

MORE ON INNER BOUNDS FOR THE EXTREMAL EIGENVALUES OF REAL SYMMETRIC MATRICES

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ABSTRACT. We provide a detailed analysis of simple subspace approximations, in order to derive formula, that may be used to compute the inner bounds for the extremal eigenvalues of real symmetric matrices. The addition of a rotation parameter to the basis vectors for the two dimensional subspace case, and optimization thereof, results in superior bounds, as compared to using fixed basis vectors. The results of these are comparable to the use of higher dimensional subspaces.

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1. INTRODUCTION

A fundamental problem in mathematics is the location of the spectrum $\sigma(\mathbf{A})$ of matrix $\mathbf{A} \in \mathbb{R}^n$. For large matrix dimensions it is unmanageable to solve the characteristic equation. Additional effort is required to compute the associated eigenvectors. For this reason numerical iterative procedures provide the eigenpairs, the gold standard being the QR algorithm [3]. In some cases only knowledge of the extremal eigenvalues are needed, here the power method and its variants are indispensable in achieving this [15]. For instance the approximation of the inverse of a real symmetric matrix relies on good approximations of the extremal eigenvalues [17]. So too does the PageRank [12] depend on the dominant eigenvalue of the modified transition matrix. Recently eigenvalues have been used to construct crypto portfolios, where the dominant eigenvalues of the correlation matrix indicate strong correlation between cryptocurrencies, and the least dominant ones indicate the need for diversification [13].

The Temple quotient [10] provides lower bounds for the smallest eigenvalue, and is a special case of Lehmann's method, which provides both upper and lower bounds for individual eigenvalues [16]. Weinstein [2] and Kato [5] derived bounds which depend on an approximate eigenpair. In contrast Brauer [1] provided outer bounds for $\sigma(\mathbf{A})$. The Rayleigh quotient [4] is often used to improve the

eigenvalue when an approximate eigenpair is known. The Gerschgorin circle theorem and the ovals of Cassini are especially useful for locating $\sigma(\mathbf{A})$, especially for sparse matrices [1]. The trace, being a function of the eigenvalues, yields fairly accurate bounds in certain cases. This idea has been proposed by Wolcowitz and Styan [11], improved by Sharma et al [6] and further generalized by Singh et al [7–9]. Singh et al [14] thereafter employed projection operators and trace bounds to determine bounding intervals, for the extremal eigenvalues.

2. THEORY

Let $\mathbf{A}$ be a $n \times n$ real symmetric matrix, with eigenvalues $\{\lambda_i\}_{i=1}^n$ and denote the associated set of orthonormal eigenvectors by $\{\mathbf{u}_i\}_{i=1}^n$. We shall assume that the eigenvalues are arranged in the order

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product in $\mathbb{R}^n$.

Theorem 1. For $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$, we have

$$\lambda_1 \leq \min_{\|\mathbf{x}\|_2=1} \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \leq \max_{\|\mathbf{x}\|_2=1} \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \leq \lambda_n \quad (1)$$

Proof. Let $\mathbf{x} \in \mathbb{R}^n$, where $\|\mathbf{x}\|_2 = 1$, then $\mathbf{x}$ has the component decomposition

$$\mathbf{x} = \sum_{i=1}^n \langle \mathbf{x}, \mathbf{u}_i \rangle \mathbf{u}_i,$$

where

$$\sum_{i=1}^n \langle \mathbf{x}, \mathbf{u}_i \rangle^2 = 1.$$

$$\begin{aligned} \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle &= \left\langle \sum_{i=1}^n \langle \mathbf{x}, \mathbf{u}_i \rangle \mathbf{A}\mathbf{u}_i, \sum_{j=1}^n \langle \mathbf{x}, \mathbf{u}_j \rangle \mathbf{u}_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \lambda_i \langle \mathbf{x}, \mathbf{u}_i \rangle \langle \mathbf{x}, \mathbf{u}_j \rangle \langle \mathbf{u}_i, \mathbf{u}_j \rangle \\ &= \sum_{i=1}^n \lambda_i \langle \mathbf{x}, \mathbf{u}_i \rangle^2 \\ &\geq \lambda_1 \sum_{i=1}^n \langle \mathbf{x}, \mathbf{u}_i \rangle^2 \\ &= \lambda_1. \end{aligned}$$

Thus $\min_{\|\mathbf{x}\|_2=1} \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \geq \lambda_1$, with equality being attained for $\mathbf{x} = \mathbf{u}_1$. The right hand side of (1) is proved in a similar manner. $\square$

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, $1 < k < n$, be an orthonormal set in $\mathbb{R}^n$, and $S_k = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, then S_k is a k dimensional subspace of $\mathbb{R}^n$.

Lemma 1. If $\mathbf{x} \in S_k$, $\|\mathbf{x}\|_2 = 1$, then

$$\lambda_1 \leq \min_{\substack{\|\mathbf{x}\|_2=1 \\ \mathbf{x} \in S_k}} \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \leq \max_{\substack{\|\mathbf{x}\|_2=1 \\ \mathbf{x} \in S_k}} \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \leq \lambda_n.$$

Proof. This follows from theorem 1 and the fact that S_k is a subset of $\mathbb{R}^n$. □

Let $\mathbf{x} \in S_k$, $\|\mathbf{x}\|_2 = 1$, then

$$\mathbf{x} = \frac{\sum_{i=1}^k c_i \mathbf{v}_i}{\left(\sum_{i=1}^k c_i^2\right)^{\frac{1}{2}}}.$$

To optimize $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle$, for $\mathbf{x} \in S_k$, $\|\mathbf{x}\|_2 = 1$, define

$$\begin{aligned} f(c_1, c_2, \dots, c_k) &= \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \\ &= \frac{\sum_{i=1}^k \sum_{j=1}^k c_i c_j \langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j \rangle}{\sum_{i=1}^k c_i^2}, \end{aligned} \quad (2)$$

thus

$$f(c_1, c_2, \dots, c_k) \sum_{i=1}^k c_i^2 = \sum_{i=1}^k \sum_{j=1}^k c_i c_j \langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j \rangle. \quad (3)$$

