

NEW OPERATIONAL IDENTITIES FOR MOTT POLYNOMIALS

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ABSTRACT. This article develops a collection of algebraic identities for the Sheffer sequence formed by the Mott polynomials. The analysis begins by reversing their finite power representation, thereby expressing every monomial in the Mott basis and enabling closed moment calculations. Differentiation and repeated integration formulas are subsequently developed. Whenever the resulting finite sums are hypergeometric, Zeilberger's algorithm is employed to obtain compact forms of the coefficients. Connection formulas are also established for derivatives of Hermite, ultraspherical, Bernoulli, Fibonacci-type, Lucas-type, and some convolved families. Standard Mott products and several mixed products are linearized, after which the formulas are applied to evaluate definite integrals over arbitrary finite intervals.

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1. INTRODUCTION

Special-function systems supply analytic representations for problems arising in mathematical physics, number theory, approximation, and scientific computation. Their structural and approximation features also support the construction of effective numerical procedures; representative accounts may be found in [1,2].

Recent research continues to reveal new features of structured polynomial sequences. A generating-function construction for a degenerate Sheffer family was given in [3]. In the Fibonacci–Lucas–Jacobsthal framework, Jacobsthal-type differentiation and linearization formulas were derived in [4], whereas combined Fibonacci–Lucas polynomials were considered in [5]. Generalized Jacobsthal and Jacobsthal–Lucas sequences were treated in [6], bivariate Fibonacci–Lucas extensions in [7], and further variants in [8]. Exceptional orthogonal families were analyzed in [9]. Probabilistic techniques produced explicit poly-Bernoulli and related representations [10,11], while an umbral extension of the Bernoulli family

appeared in [12]. More recently, new connection, derivative, linearization, and integral formulas for Bernoulli polynomials of the second kind were developed in [13]. Other results concern convolved Fibonacci polynomials [14], generalized Leonardo polynomials [15] and the two Lidstone-type subsequences considered in [16].

Polynomial bases are equally important in computation, where orthogonal and nonorthogonal systems enter spectral, collocation, tau, least-squares, and operational-matrix discretizations. For example, Müntz–Legendre bases were used for fractional variational equations [17]; shifted Pell bases were applied to third-order Rosenau–Hyman and Gilson–Pickering models [18]; and Chebyshev expansions were adopted for a nonlinear fractional Emden–Fowler delay equation [19]. A Hermite least-squares formulation for the fractional Bagley–Torvik problem was developed in [20]. Other examples include shifted Dickson approximation of a time-fractional Huxley model [21] and discrete Chebyshev treatment of a two-dimensional Caputo–Hadamard optimal-control problem [22]. Gegenbauer approximations coupled with neural networks were proposed for integral equations [23], and Hosoya collocation was used for trihybrid-nanofluid heat transport [24]. Fibonacci coefficient polynomials furnished a spectral discretization of the time-fractional Kuramoto–Sivashinsky equation [25]. Related schemes employ Chelyshkov polynomials [26], shifted Vieta–Fibonacci polynomials [27], and shifted companion Morgan–Voyce polynomials [28]. Together, these applications illustrate how structured bases can be tailored to differential, integral, fractional, and control problems.

The sequence introduced by G. Mott is a Sheffer family specified by an exponential generating function. Its members have recently been adopted as computational basis functions. Mott operational matrices were constructed for fractional optimal control in [29], and a Mott approximation for delayed integro-differential equations was reported in [30]. Further optimal-control applications appear in [31], including the delayed hybrid formulation examined in [32].

Here the emphasis shifts from numerical deployment to the algebraic structure of the Mott basis and its relations with other polynomial systems. Explicit inversion, moment, differentiation, and cross-basis identities remain comparatively scarce, which provides the motivation for the formulas derived in this paper.

The presentation proceeds as follows. Section 2 introduces the Mott family and the auxiliary bases, then proves an inverse relation. Section 3 addresses differentiated moments, ordinary moments, and iterated antiderivatives. Product expansions, both standard and mixed, are developed in Section 4. Section 5 gives forward and inverse derivative connections with several families. The definite-integral consequences are collected in Section 6. Finally, concluding remarks are reported in Section 7.

2. FOUNDATIONAL RELATIONS FOR THE MOTT BASIS AND AUXILIARY FAMILIES

This section presents the definitions of the Mott sequence and the related families required later. It concludes with a monomial-to-Mott inverse expansion.

2.1. Finite and generating representations of the Mott sequence. The Mott polynomials admit the following hypergeometric representation [30]:

$$\mathcal{Q}_p(\xi) = \left(-\frac{\xi}{2}\right)^p {}_3F_0\left(-p, \frac{1-p}{2}, \frac{2-p}{2}; -\frac{4}{\xi^2}\right). \quad (1)$$

Since the hypergeometric series terminates, Eq. (1) can be written in the following finite form:

$$\mathcal{Q}_p(\xi) = \left(-\frac{1}{2}\right)^p p!(p-1)! \sum_{a=0}^{\lfloor (p-1)/2 \rfloor} \frac{\xi^{p-2a}}{a!(p-a)!(p-2a-1)!}, \quad p \geq 1, \quad (2)$$

with $\mathcal{Q}_0(\xi) = 1$, where $\lfloor \cdot \rfloor$ denotes the floor function.

Equivalently, the Mott polynomial sequence is characterized by the exponential generating function

$$\exp\left(\frac{\xi(\sqrt{1-\tau^2}-1)}{\tau}\right) = \sum_{p=0}^{\infty} \mathcal{Q}_p(\xi) \frac{\tau^p}{p!}. \quad (3)$$

Thus, Eqs. (1), (2), and (3) give, respectively, the hypergeometric, finite-sum, and generating-function representations of the Mott polynomial sequence.

Remark 1. *A systematic treatment of connection and transformation formulas between Mott polynomials and various classical and generalized polynomial families appears to be limited in the existing literature. The subsequent results are intended to contribute to this direction.*

2.2. An overview of some classical and generalized bases. We now record the polynomial systems needed for the connection and linearization formulas. The normalized ultraspherical (Gegenbauer) polynomial is defined through

$$G_r^{(\lambda)}(\xi) = R_r^{(\lambda-\frac{1}{2}, \lambda-\frac{1}{2})}(\xi), \quad (4)$$

where the normalized Jacobi factor is

$$R_r^{\alpha, \beta}(\xi) = \frac{r! \Gamma(\alpha+1)}{\Gamma(r+\alpha+1)} P_r^{(\alpha, \beta)}(\xi). \quad (5)$$

