

HALPERN-TYPE ALGORITHM STRONG CONVERGENCE FOR SPLIT TRI-EQUILIBRIUM PROBLEMS IN HILBERT SPACES

WIJDAN S. ABED, AYED E. HASHOOSH*

College of Education for Pure Sciences, University of Thi-Qar, Nasiriyah, Iraq

*Corresponding author: Ayed.hashoosh@utq.edu.iq

Received Jul. 9, 2026

ABSTRACT. This study introduces a novel class of split equilibrium problems with two tri-function equilibrium problems in real Hilbert spaces related by a bounded linear operator. We present a Halpern-type split iterative algorithm that avoids the product-space technique to approximate a common solution. We prove the strong convergence of the generated sequence to a shared solution of the split tri-equilibrium problem and the fixed point problem under reasonable assumptions on the tri-function equilibrium mappings, the bounded linear operator, the nonexpansive mapping, and the control parameters. The suggested framework adds tri-function equilibrium models to the standard split equilibrium problem and unifies related topics. This leads to strong convergence results for split variational inequality and split fixed point problems as particular instances. A numerical example and structural comparison with split equilibrium algorithms demonstrate the suggested approach's efficacy and adaptability.

2020 Mathematics Subject Classification. 47H09; 47J25; 49J40; 47J20; 47H10.

Key words and phrases. split equilibrium; approximation theorem; fixed point theorem; strong convergence; trifunction; Halpern iteration.

1. INTRODUCTION

For the purpose of solving tri-function equilibrium issues in Hilbert space, this paper provides a new split equilibrium model. Through the utilisation of a bounded linear operator, the framework expands the traditional split equilibrium problem by establishing a connection between two tri-function equilibrium problems. Developing a substantially convergent iterative method that does not require product space is the objective of this project. The concept of the split equilibrium problem (SEP) was initially presented by He [3]. He utilised a bounded linear operator to establish a connection between two equilibrium issues that were posed in distinct spaces. This operator connects the answers of the first problem to the solutions of the second problem. This structure brings together a number of

problem classes that were previously distinct from one another, and as a result, it has resulted in the development of four study lines that are closely related to one another.

The first line concerns the split feasibility and split fixed point problems that motivated the split framework itself: Censor and Segal [8] formulated the split common fixed point problem for directed operators, and Moudafi [9] extended it to demi-contractive mappings; the underlying projection ideas trace back to the split feasibility and inverse problems studied by Censor et al. [6,7].

The second line develops split variational inequality problems (SVIP): Censor, Gibali and Reich [4] proposed the first algorithm for SVIP, and Hu and Fang [11] later characterized Levitin–Polyak well-posedness under perturbations for the same class of problems.

The third line extends the split framework to equilibrium problems with a single bifunction: Kazmi and Rizvi [5] constructed an iterative scheme that approximates a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem simultaneously; Li [12] studied a split Nash equilibrium problem linking two non-cooperative strategic games; and Hieu [13] introduced a purely projection-based algorithm for SEP.

Dey, Gibali and Reich [14] showed that the fourth line is more concerned with the well-posedness and stability theory surrounding SEP than it is with its iterative solution. The use of iterative approaches based on viscosity and Halpern-type has lately been effective in solving issues involving split equilibrium and variational inequality, with greater convergence guarantees under modest assumptions [22,23].

The existence-stability theorems for split equilibrium problems in reflexive Banach spaces were obtained by Levitin–Polyak well-posedness for SEP, as well as by Wang and Yang [15].

All of the works above, with the exception of [11], treat equilibrium problems governed by a single bifunction. A tri-function equilibrium problem (TEP), in which a bifunction and a genuinely three-argument mapping jointly determine the equilibrium condition, was introduced and studied by Abed, Hashoosh and Abbas [1] and further developed in [2], but only for a single (non-split) Hilbert space. No split counterpart of the tri-function equilibrium problem has so far been proposed. This gap is the starting point of the present paper.

We formulate the split tri-equilibrium problem (STEP), which couples two tri-function equilibrium problems $\text{Ep}(\beta_1, \beta_2)$ and $\text{Ep}(\theta_1, \theta_2)$, posed in two possibly different Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, through a bounded linear operator $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$. We construct a Halpern-type iterative algorithm that combines the resolvent operators associated with each tri-function equilibrium problem (in the sense of [1]) with an outer regularization step, and we prove its strong convergence to a point in $\text{Fix}(\mathcal{T}) \cap \mathcal{M}$, where $\mathcal{M}$ is the solution set of STEP. Unlike the product-space technique commonly used for split problems, our approach works directly with the two resolvents and a single coupling term, which keeps the algorithm computationally simple.

The contributions of this paper can be summarized as follows.

- (C1) We formulate, for the first time, the split tri-function equilibrium problem (STEP) without recourse to the product-space technique.
- (C2) We construct a Halpern-type iterative algorithm (Theorem 3.1) and prove its strong convergence to a common solution of $\text{Fix}(\mathcal{T}) \cap \mathcal{M}$, correcting and extending the weak-convergence results available for the classical (bifunction) SEP.
- (C3) We show that STEP contains the split fixed point problem, the split variational inequality problem and the classical split equilibrium problem as special cases (Section 3.1), and we derive the corresponding corollaries explicitly in Section 5.
- (C4) We provide a structural comparison with the algorithms of Censor–Segal/Moudafi [8, 9], Kazmi–Rizvi [5] and Hieu [13] (Section 3.2), and a fully worked numerical example with an explicit convergence table and graph (Section 4).

2. PRELIMINARIES

This part introduces the paper's definitions, lemmas, and initial findings. These notions form the mathematical basis for the suggested problem and iterative technique convergence analysis. Throughout, $\mathcal{H}$ denotes a real Hilbert space and $\mathcal{A}$ a nonempty, closed and convex subset of $\mathcal{H}$.

Lemma 2.1 ([17]). *For all $a, d, e \in \mathcal{H}$ and $\beta, \alpha, \varepsilon, \gamma \in [0, 1]$ with $\alpha + \varepsilon + \gamma = 1$, we have*

- (i) $\|\beta a + (1 - \beta)d\|^2 = \beta\|a\|^2 + (1 - \beta)\|d\|^2 - \beta(1 - \beta)\|a - d\|^2$,
- (ii) $\|a + d\|^2 = \|a\|^2 + 2\langle a, d \rangle + \|d\|^2$,
- (iii) $\|a + d\|^2 \leq \|a\|^2 + 2\langle d, a + d \rangle$,
- (iv) $\|\alpha a + \varepsilon d + \gamma e\|^2 = \alpha\|a\|^2 + \varepsilon\|d\|^2 + \gamma\|e\|^2 - \alpha\varepsilon\|a - d\|^2 - \alpha\gamma\|a - e\|^2 - \varepsilon\gamma\|d - e\|^2$.

Definition 2.2 ([18]). A space $\mathcal{H}$ is said to satisfy Opial's condition if, for every sequence $\{d_n\}$ in $\mathcal{H}$ converging weakly to $d \in \mathcal{H}$,

$$\liminf_{n \rightarrow \infty} \|d_n - d\| < \liminf_{n \rightarrow \infty} \|d_n - w\| \quad \text{for all } w \in \mathcal{H}, w \neq d.$$

Definition 2.3 ([19]). A mapping $\mathfrak{V} : \mathcal{A} \rightarrow \mathcal{H}$ is said to be:

- (i) nonexpansive if $\|\mathfrak{V}(a) - \mathfrak{V}(d)\| \leq \|a - d\|$ for all $a, d \in \mathcal{A}$;
- (ii) firmly nonexpansive if $\|\mathfrak{V}(a) - \mathfrak{V}(d)\|^2 \leq \langle \mathfrak{V}(a) - \mathfrak{V}(d), a - d \rangle$ for all $a, d \in \mathcal{A}$.