Differentiate (3) with respect to c_p , $p = 1, 2, \dots, k$ and set $\frac{\partial f}{\partial c_p} = 0$ to obtain

$$\begin{aligned} 2c_p f(c_1, c_2, \dots, c_k) &= \sum_{i=1}^k c_i \langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_p \rangle + \sum_{j=1}^k c_j \langle \mathbf{A}\mathbf{v}_p, \mathbf{v}_j \rangle \\ &= 2 \sum_{i=1}^k c_i \langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_p \rangle. \end{aligned} \quad (4)$$

The solution of the linear system from (4)

$$\sum_{i=1}^k c_i \langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_p \rangle = f c_p, \quad p = 1, 2, \dots, k$$

can be represented by the matrix vector system

$$\begin{bmatrix} \langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_1 \rangle & \langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_2 \rangle & \cdots & \langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_k \rangle \\ \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_1 \rangle & \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2 \rangle & \cdots & \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{A}\mathbf{v}_k, \mathbf{v}_1 \rangle & \langle \mathbf{A}\mathbf{v}_k, \mathbf{v}_2 \rangle & \cdots & \langle \mathbf{A}\mathbf{v}_k, \mathbf{v}_k \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

$$= f \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \quad (5)$$

Equation (5) is of the form $\mathbf{B}\mathbf{c} = f\mathbf{c}$, where $\mathbf{B}$ is the coefficient matrix in (5) and $\mathbf{c}$ refers to the vector of unknowns. Thus the extreme values of f are just the smallest and largest eigenvalues of the symmetric matrix $\mathbf{B}$. Let $\mathbb{P} = \{p_1, p_2, \dots, p_k\}$, where $p_i \in \mathbb{N}$ and $\sum_{i=1}^k p_i = n$. We shall partition $\mathbf{A}$ into blocks of size $p_i \times p_i$ along the diagonal. The set $\mathbb{P}$ shall be referred to as a natural partition of $\mathbf{A}$. If $\mathbf{A}$ is a 6×6 matrix, and $\mathbb{P} = \{1, 3, 2\}$, then $\mathbf{A}$ is partitioned as below:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{bmatrix},$$

where clearly $\mathbf{A}_{ij} = \mathbf{A}_{ji}^t$. It is now obvious that the submatrix $\mathbf{A}_{ij}$ has size $p_i \times p_j$. Let $\hat{\mathbf{e}}_{p_j}$ be the normalized vector

$$\hat{\mathbf{e}}_{p_j} = \frac{1}{\sqrt{p_j}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

of length p_j . Choose $\mathbf{v}_{p_j} \in \mathbb{R}^n$ to be the vector

$$\mathbf{v}_{p_j} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \hat{\mathbf{e}}_{p_j} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},$$

where the 0 block in the i_{th} position has length p_i and $\hat{e}_{p_j}$ is in the j_{th} position. It is then clear that

$$\mathbf{A}\mathbf{v}_{p_j} = \begin{bmatrix} \mathbf{S}_{1j} \\ \mathbf{S}_{2j} \\ \vdots \\ \mathbf{S}_{kj} \end{bmatrix},$$

where $\mathbf{S}_{ij}$ is a vector of length p_i which is equal to the sum of the columns of $\mathbf{A}_{ij}$ divided by $\sqrt{p_j}$. It is now not difficult to see that $\langle \mathbf{A}\mathbf{v}_{p_j}, \mathbf{v}_{p_i} \rangle = \frac{S_{ij}}{\sqrt{p_j}\sqrt{p_i}}$, where S_{ij} is the sum of the elements of $\mathbf{A}_{ij}$. We shall consider here only the cases $k = 2, 3, 4$, as closed form solutions for the extremal eigenvalues of $\mathbf{B}$ are available. We shall designate the minimum eigenvalue of $\mathbf{B}$ by β_m and the maximum eigenvalue by β_M .

Case $k = 2$

Here $S_2 = \text{span}\{\mathbf{v}_{p_1}, \mathbf{v}_{p_2}\}$,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^t & \mathbf{A}_{22} \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{B} &= \begin{bmatrix} \langle \mathbf{A}\mathbf{v}_{p_1}, \mathbf{v}_{p_1} \rangle & \langle \mathbf{A}\mathbf{v}_{p_1}, \mathbf{v}_{p_2} \rangle \\ \langle \mathbf{A}\mathbf{v}_{p_2}, \mathbf{v}_{p_1} \rangle & \langle \mathbf{A}\mathbf{v}_{p_2}, \mathbf{v}_{p_2} \rangle \end{bmatrix} \\ &= \begin{bmatrix} \frac{S_{11}}{p_1} & \frac{S_{12}}{\sqrt{p_1 p_2}} \\ \frac{S_{12}}{\sqrt{p_1 p_2}} & \frac{S_{22}}{p_2} \end{bmatrix}. \end{aligned} \quad (6)$$

The characteristic polynomial of $\mathbf{B}$ is given by

$$\lambda^2 - \left(\frac{S_{11}}{p_1} + \frac{S_{22}}{p_2} \right) \lambda + \frac{S_{11}S_{22} - S_{12}^2}{p_1 p_2}.$$

Thus

$$\beta_m = \frac{1}{2} \left[\frac{S_{11}}{p_1} + \frac{S_{22}}{p_2} - \sqrt{\left(\frac{S_{11}}{p_1} - \frac{S_{22}}{p_2} \right)^2 + \frac{4S_{12}^2}{p_1 p_2}} \right] \quad (7)$$

and

$$\beta_M = \frac{1}{2} \left[\frac{S_{11}}{p_1} + \frac{S_{22}}{p_2} + \sqrt{\left(\frac{S_{11}}{p_1} - \frac{S_{22}}{p_2} \right)^2 + \frac{4S_{12}^2}{p_1 p_2}} \right]. \quad (8)$$

