

STAR-INDUCED GEODETIC NUMBER OF GRAPHS

IMELDA S. ANIVERSARIO*, SERGIO R. CANOY JR., CATHERINE I. MERCA,
MICHAEL E. SUBIDO, FITZMEE ANN D. TAMULAK

Department of Mathematics and Statistics, MSU-Iligan Institute of Technology, Philippines

*Corresponding author: imelda.aniversario@g.msuiit.edu.ph

Received Jul. 17, 2026

ABSTRACT. This study deals with a new geodetic parameter, the star-induced geodetic number of a connected simple graph G . We investigate its properties and characterize the star-induced geodetic sets of some common graphs. The star-induced geodetic numbers of these graphs are determined. Also, characterization of graphs that admit a star-induced geodetic set are given.

2020 Mathematics Subject Classification. 05C12.

Key words and phrases. closed geodetic set; star-induced closed geodetic set; star-induced geodetic number; star-induced closed geodetic number; join; corona.

1. INTRODUCTION

The concept on star-induced geodetic numbers of graphs follows from the definition of geodetic numbers of graphs introduced by Buckley and Harary [3] and its application to game theory [4].

In this paper, the concept of star-induced geodetic sets is introduced and its properties are investigated. The authors determine the star-induced geodetic numbers of some common graphs including graphs resulting from some binary operations. Moreover, this study will show that the star-induced geodetic set is isomorphic to the star-induced closed geodetic set.

The concept of closed geodetic numbers of graphs evolves from the graphical games called the achievement and avoidance games [3]. This paper introduces new concept called the star-induced geodesic that can also be employed as additional rule in playing the achievement and avoidance games to make it more interesting and challenging. For every two vertices u and v of G , the symbol $I_G[u, v]$ is used to denote the set of vertices lying on any of the u - v geodesics. If $S \subseteq V(G)$, we define the *geodetic closure* of S to be the set $I_G[S]$ given by $I_G[S] = \bigcup \{I_G[u, v] : u, v \in S\}$. The two players select vertices of G sequentially as follows: Player A selects $v_1 \in V(G)$ and let $S_1 = \{v_1\}$. Player B selects a vertex $v_2 \neq v_1$ and let $S_2 = \{v_1, v_2\}$ and determines $I_G[S_2]$. If $I_G[S_2] \neq V(G)$, then Player A selects another

vertex $v_3 \notin I_G[S_2]$ and let $S_3 = \{v_1, v_2, v_3\}$ and determines $I_G[S_3]$. If $I_G[S_3] = V(G)$ then player A wins in the achievement game but loses in the avoidance game. Generally, the two players alternately select a vertex in this manner. That is, a player selects $v_i \notin I_G[S_{i-1}]$ and let $S_i = \{v_1, v_2, \dots, v_i\}$ and determine if $I_G[S_i] = V(G)$.

The graphs considered in this paper are undirected simple connected graphs. Readers are referred to [5], [8] for elementary graph theoretic concepts.

A *spanning subgraph* is a subgraph containing all the vertices of G . For any set S of vertices of G , the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with vertex set S . The *removal of a vertex* v from a graph G results in the subgraph $G \setminus v$ with $V(G \setminus v) = V(G) \setminus \{v\}$ and $E(G \setminus v) = \{uw \in E(G) : u \neq v \text{ and } w \neq v\}$. A vertex x of a graph G is called a *cut-vertex* if the removal of x increases the number of components of the graph G . We will use $\omega(G)$ to denote the number of components a graph G has.

Let G be a connected graph and $S \subseteq V(G)$. Then the *2-path closure* $P_2[S]_G$ of S is the set $P_2[S]_G = S \cup \{w \in I_G[u, v] \text{ for some } u, v \in S \text{ with } d_G(u, v) = 2\}$. Then set S is called the *2-path closure absorbing set* of G if $P_2[S]_G = V(G)$. A set $S \subseteq V(G)$ is called a *geodetic set* if $I_G[S] = V(G)$. The number $gn(G)$ given by $gn(G) = \min\{|S| : I_G[S] = V(G)\}$ is called the *geodetic number* of G . A geodetic set S of G for which $|S| = gn(G)$ is called the *minimum geodetic set* of G or a *geodetic basis* of G . Given a connected graph G and $S \subseteq V(G)$, the set S is a *closed geodetic cover* of G if $S = \{v_1, v_2, \dots, v_k\}$ and is obtained by choosing the vertices $v_1, v_2, \dots, v_k$ such that the following hold:

- (i) $v_1 \neq v_2$;
- (ii) $v_i \notin I_G[S_{i-1}]$ for $3 \leq i \leq k$; and
- (iii) $I_G[S_k] = V(G)$,

where $S_i = \{v_1, v_2, \dots, v_i\}$ for all $i = 1, 2, \dots, k$. If $S \subseteq V(G)$ and S satisfies (1) and (2), then S is a *closed geodetic subset* of $V(G)$. We denote by $C^*(G)$ the collection of all closed geodetic covers of G . Henceforth, $S \in C^*(G)$ means S is in a canonical form, that is, it is obtained as the set S . The *closed geodetic number* of G , is given by $cgn(G) = \min\{|S| : S \in C^*(G)\}$. A set $S \in C^*(G)$ with $|S| = cgn(G)$ is called a *closed geodetic basis* of G . A geodetic set $S \subseteq V(G)$ is called a *star-induced geodetic set* (*sig-set*) of a graph G if the set S will induce a star subgraph of G . The minimum cardinality of a star-induced geodetic set is called *star-induced geodetic number* of G , denoted by $sign(G)$. A star-induced geodetic set S with $|S| = sign(G)$ is referred to as *sign-set* of G .

