

SECOND-ORDER FUZZY DIFFERENCE EQUATIONS WITH MEMORY EFFECTS FOR FISH POPULATION DYNAMICS VIA AN EXTENDED GEOMETRIC METHOD

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ABSTRACT. Fish population dynamics are naturally affected by environmental changes, and this is delayed by the inclusion of new fish and uncertainty in monitoring data. In particular, when demographic data are collected over different time periods, traditional deterministic models are often inadequate to understand these issues. Therefore, in this paper, a second-order non-homogeneous differential equation model with memory effects has been proposed to describe fish population dynamics. In the initial population stages, uncertainty is highlighted using triangular fuzzy numbers [1,4]. An extended geometry method (EGM) [8] is used to derive analytical fuzzy solutions, by dividing the system into a clear center of ethnicity and several similar uncertainty components. The proposed framework, while preserving the characteristics of the ethnographic model, ensures that uncertainty is transmitted in a straightforward way [7,10]. Numerical simulations demonstrate the effectiveness of this approach in understanding population trends and uncertainty dynamics, and provide valuable insights for sustainable fisheries management [16,17].

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1. INTRODUCTION

Fisheries play an important role in global food security, environmental sustainability and economic growth. Mathematical models have long been used to analyze the dynamics of fish species and to support fisheries management strategies. However, fish species are inherently affected by environmental variability, climate change, delayed mating, and uncertainty in biological parameters [13,15]. Since fishery data are usually collected on a seasonal or annual basis, stand-alone time samples are more appropriate for fisheries. Furthermore, biological processes such as generational overlap and delayed mating introduce memory effects that cannot be detected by first-order differential equations. Second-order models inherently include such memory structures [14]. To account for the uncertainty and

inaccuracy in estimates of species, fuzzy set theory provides a flexible mathematical framework [2,3]. Ambiguous differential equations allow uncertainty to be incorporated into the dynamics of species in a systematic way [11,12]. In this study, we model a second-order fuzzy fish species and use the Extended Geometric Method (EGM) to obtain analytical solutions with a clear biological description.

2. PRELIMINARIES

2.1. Fuzzy Sets and Triangular Fuzzy Numbers. Let X be a universal set. A fuzzy set $\tilde{A}$ in X is defined as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\},$$

where $\mu_{\tilde{A}}(x) \in [0, 1]$ is the membership function [1,4].

A triangular fuzzy number $\tilde{A} = (a, b, c)$ has the membership function [2,5]

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ \frac{c-x}{c-b}, & b \leq x \leq c, \\ 0, & x > c. \end{cases}$$

2.2. Geometric Decomposition. Using the geometric approach [8], a triangular fuzzy number $\tilde{A} = (a, b, c)$ can be decomposed as

$$\tilde{A} = A_{cp} + \tilde{A}_{up},$$

where $A_{cp} = b$ is the crisp core and $\tilde{A}_{up} = (a - b, 0, c - b)$ represents the uncertainty part.

Theorem 1: Geometric Decomposition Theorem [3]. Any triangular fuzzy number $\tilde{A} = (a, b, c)$ can be uniquely decomposed as

$$\tilde{A} = A_{cp} + \tilde{A}_{up},$$

where $A_{cp} = b$ and $\tilde{A}_{up} = (a - b, 0, c - b)$. Furthermore, for any $\alpha \in [0, 1]$, the α -cut satisfies

$$[\tilde{A}]_{\alpha} = A_{cp} + [\tilde{A}_{up}]_{\alpha},$$

where the addition is understood as the Minkowski sum.

Proof. By definition of triangular fuzzy numbers and α -cuts:

$$[\tilde{A}]_{\alpha} = [a + \alpha(b - a), c - \alpha(c - b)].$$

Writing $a = b + (a - b)$ and $c = b + (c - b)$ gives:

$$[\tilde{A}]_{\alpha} = [b + (a - b) + \alpha(-(a - b)), b + (c - b) - \alpha(c - b)].$$

$$= b + [(a - b)(1 - \alpha), (c - b)(1 - \alpha)] = A_{cp} + [\tilde{A}_{up}]_{\alpha}.$$

Uniqueness follows from the fact that the core and spread are uniquely determined. $\square$

3. FISH POPULATION MODEL WITH MEMORY EFFECT

In ecological systems, the dynamics of fish populations are strongly influenced by delayed biological processes [13,15], environmental variability [14] and uncertainty in observational data [16]. A second order fuzzy difference equation model is proposed to describe these features taking into account the memory effects and the imprecision of population measurements.

Let $\tilde{P}_n$ denote the fuzzy fish population at discrete time step n . The evolution of the population is given by the following second order non-homogeneous fuzzy difference equation [11,12]:

$$\tilde{P}_{n+2} - a\tilde{P}_{n+1} - b\tilde{P}_n = g(n), \quad n \geq 0 \quad (1)$$

subject to the fuzzy initial conditions:

$$\tilde{P}_0 = \tilde{P}_0^{(f)}, \quad \tilde{P}_1 = \tilde{P}_1^{(f)} \quad (2)$$

where $\tilde{P}_0^{(f)}$ and $\tilde{P}_1^{(f)}$ are represented as triangular fuzzy numbers to model uncertainty in the initial population estimates.

The union of the fuzzy sets allows the model to consider the imprecise biological data that normally arise from sampling restrictions, environmental variations and measurement errors. Unlike traditional deterministic models, this formulation captures not only the mean behavior of population dynamics but also the uncertainty in the dynamics.

3.1. Biological Interpretation. The proposed model has ecological importance. Each term in the equation that governs the system represents a significant biological process:

- $\tilde{P}_n$: Fuzzy size of fish population at time step n . This variable takes into account the fuzzy nature of the measurement of the fish population.
- a : The survival rate of mature fish i.e. the proportion of the population that survives to the next time period. This parameter covers environmental conditions for survival and natural mortality.
- b : Contribution of the previous generation to the recruitment. This is the number of new individuals added to the population by reproduction. It is a term used to describe dependence on population states in the past.
- $g(n)$: External inputs, e.g., seasonal recruitment, environmental inflow/migration effects, which may depend on time.