The first two Chebyshev kinds arise through limiting ultraspherical cases, whereas the third and fourth kinds correspond to nonsymmetric Jacobi specializations. For $T_r(\xi)$, $U_r(\xi)$, $V_r(\xi)$, and $W_r(\xi)$, their trigonometric descriptions read

$$T_r(\cos \phi) = \cos(r \phi), \quad U_r(\cos \phi) = \frac{\sin((r+1)\phi)}{\sin \phi}, \quad (6)$$

$$V_r(\cos \phi) = \frac{\cos((r+\frac{1}{2})\phi)}{\cos(\frac{\phi}{2})}, \quad W_r(\cos \phi) = \frac{\sin((r+\frac{1}{2})\phi)}{\sin(\frac{\phi}{2})}. \quad (7)$$

Remark 2. Each of the four Chebyshev systems obeys

$$C_\ell(\xi) = 2\xi C_{\ell-1}(\xi) - C_{\ell-2}(\xi), \quad (8)$$

with initial values determined by the chosen kind. This common recurrence leads to the moment relation

$$\xi^q C_r(\xi) = \frac{1}{2^q} \sum_{\ell=0}^q \binom{q}{\ell} C_{q+r-2\ell}(\xi). \quad (9)$$

The generalized Fibonacci and Lucas sequences used below are generated by

$$\mathcal{F}_r^{a,b}(\xi) = a\xi \mathcal{F}_{r-1}^{a,b}(\xi) + b\mathcal{F}_{r-2}^{a,b}(\xi), \quad \mathcal{F}_0^{a,b}(\xi) = 1, \quad \mathcal{F}_1^{a,b}(\xi) = a\xi, \quad r \geq 2, \quad (10)$$

$$\mathcal{L}_r^{\bar{a},\bar{b}}(\xi) = \bar{a}\xi \mathcal{L}_{r-1}^{\bar{a},\bar{b}}(\xi) + \bar{b}\mathcal{L}_{r-2}^{\bar{a},\bar{b}}(\xi), \quad \mathcal{L}_0^{\bar{a},\bar{b}}(\xi) = 2, \quad \mathcal{L}_1^{\bar{a},\bar{b}}(\xi) = \bar{a}\xi, \quad r \geq 2, \quad (11)$$

where $a, b, \bar{a}$, and $\bar{b}$ are nonzero constants. The corresponding moment expansions are

$$\xi^s \mathcal{F}_q^{a,b}(\xi) = \sum_{r=0}^s \binom{s}{r} a^{-s} (-b)^r \mathcal{F}_{q+s-2r}^{a,b}(\xi), \quad (12)$$

$$\xi^s \mathcal{L}_q^{\bar{a},\bar{b}}(\xi) = \sum_{r=0}^s \binom{s}{r} \bar{a}^{-s} (-\bar{b})^r \mathcal{L}_{q+s-2r}^{\bar{a},\bar{b}}(\xi). \quad (13)$$

Two additional symmetric families are the convolved Fibonacci and convolved Pell polynomials. The first is governed by [14]

$$p\mathcal{CF}_p^\kappa(\xi) - (\kappa + p - 1)\xi\mathcal{CF}_{p-1}^\kappa(\xi) - (2\kappa + p - 2)\mathcal{CF}_{p-2}^\kappa(\xi) = 0, \quad \mathcal{CF}_0^\kappa(\xi) = 1, \quad \mathcal{CF}_1^\kappa(\xi) = \kappa\xi, \quad p \geq 2, \quad (14)$$

while the second satisfies [33]

$$p\mathcal{CP}_p^\sigma(\xi) - 2(\sigma + p - 1)\xi\mathcal{CP}_{p-1}^\sigma(\xi) - (2\sigma + p - 2)\mathcal{CP}_{p-2}^\sigma(\xi) = 0, \quad \mathcal{CP}_0^\sigma(\xi) = 1, \quad \mathcal{CP}_1^\sigma(\xi) = 2\sigma\xi, \quad p \geq 2. \quad (15)$$

2.3. Inverse transformation from powers to Mott polynomials. To express the monomials by Mott polynomials, an explicit inverse relation is required. The finite-sum evaluation below provides the key step.

Lemma 1. Let a and p be arbitrary positive integers. Then

$$\sum_{c=0}^p \frac{(-1)^c (a-c-1)!}{c! (p-c)! (a-p-c)!} = \begin{cases} \frac{1}{a}, & p = 0, \\ 0, & p \geq 1. \end{cases}$$

Proof. The assertion is direct when $p = 0$. For $p \geq 1$, set

$$\mathcal{A}_{p,a} = \sum_{c=0}^{p-1} \frac{(-1)^c (a-c-1)!}{c! (p-c)! (a-p-c)!}, \quad p \geq 1. \quad (16)$$

After the factorial factors are reorganized and the index is shifted, $\mathcal{A}_{p,a}$ becomes

$$\mathcal{A}_{p,a} = \sum_{c=0}^p \frac{(-1)^{c+1} (a-c)!}{(c-1)! (p-c+1)! (a-p-c+1)!}.$$

Applying Zeilberger's algorithm [34] to the finite summand gives

$$(a - 2p - 3)(a - 2p - 2) \mathcal{A}_{p+1,a} + (p + 2)(a - p - 2) \mathcal{A}_{p+2,a} = 0, \mathcal{A}_{1,a} = 1. \quad (17)$$

Solving (17) subject to the initial value yields

$$\mathcal{A}_{p,a} = \frac{(-1)^{p+1}(a - p - 1)!}{p!(a - 2p)!}, \quad p \geq 1. \quad (18)$$

This establishes the lemma. $\square$

Theorem 1. For each positive integer a , the power ξ^a has the Mott expansion

$$\xi^a = \sum_{\ell=0}^{\lfloor a/2 \rfloor} \frac{(-1)^{a+\ell} 2^{a-2\ell} (a - 2\ell)_{2\ell+1} (1 + a - 2\ell)_{\ell-1}}{\ell!} \mathcal{Q}_{a-2\ell}(\xi). \quad (19)$$

Proof. Introduce the polynomial

$$\zeta_a(\xi) = \sum_{\ell=0}^{\lfloor a/2 \rfloor} \frac{(-1)^{a+\ell} 2^{a-2\ell} (a - 2\ell)_{2\ell+1} (1 + a - 2\ell)_{\ell-1}}{\ell!} \mathcal{Q}_{a-2\ell}(\xi), \quad (20)$$

and will show that

$$\zeta_a(\xi) = \xi^a.$$

Inserting (2) gives

$$\zeta(a, \xi) = \sum_{\ell=0}^{\lfloor a/2 \rfloor} \frac{(-1)^\ell a! (a - \ell - 1)!}{\ell!} \sum_{q=0}^{\lfloor \frac{a-2\ell-1}{2} \rfloor} \frac{\xi^{a-2\ell-2q}}{q! (a - 2\ell - 2q - 1)! (a - 2\ell - q)!}, \quad (21)$$

which can be written as

$$\zeta(a, \xi) = a! \sum_{s=0}^{\lfloor \frac{a-1}{2} \rfloor} \frac{\xi^{a-2s}}{(a - 2s - 1)!} \sum_{\ell=0}^s \frac{(-1)^\ell (a - \ell - 1)!}{\ell! (a - \ell - s)! (s - \ell)!}, \quad (22)$$

or in the form

$$\zeta_a(\xi) = \xi^a + a! \sum_{s=1}^{\lfloor \frac{a}{2} \rfloor} \frac{\xi^{a-2s}}{(a - 2s - 1)!} \sum_{\ell=0}^s \frac{(-1)^\ell (a - \ell - 1)!}{\ell! (a - \ell - s)! (s - \ell)!}. \quad (23)$$