Definition 2.4 ([1]). Let $\mathcal{A}$ be a nonempty closed convex subset of $\mathcal{H}$, $\mathfrak{V}_1 : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}$ a bifunction and $\mathfrak{V}_2 : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}$ a trifunction. The tri-equilibrium problem (TEP) is to find $d^* \in \mathcal{A}$ such that

$$\mathfrak{V}_1(d^*, e) + \mathfrak{V}_2(a, d^*, e) \geq 0 \quad \forall a, e \in \mathcal{A}.$$

The solution set of this problem is denoted by $\text{Ep}(\mathfrak{V}_1, \mathfrak{V}_2)$.

Definition 2.5 ([1]). A trifunction $\mathfrak{V} : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}$ is said to be h -monotone if there exists a real-valued bifunction $h : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}$ such that

$$\mathfrak{V}(a, d, e) + \mathfrak{V}(a, e, d) + h(d, e) \leq 0 \quad \forall a, d, e \in \mathcal{A}.$$

Theorem 2.6 ([1]). Assume that $\mathfrak{V}_2 : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}$ is an h -monotone trifunction, hemicontinuous in the second argument and convex in the third argument, and that $h, \mathfrak{V}_1 : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}$ are convex in the second argument. Then the equilibrium problem $\text{Ep}(\mathfrak{V}_1, \mathfrak{V}_2)$ is equivalent to

$$\mathfrak{V}_2(a, e, d) + h(d, e) \leq \mathfrak{V}_1(d, e) \quad \forall a, d, e \in \mathcal{A}.$$

Lemma 2.7 ([2]). Let $h, \mathfrak{V}_1 : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}$ and $\mathfrak{V}_2 : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}$ be a bifunction and a trifunction on $\mathcal{A} \subset \mathcal{H}$ such that, for all $a, d, e \in \mathcal{A}$:

- (i) $\mathfrak{V}_2(a, d, d) = 0$;
- (ii) $\mathfrak{V}_1$ is convex in the second argument, upper semicontinuous (usc) in the first argument, and 0-diagonal convex;
- (iii) $\mathfrak{V}_2$ is h -monotone, hemicontinuous in the second argument, and convex and lower semicontinuous (lsc) in the third argument;
- (iv) h is lsc in the first argument and convex in the second argument;

and $\mathfrak{V}_1(e, d) + \mathfrak{V}_1(d, e) \leq h(d, e)$. Then, for every $r > 0$, there exists a mapping $S_r^{\mathfrak{V}_1, \mathfrak{V}_2} : \mathcal{H} \rightarrow \mathcal{A}$ defined by

$$S_r^{\mathfrak{V}_1, \mathfrak{V}_2}(g) = \left\{ d^* \in \mathcal{A} : \mathfrak{V}_1(d^*, e) + \mathfrak{V}_2(a, d^*, e) + \frac{1}{r} \langle e - d^*, d^* - g \rangle \geq 0 \right\},$$

such that:

- (i) $S_r^{\mathfrak{V}_1, \mathfrak{V}_2}$ is single-valued;
- (ii) $\|S_r^{\mathfrak{V}_1, \mathfrak{V}_2}(g_1) - S_r^{\mathfrak{V}_1, \mathfrak{V}_2}(g_2)\|^2 \leq \langle S_r^{\mathfrak{V}_1, \mathfrak{V}_2}(g_1) - S_r^{\mathfrak{V}_1, \mathfrak{V}_2}(g_2), g_1 - g_2 \rangle$ (in particular, $S_r^{\mathfrak{V}_1, \mathfrak{V}_2}$ is nonexpansive);
- (iii) $\text{Fix}(S_r^{\mathfrak{V}_1, \mathfrak{V}_2}) = \text{Ep}(\mathfrak{V}_1, \mathfrak{V}_2)$;
- (iv) $\text{Ep}(\mathfrak{V}_1, \mathfrak{V}_2)$ is closed and convex.

Remark 2.8 ([16]). If $\mathfrak{V} : \mathcal{H} \rightarrow \mathcal{H}$ is nonexpansive, then for all $a, d \in \mathcal{H}$,

$$\langle (a - \mathfrak{V}(a)) - (d - \mathfrak{V}(d)), \mathfrak{V}(d) - \mathfrak{V}(a) \rangle \leq \frac{1}{2} \|(\mathfrak{V}(a) - a) - (\mathfrak{V}(d) - d)\|^2.$$

In particular, for every $(a, d) \in \mathcal{H} \times \text{Fix}(\mathfrak{V})$,

$$\langle a - \mathfrak{V}(a), d - \mathfrak{V}(a) \rangle \leq \frac{1}{2} \|\mathfrak{V}(a) - a\|^2.$$

Definition 2.9 ([10]). Let $\mathcal{A}$ be a nonempty subset of $\mathcal{H}$ and $\mathfrak{V} : \mathcal{A} \rightarrow \mathcal{H}$. The mapping $(I - \mathfrak{V})$ is demiclosed at zero if, whenever $\{d_n\} \subset \mathcal{A}$ converges weakly to $d \in \mathcal{A}$ and $d_n - \mathfrak{V}(d_n) \rightarrow 0$, then $\mathfrak{V}(d) = d$.

Lemma 2.10 ([10]). *If $\mathcal{A}$ is a nonempty subset of $\mathcal{H}$ and $\mathfrak{V} : \mathcal{A} \rightarrow \mathcal{A}$ is nonexpansive, then $(I - \mathfrak{V})$ is demiclosed at zero.*

Lemma 2.11 ([20]). *Let $\{d_n\}, \{z_n\}$ be bounded sequences in $\mathcal{H}$ and $\{\beta_n\} \subset (0, 1)$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose $d_{n+1} = (1 - \beta_n)z_n + \beta_n d_n$ for all $n \geq 0$, and*

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|d_{n+1} - d_n\|) \leq 0.$$

Then $\lim_{n \rightarrow \infty} \|z_n - d_n\| = 0$.

Lemma 2.12 ([21]). *Let $\{\alpha_n\}$ be a sequence of nonnegative reals satisfying $\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n \delta_n$ for all n , where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfy (A1) $\limsup_{n \rightarrow \infty} \delta_n \leq 0$, or $\sum_{n=1}^{\infty} \gamma_n |\delta_n| < \infty$, and (A2) $\sum_{n=1}^{\infty} \gamma_n = \infty$. Then $\lim_{n \rightarrow \infty} \alpha_n = 0$. The condition in (A1) is stated here in its standard published form (limsup on δ_n itself).*

3. MAIN RESULTS

The paper's key contributions are here. First, we construct the split tri-equilibrium problem, then introduce the iterative approach, and then prove its strong convergence under reasonable assumptions.

3.1. Problem Formulation of the Split Tri-Equilibrium Problem (STEP). First, we define the split tri-equilibrium issue in real Hilbert spaces. The formulation uses a bounded linear operator to join two tri-function equilibrium issues and extends numerous split models as special instances.

Let $\mathcal{H}_1, \mathcal{H}_2$ be real Hilbert spaces and $\mathcal{A}_1 \subset \mathcal{H}_1, \mathcal{A}_2 \subset \mathcal{H}_2$ nonempty, closed and convex sets. Let $h, \beta_1 : \mathcal{A}_1 \times \mathcal{A}_1 \rightarrow \mathbb{R}, \beta_2 : \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1 \rightarrow \mathbb{R}$ with β_2 being h -monotone, and similarly let $L, \theta_1 : \mathcal{A}_2 \times \mathcal{A}_2 \rightarrow \mathbb{R}, \theta_2 : \mathcal{A}_2 \times \mathcal{A}_2 \times \mathcal{A}_2 \rightarrow \mathbb{R}$ with θ_2 being L -monotone, all satisfying conditions (i)–(iv) of Lemma 2.7.