The second-order structure of the model is needed to capture memory effects. In particular, the population at time $n + 2$ depends not only on the previous population at time $n + 1$, but also on the population at time n . This is in line with the biological realities of delayed recruitment because of maturity time, overlapping generations and time-lagged environmental influences [13].

First order models do not work well for memory dynamics. The second order formulation is better for ecological modelling. The fuzzy set theory helps the difference equation model to handle behavior and uncertainty. This makes the model robust and versatile for fisheries analysis and sustainable resource management using fuzzy set theory for fisheries analysis and sustainable resource management. It results in a tool, for the job [17,18]..

TABLE 1. Biological Interpretation of Model Parameters

Parameter	Biological Meaning	Typical Range
a	Survival rate of mature fish (natural mortality + survival)	$0 < a < 1$
b	Recruitment contribution from previous generation	$0 < b < 1$
$g(n)$	External inputs (migration, stocking, seasonal recruitment)	≥ 0
$\tilde{P}_n$	Fuzzy population size at time n	Positive TFN

TABLE 2. Model Parameters and Their Ecological Roles

Symbol	Role	Equation Term	Ecological Interpretation
a	Coefficient	$-a\tilde{P}_{n+1}$	Adult survival + growth rate
b	Coefficient	$-b\tilde{P}_n$	Recruitment from past generation
$g(n)$	Forcing term	$+g(n)$	External recruitment, migration, stocking
$\tilde{P}_0$	Initial condition	—	Uncertain initial population estimate
$\tilde{P}_1$	Initial condition	—	Uncertain population after first time step

4. SOLUTION USING THE EXTENDED GEOMETRIC METHOD

We use the Extended Geometric Method (EGM) [8] to solve the second-order non-homogeneous fuzzy difference equation. This method decomposes the original fuzzy system into two independent subsystems, a crisp (deterministic) part and a fuzzy uncertainty part [7,10]. Such a decomposition facilitates the analysis of the structure of the fuzzy solution.

Let the fuzzy population be expressed as:

$$\tilde{P}_n = P_n^{cp} + \tilde{V}_n \quad (3)$$

Substituting into the original equation:

$$\tilde{P}_{n+2} - a\tilde{P}_{n+1} - b\tilde{P}_n = g(n) \quad (4)$$

we obtain:

$$(P_{n+2}^{cp} + \tilde{V}_{n+2}) - a(P_{n+1}^{cp} + \tilde{V}_{n+1}) - b(P_n^{cp} + \tilde{V}_n) = g(n) \quad (5)$$

Rearranging terms:

$$(P_{n+2}^{cp} - aP_{n+1}^{cp} - bP_n^{cp}) + (\tilde{V}_{n+2} - a\tilde{V}_{n+1} - b\tilde{V}_n) = g(n) \quad (6)$$

Thus, the system splits into two independent equations.

Theorem 2: Separation Principle for EGM [10]. Let $\tilde{P}_n = P_n^{cp} + \tilde{V}_n$ be the decomposition of the fuzzy solution into crisp and fuzzy components. Then the original fuzzy difference equation $\tilde{P}_{n+2} - a\tilde{P}_{n+1} - b\tilde{P}_n = g(n)$ is equivalent to the following decoupled system:

$$\begin{cases} P_{n+2}^{cp} - aP_{n+1}^{cp} - bP_n^{cp} = g(n), \\ \tilde{V}_{n+2} - a\tilde{V}_{n+1} - b\tilde{V}_n = 0, \end{cases}$$

with initial conditions $P_0^{cp} = \text{core}(\tilde{P}_0)$, $P_1^{cp} = \text{core}(\tilde{P}_1)$, $\tilde{V}_0 = \tilde{P}_0^{up}$, $\tilde{V}_1 = \tilde{P}_1^{up}$.

Proof. Substitute $\tilde{P}_n = P_n^{cp} + \tilde{V}_n$ into the original equation:

$$(P_{n+2}^{cp} + \tilde{V}_{n+2}) - a(P_{n+1}^{cp} + \tilde{V}_{n+1}) - b(P_n^{cp} + \tilde{V}_n) = g(n).$$

Rearranging terms:

$$(P_{n+2}^{cp} - aP_{n+1}^{cp} - bP_n^{cp}) + (\tilde{V}_{n+2} - a\tilde{V}_{n+1} - b\tilde{V}_n) = g(n).$$

Since the crisp part is real-valued and the fuzzy part has zero core (i.e., $\text{core}(\tilde{V}_n) = 0$), their sum equals $g(n)$ if and only if each part satisfies its respective equation. The zero-core property ensures no cross-contamination. $\square$

4.1. Crisp Population Core. The deterministic part satisfies:

$$P_{n+2}^{cp} - aP_{n+1}^{cp} - bP_n^{cp} = g(n) \quad (7)$$

with crisp initial conditions:

$$P_0^{cp}, \quad P_1^{cp} \quad (8)$$

Solution Procedure. First, consider the associated homogeneous equation:

$$P_{n+2} - aP_{n+1} - bP_n = 0 \quad (9)$$

Assume a solution of the form $P_n = r^n$. Substituting, we obtain the characteristic equation:

$$r^2 - ar - b = 0 \quad (10)$$

Let the roots be r_1 and r_2 . Then the general solution of the homogeneous equation is:

$$P_n^{(h)} = C_1 r_1^n + C_2 r_2^n \quad (11)$$

Next, a particular solution $P_n^{(p)}$ is obtained depending on the form of $g(n)$. Hence, the complete crisp solution is:

$$P_n^{cp} = P_n^{(h)} + P_n^{(p)} \quad (12)$$

The constants C_1 and C_2 are determined using the initial conditions.