2. KNOWN RESULTS

In this section, we review several established results that serve as fundamental tools in proving the main theorems regarding the star-induced geodetic number of a graph.

Theorem 2.1. [1] Let $G = K_{m,n}$, where $m, n \geq 2$, and let U and W be the partite sets of G . Let $S \subseteq V(G)$. Then $S \in C^*(G)$ if and only if S is any of the following:

- (i) $S = U$;
- (ii) $S = W$;
- (iii) $S = U \cup \{w\}$ for any $w \in W$;
- (iv) $S = W \cup \{u\}$ for any $u \in U$.

Corollary 2.2. [1] If $m, n \geq 2$, $cgn(K_{m,n}) = \min\{m, n\}$.

Lemma 2.3. [8] Every geodetic cover of a connected graph G contains all its extreme vertices.

Theorem 2.4. [1] Let H be a connected noncomplete graph and $G = H + K_p$. If S is a closed geodetic basis of G , then $S \subseteq V(H)$ and S is a 2-path closure absorbing set in H .

In the succeeding sections, we present and prove results concerning the star-induced geodetic numbers of several well-known classes of graphs.

3. STAR-INDUCED GEODETIC NUMBERS OF SOME GRAPHS

Theorem 3.1. Let G be a connected graph on $n \geq 2$ vertices. Then G admits a star-induced geodetic set if and only if G has a star-induced closed geodetic set. In this case, $sicgn(G) = sign(G)$.

Proof: Suppose G admits a star-induced geodetic set. Clearly, G has a star-induced closed geodetic set.

For the converse, suppose G has a star-induced closed geodetic set S . Then $\langle S \rangle = K_{1,r}$ for some $1 \leq r \leq n - 1$. Let v_0 be the central vertex of $\langle S \rangle$ and let $S = \{v_0, v_1, v_2, \dots, v_r\}$. Let $S_0 = \{v_0\}$ and set $S_j = \{v_0, v_1, \dots, v_j\}$, for $1 \leq j \leq r$. Since $\langle S \rangle$ is a star, $(S \setminus S_j) \cap I_G[S_{j+1}] = \emptyset$. Hence, $S = S_{r+1} = \{v_0, v_1, \dots, v_r\}$ is a closed geodetic set of G . Since $\langle S \rangle$ is a star, S_0 is a star-induced closed geodetic set of G . $\square$

Proposition 3.2. Let G be a connected graph on $n \geq 2$ vertices and suppose that G admits a star-induced geodetic set. Then every star-induced geodetic set of G is a 2-path closure absorbing set. If, in addition, $G \neq K_2$ and S is a star-induced geodetic set with central vertex v , then $S \setminus \{v\}$ is a 2-path closure absorbing set that induces a non-trivial empty graph.

Proof: Suppose S is a star-induced geodetic set of G . If $S = V(G)$, then we are done. So suppose $S \neq V(G)$. Let $x \in S$ be the central vertex of $\langle S \rangle$ and let $v \in V(G) \setminus S$. Since S is a geodetic set, there exist $a, b \in S$ such that $v \in I_G[a, b]$. Clearly, $a \neq v$ and $b \neq v$. Hence, since $\langle S \rangle$ is a star, $d_G(a, b) = 2$. Therefore, S is a 2-path closure absorbing set of G . Suppose $G \neq K_2$. Then $|S| \geq 3$. Clearly, if v is a central vertex of G , $S \setminus \{v\}$ is a 2-path closure absorbing set that induces a non-trivial empty graph. $\square$

Theorem 3.3. Let G be connected graph on $n \geq 2$ vertices and suppose that G admits a star-induced geodetic set. Then $2 \leq sign(G) \leq n$. Moreover,

- (i) $\text{sign}(G) = 2$ if and only if $G = K_2$;
- (ii) $\text{sign}(G) = 3$ if and only if there exists a subgraph H of G of order $n - 3$ and $v \in V(G) \setminus V(H)$ such that G is obtained from $(v + \overline{K}_2) \cup H$ by adding the edges of $\overline{K}_2 + H$ and $\langle V(H) \cup \{v\} \rangle$;
- (iii) $\text{sign}(G) = n$ if and only if $G = K_{1,n-1}$.

Proof: Since every geodetic set in a non-trivial connected graph has at least two elements, it follows that $2 \leq \text{sign}(G) \leq n$.

- (i) Suppose that $\text{sign}(G) = 2$, say $S = \{x, y\}$ is a star-induced geodetic set of G . Since $\langle S \rangle$ is a star, it follows that $\langle S \rangle = K_2$. Since $I_G[x, y] = \{x, y\}$, $G = K_2$.

For the converse, suppose $G = P_2$. Clearly, $\text{sign}(G) = 2$.

