1 > 0$ such that

$$\begin{aligned}
&\left\| \int_{a}^{\infty} [S_{\alpha,\beta}(y+s) - S_{\alpha,\beta}(y)] f(t-y,u(t-y))dy + \int_{0}^{s} S_{\alpha,\beta}(y) f(t+s-y,u(t+s-y))dy \right\| \\
\leqslant\ & C\,k \left(\int_{a}^{\infty} \frac{2}{1+|\omega|(y^{\alpha+1}+\gamma y^{\beta})} dy + \int_{0}^{s} \frac{1}{1+|\omega|(y^{\alpha+1}+\gamma y^{\beta})} dy \right) \\
\leqslant\ & \frac{\varepsilon}{2}, \text{ for } s \text{ small enough and } s \leqslant \delta_1.
\end{aligned}$$

Moreover, we can choose $\delta_2 > 0$ such that $\|[S_{\alpha,\beta}(y+s) - S_{\alpha,\beta}(y)] f(t-y,u(t-y))\| \leqslant \frac{\varepsilon}{2a}$ for s small enough and $s \leqslant \delta_2$ since $S_{\alpha,\beta}(t)$ is compact for $t > 0$ and $\sup\limits_{t\in\mathbb{R}} \sup\limits_{\|x\|\leqslant r} \|f(t,x)\|$ is convergent on $\mathbb{R}$ for all $r > 0$.

Consequently $\|v(t+s) - v(t)\| \leqslant \varepsilon$ for s small enough and independent of $u \in \overline{B}(0,r)$ in the space $C_h(\mathbb{X})$.

Finally, $\dfrac{\|v(t)\|}{h(t)} \leqslant \dfrac{C\,k}{h(t)} \displaystyle\int_{-\infty}^{t} \frac{1}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})} ds.$

Since $h(t) \to \infty$ as $|t| \to \infty$ and $\displaystyle\int_{-\infty}^{t} \frac{1}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})} ds < +\infty$, we conclude that

$\dfrac{\|v(t)\|}{h(t)} \to 0 \; as \; |t| \to \infty$ and this convergence is independent of $u \in \overline{B}(0,r)$ in the space $C_h(\mathbb{X})$.
Hence by Lemma 2.51 we have that V is relatively compact set in $C_h(\mathbb{X})$.

Step 4: We will show that $\{u^\lambda : u^\lambda = \lambda\, F(u^\lambda),\ 0 < \lambda < 1\}$ is bounded. Indeed , assume that $u^\lambda(.)$ is a solution of the equation $u^\lambda = \lambda\, F(u^\lambda)$ for some $\lambda \in (0,1)$. Then

$$\begin{aligned}
\|u^\lambda\| &= \lambda \left\| \int_{-\infty}^{t} S_{\alpha,\beta}(t-s) f(s, u^\lambda(s)) ds \right\| \\
&\leqslant \lambda\, k \left\| \int_{-\infty}^{t} S_{\alpha,\beta}(t-s) ds \right\| \\
&\leqslant \lambda\, C\, k \int_{-\infty}^{t} \frac{1}{1 + |\omega|((t-s)^{\alpha+1} + \gamma(t-s)^{\beta})} ds \\
&= \lambda\, C\, k \int_{0}^{+\infty} \frac{1}{1 + |\omega|(s^{\alpha+1} + \gamma s^{\beta})} ds.
\end{aligned}$$

Since $I_{\alpha,\beta} = \displaystyle\int_{0}^{+\infty} \frac{1}{1 + |\omega|(s^{\alpha+1} + \gamma s^{\beta})} ds$ is convergent on $\mathbb{R}$, then we obtain

$$\|u^\lambda\| \leqslant \lambda\, C\, k\, I_{\alpha,\beta} < \infty.$$

Consequently $\{u^\lambda : u^\lambda = \lambda\, F(u^\lambda),\ 0 < \lambda < 1\}$ is bounded.

Step 5: It follows from Theorem 2.50 that the function $t \to f(t, u(t))$ belongs to $MPAAS^p(\mathbb{R}, \mathbb{X}, \mu)$. Consider the map $F : MPAA(\mathbb{R}, \mathbb{X}, \mu) \to MPAA(\mathbb{R}, \mathbb{X}, \mu)$. From Lemma 3.5 we infer that $F(MPAA(\mathbb{R}, \mathbb{X}, \mu)) \subset MPAA(\mathbb{R}, \mathbb{X}, \mu)$. Since $MPAA(\mathbb{R}, \mathbb{X}, \mu)$ is a closed subspace of $C_h(\mathbb{X})$, it follows from the previous steps of this proof that F is completely continuous map. Then, by the Leray-Schauder Alternative Theorem we have that F has a fixed point $u \in MPAA(\mathbb{R}, \mathbb{X}, \mu)$. □