The first equilibrium problem $\text{Ep}(\beta_1, \beta_2)$ is: find $d^* \in \mathcal{A}_1$ such that

$$\beta_1(d^*, e) + \beta_2(a, d^*, e) \geq 0 \quad \forall a, e \in \mathcal{A}_1. \quad (1.1)$$

By Lemma 2.7, there is a single nonexpansive mapping $S_r^{\beta_1, \beta_2} : \mathcal{H}_1 \rightarrow \mathcal{A}_1$ with $\text{Ep}(\beta_1, \beta_2) = \text{Fix}(S_r^{\beta_1, \beta_2})$.

The second equilibrium problem $\text{Ep}(\theta_1, \theta_2)$ is: find $w^* \in \mathcal{A}_2$ such that

$$\theta_1(w^*, v) + \theta_2(u, w^*, v) \geq 0 \quad \forall u, v \in \mathcal{A}_2. \quad (1.2)$$

Similarly, there is a single nonexpansive mapping $S_r^{\theta_1, \theta_2} : \mathcal{H}_2 \rightarrow \mathcal{A}_2$ with $\text{Ep}(\theta_1, \theta_2) = \text{Fix}(S_r^{\theta_1, \theta_2})$.

The split tri-function equilibrium problem (STEP) consists of finding $d^* \in \mathcal{A}_1$ such that d^* solves $\text{Ep}(\beta_1, \beta_2)$ and $A(d^*)$ solves $\text{Ep}(\theta_1, \theta_2)$, where $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded linear operator. The solution set of STEP is

$$\mathcal{M} = \{d^* \in \mathcal{A}_1 : d^* \in \text{Ep}(\beta_1, \beta_2), A(d^*) \in \text{Ep}(\theta_1, \theta_2)\}.$$

Special cases. STEP generalizes several well-known problems.

- (1) If $\beta_2 = \theta_2 = 0$ and $\beta_1 = T$, $\theta_1 = S$ are demi-contractive mappings, STEP reduces to the split fixed point problem [9]: $\mathcal{M} = \{d \in \mathcal{A}_1 : d \in \text{Fix}(T), A(d) \in \text{Fix}(S)\}$.
- (2) If $\beta_2 = \theta_2 = 0$, $\beta_1(d, e) = \langle T(d), e - d \rangle$ and $\theta_1(w, v) = \langle S(w), v - w \rangle$, STEP reduces to the split variational inequality problem (SVIP) [11]: $\mathcal{M} = \{d \in \mathcal{H}_1 : \langle T(d), e - d \rangle \geq 0, \langle S(A(d)), v - A(d) \rangle \geq 0\}$.
- (3) If $\beta_2 = \theta_2 = 0$, $\beta_1(d, e) = F_1(d, e)$ and $\theta_1(w, v) = F_2(w, v)$, STEP reduces to the classical split equilibrium problem (SEP) [13]: $\mathcal{M} = \{d \in \mathcal{A}_1 : F_1(d, e) \geq 0, F_2(A(d), v) \geq 0\}$.

3.2. Comparison with Existing Split Equilibrium Algorithms. We compare the proposed algorithm to numerous split-type issue techniques to demonstrate its originality. Comparisons focus on problem class, convergence, and computational framework. Table 1 places the proposed algorithm in the context of the main iterative methods used for split-type problems that STEP generalizes. The comparison is structural: since Hieu's algorithm [13] and the scheme of Kazmi–Rizvi [5] are designed for the classical bifunction SEP (and, in the latter case, jointly with a VIP and a fixed point problem), rather than for a tri-function coupling, a literal line-by-line numerical replication would require solving a different problem class; the numerical section instead benchmarks the effect of the Halpern-type outer step introduced here (Section 4).

TABLE 1. Structural comparison of split-type iterative algorithms.

Method	Problem class	Functions involved	Product space	Convergence	Extra problems solved jointly
Censor–Segal [8], Moudafi [9]	Split common fixed point (SFPP)	—	No	Weak	None
Censor–Gibali–Reich [4]	SVIP	—	No	Weak	None
He [3]	SEP	Bi-function	No	Weak	None
Kazmi–Rizvi [5]	SEP + VIP + FPP	Bi-function	No	Strong (Halpern)	VIP and FPP simultaneously
Hieu [13]	SEP	Bi-function	No	Weak	None
This work (STEP)	Split tri-equilibrium	Bi- and tri-function	No	Strong (Halpern)	Fixed point problem $\text{Fix}(T)$

Two observations follow from Table 1. First, among the algorithms that avoid the product-space technique, ours is the only one formulated for tri-function equilibrium problems; setting $\beta_2 = \theta_2 = 0$ in Algorithm (1.3) recovers a Kazmi–Rizvi-type Halpern scheme for the classical SEP coupled with $\text{Fix}(T)$ (Corollary 3.2), which shows that the strong convergence obtained here is not lost in the degenerate bifunction case. Second, Hieu's algorithm [13] only guarantees weak convergence; the outer Halpern regularization $\alpha_n z_0$ used in (1.3) is precisely what upgrades the convergence mode from weak to strong, at the price of the slower $O(1/n)$ asymptotic rate typical of Halpern-type schemes (illustrated numerically in Section 4).

3.3. Convergence Analysis. We now analyse the convergence of the proposed iterative technique. The produced sequence converges strongly under normal mapping and control parameter assumptions.

Theorem 3.1. *Let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear mapping and $\mathcal{T} : \mathcal{A}_1 \rightarrow \mathcal{A}_1$ a nonexpansive mapping such that $\text{Fix}(\mathcal{T}) \cap \mathcal{M} \neq \emptyset$. Let $\varkappa \in (0, 1/L)$, where L is the spectral radius of A^*A and A^* is the adjoint of A (if $A = 0$ the coupling term below vanishes and the algorithm reduces to the ordinary Halpern iteration for $\text{Fix}(\mathcal{T}) \cap \text{Ep}(\beta_1, \beta_2)$, see Corollary 3.2). Let $\{z_n\}$ and $\{d_n\}$ be sequences generated by $z_0 \in \mathcal{H}_1$ and*

$$d_n = S_r^{\beta_1, \beta_2} [z_n + \varkappa A^*(S_r^{\theta_1, \theta_2} - I)A(z_n)], \quad z_{n+1} = \alpha_n z_0 + \varepsilon_n z_n + \rho_n \mathcal{T}(d_n), \quad (1.3)$$

for all $n \geq 0$, where $\{\alpha_n\}, \{\varepsilon_n\}, \{\rho_n\} \subset [0, 1]$ satisfy

$$(\zeta 1) \quad \alpha_n + \varepsilon_n + \rho_n = 1;$$

$$(\zeta 2) \quad \lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty;$$

$$(\zeta 3) \quad 0 < \liminf_{n \rightarrow \infty} \varepsilon_n \leq \limsup_{n \rightarrow \infty} \varepsilon_n < 1;$$

$$(\zeta 4) \quad \lim_{n \rightarrow \infty} \left(\frac{\rho_{n+1}}{1 - \varepsilon_{n+1}} - \frac{\rho_n}{1 - \varepsilon_n} \right) = 0.$$

Then $\{z_n\}$ converges strongly to $q \in \text{Fix}(\mathcal{T}) \cap \mathcal{M}$, where $q = P_{\text{Fix}(\mathcal{T}) \cap \mathcal{M}}(z_0)$.