In the special case when $p_1 = p_2 = \frac{n}{2}$, for even size matrices, equations (7) and (8) simplify to [9]

$$\beta_m = \frac{1}{n} \left[S_{11} + S_{22} - \sqrt{(S_{11} - S_{22})^2 + 4S_{12}^2} \right] \quad (9)$$

and

$$\beta_M = \frac{1}{n} \left[S_{11} + S_{22} + \sqrt{(S_{11} - S_{22})^2 + 4S_{12}^2} \right]. \quad (10)$$

Optimization

For the $k = 2$ case we may introduce an optimization parameter θ to $\mathbf{v}_{p_1}$. One can similarly do the same for $\mathbf{v}_{p_2}$. For this we shall assume that $p_1 \geq 2$. Choose indices $r < s \leq p_1$ and define $\mathbf{v}_{p_1}$ by

$$\mathbf{v}_{p_1} = \frac{1}{\sqrt{p_1}} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \sqrt{2} \cos \theta \\ 1 \\ \vdots \\ 1 \\ \sqrt{2} \sin \theta \\ 1 \\ \vdots \\ 1 \\ \mathbf{0} \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{position } r \\ \leftarrow \text{position } s \end{array}$$

Note that $\mathbf{v}_{p_1}$ is still a unit vector in $\mathbb{R}^n$, so the effect of θ is to rotate $\mathbf{v}_{p_1}$ in $\mathbb{R}^n$. Thus the subspace $S_2^{rs} = \text{span}\{\mathbf{v}_{p_1}, \mathbf{v}_{p_2}\}$ varies with θ . The question is whether there is a value of θ , other than $\theta = \frac{\pi}{4}$ that yields superior bounds. We shall show that this is possible. Note that $\mathbf{v}_{p_1}$ may be written as

$$\mathbf{v}_{p_1} = \begin{bmatrix} \hat{\mathbf{e}}_{p_1} + \frac{\sqrt{2} \cos \theta - 1}{\sqrt{p_1}} \mathbf{e}_r + \frac{\sqrt{2} \sin \theta - 1}{\sqrt{p_1}} \mathbf{e}_s \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{e}_k$ is the standard unit vector in $\mathbb{R}^{p_1}$, with unity in the k_{th} position and zeroes elsewhere. Calculating the eigenvalues using (6) yields

$$\beta_-(\theta) = \beta_m(\theta) = \frac{1}{2} [\langle \mathbf{A} \mathbf{v}_{p_1}, \mathbf{v}_{p_1} \rangle + \langle \mathbf{A} \mathbf{v}_{p_2}, \mathbf{v}_{p_2} \rangle - D] \quad (11)$$

and

$$\beta_+(\theta) = \beta_M(\theta) = \frac{1}{2} [\langle \mathbf{A} \mathbf{v}_{p_1}, \mathbf{v}_{p_1} \rangle + \langle \mathbf{A} \mathbf{v}_{p_2}, \mathbf{v}_{p_2} \rangle + D], \quad (12)$$

where

$$D = \sqrt{(\langle \mathbf{A} \mathbf{v}_{p_1}, \mathbf{v}_{p_1} \rangle - \langle \mathbf{A} \mathbf{v}_{p_2}, \mathbf{v}_{p_2} \rangle)^2 + 4 \langle \mathbf{A} \mathbf{v}_{p_1}, \mathbf{v}_{p_2} \rangle^2}$$

For convenience we define $T_{11} = \langle \mathbf{A}\mathbf{v}_{p_1}, \mathbf{v}_{p_1} \rangle$, $T_{22} = \langle \mathbf{A}\mathbf{v}_{p_2}, \mathbf{v}_{p_2} \rangle$ and $T_{12} = \langle \mathbf{A}\mathbf{v}_{p_1}, \mathbf{v}_{p_2} \rangle$, so that

$$\beta_{\mp}(\theta) = \frac{1}{2} \left[T_{11} + T_{22} \mp \sqrt{(T_{11} - T_{22})^2 + 4T_{12}^2} \right] \quad (13)$$

Differentiating (13) with respect to θ yields the expression

$$\beta'_{\mp}(\theta) = \frac{1}{2} \left[T'_{11} \mp \frac{(T_{11} - T_{22}) T'_{11} + 4T_{12} T'_{12}}{\sqrt{(T_{11} - T_{22})^2 + 4T_{12}^2}} \right]. \quad (14)$$

Critical points are obtained by setting (14) to zero to yield the equation

$$T_{12} [T_{11}'^2 T_{12} - 2T_{11}' T_{12}' (T_{11} - T_{22}) - 4T_{12} T_{12}'^2] = 0. \quad (15)$$

We shall assume that $T_{12} \neq 0$, since this implies that $\mathbf{A}\mathbf{v}_{p_1} \perp \mathbf{v}_{p_2}$ and $\mathbf{v}_{p_2}$ has fixed orientation. Additionally this is the trivial case as from (6) the eigenvalues are just the diagonal elements. Let $T = \langle \mathbf{A}\mathbf{v}'_{p_1}, \mathbf{v}_{p_1} \rangle$, then $T'_{11} = 2T$, and from (15) we are required to find the zeroes of

$$f(\theta) = T_{12}(T^2 - T_{12}'^2) - T T_{12}'(T_{11} - T_{22}). \quad (16)$$

Now

$$\begin{aligned} \mathbf{A}\mathbf{v}_{p_1} &= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^t & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{e}}_{p_1} + \frac{\sqrt{2} \cos \theta - 1}{\sqrt{p_1}} \mathbf{e}_r + \frac{\sqrt{2} \sin \theta - 1}{\sqrt{p_1}} \mathbf{e}_s \\ \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}_{11} \hat{\mathbf{e}}_{p_1} + \frac{\sqrt{2} \cos \theta - 1}{\sqrt{p_1}} \mathbf{A}_{11} \mathbf{e}_r + \frac{\sqrt{2} \sin \theta - 1}{\sqrt{p_1}} \mathbf{A}_{11} \mathbf{e}_s \\ \mathbf{A}_{12}^t \hat{\mathbf{e}}_{p_1} + \frac{\sqrt{2} \cos \theta - 1}{\sqrt{p_1}} \mathbf{A}_{12}^t \mathbf{e}_r + \frac{\sqrt{2} \sin \theta - 1}{\sqrt{p_1}} \mathbf{A}_{12}^t \mathbf{e}_s \end{bmatrix}. \end{aligned} \quad (17)$$