The inner coefficient is evaluated by Lemma 1; hence

$$\zeta_a(\xi) = \xi^a.$$

The asserted identity follows. $\square$

3. DIFFERENTIAL AND INTEGRAL FORMULAS IN THE MOTT BASIS

We next derive Mott-basis formulas for differentiated moments, and iterated antiderivatives.

3.1. Higher derivatives of polynomial moments. The next theorem gives a differentiated Mott moment directly in the original basis.

Theorem 2. Assume that s, w , and r are positive integers and that $s + w \geq r$. Then

$$\begin{aligned} D^r (\xi^s \mathcal{Q}_w(\xi)) &= (w + s)! \sum_{\ell=0}^{\lfloor \frac{w+s-r}{2} \rfloor} \frac{(-1)^{s+\ell-r} 2^{s-2\ell-r} (w + s - \ell - r - 1)!}{\ell! (w + s - 2\ell - r - 1)! (w + s - 2\ell - r)!} \\ &\quad \times {}_4F_3 \left(\begin{matrix} -\ell, \frac{1}{2} - \frac{w}{2}, 1 - \frac{w}{2}, -w \\ -\frac{w+s}{2}, \frac{1}{2} - \frac{w+s}{2}, 1 - w - s + \ell + r \end{matrix} \middle| 1 \right) \mathcal{Q}_{w+s-r-2\ell}(\xi). \end{aligned} \quad (24)$$

Proof. After taking r derivatives of the finite sum (2) gives

$$D^r (\xi^s \mathcal{Q}_w(\xi)) = \left(-\frac{1}{2}\right)^w (w - 1)! w! \sum_{b=0}^{\lfloor \frac{w+s-r}{2} \rfloor} \frac{(1 + w + s - r - 2b)_r}{(w - 1 - 2b)! (w - b)! b!} \xi^{w+s-2b-r}. \quad (25)$$

Expanding each residual power through (19) produces

$$\begin{aligned} D^r (\xi^s \mathcal{Q}_w(\xi)) &= \left(-\frac{1}{2}\right)^w (w - 1)! w! \sum_{b=0}^{\lfloor \frac{w+s-r}{2} \rfloor} \frac{(1 + w + s - r - 2b)_r}{(w - 1 - 2b)! (w - b)! b!} \\ &\quad \times \sum_{\ell=0}^{\lfloor \frac{w+s-r}{2} \rfloor - b} \frac{(-1)^{w+s-r+\ell} 2^{w+s-r-2b-2\ell} (w + s - r - 2b - 2\ell)_{2\ell+1} (1 + w + s - r - 2b - 2\ell)_{\ell-1}}{\ell!} \\ &\quad \times \mathcal{Q}_{w+s-r-2b-2\ell}(\xi). \end{aligned} \quad (26)$$

Collecting terms of the same Mott degree after reversing the two finite sums yields

$$\begin{aligned} D^r (\xi^s \mathcal{Q}_w(\xi)) &= w!(w - 1)! \sum_{\ell=0}^{\lfloor \frac{w+s-r}{2} \rfloor} \frac{1}{(w + s - 2\ell - r - 1)! (w + s - 2\ell - r)!} \\ &\quad \times \sum_{b=0}^{\ell} \frac{(-1)^{s+\ell-r-b} 2^{s-2\ell-r} (w + s - 2b)! (w + s - \ell - r - b - 1)!}{(w - b)! (\ell - b)! b! (w - 2b - 1)!} \mathcal{Q}_{w+s-r-2\ell}(\xi). \end{aligned} \quad (27)$$

Writing the finite inner coefficient as a terminating hypergeometric series leads to

$$\begin{aligned} D^r (\xi^s \mathcal{Q}_w(\xi)) &= (w + s)! \sum_{\ell=0}^{\lfloor \frac{w+s-r}{2} \rfloor} \frac{(-1)^{s+\ell-r} 2^{s-2\ell-r} (w + s - \ell - r - 1)!}{\ell! (w + s - 2\ell - r - 1)! (w + s - 2\ell - r)!} \\ &\quad \times {}_4F_3 \left(\begin{matrix} -\ell, \frac{1}{2} - \frac{w}{2}, 1 - \frac{w}{2}, -w \\ -\frac{w+s}{2}, \frac{1}{2} - \frac{w+s}{2}, 1 - w - s + \ell + r \end{matrix} \middle| 1 \right) \mathcal{Q}_{w+s-r-2\ell}(\xi). \end{aligned}$$

This coefficient proves the stated identity. $\square$

Remark 3. Two special choices in (24) recover an ordinary derivative formula and the standard moment formula.

Corollary 1. Let w, r be positive integers with $w \geq r$. The derivative of order r of $\mathcal{Q}_w(\xi)$ is

$$D^r \mathcal{Q}_w(\xi) = (-1)^r r w! \sum_{\ell=0}^{\lfloor \frac{w-r}{2} \rfloor} \frac{2^{-2\ell-r} (2\ell+r-1)!}{\ell! (w-2\ell-r)! (\ell+r)!} \mathcal{Q}_{w-r-2\ell}(\xi). \quad (28)$$

Proof. Choose $s = 0$ in (24). This gives

$$\begin{aligned} D^r \mathcal{Q}_w(\xi) &= w! \sum_{\ell=0}^{\lfloor \frac{w-r}{2} \rfloor} \frac{(-1)^{\ell-r} 2^{-2\ell-r} (w-\ell-r-1)!}{\ell! (w-2\ell-r-1)! (w-2\ell-r)!} \\ &\quad \times {}_3F_2 \left(\begin{matrix} -\ell, 1 - \frac{w}{2}, -w \\ -\frac{w}{2}, 1 - w + \ell + r \end{matrix} \middle| 1 \right) \mathcal{Q}_{w-r-2\ell}(\xi). \end{aligned} \quad (29)$$