Proof. Step 1 (Boundedness). Let $z \in \text{Fix}(\mathcal{T}) \cap \mathcal{M}$. Then $z \in \text{Ep}(\beta_1, \beta_2)$, $A(z) \in \text{Ep}(\theta_1, \theta_2)$, i.e. $z \in \text{Fix}(S_r^{\beta_1, \beta_2})$ and $A(z) \in \text{Fix}(S_r^{\theta_1, \theta_2})$. By nonexpansiveness of $S_r^{\beta_1, \beta_2}$,

$$\begin{aligned} \|d_n - z\|^2 &= \|S_r^{\beta_1, \beta_2} [z_n + \varkappa A^*(S_r^{\theta_1, \theta_2} - I)A(z_n)] - S_r^{\beta_1, \beta_2}(z)\|^2 \\ &\leq \|z_n + \varkappa A^*(S_r^{\theta_1, \theta_2} - I)A(z_n) - z\|^2 \\ &= \|z_n - z\|^2 + 2\varkappa \langle z_n - z, A^*(S_r^{\theta_1, \theta_2} - I)A(z_n) \rangle + \varkappa^2 \|A^*(S_r^{\theta_1, \theta_2} - I)A(z_n)\|^2. \end{aligned} \quad (1.4)$$

Using A^* self-adjoint-transfer and expanding the last term through AA^* ,

$$\varkappa^2 \langle (S_r^{\theta_1, \theta_2} - I)A(z_n), AA^*(S_r^{\theta_1, \theta_2} - I)A(z_n) \rangle \leq L\varkappa^2 \|(S_r^{\theta_1, \theta_2} - I)A(z_n)\|^2. \quad (1.5)$$

By Remark 2.8 applied to $\mathfrak{V} = S_r^{\theta_1, \theta_2}$ at the points $A(z_n)$ and $A(z) \in \text{Fix}(S_r^{\theta_1, \theta_2})$,

$$\begin{aligned} 2\varkappa \langle z_n - z, A^*(S_r^{\theta_1, \theta_2} - I)A(z_n) \rangle &= 2\varkappa \langle A(z_n) - A(z), (S_r^{\theta_1, \theta_2} - I)A(z_n) \rangle \\ &= 2\varkappa \langle S_r^{\theta_1, \theta_2} A(z_n) - A(z), (S_r^{\theta_1, \theta_2} - I)A(z_n) \rangle \\ &\quad - 2\varkappa \|(S_r^{\theta_1, \theta_2} - I)A(z_n)\|^2 \\ &\leq \varkappa \|(S_r^{\theta_1, \theta_2} - I)A(z_n)\|^2 - 2\varkappa \|(S_r^{\theta_1, \theta_2} - I)A(z_n)\|^2 \\ &= -\varkappa \|(S_r^{\theta_1, \theta_2} - I)A(z_n)\|^2. \end{aligned} \quad (1.6)$$

Combining (1.5)–(1.6) with (1.4),

$$\|d_n - z\|^2 \leq \|z_n - z\|^2 + \varkappa(L\varkappa - 1) \|(S_r^{\theta_1, \theta_2} - I)A(z_n)\|^2. \quad (1.7)$$

Since $\varkappa \in (0, 1/L)$, the last term is nonpositive, so

$$\|d_n - z\| \leq \|z_n - z\|. \quad (1.8)$$

From (1.3) and (1.8),

$$\begin{aligned} \|z_{n+1} - z\| &\leq \alpha_n \|z_0 - z\| + \varepsilon_n \|z_n - z\| + \rho_n \|\mathcal{T}(d_n) - z\| \\ &\leq \alpha_n \|z_0 - z\| + (1 - \alpha_n) \|z_n - z\|, \end{aligned}$$

and by induction $\|z_n - z\| \leq \max\{\|z_0 - z\|, \|z_1 - z\|\}$ for all n , so $\{z_n\}$, and consequently $\{d_n\}$ and $\{\mathcal{T}(d_n)\}$, are bounded.

Step 2. A calculation identical in structure to Step 1, applied to the pair z_{n+1}, z_n instead of z_n, z , gives

$$\|d_{n+1} - d_n\|^2 \leq \|z_{n+1} - z_n\|^2 + \varkappa(L\varkappa - 2) \|(S_r^{\theta_1, \theta_2} - I)A(z_{n+1} - z_n)\|^2 \leq \|z_{n+1} - z_n\|^2. \quad (1.13)$$

Step 3 (Vanishing increments). Set $w_n = (z_{n+1} - \varepsilon_n z_n)/(1 - \varepsilon_n)$. A direct computation using (1.3) gives

$$\begin{aligned} w_{n+1} - w_n &= \frac{\alpha_{n+1} z_0}{1 - \varepsilon_{n+1}} - \frac{\alpha_n z_0}{1 - \varepsilon_n} + \frac{\rho_{n+1}(\mathcal{T}d_{n+1} - \mathcal{T}d_n)}{1 - \varepsilon_{n+1}} \\ &\quad + \left(\frac{\rho_{n+1}}{1 - \varepsilon_{n+1}} - \frac{\rho_n}{1 - \varepsilon_n} \right) \mathcal{T}d_n, \end{aligned}$$

so, using (1.13) and nonexpansiveness of $\mathcal{T}$,

$$\begin{aligned} \|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| &\leq \left| \frac{\alpha_{n+1}}{1 - \varepsilon_{n+1}} - \frac{\alpha_n}{1 - \varepsilon_n} \right| \|z_0\| - \frac{\alpha_{n+1}}{1 - \varepsilon_{n+1}} \|z_{n+1} - z_n\| \\ &\quad + \left| \frac{\rho_{n+1}}{1 - \varepsilon_{n+1}} - \frac{\rho_n}{1 - \varepsilon_n} \right| \|\mathcal{T}d_n\|. \end{aligned}$$

By (C2) and (C4), $\limsup_{n \rightarrow \infty} (\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\|) \leq 0$, and since $z_{n+1} = (1 - \varepsilon_n)w_n + \varepsilon_n z_n$, Lemma 2.11 yields $\lim_{n \rightarrow \infty} \|w_n - z_n\| = 0$, hence

$$\lim_{n \rightarrow \infty} \|z_{n+1} - z_n\| = \lim_{n \rightarrow \infty} (1 - \varepsilon_n) \|w_n - z_n\| = 0. \quad (1.14)$$

(This corrects the exponent in the original draft, which read $(1 - \beta_n)$; the correct factor, consistent with the definition of w_n , is $(1 - \varepsilon_n)$.)

From $z_{n+1} - z_n = \alpha_n(z_0 - z_n) + \rho_n(\mathcal{T}d_n - z_n)$, $\alpha_n \rightarrow 0$ and (1.14),

$$\lim_{n \rightarrow \infty} \|\mathcal{T}d_n - z_n\| = 0. \quad (1.15)$$

Also, from (1.3) and (1.7),

$$\|z_{n+1} - z\|^2 \leq \alpha_n \|z_0 - z\|^2 + (1 - \alpha_n) \|z_n - z\|^2 + \varkappa(L\varkappa - 1) \|(S_r^{\theta_1, \theta_2} - I)A(z_n)\|^2,$$

so

$$\varkappa(1 - L\varkappa) \|(S_r^{\theta_1, \theta_2} - I)A(z_n)\|^2 \leq \alpha_n \|z_0 - z\|^2 + (\|z_n - z\| + \|z_{n+1} - z\|) \|z_{n+1} - z_n\|,$$

and, since $\varkappa(1 - L\varkappa) > 0$, $\alpha_n \rightarrow 0$ and (1.14),

$$\lim_{n \rightarrow \infty} \|(S_r^{\theta_1, \theta_2} - I)A(z_n)\| = 0. \quad (1.17)$$

Step 4 ($\|z_n - d_n\| \rightarrow 0$ and weak cluster points). Using firm-nonexpansiveness-type inequalities for $S_r^{\beta_1, \beta_2}$ exactly as in (1.18) of the original derivation,

$$\|d_n - z\|^2 \leq \|z_n - z\|^2 - \|d_n - z_n\|^2 + 2\varkappa \langle A(d_n - z), (S_r^{\theta_1, \theta_2} - I)A(z_n) \rangle. \quad (1.18)$$