It thus follows from (17) and the definition of T_{12} that

$$\sqrt{p_1 p_2} T_{12} = S_{12} + (\sqrt{2} \cos \theta - 1) R_{12}^r + (\sqrt{2} \sin \theta - 1) R_{12}^s,$$

where R_{12}^r is the sum of the elements in row r of $\mathbf{A}_{12}$ and R_{12}^s is the sum of the elements in row s of $\mathbf{A}_{12}$. Similarly we can show that

$$\begin{aligned} \sqrt{p_1 p_2} T_{12}' &= -\sqrt{2} \sin \theta R_{12}^r + \sqrt{2} \cos \theta R_{12}^s, \\ p_1 T_{11} &= S_{11} + 2(\sqrt{2} \cos \theta - 1) R_{11}^r + 2(\sqrt{2} \sin \theta - 1) R_{11}^s + (\sqrt{2} \cos \theta - 1)^2 \mathbf{A}_{11}^{rr} \\ &\quad + (\sqrt{2} \sin \theta - 1)^2 \mathbf{A}_{11}^{ss} + 2(\sqrt{2} \cos \theta - 1)(\sqrt{2} \sin \theta - 1) \mathbf{A}_{11}^{rs}, \\ p_1 T &= -\sqrt{2} \sin \theta R_{11}^r + \sqrt{2} \cos \theta R_{11}^s + (-\sin 2\theta + \sqrt{2} \sin \theta) \mathbf{A}_{11}^{rr} + (\sin 2\theta - \sqrt{2} \cos \theta) \mathbf{A}_{11}^{ss} \\ &\quad + (2 \cos 2\theta + \sqrt{2}(\sin \theta - \cos \theta)) \mathbf{A}_{11}^{rs}, \end{aligned}$$

and

$$p_2 T_{22} = S_{22}.$$

Here R_{11}^r and R_{11}^s are the sum of the elements of row r and row s of $\mathbf{A}_{11}$, respectively. It is obvious that $\mathbf{A}_{11}^{rr} = a_{rr}$ and $\mathbf{A}_{11}^{ss} = a_{ss}$. Once the zeroes of $f(\theta)$ on $[0, 2\pi]$ are determined, by using a root finder, then the corresponding minimum value for $\beta_-(\theta)$ from (11), and maximum value for $\beta_+(\theta)$ from (12) are evaluated.

Case $k = 3$

Here $S_3 = \text{span}\{\mathbf{v}_{p_1}, \mathbf{v}_{p_2}, \mathbf{v}_{p_3}\}$,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{12}^t & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{13}^t & \mathbf{A}_{23}^t & \mathbf{A}_{33} \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{B} &= \begin{bmatrix} \langle \mathbf{A}\mathbf{v}_{p_1}, \mathbf{v}_{p_1} \rangle & \langle \mathbf{A}\mathbf{v}_{p_1}, \mathbf{v}_{p_2} \rangle & \langle \mathbf{A}\mathbf{v}_{p_1}, \mathbf{v}_{p_3} \rangle \\ \langle \mathbf{A}\mathbf{v}_{p_2}, \mathbf{v}_{p_1} \rangle & \langle \mathbf{A}\mathbf{v}_{p_2}, \mathbf{v}_{p_2} \rangle & \langle \mathbf{A}\mathbf{v}_{p_2}, \mathbf{v}_{p_3} \rangle \\ \langle \mathbf{A}\mathbf{v}_{p_3}, \mathbf{v}_{p_1} \rangle & \langle \mathbf{A}\mathbf{v}_{p_3}, \mathbf{v}_{p_2} \rangle & \langle \mathbf{A}\mathbf{v}_{p_3}, \mathbf{v}_{p_3} \rangle \end{bmatrix} \\ &= \begin{bmatrix} \frac{S_{11}}{p_1} & \frac{S_{12}}{\sqrt{p_1 p_2}} & \frac{S_{13}}{\sqrt{p_1 p_3}} \\ \frac{S_{12}}{\sqrt{p_1 p_2}} & \frac{S_{22}}{p_2} & \frac{S_{23}}{\sqrt{p_2 p_3}} \\ \frac{S_{13}}{\sqrt{p_1 p_3}} & \frac{S_{23}}{\sqrt{p_2 p_3}} & \frac{S_{33}}{p_3} \end{bmatrix}. \end{aligned}$$

The characteristic polynomial of $\mathbf{B}$ is given by

$$\lambda^3 + b\lambda^2 + c\lambda + d, \quad (18)$$

where

$$\begin{aligned} b &= -\text{tr}(\mathbf{B}) \\ &= -\left(\frac{S_{11}}{p_1} + \frac{S_{22}}{p_2} + \frac{S_{33}}{p_3} \right), \end{aligned}$$

$$c = \frac{S_{11}S_{22} - S_{12}^2}{p_1 p_2} + \frac{S_{11}S_{33} - S_{13}^2}{p_1 p_3} + \frac{S_{22}S_{33} - S_{23}^2}{p_2 p_3}$$

and

$$d = \frac{S_{11}S_{23}^2 + S_{22}S_{13}^2 + S_{33}S_{12}^2 - S_{11}S_{22}S_{33} - 2S_{12}S_{13}S_{23}}{p_1 p_2 p_3}.$$