- (ii) Suppose $\text{sign}(G) = 3$ and let S be a star-induced geodetic set of G . Then $\langle S \rangle = K_{1,2}$. Let v be the central vertex of $\langle S \rangle$ and let $x, y \in S \setminus \{v\}$. Then $\langle \{x, y\} \rangle = \overline{K}_2$. Let $H = \langle V(G) \setminus S \rangle$. Let $z \in V(H)$. Since S is a 2-path closure absorbing set (Proposition 3.2), $zx, zy \in E(G)$. Hence, $\overline{K}_2 + H$ is a subgraph of G . Thus, G is the graph with $V(G) = \{v\} \cup V(\overline{K}_2 + H)$ and $E(G) = E(v + \overline{K}_2) \cup E(\overline{K}_2 + H) \cup E(\langle V(H) \cup \{v\} \rangle)$, as described.

For the converse, suppose that G is the graph described. Let $S = V(v + \overline{K}_2)$. Then $\langle S \rangle = K_{1,2} = P_3$. Since $V(H) \subseteq I_G[S]$, S is a star-induced geodetic set of G . By (i), $n \geq 4$, $\text{sign}(G) = |S| = 3$.

- (iii) Suppose that $\text{sign}(G) = n$. Then the star-induced geodetic set of G is the vertex set of G , that is, $S = \{v_1, v_2, \dots, v_n\} = V(G)$. It follows that $\langle S \rangle \cong K_{1,n-1}$. Thus, $G = K_{1,n-1}$.

For the converse, suppose that $G = K_{1,n-1}$. Let v_0 be the central vertex of G and let S be a star-induced closed geodetic set of G . Let $S = \{v_1, v_2, \dots, v_k\}$ be a canonical form of S as an element of $C^*(G)$. Since $\langle S \rangle$ is a tree, it follows that $v_1 = v_0$. Suppose $k < n$, that is, $S \neq V(G)$, say $x \in V(G) \setminus S$. Then $d_G(x, v_1) = 1$ and $d_G(x, v_j) = 2$ for all $j \in \{2, 3, 4, 5, \dots, k\}$. It follows that $x \notin I_G(v_i, v_j)$ for any two distinct vertices $v_i, v_j \in \{v_2, v_3, \dots, v_k\}$. Consequently, S is not a geodetic set of G , a contradiction. Therefore, $S = V(G)$ and $\text{sign}(G) = n$. $\square$

The next result follows from Theorem 3.1

Corollary 3.4. *Let n be a positive integer. Then each of the following statements holds:*

- (i) For $n \geq 2$, P_n admits a star-induced geodetic set if and only if $n = 2$ or $n = 3$.
- (ii) For $n \geq 3$, C_n admits a star-induced geodetic set if and only if $n = 4$.
- (iii) K_n does not admit a star-induced geodetic set for all $n \geq 3$.

Proposition 3.5. *Let T be a tree on $n \geq 2$. Then T admits a star-induced geodetic set if and only if T is a star.*

Proof: Suppose T is tree that admits a star-induced geodetic set. If $n \in \{2, 3\}$, then $T \in \{P_2, P_3\}$. Suppose $n \geq 4$ and suppose $\text{sign}(T) < n$. Let S be a sig-set of T . Then $S \neq V(T)$. Let $x \in V(T) \setminus S$.

Since S is a geodetic set, there exists $a, b \in S$ such that $d_G(a, b) = 2$ and $x \in I_G[a, b]$. If v is the central vertex of $\langle S \rangle$, then $\langle \{v, a, b, x\} \rangle = C_4$, contrary to the fact that T is a tree. Thus, $\text{sign}(T) = n$. By Theorem 4.4, T is a star.

For the converse, suppose T is a star. Then, by Theorem 3.2, $\text{sign}(T) = n$. This implies that T admits a star-induced geodetic set. $\square$

Remark 3.6. Let G be a connected graph.

- (i) Every star-induced geodetic set of G contains its extreme vertices.
- (ii) Every end-vertex belongs to every star-induced geodetic set of G .

Lemma 3.7. Let G be a connected graph with a cut-vertex. If $S \subseteq V(G)$ is a star-induced geodetic set of G and x is a cut-vertex of G , then every component of $G \setminus x$ contains a vertex in S .

Proof: Let S be a star-induced geodetic set of a connected graph G and $x \in V(G)$ be a cut-vertex of G . Suppose that $C_1, C_2, \dots, C_n$ are the components of $G - x$. It remains to show that every component in $G - x$ contains an element in S . Assume that there exists a component, say C_1 , of $G - x$ such that C_1 does not contain a vertex of S . Let $x_0 \in V(C_1)$. Since S is a geodetic set and $x_0 \notin S$, there exist $u, v \in S$ such that x_0 lies in some u - v geodesic. This is impossible because $u, v \notin V(C_1)$. Hence, C_1 must contain a vertex in S . $\square$

Example 3.8. Consider the graph G shown in Figure 1. Let $S = \{v_1, v_2, v_3, v_5\}$. Then S is a star-induced closed geodetic set of G . Moreover, every component of $G - v_2$ contains at least one vertex of S .