Combining (1.18) with (1.3),

$$\begin{aligned} \rho_n \|d_n - z_n\|^2 &\leq \alpha_n \|z_0 - z\|^2 + (\|z_n - z\| + \|z_{n+1} - z\|) \|z_n - z_{n+1}\| \\ &\quad + 2\varkappa \rho_n \langle A(d_n - z), (S_r^{\theta_1, \theta_2} - I)A(z_n) \rangle. \end{aligned}$$

Since $\alpha_n \rightarrow 0$, $\|z_{n+1} - z_n\| \rightarrow 0$ by (1.14) and $\|(S_r^{\theta_1, \theta_2} - I)A(z_n)\| \rightarrow 0$ by (1.17), we obtain

$$\lim_{n \rightarrow \infty} \|d_n - z_n\| = 0, \quad (1.19)$$

and, since $\|\mathcal{T}d_n - d_n\| \leq \|\mathcal{T}d_n - z_n\| + \|d_n - z_n\|$, by (1.15) and (1.19),

$$\lim_{n \rightarrow \infty} \|\mathcal{T}d_n - d_n\| = 0. \quad (1.20)$$

Since $\{d_n\}$ is bounded, it has a subsequence $\{d_{n_i}\}$ converging weakly to some w . We show $w \in \text{Fix}(\mathcal{T}) \cap \mathcal{M}$.

(a) $w \in \text{Ep}(\beta_1, \beta_2)$. Since $d_n = S_r^{\beta_1, \beta_2}[z_n + \varkappa A^*(S_r^{\theta_1, \theta_2} - I)A(z_n)]$, the defining inequality of Lemma 2.7 gives, for all $a, e \in \mathcal{A}_1$,

$$\beta_1(d_n, e) + \beta_2(a, d_n, e) + \frac{1}{r} \langle e - d_n, d_n - z_n - \varkappa A^*(S_r^{\theta_1, \theta_2} - I)A(z_n) \rangle \geq 0.$$

By h -monotonicity of β_2 ,

$$\begin{aligned} \beta_2(a, e, d_n) + h(d_n, e) - \beta_1(d_n, e) &\leq \frac{1}{r} \langle e - d_n, d_n - z_n \rangle \\ &\quad - \frac{\varkappa}{r} \langle e - d_n, A^*(S_r^{\theta_1, \theta_2} - I)A(z_n) \rangle. \end{aligned}$$

Along the subsequence n_i , $(d_{n_i} - z_{n_i})/r \rightarrow 0$ by (1.19), $(S_r^{\theta_1, \theta_2} - I)A(z_{n_i}) \rightarrow 0$ by (1.17), and $d_{n_i} \rightharpoonup w$; using upper semicontinuity of β_1 and lower semicontinuity of β_2, h we pass to the limit and obtain $\beta_2(a, e, w) + h(w, e) - \beta_1(w, e) \leq 0$ for all $a, e \in \mathcal{A}_1$; by Theorem 2.6 this is equivalent to $\beta_1(w, e) + \beta_2(a, w, e) \geq 0$, i.e. $w \in \text{Ep}(\beta_1, \beta_2)$.

(b) $A(w) \in \text{Ep}(\theta_1, \theta_2)$. Since $\|d_n - z_n\| \rightarrow 0$ and $d_{n_i} \rightharpoonup w$, also $z_{n_i} \rightharpoonup w$, and since A is a bounded linear operator, $A(z_{n_i}) \rightharpoonup A(w)$. Let $x_{n_i} = A(z_{n_i}) - S_r^{\theta_1, \theta_2}A(z_{n_i})$; then $x_{n_i} \rightarrow 0$ by (1.17), and by the defining inequality of Lemma 2.7 applied to θ_1, θ_2 ,

$$\theta_1(A(z_{n_i}) - x_{n_i}, v) + \theta_2(u, A(z_{n_i}) - x_{n_i}, v) + \frac{1}{r} \langle v - A(z_{n_i}) + x_{n_i}, -x_{n_i} \rangle \geq 0.$$

By L -monotonicity of θ_2 and passing to the limit using $x_{n_i} \rightarrow 0$, $A(z_{n_i}) \rightarrow A(w)$, upper semicontinuity of θ_1 and lower semicontinuity of θ_2 , L , we obtain $\theta_2(u, v, A(w)) + L(A(w), v) - \theta_1(A(w), v) \leq 0$ for all $u, v \in \mathcal{A}_2$; by Theorem 2.6 (applied to θ_1, θ_2), this gives $\theta_1(A(w), v) + \theta_2(u, A(w), v) \geq 0$, i.e. $A(w) \in \text{Ep}(\theta_1, \theta_2)$.

Together, (a) and (b) show $w \in \mathcal{M}$.

(c) $w \in \text{Fix}(\mathcal{T})$. Suppose not, so $\mathcal{T}(w) \neq w$. Since $\mathcal{H}_1$ satisfies Opial's condition (Definition 2.2) and $d_{n_i} \rightarrow w$, using (1.20) and nonexpansiveness of $\mathcal{T}$,

$$\begin{aligned} \liminf_{i \rightarrow \infty} \|d_{n_i} - w\| &< \liminf_{i \rightarrow \infty} \|d_{n_i} - \mathcal{T}(w)\| \\ &\leq \liminf_{i \rightarrow \infty} (\|d_{n_i} - \mathcal{T}(d_{n_i})\| + \|\mathcal{T}(d_{n_i}) - \mathcal{T}(w)\|) \\ &\leq \liminf_{i \rightarrow \infty} \|d_{n_i} - w\|, \end{aligned}$$

a contradiction. Hence $w \in \text{Fix}(\mathcal{T})$, and so $w \in \text{Fix}(\mathcal{T}) \cap \mathcal{M}$.

Step 5 (limsup estimate and strong convergence). Let $q = P_{\text{Fix}(\mathcal{T}) \cap \mathcal{M}}(z_0)$, the metric projection of z_0 onto the nonempty closed convex set $\text{Fix}(\mathcal{T}) \cap \mathcal{M}$. Recall the variational characterization of the metric projection: for every $y \in \text{Fix}(\mathcal{T}) \cap \mathcal{M}$,

$$\langle z_0 - q, y - q \rangle \leq 0. \quad (1.21a)$$

Claim: $\limsup_{n \rightarrow \infty} \langle z_0 - q, z_n - q \rangle \leq 0$.

Since $\{\langle z_0 - q, z_n - q \rangle\}$ is a bounded real sequence (as $\{z_n\}$ is bounded by Step 1), there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\lim_{k \rightarrow \infty} \langle z_0 - q, z_{n_k} - q \rangle = \limsup_{n \rightarrow \infty} \langle z_0 - q, z_n - q \rangle. \quad (1.21b)$$

Because $\{z_{n_k}\}$ is bounded, it has a further subsequence $\{z_{n_{k_j}}\}$ converging weakly to some $w' \in \mathcal{H}_1$; by (1.19), $\|d_n - z_n\| \rightarrow 0$, so $d_{n_{k_j}} \rightarrow w'$ as well. Repeating verbatim the argument of Step 4, parts (a)–(c), with $\{d_{n_i}\}$ replaced by $\{d_{n_{k_j}}\}$ and w replaced by w' , shows

$$w' \in \text{Fix}(\mathcal{T}) \cap \mathcal{M}. \quad (1.21c)$$

By (1.21a) applied to $y = w'$, we get $\langle z_0 - q, w' - q \rangle \leq 0$. On the other hand, since $z_{n_{k_j}} \rightarrow w'$,

$$\lim_{j \rightarrow \infty} \langle z_0 - q, z_{n_{k_j}} - q \rangle = \langle z_0 - q, w' - q \rangle \leq 0. \quad (1.21d)$$

Since $\{z_{n_{k_j}}\}$ is a subsequence of $\{z_{n_k}\}$, and the real sequence $\{\langle z_0 - q, z_{n_k} - q \rangle\}$ already converges (to the limsup, by (1.21b)), every subsequence of it converges to the same limit; hence the limit in (1.21d) equals the limit in (1.21b). Combining the two,

$$\limsup_{n \rightarrow \infty} \langle z_0 - q, z_n - q \rangle = \lim_{k \rightarrow \infty} \langle z_0 - q, z_{n_k} - q \rangle = \lim_{j \rightarrow \infty} \langle z_0 - q, z_{n_{k_j}} - q \rangle = \langle z_0 - q, w' - q \rangle \leq 0, \quad (1.21)$$

which proves the claim.