It remains to evaluate the terminating coefficient in (29). Apply Zeilberger's algorithm [34] to

$$\mathcal{H}_{\ell,w,r} = {}_3F_2 \left(\begin{matrix} -\ell, 1 - \frac{w}{2}, -w \\ -\frac{w}{2}, 1 - w + \ell + r \end{matrix} \middle| 1 \right).$$

The resulting recurrence for $\mathcal{H}_{\ell,w,r}$ is

$$\begin{aligned} &(-2\ell+w-r-2)(-2\ell+w-r-1)(\ell+r+1) \mathcal{H}_{\ell+1,w,r} \\ &+ (-\ell+w-r-1)(2\ell+r)(2\ell+r+1) \mathcal{H}_{\ell,w,r} = 0, \quad \mathcal{H}_{0,w,r} = 1. \end{aligned} \quad (30)$$

The initial condition together with (30) gives

$${}_3F_2 \left(\begin{matrix} -\ell, 1 - \frac{w}{2}, -w \\ -\frac{w}{2}, 1 - w + \ell + r \end{matrix} \middle| 1 \right) = \frac{(-1)^\ell r (r+\ell+1)_{\ell-1}}{(w-r-2\ell)_\ell}. \quad (31)$$

Inserting this evaluation into (29) therefore produces the reduced derivative expression

$$D^r \mathcal{Q}_w(\xi) = (-1)^r r w! \sum_{\ell=0}^{\lfloor \frac{w-r}{2} \rfloor} \frac{2^{-2\ell-r} (2\ell+r-1)!}{\ell! (w-2\ell-r)! (\ell+r)!} \mathcal{Q}_{w-r-2\ell}(\xi). \quad (32)$$

□

Corollary 2. For positive integers w and s , the product with ξ^s expands as

$$\begin{aligned} \xi^s \mathcal{Q}_w(\xi) &= (-1)^s 2^s (w+s)! \sum_{\ell=0}^{\lfloor \frac{w+s}{2} \rfloor} \frac{(-1)^\ell 2^{-2\ell} (w+s-\ell-1)!}{(w+s-2\ell-1)! (w+s-2\ell)! \ell!} \\ &\quad \times {}_4F_3 \left(\begin{matrix} -\ell, \frac{1}{2} - \frac{w}{2}, 1 - \frac{w}{2}, -w \\ -\frac{w+s}{2}, \frac{1}{2} - \frac{w+s}{2}, 1 - w - s + \ell \end{matrix} \middle| 1 \right). \end{aligned} \quad (33)$$

Proof. Setting $r = 0$ in (24) yields immediately formula (33). □

3.2. Repeated integrals for Mott polynomials. We now construct a finite Mott expansion for an antiderivative taken r times.

Theorem 3. Denote the r -fold antiderivative of $\mathcal{Q}_q(\xi)$ by

$$\mathfrak{I}_q^r(\xi) = \overbrace{\int \int \dots \int}^{r \text{ times}} \mathcal{Q}_q(\xi) (d\xi)^r,$$

then we have

$$\mathfrak{I}_q^r(\xi) = r! q! \sum_{b=0}^{\lfloor \frac{q+r}{2} \rfloor} \frac{(-1)^{r+b} 2^{-2b+r} (r-2b+1)_{b-1}}{b! (q+r-2b)!} \mathcal{Q}_{q+r-2b}(\xi) + \rho_{r-1}(\xi), \quad (34)$$

where $\rho_{r-1}(\xi)$ is the integration polynomial and has degree no greater than $(r-1)$.

Proof. Integrating the finite representation in (2), repeated r times, yields

$$\mathfrak{I}_q^r(\xi) = \left(-\frac{1}{2}\right)^q (q-1)! q! \sum_{s=0}^{\lfloor \frac{q}{2} \rfloor} \frac{\xi^{q-2s+r}}{(q-2s-1)! (q-s)! s! (1+q-2s)_r} + \rho_{r-1}(\xi). \quad (35)$$

Here $\deg \rho_{r-1} \leq r-1$. Expanding each power through (19) gives

$$\begin{aligned} \mathfrak{I}_q^r(\xi) &= \left(-\frac{1}{2}\right)^q (q-1)! q! \sum_{s=0}^{\lfloor \frac{q}{2} \rfloor} \frac{1}{(q-2s-1)! (q-s)! s! (1+q-2s)_r} \\ &\quad \times \sum_{b=0}^{\lfloor \frac{q-2s+r}{2} \rfloor} \frac{(-1)^{q+b+r} 2^{q+r-2b-2s} (q+r-2b-2s)_{2b+1} (1+q+r-2b-2s)_{b-1}}{b!} \mathcal{Q}_{q+r-2b-2s}(\xi) \\ &\quad + \rho_{r-1}(\xi). \end{aligned} \quad (36)$$

Reversing the order of these finite sums and collecting equal degrees gives

$$\begin{aligned} \mathfrak{I}_q^r(\xi) &= (-1)^r 2^r (q-1)! q! \sum_{b=0}^{\lfloor \frac{q}{2} \rfloor} \frac{2^{-2b}}{(q-2b+r-1)! (q-2b+r)!} \\ &\quad \times \sum_{w=0}^b \frac{(-1)^{b+w} (q-2w)(q-b-w+r-1)!}{(q-w)! (b-w)! w!} \mathcal{Q}_{q+r-2b}(\xi) + \rho_{r-1}(\xi). \end{aligned} \quad (37)$$

For a symbolic evaluation of the inner coefficient, define

$$\mathcal{V}_{b,q,r} = \sum_{w=0}^b \frac{(-1)^{b+w} (q-2w)(q-b-w+r-1)!}{(q-w)! (b-w)! w!}. \quad (38)$$

Zeilberger's procedure [34] then implies the following recursive formula

$$(b+1)(r-b-1)(q+r-2b-1)! \mathcal{V}_{b+1,q,r} - (2b-r+1)(r-2b)(q+r-2b-3)! \mathcal{V}_{b,q,r} = 0, \quad (39)$$

with the initial value

$$\mathcal{V}_{0,q,r} = \frac{q(q+r-1)!}{q!}. \quad (40)$$

Its solution is

$$\mathcal{V}_{b,q,r} = \frac{(-1)^b r(r-2b+1)_{b-1} (-2b+q+r-1)!}{b!(q-1)!}. \quad (41)$$

The reduced result follows upon inserting (41) into (37) to obtain the reduced expansion

$$\mathfrak{I}_q^r(\xi) = r q! \sum_{b=0}^{\lfloor \frac{q+r}{2} \rfloor} \frac{(-1)^{r+b} 2^{-2b+r} (r-2b+1)_{b-1}}{b!(q+r-2b)!} \mathcal{Q}_{q+r-2b}(\xi) + \rho_{r-1}(\xi). \quad (42)$$

Thus, the required representation is obtained. $\square$

4. STANDARD AND MIXED PRODUCT EXPANSIONS

The standard Mott product formula is considered first. It is followed by mixed linearizations involving classical and generalized polynomial systems.