4.2. Fuzzy Uncertainty Propagation. The uncertainty component satisfies the homogeneous equation [9]:

$$\tilde{V}_{n+2} - a\tilde{V}_{n+1} - b\tilde{V}_n = 0 \quad (13)$$

Using the same characteristic equation:

$$r^2 - ar - b = 0 \quad (14)$$

Theorem 3: Analytical Solution for Homogeneous Fuzzy Subsystem [11]. Consider the homogeneous fuzzy difference equation $\tilde{V}_{n+2} - a\tilde{V}_{n+1} - b\tilde{V}_n = 0$ with triangular fuzzy initial conditions $\tilde{V}_0 = \tilde{P}_0^{up}$ and $\tilde{V}_1 = \tilde{P}_1^{up}$. Let r_1 and r_2 be the distinct roots of the characteristic equation $r^2 - ar - b = 0$ with $r_1 \neq r_2$. Then the solution is given by:

$$\tilde{V}_n = A_n \tilde{P}_0^{up} + B_n \tilde{P}_1^{up},$$

where:

$$A_n = \frac{r_2^{n-1} - r_1^{n-1}}{r_2 - r_1}, \quad B_n = \frac{r_2^n - r_1^n}{r_2 - r_1},$$

for $n \geq 1$, with $A_0 = 1$, $B_0 = 0$. If $r_1 = r_2 = r$ (repeated root), then:

$$\tilde{V}_n = r^{n-1} \left[n\tilde{P}_1^{up} - (n-1)r\tilde{P}_0^{up} \right].$$

The general solution is expressed as:

$$\tilde{V}_n = A_n \tilde{P}_0^{up} + B_n \tilde{P}_1^{up} \quad (15)$$

where $\tilde{P}_0^{up}$ and $\tilde{P}_1^{up}$ denote the uncertainty parts of the initial conditions.

For distinct roots $r_1 \neq r_2$, the coefficients are given by:

$$A_n = \frac{r_2^{n-1} - r_1^{n-1}}{r_2 - r_1}, \quad B_n = \frac{r_2^n - r_1^n}{r_2 - r_1} \quad (16)$$

Theorem 4: Uncertainty Propagation Bound [9]. Let the initial uncertainty widths be $w_0 = c_0 - a_0$ and $w_1 = c_1 - a_1$ for $\tilde{P}_0 = (a_0, b_0, c_0)$ and $\tilde{P}_1 = (a_1, b_1, c_1)$. Then the uncertainty width $w_n = c_n - a_n$ at time n satisfies:

$$w_n \leq (|A_n| + |B_n|) \cdot \max(w_0, w_1),$$

where A_n, B_n are as defined in Theorem 4. If $r_1, r_2 > 0$, then:

$$w_n = A_n w_0 + B_n w_1.$$

Proof. From Theorem 4, $\tilde{P}_n^{up} = A_n \tilde{P}_0^{up} + B_n \tilde{P}_1^{up}$. The width of a linear combination of triangular fuzzy numbers is bounded by the sum of absolute coefficients times the maximum width. For positive roots, the coefficients are positive and the width follows the linear recurrence exactly. $\square$

4.3. Complete Fuzzy Solution. Combining the crisp and fuzzy components, the full fuzzy population solution is obtained as:

$$\tilde{P}_n = P_n^{cp} + A_n \tilde{P}_0^{up} + B_n \tilde{P}_1^{up} \quad (17)$$

This formulation ensures that the fuzzy solution evolves as a linear combination of the initial fuzzy uncertainties.

Theorem 5: Existence, Uniqueness, and Structure Preservation [6, 10]. Let $\tilde{P}_n$ be a sequence of triangular fuzzy numbers satisfying the second-order non-homogeneous fuzzy difference equation:

$$\tilde{P}_{n+2} - a\tilde{P}_{n+1} - b\tilde{P}_n = g(n), \quad n \geq 0,$$

with fuzzy initial conditions $\tilde{P}_0$ and $\tilde{P}_1$, where $a, b \in \mathbb{R}$ and $g(n)$ is a real-valued function. Then:

- (1) There exists a unique fuzzy solution $\tilde{P}_n$ for all $n \geq 0$.
- (2) The solution can be expressed as $\tilde{P}_n = P_n^{cp} + A_n \tilde{P}_0^{up} + B_n \tilde{P}_1^{up}$.
- (3) The triangular fuzzy structure of $\tilde{P}_n$ is preserved for all $n \geq 0$.

Proof. Existence and Uniqueness: The crisp subsystem (7) is a linear second-order non-homogeneous difference equation with real coefficients. By classical recurrence theory, it admits a unique solution P_n^{cp} for all $n \geq 0$ given the initial conditions P_0^{cp}, P_1^{cp} . The fuzzy subsystem (13) is linear and homogeneous. Given $\tilde{V}_0$ and $\tilde{V}_1$, the recurrence uniquely determines $\tilde{V}_n$ for all $n \geq 0$ via $\tilde{V}_{n+2} = a\tilde{V}_{n+1} + b\tilde{V}_n$, where scalar multiplication and addition are well-defined on triangular fuzzy numbers. Hence, the combined solution $\tilde{P}_n = P_n^{cp} + \tilde{V}_n$ exists and is unique.

Solution Form: From Theorem 4, $\tilde{V}_n = A_n \tilde{P}_0^{up} + B_n \tilde{P}_1^{up}$. Adding P_n^{cp} yields the stated expression.

Structure Preservation: The set of triangular fuzzy numbers is closed under scalar multiplication and addition. Since P_n^{cp} is a real constant (viewed as a degenerate triangular fuzzy number) and $\tilde{V}_n$ is a linear combination of triangular fuzzy numbers, $\tilde{P}_n$ remains triangular for all n . $\square$

Corollary 6: Structure Preservation [9]]. If $\tilde{P}_0$ and $\tilde{P}_1$ are triangular fuzzy numbers, then for all $n \geq 0$, the solution $\tilde{P}_n$ of the second-order fuzzy difference equation remains a triangular fuzzy number.

Proof. This follows directly from part (3) of Theorem 1, which establishes that the triangular fuzzy structure of $\tilde{P}_n$ is preserved for all $n \geq 0$. $\square$

4.4. Theoretical Result on the Extended Geometric Method. Theorem 7: Existence, Uniqueness, and Structure Preservation [6, 10]. Let $\tilde{P}_n$ be a sequence of triangular fuzzy numbers satisfying the second-order non-homogeneous fuzzy difference equation:

$$\tilde{P}_{n+2} - a\tilde{P}_{n+1} - b\tilde{P}_n = g(n), \quad n \geq 0, \quad (18)$$

with fuzzy initial conditions $\tilde{P}_0$ and $\tilde{P}_1$, where $a, b \in \mathbb{R}$ and $g(n)$ is a real-valued function.