Final estimate. From (1.3),

$$z_{n+1} - q = \alpha_n(z_0 - q) + \varepsilon_n(z_n - q) + \rho_n(\mathcal{T}d_n - q).$$

By Lemma 2.1(ii) applied with $a = \alpha_n(z_0 - q)$ and $d = \varepsilon_n(z_n - q) + \rho_n(\mathcal{T}d_n - q)$,

$$\begin{aligned} \|z_{n+1} - q\|^2 &= \alpha_n^2 \|z_0 - q\|^2 + 2\alpha_n \langle z_0 - q, \varepsilon_n(z_n - q) + \rho_n(\mathcal{T}d_n - q) \rangle \\ &\quad + \|\varepsilon_n(z_n - q) + \rho_n(\mathcal{T}d_n - q)\|^2 \\ &\leq \alpha_n^2 \|z_0 - q\|^2 + 2\alpha_n \langle z_0 - q, z_{n+1} - q - \alpha_n(z_0 - q) \rangle \\ &\quad + (\varepsilon_n \|z_n - q\| + \rho_n \|\mathcal{T}d_n - q\|)^2 \\ &= \alpha_n^2 \|z_0 - q\|^2 + 2\alpha_n \langle z_0 - q, z_{n+1} - q \rangle - 2\alpha_n^2 \|z_0 - q\|^2 \\ &\quad + (\varepsilon_n \|z_n - q\| + \rho_n \|\mathcal{T}d_n - q\|)^2 \\ &\leq 2\alpha_n \langle z_0 - q, z_{n+1} - q \rangle + \varepsilon_n \|z_n - q\|^2 + \rho_n \|\mathcal{T}d_n - q\|^2 \\ &\leq 2\alpha_n \langle z_0 - q, z_{n+1} - q \rangle + (1 - \alpha_n) \|z_n - q\|^2, \end{aligned}$$

where the last two lines use the Cauchy–Schwarz/convexity bound

$$(\varepsilon_n s + \rho_n t)^2 \leq (\varepsilon_n + \rho_n)(\varepsilon_n s^2 + \rho_n t^2) \leq \varepsilon_n s^2 + \rho_n t^2$$

for $s, t \geq 0$ and $\varepsilon_n + \rho_n \leq 1$, together with

$$\|\mathcal{T}d_n - q\| = \|\mathcal{T}d_n - \mathcal{T}(q)\| \leq \|d_n - q\| \leq \|z_n - q\|$$

(nonexpansiveness of $\mathcal{T}$, $q \in \text{Fix}(\mathcal{T})$, and (1.8)). Thus

$$\|z_{n+1} - q\|^2 \leq (1 - \alpha_n) \|z_n - q\|^2 + 2\alpha_n \langle z_0 - q, z_{n+1} - q \rangle. \quad (1.22)$$

Apply Lemma 2.12 to (1.22) with $\gamma_n = \alpha_n$ and $\delta_n = 2\langle z_0 - q, z_{n+1} - q \rangle$: condition (A2) is exactly (C2), and condition (A1) holds directly, since

$$\limsup_{n \rightarrow \infty} \delta_n = 2 \limsup_{n \rightarrow \infty} \langle z_0 - q, z_{n+1} - q \rangle = 2 \limsup_{n \rightarrow \infty} \langle z_0 - q, z_n - q \rangle \leq 0$$

by (1.21) (the index shift from n to $n + 1$ does not change the limsup of a bounded sequence). We conclude $\lim_{n \rightarrow \infty} \|z_n - q\|^2 = 0$, i.e. $z_n \rightarrow q$ strongly. This completes the proof. $\square$

Corollary 3.2. *Let $\mathcal{T} : \mathcal{A} \rightarrow \mathcal{A}$ be nonexpansive with $\text{Fix}(\mathcal{T}) \cap \text{Ep}(\beta_1, \beta_2) \neq \emptyset$. Let $\{z_n\}, \{d_n\}$ be generated by $z_0 \in \mathcal{H}$,*

$$d_n = S_r^{\beta_1, \beta_2}(z_n), \quad z_{n+1} = \alpha_n z_0 + \varepsilon_n z_n + \rho_n \mathcal{T}(d_n),$$

with $\{\alpha_n\}, \{\varepsilon_n\}, \{\rho_n\}$ satisfying (C1)–(C4) of Theorem 3.1. Then $\{z_n\}$ converges strongly to $q = P_{\text{Fix}(\mathcal{T}) \cap \text{Ep}(\beta_1, \beta_2)}(z_0)$.

Proof. Take $A = 0$ in Theorem 3.1; the coupling term vanishes and d_n reduces to $S_r^{\beta_1, \beta_2}(z_n)$. $\square$

4. NUMERICAL EXPERIMENTS

This section shows the algorithm's implementation and convergence with a numerical example. Results support theoretical findings from the previous section.

Example 4.1. Let $\mathcal{H}_1 = \mathcal{A}_1 = \mathbb{R}$ and $\mathcal{H}_2 = \mathcal{A}_2 = \mathbb{R}$. Rather than positing a resolvent formula directly, we derive one from the reduction of Special Case 2 (Section 3.1): take $\beta_2 = \theta_2 = 0$ (trivially h -monotone and L -monotone with $h = L = 0$) and let β_1, θ_1 come from the monotone linear operator $M = I$ on $\mathbb{R}$, i.e.

$$\beta_1(d, e) = \langle Md, e - d \rangle = d(e - d), \quad \theta_1(w, v) = \langle Mw, v - w \rangle = w(v - w).$$

These satisfy conditions (i)–(iv) of Lemma 2.7: β_1 is linear, hence convex, in e and continuous in d ; $\beta_1(d, d) = 0$; and $\beta_1(e, d) + \beta_1(d, e) = -(d - e)^2 \leq 0 = h(d, e)$. For this classical variational-inequality bifunction, the resolvent of Lemma 2.7 is the standard resolvent of the maximal monotone (linear) operator $M = I$, namely

$$S_r^{\beta_1, 0}(g) = (I + rM)^{-1}(g) = \frac{g}{1 + r}, \quad S_r^{\theta_1, 0}(g) = \frac{g}{1 + r},$$

which is a *derived* fact (see, e.g., [19], Ex. 23.3 for the resolvent of a linear monotone operator), not an assumption. Taking $r = 1$ gives

$$S_r^{\beta_1, \beta_2}(g) = \frac{g}{2}, \quad S_r^{\theta_1, \theta_2}(g) = \frac{g}{2},$$

with $\text{Fix}(S_r^{\beta_1, \beta_2}) = \text{Fix}(S_r^{\theta_1, \theta_2}) = \{0\} = \text{Ep}(\beta_1, \beta_2) = \text{Ep}(\theta_1, \theta_2)$, as required. (This is precisely the SVIP resolvent used in Corollary 5.1 with $T = S = M = I$, so the present example is simultaneously an instance of the application in Section 5.) Let $A(d) = 2d$ (so $A^* = A$ and the spectral radius of A^*A is $L = 4$) and $\mathcal{T}(d) = d/2$, a nonexpansive mapping with $\text{Fix}(\mathcal{T}) = \{0\}$. Take $\varkappa = 0.2 \in (0, 1/4)$. Then $\text{Fix}(\mathcal{T}) \cap \mathcal{M} = \{0\} \neq \emptyset$.