4.1. Expansion of a product of two Mott polynomials.

Theorem 4. *If a and p are positive integers, then the product of the corresponding Mott polynomials is*

$$\begin{aligned} \mathcal{Q}_a(\xi) \mathcal{Q}_p(\xi) &= (p-1)! p! \sum_{w=0}^{\lfloor \frac{a+p}{2} \rfloor} \frac{(-1)^w 4^{-w}}{(a+p-2w-1)!(a+p-2w)!} \\ &\times \sum_{s=0}^w \frac{(-1)^s (a+p-2s)!(a+p-w-s-1)!}{(p-s)! s! (p-2s-1)!(w-s)!} \\ &\times {}_4F_3 \left(\begin{matrix} -w+s, \frac{1}{2} - \frac{a}{2}, 1 - \frac{a}{2}, -a \\ -\frac{a}{2} - \frac{p}{2} + s, \frac{1}{2} - \frac{a}{2} - \frac{p}{2} + s, 1 - a - p + w + s \end{matrix} \middle| 1 \right) \mathcal{Q}_{a+p-2w}(\xi). \end{aligned} \quad (43)$$

Proof. Insert the finite power representation of $\mathcal{Q}_a(\xi)$ to obtain

$$\mathcal{Q}_a(\xi) \mathcal{Q}_p(\xi) = \sum_{s=0}^{\lfloor \frac{p}{2} \rfloor} R_{s,p} \xi^{p-2s} \mathcal{Q}_a(\xi), \quad (44)$$

where

$$R_{s,p} = \frac{(-\frac{1}{2})^p (p-1)! p!}{(p-2s-1)!(p-s)! s!}.$$

For later substitution, write (33) in the form

$$\xi^s \mathcal{Q}_a(\xi) = \sum_{r=0}^{\lfloor \frac{s+a}{2} \rfloor} \mathcal{S}_{r,s,a} \mathcal{Q}_{a+s-2r}(\xi), \quad (45)$$

with

$$\mathcal{S}_{r,s,a} = \frac{(-1)^{r+s} 2^{-2r+s} (a+s)!(a-r+s-1)!}{r!(a-2r+s-1)!(a-2r+s)!} {}_4F_3 \left(\begin{matrix} -r, \frac{1}{2} - \frac{a}{2}, 1 - \frac{a}{2}, -a \\ 1 - a + r - s, -\frac{a}{2} - \frac{s}{2}, \frac{1}{2} - \frac{a}{2} - \frac{s}{2} \end{matrix} \middle| 1 \right).$$

Using (44) in (45) produces

$$\mathcal{Q}_a(\xi) \mathcal{Q}_p(\xi) = \sum_{s=0}^{\lfloor \frac{p}{2} \rfloor} R_{s,p} \sum_{r=0}^{\lfloor \frac{p-2s+a}{2} \rfloor} S_{r,p-2s,a} \mathcal{Q}_{a+p-2s-2r}(\xi). \quad (46)$$

Reversing the finite summation order transforms the preceding formula into

$$\mathcal{Q}_a(\xi) \mathcal{Q}_p(\xi) = \sum_{w=0}^{\lfloor \frac{a+p}{2} \rfloor} \left(\sum_{s=0}^w R_{s,p} S_{w-s,p-2s,a} \right) \mathcal{Q}_{a+p-2w}(\xi), \quad (47)$$

which gives

$$\begin{aligned} \mathcal{Q}_a(\xi) \mathcal{Q}_p(\xi) &= (p-1)! p! \sum_{w=0}^{\lfloor \frac{a+p}{2} \rfloor} \frac{(-1)^w 4^{-w}}{(a+p-2w-1)! (a+p-2w)!} \\ &\times \sum_{s=0}^w \frac{(-1)^s (a+p-2s)! (a+p-w-s-1)!}{(p-s)! s! (p-2s-1)! (w-s)!} \\ &\times {}_4F_3 \left(\begin{matrix} -w+s, \frac{1}{2} - \frac{a}{2}, 1 - \frac{a}{2}, -a \\ -\frac{a}{2} - \frac{p}{2} + s, \frac{1}{2} - \frac{a}{2} - \frac{p}{2} + s, 1 - a - p + w + s \end{matrix} \middle| 1 \right) \mathcal{Q}_{a+p-2w}(\xi). \end{aligned}$$

This proves the theorem. $\square$

4.2. Products coupling Mott and auxiliary bases. We next linearize Mott products with the unified Chebyshev system from (8) and with the generalized Fibonacci and Lucas systems defined in (10) and (11).

Theorem 5. For $r \in \mathbb{N}$ and $v \in \mathbb{Z}$,

$$\begin{aligned} \mathcal{Q}_r(\xi) \mathcal{F}_v^{a,b}(\xi) &= \left(-\frac{1}{2}\right)^r r! (r-1)! \sum_{q=0}^r \frac{a^{-r} (-b)^q}{q! r! (r-q)!} \left[\frac{{}_2F_0 \left(\begin{matrix} 1-q, 1-r, 1+q-r \end{matrix} \middle| \frac{a^2}{b} \right)}{b} \right. \\ &\quad \left. + r {}_3F_0 \left(\begin{matrix} -q, q-r, -r \end{matrix} \middle| \frac{a^2}{b} \right) \right] \mathcal{F}_{r+v-2q}^{a,b}(\xi). \end{aligned} \quad (48)$$

Proof. Substitution of the Mott series (2) gives

$$\mathcal{Q}_r(\xi) \mathcal{F}_v^{a,b}(\xi) = \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \frac{\left(-\frac{1}{2}\right)^r (r-1)! r!}{k! (r-2k-1)! (r-k)!} \xi^{r-2k} \mathcal{F}_v^{a,b}(\xi). \quad (49)$$

Apply (12) separately to every term. The result is

$$\mathcal{Q}_r(\xi) \mathcal{F}_v^{a,b}(\xi) = \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \frac{\left(-\frac{1}{2}\right)^r (r-1)! r!}{k! (r-2k-1)! (r-k)!} \sum_{\ell=0}^{r-k} a^{2k-r} (-b)^\ell \binom{r-2k}{\ell} \mathcal{F}_{r+v-2k-2\ell}^{a,b}(\xi). \quad (50)$$

Grouping terms carrying the same polynomial index yields

$$\mathcal{Q}_r(\xi) \mathcal{F}_v^{a,b}(\xi) = \left(-\frac{1}{2}\right)^r r! (r-1)! \sum_{q=0}^r \left(\sum_{k=0}^q \frac{a^{2k-r} (-b)^{q-k} \binom{r-2k}{q-k}}{k! (r-k)! (r-2k-1)!} \right) \mathcal{F}_{r+v-2q}^{a,b}(\xi). \quad (51)$$