Let

$$p = c - \frac{b^2}{3} \quad (19)$$

$$q = \frac{2b^3}{27} - \frac{bc}{3} + d. \quad (20)$$

If $p = 0$, then

$$\beta_m = \beta_M = \sqrt[3]{-q} - \frac{b}{3}. \quad (21)$$

If $p \neq 0$, then from Viète's formula [18] we have

$$\beta_m = 2\sqrt{\frac{-p}{3}} \cos\left(\frac{1}{3} \cos^{-1} \sqrt{\frac{-27q^2}{4p^3}} - \frac{4\pi}{3}\right) - \frac{b}{3} \quad (22)$$

and

$$\beta_M = 2\sqrt{\frac{-p}{3}} \cos\left(\frac{1}{3} \cos^{-1} \sqrt{\frac{-27q^2}{4p^3}}\right) - \frac{b}{3}. \quad (23)$$

It is simple to verify the case $p = 0$, since from (19) we have $c = \frac{b^2}{3}$ and from (20) $q = -\frac{b^3}{27} + d$. The characteristic polynomial (18) then simplifies to

$$\lambda^3 + b\lambda^2 + \frac{b^2}{3}\lambda + q + \frac{b^3}{27}. \quad (24)$$

Using the substitution $\lambda = u - \frac{b}{3}$ the characteristic equation from (24) yields $u^3 + q = 0$, from which the solution (21) follows. We shall show that $p < 0$ so that equations (22) and (23) are indeed real solutions.

$$\begin{aligned} p &= c - \frac{b^2}{3} \\ &= -\left(\frac{S_{12}^2}{p_1 p_2} + \frac{S_{13}^2}{p_1 p_3} + \frac{S_{23}^2}{p_2 p_3}\right) - \frac{1}{3} \left[\frac{S_{11}^2}{p_1^2} + \frac{S_{22}^2}{p_2^2} + \frac{S_{33}^2}{p_3^2} - \left(\frac{S_{11} S_{22}}{p_1 p_2} + \frac{S_{11} S_{33}}{p_1 p_3} + \frac{S_{22} S_{33}}{p_2 p_3} \right) \right] \\ &= -\left(\frac{S_{12}^2}{p_1 p_2} + \frac{S_{13}^2}{p_1 p_3} + \frac{S_{23}^2}{p_2 p_3}\right) - \frac{1}{6} \left[\left(\frac{S_{11}}{p_1} - \frac{S_{22}}{p_2} \right)^2 + \left(\frac{S_{11}}{p_1} - \frac{S_{33}}{p_3} \right)^2 + \left(\frac{S_{22}}{p_2} - \frac{S_{33}}{p_3} \right)^2 \right] \\ &= -\left[\sum_{\substack{i,j=1 \\ i < j}}^3 \frac{S_{ij}^2}{p_i p_j} + \frac{1}{6} \sum_{\substack{i,j=1 \\ i < j}}^3 \left(\frac{S_{ii}}{p_i} - \frac{S_{jj}}{p_j} \right)^2 \right] \\ &< 0. \end{aligned}$$

Case $k = 4$

Here $S_4 = \text{span}\{\mathbf{v}_{p_1}, \mathbf{v}_{p_2}, \mathbf{v}_{p_3}, \mathbf{v}_{p_4}\}$,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \mathbf{A}_{14} \\ \mathbf{A}_{12}^t & \mathbf{A}_{22} & \mathbf{A}_{23} & \mathbf{A}_{24} \\ \mathbf{A}_{13}^t & \mathbf{A}_{23}^t & \mathbf{A}_{33} & \mathbf{A}_{34} \\ \mathbf{A}_{14}^t & \mathbf{A}_{24}^t & \mathbf{A}_{34}^t & \mathbf{A}_{44} \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} \langle \mathbf{A} \mathbf{v}_{p_1}, \mathbf{v}_{p_1} \rangle & \langle \mathbf{A} \mathbf{v}_{p_1}, \mathbf{v}_{p_2} \rangle & \langle \mathbf{A} \mathbf{v}_{p_1}, \mathbf{v}_{p_3} \rangle & \langle \mathbf{A} \mathbf{v}_{p_1}, \mathbf{v}_{p_4} \rangle \\ \langle \mathbf{A} \mathbf{v}_{p_2}, \mathbf{v}_{p_1} \rangle & \langle \mathbf{A} \mathbf{v}_{p_2}, \mathbf{v}_{p_2} \rangle & \langle \mathbf{A} \mathbf{v}_{p_2}, \mathbf{v}_{p_3} \rangle & \langle \mathbf{A} \mathbf{v}_{p_2}, \mathbf{v}_{p_4} \rangle \\ \langle \mathbf{A} \mathbf{v}_{p_3}, \mathbf{v}_{p_1} \rangle & \langle \mathbf{A} \mathbf{v}_{p_3}, \mathbf{v}_{p_2} \rangle & \langle \mathbf{A} \mathbf{v}_{p_3}, \mathbf{v}_{p_3} \rangle & \langle \mathbf{A} \mathbf{v}_{p_3}, \mathbf{v}_{p_4} \rangle \\ \langle \mathbf{A} \mathbf{v}_{p_4}, \mathbf{v}_{p_1} \rangle & \langle \mathbf{A} \mathbf{v}_{p_4}, \mathbf{v}_{p_2} \rangle & \langle \mathbf{A} \mathbf{v}_{p_4}, \mathbf{v}_{p_3} \rangle & \langle \mathbf{A} \mathbf{v}_{p_4}, \mathbf{v}_{p_4} \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \frac{S_{11}}{p_1} & \frac{S_{12}}{\sqrt{p_1 p_2}} & \frac{S_{13}}{\sqrt{p_1 p_3}} & \frac{S_{14}}{\sqrt{p_1 p_4}} \\ \frac{S_{12}}{\sqrt{p_1 p_2}} & \frac{S_{22}}{p_2} & \frac{S_{23}}{\sqrt{p_2 p_3}} & \frac{S_{24}}{\sqrt{p_2 p_4}} \\ \frac{S_{13}}{\sqrt{p_1 p_3}} & \frac{S_{23}}{\sqrt{p_2 p_3}} & \frac{S_{33}}{p_3} & \frac{S_{34}}{\sqrt{p_3 p_4}} \\ \frac{S_{14}}{\sqrt{p_1 p_4}} & \frac{S_{24}}{\sqrt{p_2 p_4}} & \frac{S_{34}}{\sqrt{p_3 p_4}} & \frac{S_{44}}{p_4} \end{bmatrix}.$$