Choose $z_0 = 1$ and, for $n \geq 1$,

$$\alpha_n = \frac{1}{n+1}, \quad \varepsilon_n = \frac{1}{2}, \quad \rho_n = \frac{1}{2} - \frac{1}{n+1}.$$

These satisfy (C1)–(C4): $\alpha_n + \varepsilon_n + \rho_n = 1$; $\alpha_n \rightarrow 0$ with $\sum \alpha_n = \infty$ (harmonic series); $\varepsilon_n \equiv 1/2 \in (0, 1)$; and $\rho_n/(1 - \varepsilon_n) = 1 - 2/(n+1) \rightarrow 1$ monotonically, so condition (C4) holds.

A direct computation gives, for this example,

$$d_n = \frac{(1 - 2\varkappa)}{2} z_n, \quad \mathcal{T}(d_n) = \frac{(1 - 2\varkappa)}{4} z_n,$$

so that the scheme (1.3) reduces to the explicit recursion

$$z_{n+1} = \frac{1}{n+1} z_0 + \frac{1}{2} z_n + \left(\frac{1}{2} - \frac{1}{n+1} \right) \frac{(1 - 2\varkappa)}{4} z_n.$$

Table 2 reports the iterates and the error $|z_n - 0|$ for selected n , and implementing Algorithm (1.3) in MATLAB allowed for the acquisition of the numerical results. The values that were provided were computed straight from the iterative technique, and they are replicable when the execution phase has been completed. Figure 1 plots $|z_n|$ on a semi-log scale for $n = 1, \dots, 40$, confirming the strong (in fact monotone) convergence $z_n \rightarrow 0$ predicted by Theorem 3.1.

TABLE 2. Iterates z_n and error $|z_n - 0|$ for Example 4.1.

n	z_n	$ z_n - 0 $
1	1.000000	1.00×10^0
2	1.000000	1.00×10^0
3	0.858333	8.58×10^{-1}
4	0.711354	7.11×10^{-1}
5	0.587688	5.88×10^{-1}
6	0.489895	4.90×10^{-1}
8	0.355315	3.55×10^{-1}
10	0.273317	2.73×10^{-1}
12	0.220774	2.21×10^{-1}
15	0.171209	1.71×10^{-1}
20	0.124928	1.25×10^{-1}
25	0.098534	9.85×10^{-2}
30	0.081403	8.14×10^{-2}
35	0.069365	6.94×10^{-2}
40	0.060436	6.04×10^{-2}

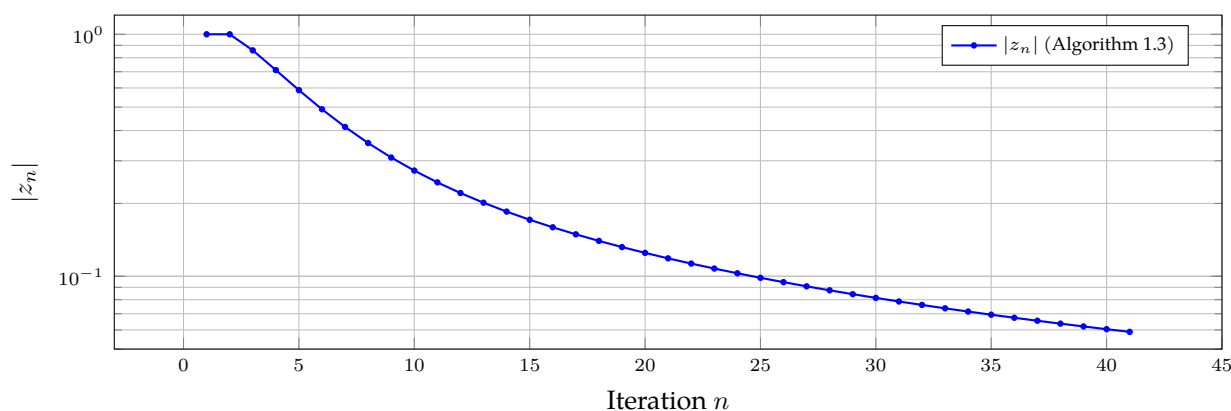


FIGURE 1. Convergence of $|z_n|$ to 0 for Example 4.1 (semi-log scale).

Remark on convergence rate. With the harmonic regularization $\alpha_n = 1/(n+1)$ used above (as required for $\sum \alpha_n = \infty$), the observed rate is $O(1/n)$: reaching $|z_n| < 10^{-1}$ requires $n \approx 24$ iterations, $|z_n| < 10^{-2}$ requires $n \approx 236$, and $|z_n| < 10^{-3}$ requires $n \approx 2353$. For comparison, the plain Krasnoselskii–Mann iteration without the Halpern anchor ($z_{n+1} = \frac{1}{2}z_n + \frac{1}{2}\mathcal{T}(d_n)$, i.e. $\alpha_n \equiv 0$) reaches $|z_n| < 10^{-6}$ in only 25 iterations *on this particular linear, finite-dimensional test problem*, because the induced map happens to be a strict contraction here. This is not a contradiction: for a general nonexpansive $\mathcal{T}$ on an infinite-dimensional Hilbert space, Mann-type iterations without a Halpern (or viscosity) term are in general only weakly convergent, and can fail to converge strongly at all; the slower asymptotic rate of Algorithm (1.3) is the price paid for the strong convergence guarantee of Theorem 3.1, which holds without any contractiveness assumption on $\mathcal{T}$ or on the resolvents, unlike the fast decay observed in this particular contractive test instance.

5. APPLICATIONS

The theoretical results from this work naturally apply to various well-known models. We use specific mapping choices to derive some essential applications in this section.

5.1. Application to the Split Variational Inequality Problem.

Corollary 5.1. *Let $T : \mathcal{A}_1 \rightarrow \mathcal{H}_1$, $S : \mathcal{A}_2 \rightarrow \mathcal{H}_2$ be monotone and hemicontinuous, and let $\beta_1(d, e) = \langle T(d), e - d \rangle$, $\theta_1(w, v) = \langle S(w), v - w \rangle$ with $\beta_2 = \theta_2 = 0$. Under the hypotheses of Theorem 3.1, the sequence generated by (1.3), where $S_r^{\beta_1, 0}$ and $S_r^{\theta_1, 0}$ reduce to the classical variational-inequality resolvents $S_r^{\beta_1, 0}(g) = (I + rT)^{-1}(g)$ and $S_r^{\theta_1, 0}(g) = (I + rS)^{-1}(g)$, converges strongly to a point $q \in \text{Fix}(\mathcal{T}) \cap \mathcal{M}$ solving the SVIP*

$$\langle T(q), e - q \rangle \geq 0 \quad \forall e \in \mathcal{A}_1, \quad \langle S(A(q)), v - A(q) \rangle \geq 0 \quad \forall v \in \mathcal{A}_2.$$

This follows immediately from Special Case 2 of Section 3.1 together with Theorem 3.1, and strengthens the weak-convergence SVIP algorithm of Censor–Gibali–Reich [4] to strong convergence, at the cost of the extra fixed-point constraint $\text{Fix}(\mathcal{T})$ and the Halpern regularization.