The inner finite coefficient has the equivalent hypergeometric form

$$\begin{aligned} \mathcal{Q}_r(\xi) \mathcal{F}_v^{a,b}(\xi) &= \left(-\frac{1}{2}\right)^r r! (r-1)! \sum_{q=0}^r \frac{a^{-r} (-b)^q}{q! r! (r-q)!} \left[\frac{{}_3F_0 \left(\begin{matrix} 1-q, 1-r, 1+q-r \\ - \end{matrix} \middle| \frac{a^2}{b} \right)}{b} \right. \\ &\quad \left. + r {}_3F_0 \left(\begin{matrix} -q, q-r, -r \\ - \end{matrix} \middle| \frac{a^2}{b} \right) \right] \mathcal{F}_{r+v-2q}^{a,b}(\xi). \end{aligned}$$

□

Theorem 6. For $r \in \mathbb{N}$ and $v \in \mathbb{Z}$, the following identity holds:

$$\begin{aligned} \mathcal{Q}_r(\xi) \mathcal{L}_v^{\bar{a}, \bar{b}}(\xi) &= \left(-\frac{1}{2}\right)^r r! (r-1)! \sum_{q=0}^r \frac{\bar{a}^{-r} (-\bar{b})^q}{q! r! (r-q)!} \left[\frac{{}_3F_0 \left(\begin{matrix} 1-q, 1-r, 1+q-r \\ - \end{matrix} \middle| \frac{\bar{a}^2}{\bar{b}} \right)}{\bar{b}} \right. \\ &\quad \left. + r {}_3F_0 \left(\begin{matrix} -q, q-r, -r \\ - \end{matrix} \middle| \frac{\bar{a}^2}{\bar{b}} \right) \right] \mathcal{L}_{r+v-2q}^{\bar{a}, \bar{b}}(\xi). \end{aligned} \quad (52)$$

Proof. The claim follows by repeating the finite-series substitution and moment expansion used for Theorem 5. □

Theorem 7. Let r be positive and v be any integer. Then

$$\begin{aligned} \mathcal{Q}_r(\xi) C_v(\xi) &= (-1)^r r! (r-1)! \sum_{q=0}^r \frac{4^{-r}}{q! r! (r-q)!} \left[-8 q r (r-q) {}_3F_0 \left(\begin{matrix} 1-q, 1-r, 1+q-r \\ - \end{matrix} \middle| -4 \right) \right. \\ &\quad \left. + r {}_3F_0 \left(\begin{matrix} -q, q-r, -r \\ - \end{matrix} \middle| -4 \right) \right] C_{r+v-2q}(\xi). \end{aligned} \quad (53)$$

Proof. Use the recurrence and initial values of the selected Chebyshev kind, then perform the summation rearrangement from Theorem 5. □

5. DIFFERENTIAL RELATIONS AMONG POLYNOMIAL FAMILIES

This section connects derivatives of several polynomial families to the Mott basis and then derives inverse derivative transformations.

5.1. Forward transformations into the Mott basis. We first give finite Mott representations for selected higher-order derivatives.

Theorem 8. Suppose $r, q \in \mathbb{N}$ with $q \geq r$. For the Hermite family,

$$D^r H_q(\xi) = (-1)^{q-r} 2^{2q-r} q! \sum_{\ell=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{(-1)^\ell 2^{-2\ell} (q-r-\ell-1)! {}_1F_1(-\ell; 1-q+\ell+r; \frac{1}{4})}{\ell! (q-r-2\ell-1)! (q-r-2\ell)!} \times \mathcal{Q}_{q-r-2\ell}(\xi). \quad (54)$$

Proof. Differentiate the Hermite power sum term by term to obtain

$$D^r H_q(\xi) = q! \sum_{s=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{(-1)^s 2^{q-2s}}{s! (q-r-2s)!} \xi^{q-r-2s}. \quad (55)$$

Now use (19) to replace every monomial by Mott polynomials:

$$D^r H_q(\xi) = q! \sum_{s=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{(-1)^s 2^{q-2s}}{s! (q-r-2s)!} \times \sum_{\tau=0}^{\lfloor \frac{q-r}{2} \rfloor - s} \frac{(-1)^{q-r+\tau} 2^{q-r-2s-2\tau} (q-r-2s-2\tau)_{2\tau+1} (1+q-r-2s-2\tau)_{\tau-1}}{\tau!} \mathcal{Q}_{q-r-2s-2\tau}(\xi). \quad (56)$$

Collecting terms with a common Mott index gives the equivalent expression

$$D^r H_q(\xi) = q! \sum_{\ell=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{(-1)^{q-r+\ell}}{(q-r-2\ell)! (q-r-2\ell-1)!} \times \sum_{s=0}^{\ell} \frac{2^{2q-r-2(\ell+s)} (q-r-\ell-s-1)!}{(\ell-s)! s!} \mathcal{Q}_{q-r-2\ell}(\xi), \quad (57)$$

and therefore we have

$$D^r H_q(\xi) = (-1)^{q-r} 2^{2q-r} q! \sum_{\ell=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{(-1)^\ell 2^{-2\ell} (q-r-\ell-1)! {}_1F_1(-\ell; 1-q+\ell+r; \frac{1}{4})}{\ell! (q-r-2\ell-1)! (q-r-2\ell)!} \times \mathcal{Q}_{q-r-2\ell}(\xi).$$

The asserted identity follows. $\square$

Theorem 9. For $r, q \in \mathbb{N}$ and $q \geq r$, the normalized ultraspherical derivative expands as

$$D^r G_q^{(\lambda)}(\xi) = \frac{(-1)^{q-r} 2^{-1+2q-r+2\lambda} q! \Gamma(q+\lambda) \Gamma(\lambda + \frac{1}{2})}{\sqrt{\pi} \Gamma(q+2\lambda)} \times \sum_{\ell=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{(-1)^\ell 2^{-2\ell} (q-\ell-r-1)! {}_1F_2 \left(\begin{matrix} -\ell \\ 1-q+\ell+r, 1-q-\lambda \end{matrix} \middle| -\frac{1}{4} \right)}{\ell! (q-2\ell-r-1)! (q-2\ell-r)!} \mathcal{Q}_{q-r-2\ell}(\xi). \quad (58)$$

Proof. Differentiate the ultraspherical finite series and invoke (19); the remaining collection of indices is identical to that in Theorem 8. $\square$