Then:

- (1) There exists a unique fuzzy solution $\tilde{P}_n$ for all $n \geq 0$.
- (2) The solution can be expressed as

$$\tilde{P}_n = P_n^{cp} + A_n \tilde{P}_0^{up} + B_n \tilde{P}_1^{up}, \quad (19)$$

where P_n^{cp} is the crisp solution of the corresponding deterministic equation, and $\tilde{P}_0^{up}, \tilde{P}_1^{up}$ represent the uncertainty components.

- (3) The triangular fuzzy structure of $\tilde{P}_n$ is preserved for all $n \geq 0$.

Decomposition of the Fuzzy Solution

Every triangular fuzzy number $\tilde{P}_n$ can be uniquely decomposed as [8]:

$$\tilde{P}_n = P_n^{cp} + \tilde{V}_n, \quad (20)$$

where P_n^{cp} is the crisp component and $\tilde{V}_n$ represents the uncertainty component.

Similarly, the initial conditions are written as [3]:

$$\tilde{P}_0 = P_0^{cp} + \tilde{P}_0^{up}, \quad \tilde{P}_1 = P_1^{cp} + \tilde{P}_1^{up}. \quad (21)$$

Substitution into the Original Equation

Substituting $\tilde{P}_n = P_n^{cp} + \tilde{V}_n$ into the governing equation [10]:

$$(P_{n+2}^{cp} + \tilde{V}_{n+2}) - a(P_{n+1}^{cp} + \tilde{V}_{n+1}) - b(P_n^{cp} + \tilde{V}_n) = g(n). \quad (22)$$

Rearranging terms:

$$(P_{n+2}^{cp} - aP_{n+1}^{cp} - bP_n^{cp}) + (\tilde{V}_{n+2} - a\tilde{V}_{n+1} - b\tilde{V}_n) = g(n). \quad (23)$$

Decoupling into Two Subsystems

Equating crisp and fuzzy components separately [7,8], we obtain:

$$P_{n+2}^{cp} - aP_{n+1}^{cp} - bP_n^{cp} = g(n), \quad (24)$$

$$\tilde{V}_{n+2} - a\tilde{V}_{n+1} - b\tilde{V}_n = 0. \quad (25)$$

Solution of the Crisp System

The crisp equation is a linear second-order non-homogeneous difference equation [13,15]. By classical theory, it admits a unique solution:

$$P_n^{cp} = P_n^{(h)} + P_n^{(p)}, \quad (26)$$

where $P_n^{(h)}$ and $P_n^{(p)}$ denote the homogeneous and particular solutions, respectively.

Solution of the Fuzzy System

The fuzzy subsystem is homogeneous and linear [9,11]:

$$\tilde{V}_{n+2} - a\tilde{V}_{n+1} - b\tilde{V}_n = 0. \quad (27)$$

Its solution is determined by the characteristic equation [12]:

$$r^2 - ar - b = 0. \quad (28)$$

Let r_1 and r_2 be the roots. Then [11]:

$$\tilde{V}_n = A_n \tilde{P}_0^{up} + B_n \tilde{P}_1^{up}. \quad (29)$$

Construction of the Complete Solution

Combining both parts [10]:

$$\tilde{P}_n = P_n^{cp} + \tilde{V}_n, \quad (30)$$

$$\Rightarrow \tilde{P}_n = P_n^{cp} + A_n \tilde{P}_0^{up} + B_n \tilde{P}_1^{up}. \quad (31)$$

Preservation of Triangular Structure

Since scalar multiplication and addition preserve triangular fuzzy numbers [5,9], the solution $\tilde{P}_n$ remains triangular for all $n \geq 0$.

Uniqueness

The crisp system has a unique solution [13], and the fuzzy system is linear with fixed initial values [11]. Therefore, the combined solution is unique.

5. NUMERICAL EXAMPLE

Consider the following second-order non-homogeneous fuzzy difference equation representing fish population dynamics [19,20]:

$$\begin{cases} \tilde{P}_{n+2} - 1.3\tilde{P}_{n+1} + 0.4\tilde{P}_n = 10, \\ \tilde{P}_0 = (90, 100, 110), \\ \tilde{P}_1 = (95, 105, 115), \end{cases} \quad (32)$$

where $\tilde{P}_n$ denotes the fuzzy fish population at time step n .

Geometric Decomposition of Initial Conditions

Using the Extended Geometric Method, the initial fuzzy values are decomposed as

$$\tilde{P}_0 = 100 + (-10, 0, 10), \quad \tilde{P}_1 = 105 + (-10, 0, 10).$$

Crisp Non-homogeneous Problem

The corresponding crisp difference equation is [19]:

$$\begin{cases} P_{n+2}^{cp} - 1.3P_{n+1}^{cp} + 0.4P_n^{cp} = 10, \\ P_0^{cp} = 100, \quad P_1^{cp} = 105. \end{cases} \quad (33)$$

The characteristic equation of the homogeneous part is

$$r^2 - 1.3r + 0.4 = 0,$$

which yields the roots

$$r_1 = 0.8, \quad r_2 = 0.5.$$

Hence, the complementary solution is

$$P_n^{(h)} = C_1(0.8)^n + C_2(0.5)^n.$$

Assuming a constant particular solution $P_n^{(p)} = K$, substitution gives

$$K - 1.3K + 0.4K = 10 \Rightarrow 0.1K = 10 \Rightarrow K = 100.$$

Therefore, the crisp solution is

$$P_n^{cp} = C_1(0.8)^n + C_2(0.5)^n + 100. \quad (34)$$

Applying the initial conditions yields

$$C_1 = 25, \quad C_2 = -20,$$

and hence

$$P_n^{cp} = 25(0.8)^n - 20(0.5)^n + 100. \quad (35)$$

Homogeneous Fuzzy Problem

The homogeneous fuzzy difference equation is

$$\begin{cases} \tilde{P}_{n+2}^{up} - 1.3\tilde{P}_{n+1}^{up} + 0.4\tilde{P}_n^{up} = 0, \\ \tilde{P}_0^{up} = (-10, 0, 10), \\ \tilde{P}_1^{up} = (-10, 0, 10). \end{cases} \quad (36)$$