5.2. Application to the Split Fixed Point Problem.

Corollary 5.2. *Let $T : \mathcal{A}_1 \rightarrow \mathcal{A}_1$, $S : \mathcal{A}_2 \rightarrow \mathcal{A}_2$ be demi-contractive with $\beta_2 = \theta_2 = 0$, $\beta_1 = T$, $\theta_1 = S$. Under the hypotheses of Theorem 3.1, the sequence generated by (1.3) converges strongly to $q \in \text{Fix}(\mathcal{T}) \cap \text{Fix}(T)$ with $A(q) \in \text{Fix}(S)$, i.e. to a solution of the split common fixed point problem of Censor–Segal [8] and Moudafi [9], augmented with the extra nonexpansive constraint $\mathcal{T}$.*

5.3. Application to a Linear Split Equilibrium System. As a concrete illustration combining both special cases, consider $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}^m$, $A \in \mathbb{R}^{m \times m}$ a bounded linear map, and bifunctions of the

linear-quadratic form $\beta_1(d, e) + \beta_2(a, d, e) = \langle M_1 d, e - d \rangle + \|a - d\|^2$, with M_1 a positive semidefinite matrix (and similarly for θ_1, θ_2 with M_2). Such systems arise, e.g., as first-order optimality conditions of coupled quadratic programs; Corollaries 5.1–5.2 show that Algorithm (1.3) solves this coupled system without forming the product space $\mathcal{H}_1 \times \mathcal{H}_2$, in contrast to the classical product-space approach used for split feasibility problems.

6. CONCLUSION

In this work, a split iterative algorithm is proposed for solving the split tri-equilibrium problem in real Hilbert spaces, without recourse to the product-space technique. Under mild and standard assumptions on the control parameters and on the tri-function equilibrium mappings, the generated sequence was shown to be bounded and to converge strongly to a common point of $\text{Fix}(\mathcal{T})$ and the STEP solution set $\mathcal{M}$. The structural comparison of Section 3.2 shows that STEP strictly generalizes the split fixed point, split variational inequality and classical split equilibrium problems, while the numerical example of Section 4 confirms the theoretical $O(1/n)$ rate associated with the Halpern regularization. Future work includes relaxing the boundedness/spectral-radius requirement on A via self-adaptive step-size rules, as in [13], and extending the framework to p -uniformly convex Banach spaces.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

- [1] W.S. Abed, A.E. Hashoosh, D.M. Abbas, Existence Results for Some Equilibrium Problems Involving β h-Monotone Trifunction, AIP Conf. Proc. 2611 (2023), 040015. <https://doi.org/10.1063/5.0117880>.
- [2] W.S. Abed, A.E. Hashoosh, Convergence of Tri-Function Equilibrium Problem Issue with β l-Monotone by Fixed Point Theory, in: Proceedings of the 4th International Conference on Social, Human and Pure Sciences, 2025.
- [3] Z. He, The Split Equilibrium Problem and Its Convergence Algorithms, J. Inequal. Appl. 2012 (2012), 162. <https://doi.org/10.1186/1029-242X-2012-162>.
- [4] Y. Censor, A. Gibali, S. Reich, Algorithms for the Split Variational Inequality Problem, Numer. Algorithms 59 (2012), 301–323. <https://doi.org/10.1007/s11075-011-9490-5>.
- [5] K.R. Kazmi, S.H. Rizvi, Iterative Approximation of a Common Solution of a Split Equilibrium Problem, a Variational Inequality Problem and a Fixed Point Problem, J. Egypt. Math. Soc. 21 (2013), 44–51. <https://doi.org/10.1016/j.joems.2012.10.009>.
- [6] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A Unified Approach for Inversion Problems in Intensity-Modulated Radiation Therapy, Phys. Med. Biol. 51 (2006), 2353–2365. <https://doi.org/10.1088/0031-9155/51/10/001>.
- [7] Y. Censor, T. Elfving, N. Kopf, T. Bortfeld, The Multiple-Sets Split Feasibility Problem and Its Applications for Inverse Problems, Inverse Probl. 21 (2005), 2071–2084. <https://doi.org/10.1088/0266-5611/21/6/017>.

- [8] Y. Censor, A. Segal, The Split Common Fixed Point Problem for Directed Operators, *J. Convex Anal.* 16 (2009), 587–600. <https://journalofconvexanalysis.com/articles/jca16031>.
- [9] A. Moudafi, The Split Common Fixed-Point Problem for Demicontractive Mappings, *Inverse Probl.* 26 (2010), 055007. <https://doi.org/10.1088/0266-5611/26/5/055007>.
- [10] Q.-B. Zhang, C.-Z. Cheng, Strong Convergence Theorem for a Family of Lipschitz Pseudocontractive Mappings in a Hilbert Space, *Math. Comput. Model.* 48 (2008), 480–485. <https://doi.org/10.1016/j.mcm.2007.09.014>.
- [11] R. Hu, Y.-P. Fang, Characterizations of Levitin–Polyak Well-Posedness by Perturbations for the Split Variational Inequality Problem, *Optimization* 65 (2016), 1717–1732. <https://doi.org/10.1080/02331934.2016.1166500>.
- [12] J. Li, Split Nash Equilibria for Related Games and Applications to Economics, arXiv:1610.05409, 2016. <https://doi.org/10.48550/arXiv.1610.05409>.
- [13] D.V. Hieu, Projection Methods for Solving Split Equilibrium Problems, *J. Ind. Manag. Optim.* 16 (2020), 2331–2349. <https://doi.org/10.3934/jimo.2019056>.
- [14] S. Dey, A. Gibali, S. Reich, Levitin–Polyak Well-Posedness for Split Equilibrium Problems, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* 117 (2023), 88. <https://doi.org/10.1007/s13398-023-01416-8>.
- [15] G. Wang, X. Yang, Existence-Stability Theorems for Split Equilibrium Problems in Reflexive Banach Spaces, *J. Fixed Point Theory Appl.* 27 (2025), 13. <https://doi.org/10.1007/s11784-025-01166-7>.
- [16] G. Crombez, A Geometrical Look at Iterative Methods for Operators with Fixed Points, *Numer. Funct. Anal. Optim.* 26 (2005), 157–175. <https://doi.org/10.1081/nfa-200063882>.
- [17] E. Kreyszig, *Advanced Engineering Mathematics*, John Wiley and Sons, 2011.
- [18] Z. Opial, Weak Convergence of the Sequence of Successive Approximations for Nonexpansive Mappings, *Bull. Am. Math. Soc.* 73 (1967), 591–597. <https://doi.org/10.1090/S0002-9904-1967-11761-0>.
- [19] H.H. Bauschke, P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer International Publishing, 2017. <https://doi.org/10.1007/978-3-319-48311-5>.
- [20] T. Suzuki, Strong Convergence of Krasnoselskii and Mann’s Type Sequences for One-Parameter Nonexpansive Semigroups Without Bochner Integrals, *J. Math. Anal. Appl.* 305 (2005), 227–239. <https://doi.org/10.1016/j.jmaa.2004.11.017>.
- [21] H.-K. Xu, Iterative Algorithms for Nonlinear Operators, *J. Lond. Math. Soc.* 66 (2002), 240–256. <https://doi.org/10.1112/S0024610702003332>.
- [22] Y. Dai, G. Cai, Y. Shehu, O.S. Iyiola, Modified Halpern-Type Splitting Algorithm for Common Solutions of Equilibrium Problems and Monotone Inclusion Problems in Reflexive Banach Spaces, *Numer. Algorithms* (2026). <https://doi.org/10.1007/s11075-026-02345-4>.
- [23] L. Chen, G. Cai, P. Cholamjiak, P. Inkrong, A New Modified Halpern-Type Splitting Algorithm for Solving Monotone Inclusion Problems in Reflexive Banach Spaces, *J. Glob. Optim.* 91 (2025), 559–585. <https://doi.org/10.1007/s10898-025-01467-8>.