Theorem 10. Let $r, q \in \mathbb{N}$ with $q \geq r$. The generalized Fibonacci and Lucas families in (10)–(11) then have the derivative representations

$$\begin{aligned} D^r \mathcal{F}_q^{a,b}(\xi) &= (-1)^{q-r} 2^{q-r} a^q q! \sum_{\ell=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{(-1)^\ell 2^{-2\ell} (q-\ell-r-1)!}{\ell! (q-2\ell-r-1)! (q-2\ell-r)!} \\ &\quad \times {}_1F_2 \left(\begin{matrix} -\ell \\ -q, 1-q+\ell+r \end{matrix} \middle| \frac{b}{a^2} \right) \mathcal{Q}_{q-r-2\ell}(\xi), \end{aligned} \quad (59)$$

$$\begin{aligned} D^r \mathcal{L}_q^{\bar{a},\bar{b}}(\xi) &= (-1)^{q-r} 2^{q-r} \bar{a}^q q! \sum_{\ell=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{(-1)^\ell 2^{-2\ell} (q-\ell-r-1)!}{\ell! (q-2\ell-r-1)! (q-2\ell-r)!} \\ &\quad \times {}_1F_2 \left(\begin{matrix} -\ell \\ 1-q, 1-q+\ell+r \end{matrix} \middle| \frac{\bar{b}}{\bar{a}^2} \right) \mathcal{Q}_{q-r-2\ell}(\xi). \end{aligned} \quad (60)$$

Proof. Insert the power formulas for both generalized families and repeat the Mott inversion and index collection used in Theorem 8. $\square$

Theorem 11. Assume again that $r, q \in \mathbb{N}$ and $q \geq r$. The convolved Fibonacci and convolved Pell systems from (14) and (15) admit the following derivative expansions in the Mott basis:

$$\begin{aligned} D^r \mathcal{CF}_q^\kappa(\xi) &= \frac{\Gamma(q+\kappa)}{\Gamma(\kappa)} \sum_{\ell=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{(-1)^{q+\ell-r} 2^{q-2\ell-r} (q-\ell-r-1)!}{\ell! (q-2\ell-r-1)! (q-2\ell-r)!} \\ &\quad \times {}_1F_2 \left(\begin{matrix} -\ell \\ 1-q+\ell+r, 1-q-\kappa \end{matrix} \middle| 1 \right) \mathcal{Q}_{q-r-2\ell}(\xi), \end{aligned} \quad (61)$$

$$\begin{aligned} D^r \mathcal{CP}_q^\sigma(\xi) &= \frac{\Gamma(q+\sigma)}{\Gamma(\sigma)} \sum_{\ell=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{(-1)^{q+\ell-r} 2^{2q-2\ell-r} (q-\ell-r-1)!}{\ell! (q-2\ell-r-1)! (q-2\ell-r)!} \\ &\quad \times {}_1F_2 \left(\begin{matrix} -\ell \\ 1-q-\sigma, 1-q+\ell+r \end{matrix} \middle| \frac{1}{4} \right) \mathcal{Q}_{q-r-2\ell}(\xi). \end{aligned} \quad (62)$$

Proof. Substitute the two convolved power expansions and apply the inversion argument established in Theorem 8. $\square$

5.2. Reverse transformations to classical bases. In the opposite direction, derivatives of Mott polynomials are expanded in familiar classical bases.

Theorem 12. For $q, r \in \mathbb{N}$ with $q \geq r$, the Bernoulli-basis representation is

$$\begin{aligned} D^r \mathcal{Q}_q(\xi) = & \left(-\frac{1}{2} \right)^q q! \left[\sum_{\ell=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{{}_4F_1 \left(\begin{matrix} -\ell, 1 - \frac{q}{2}, -q, -\frac{1}{2} - \ell \\ -\frac{q}{2} \end{matrix} \middle| -4 \right)}{(2\ell+1)!(q-2\ell-r)!} B_{q-r-2\ell}(\xi) \right. \\ & \left. + \sum_{\ell=0}^{\lfloor \frac{q-r-1}{2} \rfloor} \frac{(-2+q-2\ell)(q-1)!}{(\ell+1)!(q-\ell-1)!} + \frac{{}_4F_1 \left(\begin{matrix} -\ell-1, 1 - \frac{q}{2}, -q, -\frac{1}{2} - \ell \\ -\frac{q}{2} \end{matrix} \middle| -4 \right)}{(2\ell+2)!} B_{q-r-2\ell-1}(\xi) \right]. \end{aligned} \quad (63)$$

Proof. Differentiating (2) term by term gives

$$D^r \mathcal{Q}_q(\xi) = \left(-\frac{1}{2} \right)^q (q-1)! q! \sum_{s=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{(1+q-r-2s)_r}{(q-2s-1)!(q-s)! s!} \xi^{q-r-2s}. \quad (64)$$

Next substitute the Bernoulli monomial inversion from [35]. This yields

$$D^r \mathcal{Q}_q(\xi) = \left(-\frac{1}{2} \right)^q (q-1)! q! \sum_{s=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{(q-2s)}{(q-r-2s+1)!(q-s)! s!} \sum_{\tau=0}^{q-2s-r} \binom{1+q-r-2s}{q-r-2s-\tau} B_{q-r-2s-\tau}(\xi). \quad (65)$$

Separating the resulting terms according to parity gives

$$\begin{aligned} D^r \mathcal{Q}_q(\xi) = & \left(-\frac{1}{2} \right)^q (q-1)! q! \left[\sum_{\ell=0}^{\lfloor \frac{q-r}{2} \rfloor} \sum_{s=0}^{\ell} \frac{\binom{1+q-r-2s}{q-2\ell-r}}{(1+q-r-2s)(q-2s-1)!(q-s)! s!} B_{q-r-2\ell}(\xi) \right. \\ & \left. + \sum_{\ell=0}^{\lfloor \frac{q-r-1}{2} \rfloor} \sum_{s=0}^{\ell} \frac{\binom{1+q-r-2s}{q-2\ell-r-1}}{(1+q-r-2s)(q-2s-1)!(q-s)! s!} B_{q-r-2\ell-1}(\xi) \right]. \end{aligned} \quad (66)$$

Interchanging the finite sums and evaluating their coefficients leads to

$$D^r \mathcal{Q}_q(\xi) = \left(-\frac{1}{2} \right)^q q! \left[\sum_{\ell=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{{}_4F_1 \left(\begin{matrix} -\ell, 1 - \frac{q}{2}, -q, -\frac{1}{2} - \ell \\ -\frac{q}{2} \end{matrix} \middle| -4 \right)}{(2\ell+1)!(q-2\ell-r)!} B_{q-r-2\ell}(\xi) \right]$$

$$+ \left[\sum_{\ell=0}^{\lfloor \frac{q-r-1}{2} \rfloor} \frac{(-2+q-2\ell)(q-1)!}{(\ell+1)!(q-\ell-1)!} + \frac{{}_4F_1 \left(\begin{matrix} -\ell-1, 1-\frac{q}{2}, -q, -\frac{1}{2}-\ell \\ -\frac{q}{2} \end{matrix} \middle| -4 \right)}{(2\ell+2)!} B_{q-r-2\ell-1}(\xi) \right].$$