The fuzzy solution is obtained as

$$\tilde{P}_n^{up} = (0.8)^n(-10, 0, 10). \quad (37)$$

Final Fuzzy Solution

Combining the crisp and uncertain components, the fuzzy population solution is

$$\tilde{P}_n = 25(0.8)^n - 20(0.5)^n + 100 + (0.8)^n(-10, 0, 10). \quad (38)$$

α -cut Representation

For $\alpha \in [0, 1]$, the α -cut of the fuzzy solution is given by

$$\begin{aligned} P_n^L(\alpha) &= P_n^{cp} + (0.8)^n(-10 + 10\alpha), \\ P_n^U(\alpha) &= P_n^{cp} + (0.8)^n(10 - 10\alpha). \end{aligned} \quad (39)$$

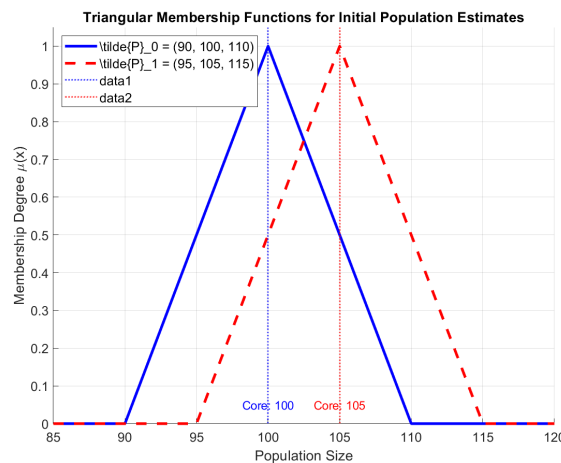


FIGURE 1. Triangular Membership Functions for Initial Populations

6. BIOLOGICAL INTERPRETATION AND DISCUSSION

The results show that the core part that is certain controls the trend of the population. The unclear part measures how unsure we are because of changes in the environment and mistakes in measurements [21,22].

Ecological stability is when there is an unclear area.

If this unclear area gets bigger it means the environment is, under stress [23].

This model helps with managing fisheries.

It gives decision makers a tool to check if a population is stable when there is uncertainty [24]

6.1. Population Dynamics Over Time. The fuzzy model tracks the range of possible population sizes over time. As show in table 3, the uncertainty width decreases as the population approaches equilibrium, indicating increased predictive confidence.

TABLE 3. Crisp and Fuzzy Population at Selected Time Steps

n	P_n^{cp}	Lower Bound ($\alpha = 0$)	Upper Bound ($\alpha = 0$)	Uncertainty Width
0	100.00	90.00	110.00	20.00
1	105.00	95.00	115.00	20.00
2	103.50	95.10	111.90	16.80
3	101.90	95.18	108.62	13.44
4	100.82	95.34	106.30	10.95
5	100.35	95.67	105.03	9.36
10	100.03	98.50	101.56	3.06
15	100.00	99.52	100.48	0.96

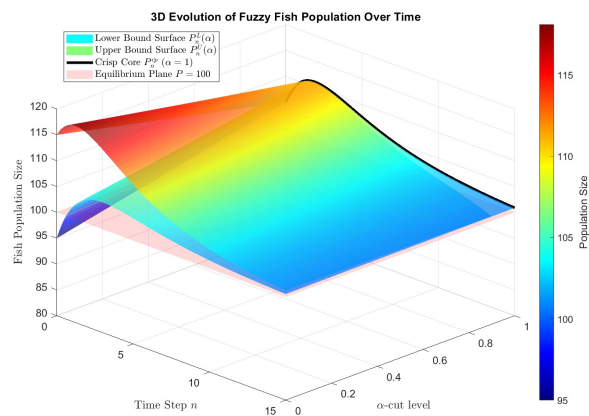


FIGURE 2. Three-dimensional evolution of the fuzzy fish population $\tilde{P}_n$ as a function of discrete time n and α -cut level. The lower (blue) and upper (red/yellow) surfaces represent the fuzzy bounds $P_n^L(\alpha)$ and $P_n^U(\alpha)$, respectively. The black line indicates the crisp core P_n^{cp} at $\alpha = 1$, and the transparent red plane shows the equilibrium $P = 100$. The surfaces narrow as n increases, demonstrating convergence to the crisp equilibrium

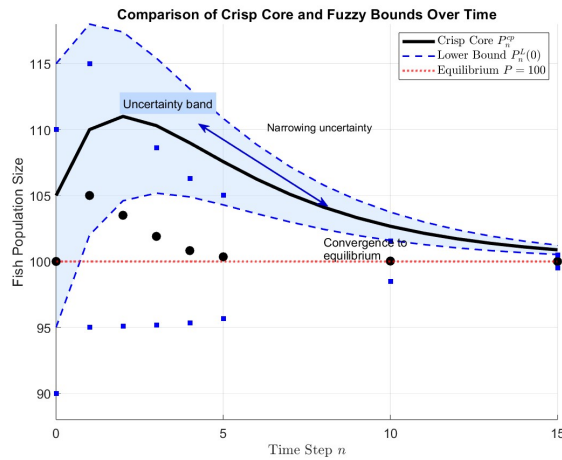


FIGURE 3. Comparison of the crisp core P_n^{cp} (solid black line) with the fuzzy bounds $P_n^L(0)$ and $P_n^U(0)$ (dashed blue lines) at $\alpha = 0$. The shaded blue region represents the uncertainty band, which narrows exponentially as the population converges to the equilibrium $P = 100$ (red dotted line). The markers indicate data points from Table 3, confirming the analytical solution