4. Example

To illustrate the result in Theorem 3.7, we consider the following fractional differential equation

$$\begin{cases}
D_t^{\alpha+1} u(t,x) + \gamma D_t^{\beta} u(t,x) = \dfrac{\partial^2}{\partial x^2} u(t,x) + \eta u(t,x) \\
\qquad\qquad\qquad\qquad + D_t^{\alpha}\left[h(t) u(t,x) + h(t) \exp(-\tau t) \sin(u(t,x))\right], \\
u(t,0) = u(t,\pi) = 0, \text{ for } t \in \mathbb{R},\ x \in [0,\pi],\ 0 < \alpha \leqslant \beta < 1,\ \eta < 0,\ \gamma > 0 \; and \; \tau > 0,
\end{cases} \tag{6}$$

where

$$h(t) = \begin{cases}
\sin\left(\dfrac{1}{2 + \cos n + \cos \pi n}\right), & n - \varepsilon_0 < t < n + \varepsilon_0,\ 0 < \varepsilon_0 < \frac{1}{2}, \text{ with } n \in \mathbb{Z}, \\
0 & \text{otherwise.}
\end{cases}$$

Let $(\mathbb{X}, \|.\|_{\mathbb{X}}) = (L^2([0,\pi]), \|.\|_{L^2})$. Define the operator $A : D(A) \subset \mathbb{X} \longrightarrow \mathbb{X}$ by

$$\begin{cases}
D(A) := \{u \in L^2([0,\pi]) : u(0) = u(\pi) = 0\} \\
Au := \frac{\partial^2}{\partial x^2} u + \eta u.
\end{cases}$$

A is ω-sectorial with $\omega = \eta < 0$ and angle $\frac{\pi}{2}$.

Let us consider the nonlinearity

$$\begin{aligned} f(t,\phi(s)) &= h(t)\phi(s) + h(t)\exp(-\tau t)\sin(\phi(s)) \\ &= g(t,\phi)(s) + \varphi(t,\phi)(s), \text{ for all } \phi \in \mathbb{X},\ t \in \mathbb{R},\ s \in [0,\pi]. \end{aligned}$$

Then the equation (6) takes the abstract form (1).

$h(t) \in AAS^2(\mathbb{R},\mathbb{R})$ but $h(t) \notin AA(\mathbb{R},\mathbb{R})$ (see [30],Example 2.3), then $g(t,\phi) \in AAS^2(\mathbb{R}\times\mathbb{X},\mathbb{X})$.

Let $h_1(t) = h(t)\exp(-\tau t)$, the function h_1 is not continuous in $\mathbb{R}$, which implies that $h_1 \notin MAA_0(\mathbb{R},\mathbb{R},\mu)$, then $\varphi(t,\phi) \notin MAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$, but $\varphi(t,\phi) \in MAA_0S^2(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ where μ is the measure and its Radon-Nikodym derivative ρ is given by:

$$\rho(t) = \begin{cases} \exp(\nu t) & if\ t < 0, \text{ with } 0 < \tau < \nu, \\ 1 & if\ t \geqslant 0. \end{cases}$$

Consequently $f \in MPAAS^2(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$.

For each $t \in \mathbb{R}$ and $\phi,\psi \in \mathbb{X}$ we have:

$$\begin{aligned} \|f(t,\phi) - f(t,\psi)\|_{L^2} &\leqslant \left(\int_0^\pi |h(t)\phi(s) - h(t)\psi(s)|^2 ds\right)^{\frac{1}{2}} \\ &+ \left(\int_0^\pi |h(t)\exp(-\tau t)\sin(\phi(s)) - h(t)\exp(-\tau t)\sin(\psi(s))|^2 ds\right)^{\frac{1}{2}} \\ &\leqslant 2|h(t)|\|\phi-\psi\|_{L^2}. \end{aligned}$$

Since $\|h(.)\|_{S^1} = \sup_{t\in\mathbb{R}} \int_t^{t+1} |h(s)|ds \leqslant 2\,\varepsilon_0$, then

$$\begin{aligned} C\,\|L_f\|_{S^1}\left(1 + \frac{\pi|\eta|^{\frac{-1}{\alpha+1}}}{(\alpha+1)\sin(\frac{\pi}{\alpha+1})}\right) &= 2\,C\,\|h(.)\|_{S^1}\left(1 + \frac{\pi|\eta|^{\frac{-1}{\alpha+1}}}{(\alpha+1)\sin(\frac{\pi}{\alpha+1})}\right) \\ &\leqslant 4\,C\,\varepsilon_0\left(1 + \frac{\pi|\eta|^{\frac{-1}{\alpha+1}}}{(\alpha+1)\sin(\frac{\pi}{\alpha+1})}\right). \end{aligned}$$

Assume that $\varepsilon_0 < \dfrac{(\alpha+1)\sin(\frac{\pi}{\alpha+1})}{4\,C\left((\alpha+1)\sin(\frac{\pi}{\alpha+1}) + \frac{\pi}{|\eta|^{\frac{1}{\alpha+1}}}\right)}$, then by Theorem 3.7, the Equation (1) has a unique mild solution in $MPAA(\mathbb{R},\mathbb{X},\mu)$.

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