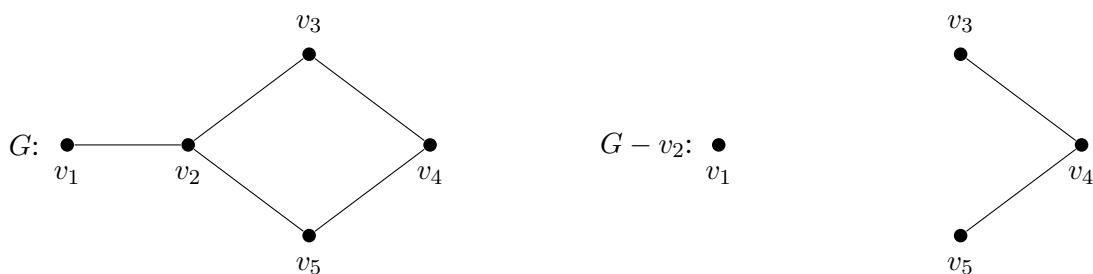


FIGURE 1. A graph G with a cut-vertex v_2 and the graph $G - v_2$. The set $S = \{v_1, v_2, v_3, v_5\}$ is a star-induced closed geodetic set of G , and every component of $G - v_2$ contains a vertex of S

Theorem 3.9. Let G be a connected graph that admits a star-induced geodetic set. If x is a cut-vertex and S is a star-induced geodetic set of G , then $x \in S$. Moreover, x is the central vertex of $\langle S \rangle$.

Proof: Let $C_1, C_2, C_3, C_4, \dots, C_n$ be the components of $G \setminus x$. By Lemma 3.7, $V(C_j) \cap S \neq \emptyset$ for each $j \in \{1, 2, \dots, k\}$. Since $\langle S \rangle$ is connected, $x \in S$. Next, let v be the central vertex of $\langle S \rangle$. If $G = K_{1,n}$, then

$x = v$. Suppose $G \neq K_{1,n}$ and suppose $x \neq v$. Then $d_G(x, v) = 1$. Let $C = C_j$ be a component of $G \setminus x$. Let $y \in V(C) \cap N_G(x)$. If $y = v$, then $v \in V(C)$. Suppose $y \neq v$. Since $x \neq v$ and $d_G(x, y) = 1$, $y \notin S$. By Theorem 3.2, there exist $z \in V(C) \cap S$ such that $y \in I_G(x, z)$ where $d_G(x, z) = 2$. This implies that $z \neq v$. Hence, $d_G(z, v) = 1$. Therefore, $v \in V(C)$ (since C is connected). Since C was arbitrary chosen, it follows that $v \in \bigcap_{i=1}^k C_i$, a contradiction. Therefore, $x = v$. $\square$

Corollary 3.10. *If G admits a star-induced geodetic set, then G contains at most a single cut-vertex.*

Proof: Suppose G has a cut-vertex. Suppose x and y are cut-vertices of G and let S be a star-induced closed geodetic set. By Theorem 3.9, $x, y \in S$ and each of x and y is the central vertex of $\langle S \rangle$. It follows that $x = y$. $\square$

Example 3.11. Consider the graph G in Figure 2, where x and y are the two cut-vertices of G . Then G has no star-induced closed geodetic set.

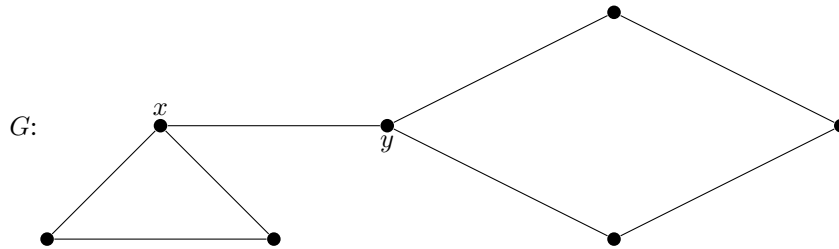


FIGURE 2. A graph without a *sicg*-set

Theorem 3.12. *Let $G = K_{m,n}$, where $m, n \geq 3$, and let X and Y be the partite sets of $V(G)$. Let $S \subseteq V(G)$. Then S is a star-induced geodetic set of G if and only if S is one of the following:*

- (i) $S = X \cup \{y\}$ for any $y \in Y$;
- (ii) $S = Y \cup \{x\}$ for any $x \in X$.

Proof: Let $G = K_{m,n}$, where $m, n \geq 3$, and let X and Y be the partite sets of $V(G)$. Suppose that S is a star-induced closed geodetic set of G . Then, $S \in C^*(G)$. By Theorem 2.1, $S = X \cup \{y\}$ for any $y \in Y$ or $S = Y \cup \{x\}$ for any $x \in X$.

For the converse, suppose first that $S = X \cup \{y\}$ for some $y \in Y$. Then $\langle S \rangle = K_{1,|X|}$. Clearly, $Y \subseteq I_G[S]$. Hence, S is a star-induced geodetic set. Similarly, $Y \cup \{x\}$ is a star-induced geodetic set of G . $\square$

Corollary 3.13. *For $m, n \geq 3$, $\text{sign}(K_{m,n}) = \min\{m, n\} + 1$.*

Proof: The proof follows immediately from Theorem 3.12 and Corollary 2.2. $\square$

4. STAR-INDUCED GEODETIC NUMBER OF THE JOIN OF GRAPHS

Theorem 4.1. *Let H be a connected noncomplete graph and $G = H + K_m$. Then S is a star-induced geodetic set of G if and only if*

- (i) S is star-induced geodetic set of H ; or
- (ii) $S = \{x\} \cup S_H$, for some $x \in V(K_m)$ and S_H is a 2-path closure absorbing set of H with $\langle S_H \rangle$ is an empty graph.