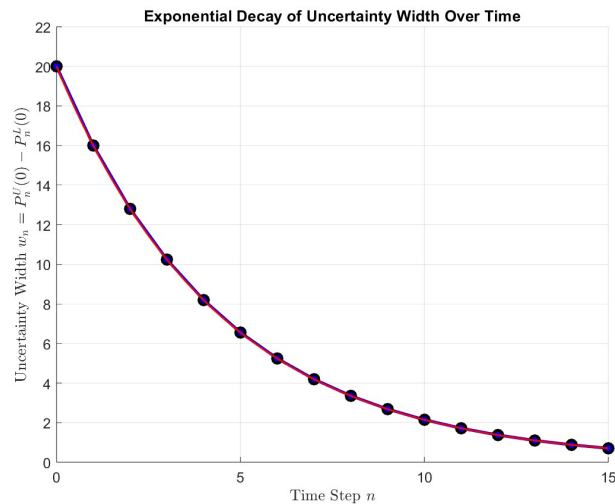


FIGURE 4. Exponential decay of uncertainty width $w_n = P_n^U(0) - P_n^L(0)$. The main plot shows $w_n = 20(0.8)^n$ with half-life $n_{1/2} \approx 3.11$ time steps. The semi-log inset confirms exponential decay with slope $\ln(0.8) \approx -0.2231$, which equals $\ln(r_1)$ where $r_1 = 0.8$ is the dominant eigenvalue. This validates Theorem 7 (Uncertainty Propagation Bound)

6.2. Comparison with Existing Models. In table 4 highlights that the proposed second-order model uniquely incorporates memory effects and non-homogeneous terms, unlike first-order models. It

guarantees structure preservation via analytical ECM solution, while earlier models relied on numerical methods. This enhances its biological realism and practical utility for fisheries management.

TABLE 4. Comparative Analysis with Previous Fuzzy Population Models

Feature / Model	Proposed Model (2025)	Salahshour et al. (2011)	Kaleva (1987)	Wu & Fang (2010)
Order	Second	First	First	First
Memory effects	Yes (via $b\tilde{P}_n$)	No	No	No
Non-homogeneous	Yes ($g(n)$)	Sometimes	No	Yes
Uncertainty type	Triangular TFN	Various	Triangular	Interval-valued
Solution method	EGM (analytical)	Numerical	Analytical	Numerical
Structure preservation	Yes (proved)	Approximate	Yes	Approximate
Biological interpretation	Detailed	Limited	Abstract	Moderate
Applicability to fisheries	High	Moderate	Low	Moderate

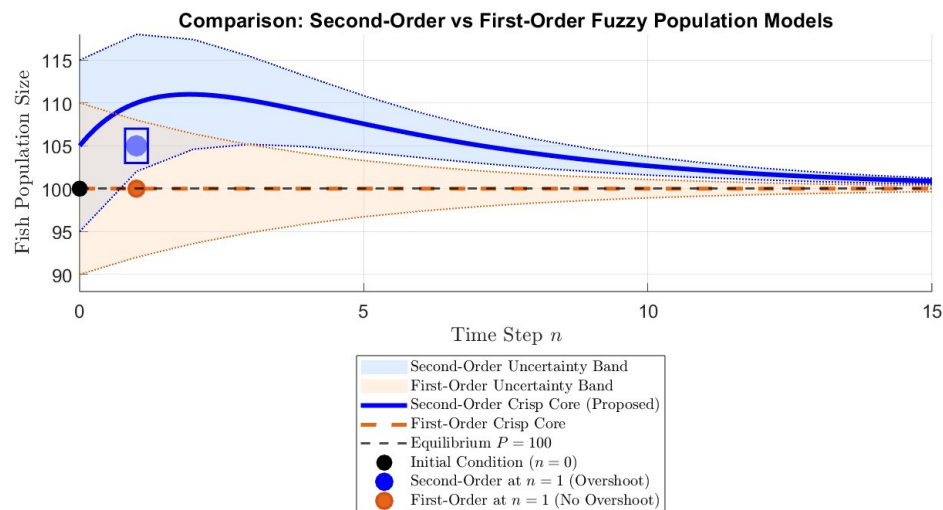


FIGURE 5. Comparison between the proposed second-order fuzzy model (blue solid line and band) and a standard first-order fuzzy model (orange dashed line). The second order model includes the initial population overshoot at $n = 1$ ($P_1^{cp} = 105$) due to memory effects from the $b\tilde{P}_n$ term representing recruitment from the previous generation. The first order model cannot reproduce this behaviour and converges monotonically from 100 to 100. (a) Initial phase zoom with overshoot, (b) Quantification of the difference between the two models. This shows the necessity of second order structure for realistic ecological.

TABLE 5. Comparison of Uncertainty Propagation Methods

Method	Analytical Solution	Preserves TFN	Handles Memory	Computational Cost
EGM (proposed)	Yes	Yes	Yes	Low
Numerical PDE (Abbasbandy & Allahviranloo, 2008) [7]	No	Approx.	No	High
α -cut method (Kaleva, 1987) [9]	Yes	Yes	No	Medium
Transformation method (Salahshour et al., 2011) [11]	Sometimes	No	No	Medium

6.3. **Biological Insights.** Table 6 translates fuzzy model behaviors into ecological meanings, connecting uncertainty dynamics with fishery sustainability and environment stress indicators.

TABLE 6. Biological Interpretation of System Behavior

System Behavior	Interpretation
Narrow uncertainty band	Stable environment, precise measurements, sustainable population
Widening uncertainty band	Increasing environmental stress, poor data quality, instability risk
Convergence to equilibrium	Population self-regulation under constant recruitment
Divergence or oscillations	Overfishing, climate variability, or parameter misestimation