The characteristic polynomial of $\mathbf{B}$ is given by

$$\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E, \quad (25)$$

where

$$B = -\text{tr}(\mathbf{B})$$

$$= -\left(\frac{S_{11}}{p_1} + \frac{S_{22}}{p_2} + \frac{S_{33}}{p_3} + \frac{S_{44}}{p_4} \right),$$

$$C = \sum_{\substack{i,j=1 \\ i < j}}^4 \frac{S_{ii}S_{jj} - S_{ij}^2}{p_i p_j},$$

$$D = \sum_{\substack{i,j,k=1 \\ k \neq i \\ k < j}}^4 \frac{S_{ii}S_{kj}^2}{p_i p_k p_j} - \sum_{\substack{i,j,k=1 \\ i < k < j}}^4 \frac{S_{ii}S_{kk}S_{jj}}{p_i p_k p_j} - 2 \sum_{\substack{i,j,k=1 \\ i < k < j}}^4 \frac{S_{ik}S_{ij}S_{kj}}{p_i p_k p_j}$$

and

$$(p_1 p_2 p_3 p_4) E = 2 \sum_{\substack{i,j,k,q=1 \\ k,j,q \neq i \\ k < j < q}}^4 S_{ii}S_{kj}S_{kq}S_{jq} - \sum_{\substack{i,j,k,q=1 \\ i < j,k < q \\ k \neq i,j}}^4 S_{ii}S_{jj}S_{kq}^2 - 2(S_{12}S_{14}S_{23}S_{34} + S_{12}S_{13}S_{24}S_{34} \\ + S_{13}S_{14}S_{23}S_{24}) + S_{12}^2S_{34}^2 + S_{13}^2S_{24}^2 + S_{14}^2S_{23}^2 + S_{11}S_{22}S_{33}S_{44}.$$

The solution of (25) shall be obtained by using Ferrari's method [19]. Define

$$a = -\frac{3B^2}{8} + C$$

$$b = \frac{B^3}{8} - \frac{BC}{2} + D \quad (26)$$

$$c = -\frac{3B^4}{256} + \frac{B^2C}{16} - \frac{BD}{4} + E.$$

If $b = 0$, then

$$\beta_m = -\sqrt{\frac{-a - \sqrt{a^2 - 4c}}{2}} - \frac{B}{4} \quad (27)$$

and

$$\beta_M = \sqrt{\frac{-a + \sqrt{a^2 - 4c}}{2}} - \frac{B}{4}. \quad (28)$$

It is simple to verify the case $b = 0$, since from (26) we have

$$D = \frac{BC}{2} - \frac{B^3}{8}, \quad (29)$$

and the characteristic polynomial (25) then simplifies to

$$\lambda^4 + B\lambda^3 + C\lambda^2 + \left(\frac{BC}{2} - \frac{B^3}{8}\right)\lambda + E. \quad (30)$$

The substitution $\lambda = u - \frac{B}{4}$ in (30) results in a characteristic polynomial in u of the form

$$u^4 + \left(\frac{-3B^3}{8} + C\right)u^2 + \left(\frac{5B^4}{256} + \frac{B^2C}{16} - \frac{B^2C}{8} + E\right). \quad (31)$$

Multiply (29) by $\frac{-B}{4}$ to get

$$\frac{-B^2C}{8} = \frac{-BD}{4} - \frac{B^4}{32}. \quad (32)$$

Substituting (32) into (31), the characteristic equation in u has the form

$$u^4 + au^2 + c = 0. \quad (33)$$

The solution of (33) together with using the transformation $\lambda = u - \frac{B}{4}$ results in (27) and (28). We shall verify that $-a \geq 0$.

$$\begin{aligned} -a &= \frac{3B^2}{8} - C \\ &= \frac{3}{8} \left(\sum_{i=1}^4 \frac{S_{ii}^2}{p_i^2} + 2 \sum_{\substack{i,j=1 \\ i < j}}^4 \frac{S_{ii}S_{jj}}{p_i p_j} \right) - \sum_{\substack{i,j=1 \\ i < j}}^4 \frac{S_{ii}S_{jj}}{p_i p_j} + \sum_{\substack{i,j=1 \\ i < j}}^4 \frac{S_{ij}^2}{p_i p_j} \\ &= \frac{1}{8} \left(3 \sum_{i=1}^4 \frac{S_{ii}^2}{p_i^2} - 2 \sum_{\substack{i,j=1 \\ i < j}}^4 \frac{S_{ii}S_{jj}}{p_i p_j} \right) + \sum_{\substack{i,j=1 \\ i < j}}^4 \frac{S_{ij}^2}{p_i p_j} \\ &= \frac{1}{8} \sum_{\substack{i,j=1 \\ i < j}}^4 (S_{ii} - S_{jj})^2 + \sum_{\substack{i,j=1 \\ i < j}}^4 \frac{S_{ij}^2}{p_i p_j}. \end{aligned}$$

It is also true that $-a \geq -\sqrt{a^2 - 4c}$ and that $a^2 - 4c \geq 0$, although this is tedious to verify. This must be the case as the eigenvalues are real. In the case that $b \neq 0$ define

$$\begin{aligned} p &= -\frac{a^2}{12} - c \\ q &= -\frac{a^3}{108} + \frac{ac}{3} - \frac{b^2}{8} \\ w &= \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \end{aligned} \quad (34)$$