Proof: Suppose G admits a star-induced geodetic set, say S . If $S \subseteq V(H)$, then $\langle S \rangle$ is a star-induced subgraph of H . Since S is a geodetic set in G , it is a geodetic set in H . Thus, H admits a star-induced geodetic set. Next, suppose that $S \cap V(K_m) \neq \emptyset$. Since S is a geodetic set, $S_H = S \cap V(H) \neq \emptyset$. Moreover, since $\langle S \rangle$ is a star, $|S \cap V(K_m)| = 1$, say $S \cap V(K_m) = \{v\}$. Thus, $S_H = S \setminus \{v\}$ and $\langle S_H \rangle$ is an empty graph. Now, let $z \in V(H) \setminus S_H$. Since S is geodetic set in G , there exist $a, b \in S_H$ such that $d_H(a, b) = 2$ and $x \in I_H[a, b]$. This implies that S_H is a 2-path closure absorbing set in H .

For the converse, suppose that H admits a star-induced geodetic set, say S^* . Pick any $x, y \in S^*$ such that $d_H(x, y) = d_G(x, y) = 2$. Then $v \in I_G[x, y]$. Hence, S^* is a star-induced geodetic set in G . Next, suppose that H has a 2-path closure absorbing set $S = \{x_1, x_2, \dots, x_k\}$ such that $\langle S \rangle$ is an empty graph. Choose any $v \in V(K_m)$ and let $S_0 = \{v\}$ and $S_j = \{v, x_1, \dots, x_j\}$ for each $j = \{1, 2, \dots, k\}$. Then $D = \{v, x_1, x_2, \dots, x_k\}$ is a geodetic set in G and $\langle D \rangle$ is a star. Thus, in either case, G admits a star-induced geodetic set. $\square$

Corollary 4.2. *Let H be a connected noncomplete graph and let n be a positive integer. Suppose $G = H + K_m$ admits a star-induced geodetic set. Then S is a star-induced geodetic set of G if and only if*

- (A) S is a star-induced geodetic set of H ; or
- (B) $S = \{v\} \cup S_H$, for some $v \in V(K_m)$ and S_H is a 2-path closure absorbing set of H with $\langle S_H \rangle$ is an empty graph.

Moreover, if G admits a star-induced geodetic set, then

- (i) $\text{sign}(G) = \text{sign}(H)$ if a set of type (B) does not exist and
- (ii) $\text{sign}(G) = \rho_2^\emptyset(H) + 1$ if a set of type (B) exists.

Proof: The first part is immediate from Theorem 4.1. Clearly, (i) holds. Suppose a set of type (B) exists. The result is clear if a set of type (A) does not exist. So suppose a set of type (A) also exists. Let S be a star-induced geodetic set of H and let v be the central vertex of $\langle S \rangle$. Note that since $H \neq K_2$, $S_H = S \setminus \{v\}$ is a 2-path closure absorbing set of H that induces a non-trivial empty graph by Proposition 3.2. Hence, $\rho_2^\emptyset(H) \leq |S_H| = \text{sign}(H) - 1$, that is, $\rho_2^\emptyset(H) + 1 \leq \text{sign}(H)$. Therefore, $\text{sign}(G) = \rho_2^\emptyset(H) + 1$. $\square$

Corollary 4.3. *Let n and m be positive integers. Then each of the following holds:*

- (i) $K_n + C_m$ admits a star-induced geodetic set if and only if m is even and $m \geq 4$. Moreover, if it does admit a star-induced geodetic set, then $\text{sign}(K_n + C_m) = \frac{m}{2} + 1$.
- (ii) $K_n + P_m$ admits a star-induced geodetic set if and only if $m = n = 1$ or m is odd and $m \geq 3$. If it admits a star-induced geodetic set, then $\text{sign}(K_n + P_m) = \lceil \frac{m}{2} \rceil + 1$.

Proof: Let n and m be positive integers.

- (i) If $m = 3$, then $K_n + C_3 = K_{n+3}$. Since it is complete graph different from K_2 , it does not admit a star-induced geodetic set. Suppose $m \geq 4$. If m is odd, then C_m does not have any 2-path closure absorbing set that induces an empty graph. By Proposition 3.2, C_m also does not admit a star-induced geodetic set. Hence, $K_n + C_m$ admits a star-induced geodetic set if and only if m is even. Since $\rho_2^\emptyset(C_m) = \frac{m}{2}$, it follows that $\text{sign}(K_n + C_m) = \frac{m}{2} + 1 = \text{sign}(K_n + C_m)$.
- (ii) If $m = 1$ and $n \geq 2$ or $m = 2$, then $K_n + P_m$ is a complete graph different from K_2 . Hence, $K_n + P_m$ does not admit a star-induced geodetic set. Suppose $m \geq 3$. If m is even, then P_m does not have any 2-path closure absorbing set that induces an empty graph. therefore, $K_n + P_m$ admits a star-induced geodetic if and only if $n = m = 1$ or m is odd and $m \geq 3$. Moreover, since $\rho_2^\emptyset(P_m) = \lceil \frac{m}{2} \rceil$, $\text{sign}(K_n + P_m) = \lceil \frac{m}{2} \rceil + 1 = \text{sign}(K_n + P_m)$. $\square$

Theorem 4.4. Let G and H be any two non-complete graphs. Then $G + H$ admits a star-induced geodetic set if and only if one of the following holds:

- (i) G admits a star-induced geodetic set.
- (ii) H admits a star-induced geodetic set.
- (iii) G has a 2-path closure absorbing set that induces an empty graph.
- (iv) H has a 2-path closure absorbing set that induces an empty graph.