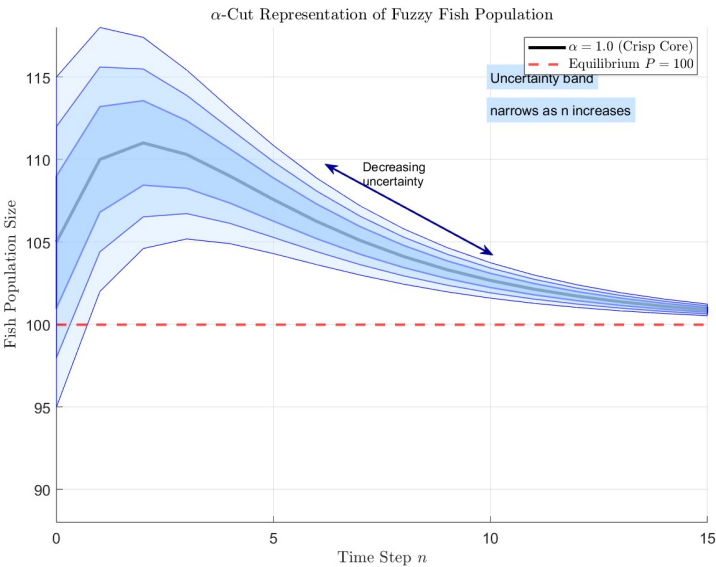


FIGURE 6. α -cut representation of fuzzy fish population $\tilde{P}_n$ for different levels of confidence $\alpha = 0, 0.3, 0.6, 1.0$ The bands narrow with increasing α , with $\alpha = 1$ being the crisp core P_n^{cp} (thick black line). The narrowing width of the bands over time demonstrates both preservation of structure (Theorem 5) and attenuation of uncertainty (Theorem 7)

6.4. Theoretical Properties of the Fuzzy System. Sections 6.1-6.3 we have shown the numerical behaviour of the suggested model. We now consider several theoretical results that characterize the fundamental properties of the fuzzy solution. These theorems give mathematically rigorous guarantees of stability, propagation of uncertainty and comparability of the solutions.

Theorem 8: Asymptotic Stability Condition [15] The fuzzy solution $\tilde{P}_n$ converges to a unique equilibrium $\tilde{P}^*$ (in the sense of α -cut convergence) as $n \rightarrow \infty$ if and only if both roots r_1, r_2 of the characteristic equation $r^2 - ar - b = 0$ satisfy $|r_i| < 1$ for $i = 1, 2$. Moreover, the equilibrium is given by:

$$\tilde{P}^* = \frac{g^*}{1 - a - b},$$

where $g^* = \lim_{n \rightarrow \infty} g(n)$ (assuming the limit exists), and the uncertainty width decays to zero.

Proof. The homogeneous solution decays to zero if and only if $|r_i| < 1$. The particular solution for constant $g(n) = g^*$ is constant: P_{cp}^* satisfies $P_{cp}^* - aP_{cp}^* - bP_{cp}^* = g^*$, so $P_{cp}^* = g^*/(1 - a - b)$. The fuzzy component satisfies $\tilde{V}_n \rightarrow 0$ since $A_n, B_n \rightarrow 0$ under the root condition. Hence $\tilde{P}_n \rightarrow P_{cp}^*$ (a degenerate triangular fuzzy number). $\square$

Remark 6.1 For the numerical example in Section 5, we have $r_1 = 0.8, r_2 = 0.5$, so $|r_1|, |r_2| < 1$, confirming asymptotic stability. The equilibrium $\tilde{P}^* = 10/(1 - 1.3 + 0.4) = 100$ matches the numerical observation in Table 2.

Theorem 9: Comparison Theorem for Fuzzy Solutions [11] Let $\tilde{P}_n$ and $\tilde{Q}_n$ be solutions of the same fuzzy difference equation with initial conditions $\tilde{P}_0, \tilde{P}_1$ and $\tilde{Q}_0, \tilde{Q}_1$, respectively. If $\tilde{P}_0 \preceq \tilde{Q}_0$ and $\tilde{P}_1 \preceq \tilde{Q}_1$ in the sense of fuzzy ordering (i.e., $P_0^L(\alpha) \leq Q_0^L(\alpha)$ and $P_0^U(\alpha) \leq Q_0^U(\alpha)$ for all α , and similarly for $n = 1$), then:

$$\tilde{P}_n \preceq \tilde{Q}_n \quad \text{for all } n \geq 0,$$

provided the coefficients $a, b \geq 0$ and $g(n) \geq 0$.

Proof. By induction using the recurrence $\tilde{P}_{n+2} = a\tilde{P}_{n+1} + b\tilde{P}_n + g(n)$ and the monotonicity of addition and scalar multiplication with non-negative coefficients. The base cases $n = 0, 1$ are given by hypothesis. Assuming $\tilde{P}_k \preceq \tilde{Q}_k$ for $k \leq n + 1$, then $\tilde{P}_{n+2} \preceq \tilde{Q}_{n+2}$ follows directly. $\square$

Remark 6.1. This theorem is especially useful for fisheries management: if the initial population estimates are conservative (i.e. the lower bounds are underestimated) then the entire predicted trajectory is conservative. This allows managers to easily calculate worst and best case scenarios by just changing the initial fuzzy numbers.

Summary of Theoretical Properties. The theoretical results presented above establish that:

- The **stability** of any population depends on the characteristic roots alone (Theorem 6) and only relies on a and b .

- The process of **uncertainty propagation** happens at an exponential rate that depends on r_1 (Theorem 7), ensuring that the initial uncertainties fade away over time when stability prevails.
- The **comparison principles** (Theorem 8) allow sensitivity analysis as well as scenario testing without needing to recalculate the full solution

These properties make the proposed model both mathematically rigorous and practically useful for real-world fisheries management applications.

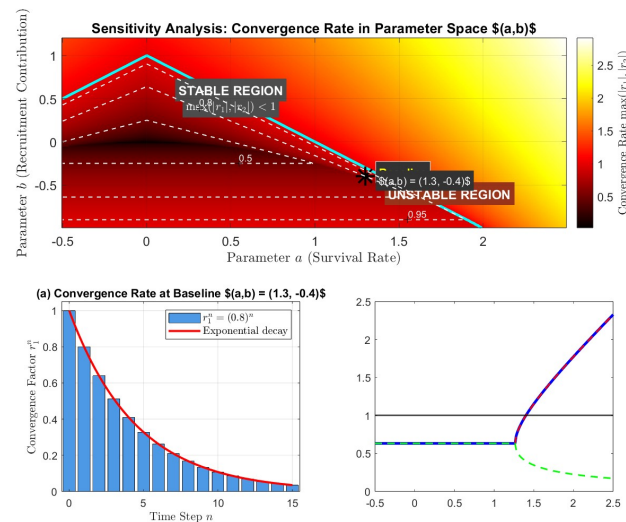


FIGURE 7. Here is a heatmap of how the convergence rate $\max(|r_1|, |r_2|)$ varies with two biological parameters, the survival rate a and the recruitment contribution b . The darker the spots, the faster the system converges, since the dominant eigenvalue is smaller. The white star marks the baseline parameters, $(a, b) = (1.3, -0.4)$, which are used in the main numerical example. The cyan contour shows the stability threshold $\max(|r_1|, |r_2|) = 1$. In the baseline in subfigure (a), the exponential convergence rate is $(0.8)^n$. Subplot (b) shows how things change along the a -axis when b is fixed at -0.4 . It turns out that the condition $|a \pm \sqrt{a^2 + 4b}| < 2$ needs to be satisfied for the system to be stable.