It is not true that w is always real. Therefore it is necessary to compute w using complex arithmetic. However any cube root in equation (34) will suffice. Further let

$$y = \begin{cases} \frac{-5a}{6} - \sqrt[3]{q} & \text{if } w = 0 \\ \frac{-5a}{6} + 2\operatorname{Re}(w) & \text{if } w \neq 0 \end{cases},$$

then

$$\beta_m = -\frac{1}{2} \left(\sqrt{a+2y} + \sqrt{-\left(3a+2y - \frac{2b}{\sqrt{a+2y}}\right)} \right) - \frac{B}{4}$$

and

$$\beta_M = \frac{1}{2} \left(\sqrt{a+2y} + \sqrt{-\left(3a+2y + \frac{2b}{\sqrt{a+2y}}\right)} \right) - \frac{B}{4}$$

Example 1. Here we consider a randomly generated real symmetric matrix of order 20. The matrix is constructed by using the Mersenne Twister pseudo-random number generator and the following syntax in Julia 1.12.3:

- (a) `rng=MersenneTwister(123)`
- (b) `Z=rand(rng, 20, 20)`
- (c) `A = Z + Z'`,

where $\mathbf{Z}' = \mathbf{Z}^t$. For matrix $\mathbf{A}$, $\lambda_1 = -2.920365$ and $\lambda_n = 20.716471$ rounded to six decimal digits.

Table 1 illustrates the $k = 2$ case for all 19 combinations of (p_1, p_2) . Here we identify $\beta_m = -0.304189$ and $\beta_M = 20.495420$. Similarly we summarize in Table 2 results for some of the 171 combinations of (p_1, p_2, p_3) for the $k = 3$ case. Here we identify $\beta_m = -1.161988$ and $\beta_M = 20.528935$. Finally Table 3 shows the $k = 4$ case for some of the 1938 combinations of (p_1, p_2, p_3, p_4) . From here we ascertain that $\beta_m = -1.428155$ and $\beta_M = 20.598374$. As expected the inner bounds are superior as k increases. In contrast in Table 4 we summarize the optimization case for $k = 2$ using only $p_1 = p_2 = 10$. This case yields $\beta_m = -0.580636$ and $\beta_M = 20.513928$. This result is superior to the $k = 2$ case for $(p_1, p_2) = (10, 10)$ from Table 1, in fact it is far better than all results from Table 1. It is clear that the $k = 2$ case with optimization is advantageous due to less computation than the $k = 3$ and $k = 4$ cases, and can yield comparable results for some of the inner bounds.

TABLE 1. Case $k = 2$

(p_1, p_2)	β_m	β_M
(01, 19)	−0.304189	20.468394
(02, 18)	0.642061	20.468497
(03, 17)	0.131881	20.468695
(04, 16)	0.083135	20.468371
(05, 15)	0.047698	20.475519
(06, 14)	−0.284566	20.468898
(07, 13)	−0.077910	20.468377
(08, 12)	0.338193	20.473706
(09, 11)	0.487113	20.470216
(10, 10)	0.693978	20.476437
(11, 09)	0.197231	20.473975
(12, 08)	0.109908	20.473867
(13, 07)	0.462609	20.476869
(14, 06)	0.489189	20.468751
(15, 05)	0.202708	20.472084
(16, 04)	0.268066	20.468516
(17, 03)	0.811370	20.472177
(18, 02)	0.093708	20.495420
(19, 01)	0.527112	20.480306

TABLE 2. Case $k = 3$

(p_1, p_2, p_3)	β_m	β_M
(01, 01, 18)	−0.507251	20.468981
(01, 05, 14)	−0.510541	20.468899
(01, 08, 11)	−0.656512	20.470223
(01, 10, 09)	−0.764338	20.474017
(01, 14, 05)	−0.564099	20.472084
(01, 17, 02)	−0.648945	20.495683
(02, 01, 17)	−0.298805	20.468709
(02, 05, 13)	−0.487447	20.468663
(02, 15, 03)	0.168181	20.472571
(03, 03, 14)	−0.422014	20.470844
(03, 04, 13)	−0.467594	20.469103
(03, 16, 01)	0.023695	20.480354
(04, 01, 15)	−1.161988	20.492591
(04, 02, 14)	−0.928680	20.469248
(04, 14, 02)	−0.193163	20.496411
(05, 03, 12)	−0.848140	20.475927
(05, 13, 02)	−0.326044	20.508792
(06, 01, 13)	−0.896792	20.469955
(06, 12, 02)	−0.684328	20.498694
(07, 01, 12)	−0.180028	20.492814
(07, 11, 02)	−0.450816	20.497427
(08, 10, 02)	−0.620515	20.509028
(09, 01, 10)	−0.350812	20.483942
(09, 09, 02)	−0.645121	20.503730
(10, 08, 02)	−0.421580	20.517356
(11, 07, 02)	−0.964545	20.514456
(12, 07, 01)	−0.422632	20.491517
(13, 05, 02)	−0.701487	20.527926
(14, 04, 02)	−0.335780	20.509160
(15, 03, 02)	−0.967758	20.528935
(16, 01, 03)	−0.688565	20.476929
(17, 02, 01)	0.056590	20.500498