Proof: Suppose $G + H$ admits a star-induced geodetic set, say S is a star-induced geodetic set of $G + H$. If $S \subseteq V(G)$, then S is a star-induced geodetic set of G . Similarly, if $S \subseteq V(H)$, then S is star-induced geodetic set of H . Thus, (i) or (ii) holds. Next, suppose that $S_G = V(G) \cap S \neq \emptyset$ and $S_H = V(H) \cap S \neq \emptyset$. Let v be the central vertex of $\langle S \rangle$. Suppose that $v \in S_G$. Since $S_H \neq \emptyset$ and $\langle S \rangle$ is a tree, $|S_G| = 1$, that is, $S_G = \{v\}$. Furthermore, because $S_H = S \setminus \{v\}$ and H is a non-trivial, $\langle S_H \rangle$ is a non-trivial empty graph. Now, let $w \in V(H) \setminus S_H$. Since S is a 2-path closure absorbing set of $G + H$, there exist $a, b \in S$ such that $d_{G+H}(a, b) = 2$ and $w \in I_G[a, b]$. Hence, $a, b \in S_H$, $d_H(a, b) = 2$, and $w \in I_H(a, b)$. This shows that S_H is a 2-path closure absorbing set of H . Similarly, if $v \in V(H)$, then G has a 2-path closure absorbing set that induces an empty graph. This shows that (iii) and (iv) also holds.

For the converse, suppose that (i) holds, that is., G admits a star-induced geodetic set, say D . Since $G \neq K_2$, $|D| \geq 3$ by Theorem 3.3(i). Pick any distinct vertices $x, y \in D$ with $d_G(x, y) = d_{G+H}(x, y) = 2$. Then $V(H) \subseteq I_G[x, y]$. This implies that D is a star-induced geodetic set of $G + H$. Similarly, $G + H$

admits a star-induced geodetic set if (ii) holds. Next, suppose that G has a 2-path closure absorbing set $D' = \{x_1, x_2, \dots, x_r\}$ such that $\langle D' \rangle$ is an empty graph. Since H is a non-complete graph, $|D'| \geq 2$. Let $p, q \in D'$ with $p \neq q$. Then $V(K_n) \subseteq I_G(p, q)$. This implies that D' is a geodetic set of G . Pick any $v_0 \in V(K_n)$ and set $D_0 = \{v_0\}$ and $D_j = \{v_0, x_1, \dots, x_j\}$ for $j \in \{1, 2, \dots, r\}$. Then $S' = D_k$ is star-induced geodetic set of G , showing that G admits a star-induced geodetic set. Similarly, G admits a star-induced geodetic set if (iv) holds. $\square$

Corollary 4.5. *Let G and H be non-complete graphs and suppose $G + H$ admits a star-induced geodetic set. Then S is a star-induced geodetic set of G if and only if it is one of the following sets:*

- (A) S is a star-induced geodetic set of G ,
- (B) S is a star-induced cgeodetic set of H ,
- (C) $S = \{v\} \cup S_G$ for some $v \in V(H)$ and 2-path closure absorbing set S_G of G that induces an empty graph, or
- (D) $S = \{v\} \cup S_H$ for some $v \in V(G)$ and 2-path closure absorbing set S_H of H that induces an empty graph,

Moreover, if $G + H$ admits a star-induced geodetic set, then

- (i) $\text{sign}(G + H) = \text{sign}(G)$ (respectively $\text{sign}(H)$) if a set of type (A) (respectively type (B)) exists and the sets of type (B), type (C), and type (D) (respectively type (A), type (C), and type (D)) do not exist. If set of type (A) and type (B) exist and sets of type (C) and type (D) do not exist, then $\text{sign}(G + H) = \min\{\text{sign}(G), \text{sign}(H)\}$;
- (ii) $\text{sign}(G + H) = \min\{\rho_2^\emptyset(G) + 1, \text{sign}(H)\}$ if sets of type (B) and type (C) exist but set of type (D) does not exist;
- (iii) $\text{sign}(G + H) = \min\{\rho_2^\emptyset(H) + 1, \text{sign}(G)\}$ if sets of type (A) and type (D) exist but set of type (C) does not exist;
- (iv) $\text{sign}(G + H) = \rho_2^\emptyset(G) + 1$ if a set of type (C) exists but sets of type (B) and (D) do not exist;
- (v) $\text{sign}(G + H) = \rho_2^\emptyset(H) + 1$ if a set of type (D) exists but sets of type (A) and (C) do not exist.
- (vi) $\text{sign}(G + H) = \min\{\rho_2^\emptyset(G) + 1, \rho_2^\emptyset(H) + 1\}$ if sets of type (C) and type (D) both exist.