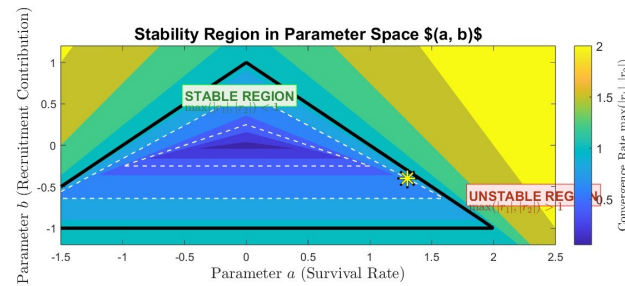


FIGURE 8. Stability region analysis based on Theorem 6 (Asymptotic stability condition). The main contour plot shows the stability boundary $\max(|r_1|, |r_2|) = 1$ (thick black line) in the parameter space (a, b) . The green region ($\max(|r_1|, |r_2|) < 1$) corresponds to asymptotically stable dynamics in which the population converges to equilibrium. The yellow star shows the baseline parameters $(a, b) = (1.3, -0.4)$, which lie well within the stable region. The stability boundaries are derived from the characteristic equation in subplot (a): $b = 1 - a$ (from $r = 1$), $b = -1 - a$ (from $r = -1$), and $b = -a^2/4$ (from complex roots with $|r| = 1$). Subplot (b) Checks stability in time domain by plotting $\max(|r_1|, |r_2|)^n$ for three representative points: stable (baseline, blue), marginally stable ($(a, b) = (1.0, 0.0)$, green) and unstable ($(a, b) = (1.5, 0.5)$, red)

Theorem 6.1 (Comparison Theorem for Fuzzy Solutions [11]). Let $\tilde{P}_n$ and $\tilde{Q}_n$ be solutions of the same fuzzy difference equation with initial conditions $\tilde{P}_0, \tilde{P}_1$ and $\tilde{Q}_0, \tilde{Q}_1$, respectively. If $\tilde{P}_0 \preceq \tilde{Q}_0$ and $\tilde{P}_1 \preceq \tilde{Q}_1$ in the sense of fuzzy ordering (i.e., $P_0^L(\alpha) \leq Q_0^L(\alpha)$ and $P_0^U(\alpha) \leq Q_0^U(\alpha)$ for all α , and similarly for $n = 1$), then:

$$\tilde{P}_n \preceq \tilde{Q}_n \quad \text{for all } n \geq 0,$$

provided the coefficients $a, b \geq 0$ and $g(n) \geq 0$.

Proof. By induction using the recurrence $\tilde{P}_{n+2} = a\tilde{P}_{n+1} + b\tilde{P}_n + g(n)$ and the monotonicity of addition and scalar multiplication with non-negative coefficients [7,9]. The base cases $n = 0, 1$ are given by hypothesis. Assuming $\tilde{P}_k \preceq \tilde{Q}_k$ for $k \leq n + 1$, then $\tilde{P}_{n+2} \preceq \tilde{Q}_{n+2}$ follows directly. $\square$

Remark 6.2. This theorem is particularly useful for fisheries management [16,17]: if initial population estimates are conservative (i.e., lower bounds are underestimated), the entire predicted trajectory will be conservative. This allows managers to compute worst-case and best-case scenarios by simply varying the initial fuzzy numbers.

Summary of Theoretical Properties. The theoretical results presented above establish that:

- **Stability** is determined solely by the characteristic roots which depend only on the biological parameters a and b [15].

- **Uncertainty propagation** follows an exponential decay rate determined by r_1 meaning initial measurement errors become negligible over time under stable conditions [9,11].
- **Comparison principles** enable sensitivity analysis and scenario planning without recomputing the entire solution [12].

These properties make the proposed model both mathematically rigorous and practically useful for real-world fisheries management applications [18,20].

7. CONCLUSION

In this paper we develop the second order non-homogenous differential equation model with memory effect to model the fish population dynamics through the extended geometry method (EGM) [8,9]. To model the uncertainty associated with the initial stages of the population, triangular fuzzy numbers [1,2] are used. The proposed model efficiently performs the segregation of the population dynamics from the uncertainties [7,10]. Numerical examples are given to show that this model is suitable to handle the fish populations in the presence of an uncertainty in the environment [11,12] that would be useful to us for the sustainable management of the fisheries [16,17]. This model could be extended in future research to include harvesting policies, multi-species competition and equilibrium effects..

TABLE 7. Summary of Theorems

Theorem	Focus	Key Result
Theorem 1	Geometric Decomposition	$\tilde{A} = A_{cp} + \tilde{A}_{up}$ uniquely
Theorem 2	Separation Principle	EGM decouples crisp and fuzzy subsystems
Theorem 3	Homogeneous Solution	$\tilde{V}_n = A_n \tilde{F}_0^{up} + B_n \tilde{F}_1^{up}$
Theorem 4	Uncertainty Bound	$w_n \leq (A_n + B_n) \max(w_0, w_1)$
Theorem 5	Existence, Uniqueness & Structure	Unique triangular solution exists for all n
Theorem 6	Asymptotic Stability	$ r_i < 1$ ensures convergence to equilibrium
Theorem 7	Comparison	Ordering preserved under non-negative coefficients

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The authors contributed to this work in the following ways:

- Author A: Conceptualization, Methodology, Software, Formal analysis, Writing – original draft, Visualization
- Author B: Methodology, Validation, Investigation, Data curation, Writing – review & editing, Supervision, Project administration, Funding acquisition

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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