TABLE 3. Case $k = 4$

(p_1, p_2, p_3, p_4)	β_m	β_M	(p_1, p_2, p_3, p_4)	β_m	β_M
(01, 01, 01, 17)	−0.655589	20.469198	(04, 02, 12, 02)	−1.141286	20.498986
(01, 01, 07, 11)	−0.692514	20.471610	(04, 07, 07, 02)	−1.088600	20.515268
(01, 01, 09, 09)	−0.813552	20.475798	(04, 09, 05, 02)	−0.860223	20.528626
(01, 02, 08, 09)	−0.772098	20.477130	(04, 11, 03, 02)	−0.967783	20.529097
(01, 03, 07, 09)	−0.830054	20.474960	(05, 02, 05, 08)	−1.194600	20.493411
(01, 04, 03, 12)	−0.938372	20.476723	(05, 03, 10, 02)	−1.088675	20.511477
(01, 04, 04, 11)	−0.931074	20.476485	(05, 04, 09, 02)	−1.009892	20.509497
(01, 05, 01, 13)	−0.896825	20.469956	(05, 08, 05, 02)	−1.000367	20.530852
(01, 07, 03, 09)	−0.765057	20.474811	(05, 10, 03, 02)	−0.971838	20.533585
(01, 09, 01, 09)	−0.802320	20.476825	(05, 12, 01, 02)	−0.417081	20.598037
(01, 10, 01, 08)	−0.906499	20.474258	(06, 04, 08, 02)	−0.701550	20.519791
(01, 10, 07, 02)	−1.015922	20.514520	(06, 08, 04, 02)	−0.684911	20.509365
(01, 10, 08, 01)	−1.017616	20.490816	(06, 09, 03, 02)	−1.021360	20.528937
(01, 16, 01, 02)	−0.739940	20.592637	(07, 08, 03, 02)	−1.001502	20.529500
(01, 17, 01, 01)	−0.937498	20.495698	(07, 09, 02, 02)	−1.016643	20.520382
(02, 01, 15, 02)	−0.896796	20.495455	(07, 10, 01, 02)	−0.604009	20.592944
(02, 02, 01, 15)	−1.210743	20.493159	(08, 07, 03, 02)	−1.122926	20.530890
(02, 04, 01, 13)	−0.932811	20.470796	(08, 08, 02, 02)	−1.095203	20.524964
(02, 11, 05, 02)	−0.724648	20.529383	(08, 09, 01, 02)	−0.722757	20.595147
(02, 13, 03, 02)	−0.987409	20.529709	(09, 06, 03, 02)	−1.199028	20.528959
(02, 14, 02, 02)	−0.953920	20.520758	(09, 07, 02, 02)	−1.135531	20.521548
(03, 01, 01, 15)	−1.259461	20.494821	(09, 08, 01, 02)	−0.800392	20.592811
(03, 02, 01, 14)	−0.721775	20.501161	(10, 01, 07, 02)	−1.084051	20.517495
(03, 02, 13, 02)	−0.464203	20.521863	(10, 07, 01, 02)	−0.677038	20.596449
(03, 02, 14, 01)	−0.133599	20.503465	(11, 05, 02, 02)	−1.415047	20.525414
(03, 03, 12, 02)	−0.933300	20.500323	(11, 06, 01, 02)	−1.125612	20.594108
(03, 05, 10, 02)	−0.973327	20.512628	(12, 03, 03, 02)	−1.087708	20.530493
(03, 08, 07, 02)	−1.080672	20.516670	(12, 04, 02, 02)	−1.169288	20.526339
(03, 09, 06, 02)	−0.853704	20.519442	(12, 05, 01, 02)	−1.004540	20.593942
(03, 13, 02, 02)	−0.908284	20.520784	(13, 04, 01, 02)	−0.817721	20.596146
(04, 01, 01, 14)	−1.164293	20.504033	(14, 01, 03, 02)	−1.060530	20.533569
(04, 01, 04, 11)	−1.428155	20.492687	(14, 03, 01, 02)	−0.535230	20.593610
(04, 01, 07, 08)	−1.390303	20.494395	(15, 01, 02, 02)	−1.072196	20.529272
(04, 01, 12, 03)	−1.255262	20.494692	(16, 01, 01, 02)	−0.956400	20.598374
(04, 01, 13, 02)	−1.212095	20.525540	(17, 01, 01, 01)	−0.428971	20.592704

TABLE 4. Case $k = 2, p_1 = p_2 = 10$

(r, s)	β_m	β_M	(r, s)	β_m	β_M
(1, 2)	0.103608	20.476951	(4, 5)	-0.104147	20.481996
(1, 3)	0.166969	20.476750	(4, 6)	0.119204	20.490599
(1, 4)	-0.519517	20.477017	(4, 7)	-0.243845	20.479538
(1, 5)	-0.206776	20.486020	(4, 8)	-0.580636	20.481859
(1, 6)	0.204319	20.485064	(4, 9)	0.053459	20.482150
(1, 7)	-0.062449	20.477421	(4, 10)	0.001026	20.478146
(1, 8)	-0.404929	20.485738	(5, 6)	0.334105	20.512496
(1, 9)	0.026226	20.479098	(5, 7)	0.001944	20.492916
(1, 10)	-0.203717	20.480690	(5, 8)	-0.252024	20.476444
(2, 3)	0.527043	20.476464	(5, 9)	0.147194	20.498594
(2, 4)	-0.214171	20.478715	(5, 10)	-0.292317	20.477595
(2, 5)	-0.067986	20.491418	(6, 7)	0.411395	20.480189
(2, 6)	0.455109	20.481491	(6, 8)	-0.066452	20.513928
(2, 7)	0.348149	20.476517	(6, 9)	0.430199	20.478254
(2, 8)	-0.350111	20.491272	(6, 10)	0.390886	20.501070
(2, 9)	0.224731	20.477318	(7, 8)	-0.402829	20.492922
(2, 10)	0.173066	20.484356	(7, 9)	0.015875	20.476838
(3, 4)	0.121158	20.478178	(7, 10)	0.227460	20.485566
(3, 5)	0.304842	20.489807	(8, 9)	-0.177455	20.498593
(3, 6)	0.672712	20.482203	(8, 10)	-0.275988	20.477431
(3, 7)	0.424809	20.476636	(9, 10)	0.202749	20.490236
(3, 8)	-0.062935	20.489645			
(3, 9)	0.452546	20.477667			
(3, 10)	0.442523	20.483248			

3. CONCLUSION

We have provided simple formulae to compute inner bounds for the extremal eigenvalues of real symmetric matrices. In addition the inclusion of a parameter for the $k = 2$ case provides good enough results compared to the higher dimensional cases. While it is possible to include an optimization parameter for the latter cases, it is not practical due to the complexity of determining the optimizing parameter for these cases.

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