Proof: One of (A), (B), (C), and (D) holds by Theorem 4.4. Next suppose that $G + H$ admits a star-induced geodetic set. Clearly, (i) holds. Suppose sets of type (B) and type (C) exist but set of type (D) does not exist. If a set of type (A) exists, say D is a star-induced geodetic set of G and v is the central vertex of $\langle D \rangle$, then $D \setminus \{v\}$ is a 2-path closure absorbing set in G and $\langle D \setminus \{v\} \rangle$ is an empty graph. Thus, $\rho_2^\emptyset(G) \leq |D \setminus \{v\}| = \text{sign}(G) - 1$. Hence, by assumption, $\text{sign}(G + H) = \min\{\rho_2^\emptyset(G) + 1, \text{sign}(H)\}$ that is, (ii) holds. Similarly, (iii) holds. Next, suppose that a set of type (C) exists but sets of type (B) and type (D) do not exist. Again, if a set of type (A) exists, then $\rho_2^\emptyset(G) + 1 \leq \text{sign}(G)$. Therefore, (iv) holds. Similarly, (v) holds. Finally, suppose that sets of type (C) and type (D) both exist. the

result holds if sets of type (A) and type (B) do not exist. If they do exist, $\rho_2^\partial(G) + 1 \leq \text{sign}(G)$ and $\rho_2^\partial(H) + 1 \leq \text{sign}(H)$. Thus, (vi) holds. $\square$

5. STAR-INDUCED GEODETIC NUMBER OF THE CORONA OF GRAPHS

This section discusses the star-induced geodetic number of the corona of two graphs G and H . The corona of graphs G and H denoted by $G \circ H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H and then joining the i th vertex of G to every vertex of the i th copy of H .

Theorem 5.1. *Let G be a connected graph and let H be any graph. Then $G \circ H$ admits a star-induced geodetic set if and only if $G = K_1$ and*

- (i) H admits a star-induced geodetic set; or
- (ii) H has a 2-path closure absorbing set that induces an empty graph.

Proof: Suppose $G \circ H$ admits a star-induced geodetic set, say S . Suppose that $G \neq K_1$. Let v be the central vertex of $\langle S \rangle$. Since $G \neq K_1$, $v \in V(G)$. Pick any $w \in N_G[v]$ and let $p \in V(H^w)$. Since $d_{G \circ H}(v, p) = 2$ and $\langle S \rangle$ is a star, $p \notin S$. Clearly, $p \notin I_{G \circ H}[a, b]$ for all $a, b \in S$, contrary to the assumption that S is a star-induced geodetic set. Thus, $G = K_1$ and $G \circ H = K_1 \circ H$. Moreover, if $G \circ H$ admits a star-induced geodetic set, then H admits a star-induced geodetic set or H has a 2-path closure absorbing set that induces an empty graph.

For the converse, suppose that $G = K_1$. Then $G \circ H = K_1 + H$. Hence, $G \circ H$ admits a star-induced geodetic set by Theorem 4.1. $\square$

Example 5.2. Consider the graphs P_3 , P_2 , and $P_3 \circ P_2$ in Figure 3.

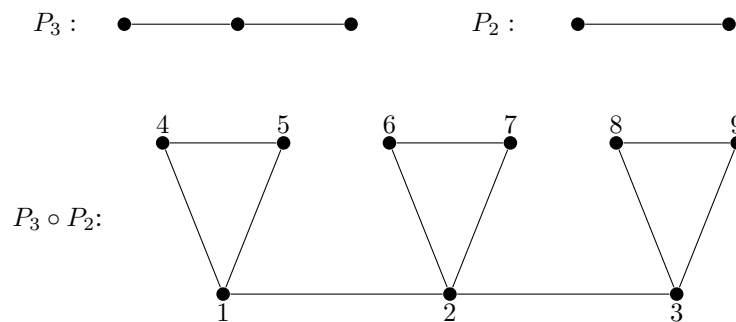


FIGURE 3. Corona of $P_3 \circ P_2$

The graphs P_3 and P_2 admit a star-induced closed geodetic set. Now, observe that the $\langle \{4, 1, 2\} \rangle$, $\langle \{5, 1, 2\} \rangle$, $\langle \{1, 2, 6, 3\} \rangle$, $\langle \{1, 2, 7, 3\} \rangle$, $\langle \{8, 3, 2\} \rangle$, and $\langle \{9, 3, 2\} \rangle$ are star-induced subgraph of $P_3 \circ P_2$ but none of these sets is a closed geodetic set. Thus, $P_3 \circ P_2$ does not admit a star-induced closed geodetic set.

Corollary 5.3. *Let G be a connected graph and let H be any graph. Then $G \circ H$ admits a star-induced geodetic set if and only if $G = K_1$ and*

- (i) *H admits a star-induced geodetic set; or*
- (ii.) *H has a 2-path closure absorbing set that induces an empty graph.*

Then $\text{sign}(G \circ H) = \text{sign}(K_1 + H)$.

6. CONCLUSION

This work establishes the basic theory of the star-induced geodetic number by investigating its properties, determining its values for several common graph classes, and characterizing graphs that admit a star-induced geodetic set. These results enrich the study of geodetic parameters and provide a foundation for further research on star-induced geodetic concepts in graph theory.

Acknowledgment. The authors would like to express their gratitude to the referees for their insightful comments and suggestions which significantly improved the paper. The authors would also like to thank the following funding agencies: Mindanao State University - Iligan Institute of Technology (MSU-IIT) and the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP), Philippines.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

- [1] I.S. Aniversario, F.P. Jamil, S.R. Canoy Jr., The Closed Geodetic Numbers of Graphs, Util. Math. 74 (2007), 3–18.
- [2] F.P. Jamil, I.S. Aniversario, S.R. Canoy Jr., The Closed Geodetic Numbers of the Corona and Composition of Graphs, Util. Math. 82 (2010), 135–153.
- [3] F. Buckley, F. Harary, Distance in Graphs, Addison-Wesley, 1990.
- [4] F. Buckley, F. Harary, Geodetic Games for Graphs, Quaest. Math. 8 (1986), 321–334.
- [5] F. Buckley, F. Harary, L.V. Quintas, Extremal Results on the Geodetic Number of a Graph, Scientia, Ser. A 2 (1988), 17–26.
- [6] O.I. Cauntongan, I.S. Aniversario, Path-Induced Closed Geodetic Numbers of Some Graphs, Adv. Appl. Discrete Math. 22 (2019), 41–53. <https://doi.org/10.17654/DM022010041>.
- [7] G.J. Chang, L.-D. Tong, H.-T. Wang, Geodetic Spectra of Graphs, Eur. J. Comb. 25 (2004), 383–391. <https://doi.org/10.1016/j.ejc.2003.09.010>.
- [8] G. Chartrand, F. Harary, P. Zhang, Geodetic Sets in Graphs, Discuss. Math. Graph Theory 20 (2000), 129–138. <https://doi.org/10.7151/dmgt.1112>.
- [9] G. Chartrand, P. Zhang, The Forcing Geodetic Number of a Graph, Discuss. Math. Graph Theory 19 (1999), 45–58. <https://doi.org/10.7151/dmgt.1084>.
- [10] S.R. Chellathurai, S.P. Vijaya, The Geodetic Domination Number for the Product of Graphs, Trans. Comb. 3 (2014), 19–30. <https://doi.org/10.22108/toc.2014.5750>.

- [11] H. Escudro, R. Gera, A. Hansberg, N. Jafari Rad, L. Volkmann, Geodetic Domination in Graphs, *J. Comb. Math. Comb. Comput.* 77 (2011), 89–101.
- [12] B. Gayathri, S. Kaspar, Connected Co-Independent Domination of a Graph, *Int. J. Contemp. Math. Sci.* 6 (2011), 423–429.
- [13] J.L. Gross, J. Yellen, *Graph Theory and Its Applications*, Chapman and Hall/CRC, 2006. <https://doi.org/10.1201/9781420057140>.
- [14] A. Hansberg, L. Volkmann, On the Geodetic and Geodetic Domination Numbers of a Graph, *Discret. Math.* 310 (2010), 2140–2146. <https://doi.org/10.1016/j.disc.2010.04.013>.
- [15] F. Harary, Convexity in Graphs: Achievement and Avoidance Games, in: *Annals of Discrete Mathematics* (20): Convexity and Graph Theory, Elsevier, 1984, pp. 323. [https://doi.org/10.1016/S0304-0208\(08\)72843-3](https://doi.org/10.1016/S0304-0208(08)72843-3).
- [16] F. Harary, *Graph Theory*, Addison-Wesley, 1969.
- [17] T.W. Haynes, S. Hedetniemi, P. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, 1998. <https://doi.org/10.1201/9781482246582>.
- [18] F.P. Jamil, M.P. Laurente, M.B. Macababat, Strongly Closed Geodetic Numbers of Graphs, *Int. Math. Forum* 5 (2010), 1277–1290.
- [19] T. Jiang, I. Pelayo, D. Pritikin, Geodesic Convexity and Cartesian Products in Graphs, preprint, 2004. <https://web.mat.upc.edu/ignacio.m.pelayo/research/cart-draft.pdf>.
- [20] A.A. Khalil, Determination and Testing the Domination Numbers of Helm Graph, Web Graph and Levi Graph Using MATLAB, *J. Educ. Sci.* 24 (2011), 103–116. <https://doi.org/10.33899/edusj.1999.58719>.
- [21] S. Pemmaraju, S. Skiena, *Computational Discrete Mathematics*, Cambridge University Press, 2003. <https://doi.org/10.1017/CB09781139164849>.
- [22] A.P. Santhakumaran, J. John, The Upper Connected Edge Geodetic Number of a Graph, *Filomat* 26 (2012), 131–141. <https://doi.org/10.2298/FIL1201131S>.
- [23] A.P. Santhakumaran, P. Titus, J. John, The Upper Connected Geodetic Number and Forcing Connected Geodetic Number of a Graph, *Discret. Appl. Math.* 157 (2009), 1571–1580. <https://doi.org/10.1016/j.dam.2008.06.005>.
- [24] A.P. Santhakumaran, S.V. Ullas Chandran, The Edge Geodetic Number and Cartesian Product of Graphs, *Discuss. Math. Graph Theory* 30 (2010), 55–73. <https://doi.org/10.7151/dmgt.1476>.
- [25] R.N. Villarante, I.S. Aniversario, Path-Induced Geodetic Numbers of Some Graphs, *J. Ultra Sci. Phys. Sci. Sect. A* 29 (2017), 197–205. <https://doi.org/10.22147/jusps-a/290502>.