Hence the required expansion follows. $\square$

Theorem 13. For $r, q \in \mathbb{N}$ and $q \geq r$, the normalized ultraspherical representation is

$$\begin{aligned} D^r \mathcal{Q}_q(\xi) &= \frac{(-1)^q 2^{1-2q+r-2\lambda} q! \sqrt{\pi}}{\Gamma(\frac{1}{2} + \lambda)} \left[\sum_{\ell=0}^{\lfloor \frac{q-r}{2} \rfloor} \frac{(q-2\ell-r+\lambda) \Gamma(q-2\ell-r+2\lambda)}{\ell! (q-2\ell-r)! \Gamma(1+q-\ell-r+\lambda)} \right. \\ &\quad \left. \times {}_4F_1 \left(\begin{matrix} -\ell, 1-\frac{q}{2}, -q, -q+\ell+r-\lambda \\ -\frac{q}{2} \end{matrix} \middle| -4 \right) G_{q-r-2\ell}^{(\lambda)}(\xi) \right]. \end{aligned} \quad (67)$$

Proof. Differentiate the Mott finite series, apply the normalized ultraspherical inversion of a monomial, and arrange the indices as in Theorem 9. $\square$

6. CLOSED INTEGRAL EVALUATIONS

The preceding algebraic expansions now yield explicit values for several integrals involving Mott polynomials.

6.1. Integrals of Mott moments.

Corollary 3. For each $p \in \mathbb{N}$,

$$\int_0^1 \mathcal{Q}_p(\xi) d\xi = \begin{cases} \frac{2^{-p} {}_3F_0 \left(-\frac{1}{2} - \frac{p}{2}, 1 - \frac{p}{2}, -p \middle| -4 \right)}{p+1}, & \text{if } p \text{ is even,} \\ -\frac{(p-1)! \Gamma\left(\frac{2+p}{2}\right)}{\sqrt{\pi} \left(\frac{1+p}{2}\right)!} - \frac{2^{-p} {}_3F_0 \left(-\frac{1}{2} - \frac{p}{2}, 1 - \frac{p}{2}, -p \middle| -4 \right)}{p+1}, & \text{if } p \text{ is odd.} \end{cases} \quad (68)$$

Proof. With $r = 0$, formula (63) reduces to the following Mott–Bernoulli connection:

$$\mathcal{Q}_p(\xi) = \left(-\frac{1}{2}\right)^p p! \sum_{\ell=0}^{\lfloor \frac{p}{2} \rfloor} \frac{{}_4F_1 \left(\begin{matrix} -\ell, 1-\frac{p}{2}, -p, -\frac{1}{2}-\ell \\ -\frac{p}{2} \end{matrix} \middle| -4 \right)}{(2\ell+1)!(p-2\ell)!} B_{p-2\ell}(\xi)$$

$$+ \left(-\frac{1}{2}\right)^p p! \sum_{\ell=0}^{\left[\frac{p-1}{2}\right]} \frac{(-2+p-2\ell)(p-1)!}{(\ell+1)!(p-\ell-1)!} + \frac{{}_4F_1\left(\begin{matrix} -\ell-1, 1-\frac{p}{2}, -p, -\frac{1}{2}-\ell \\ -\frac{p}{2} \end{matrix} \middle| -4\right)}{(2\ell+2)!} B_{p-2\ell-1}(\xi). \quad (69)$$

We shall use the Bernoulli integral

$$\int_0^1 B_r(\xi) d\xi = \delta_{r,0} = \begin{cases} 1, & r = 0, \\ 0, & r \geq 1. \end{cases}$$

Replace p with $2p$ in (69), integrate the finite sum, and use the preceding Bernoulli relation. Thus

$$\int_0^1 \mathcal{Q}_{2p}(\xi) d\xi = (2p)! 2^{-2p} \sum_{\ell=0}^p \frac{{}_4F_1\left(\begin{matrix} -\ell, 1-p, -2p, -\frac{1}{2}-\ell \\ -p \end{matrix} \middle| -4\right)}{(2\ell+1)!(2p-2\ell)!} \delta_{p-\ell,0}. \quad (70)$$

Only the surviving Kronecker-delta term remains, so

$$\int_0^1 \mathcal{Q}_{2p}(\xi) d\xi = \frac{2^{-2p} {}_3F_0\left(-\frac{1}{2}-p, 1-p, -2p \middle| -4\right)}{2p+1}. \quad (71)$$

For an odd index, the analogous calculation gives

$$\int_0^1 \mathcal{Q}_{2p+1}(\xi) d\xi = -\frac{(2p)! \Gamma\left(\frac{3}{2}+p\right)}{\sqrt{\pi} (p+1)!} - \frac{4^{-p-1} {}_3F_0\left(-1-2p, -1-p, \frac{1}{2}-p \middle| -4\right)}{p+1}. \quad (72)$$

Equations (71) and (72) may be summarized by

$$\int_0^1 \mathcal{Q}_p(\xi) d\xi = \mathfrak{h}_p,$$

where the parity-dependent coefficient $\mathfrak{h}_p$ is

$$\mathfrak{h}_p = \begin{cases} \frac{2^{-p} {}_3F_0\left(-\frac{1}{2}-\frac{p}{2}, 1-\frac{p}{2}, -p \middle| -4\right)}{p+1}, & \text{if } p \text{ is even,} \\ -\frac{(p-1)! \Gamma\left(\frac{2+p}{2}\right)}{\sqrt{\pi} \left(\frac{1+p}{2}\right)!} - \frac{2^{-p} {}_3F_0\left(-\frac{1}{2}-\frac{p}{2}, 1-\frac{p}{2}, -p \middle| -4\right)}{p+1}, & \text{if } p \text{ is odd.} \end{cases} \quad (73)$$

□

Corollary 4. *The associated Mott moment has the following integral:*

$$\int_0^1 \xi^s \mathcal{Q}_q(\xi) d\xi = (-1)^s 2^s (q+s)! \sum_{\ell=0}^{\lfloor \frac{q+s}{2} \rfloor} \frac{(-1)^\ell 2^{-2\ell} (q+s-\ell-1)!}{(q+s-2\ell-1)! (q+s-2\ell)! \ell!} \times {}_4F_3 \left(\begin{matrix} -\ell, \frac{1}{2} - \frac{q}{2}, 1 - \frac{q}{2}, -q \\ -\frac{1}{2}(q+s), \frac{1}{2} - \frac{q+s}{2}, 1 - q - s + \ell \end{matrix} \middle| 1 \right) \mathfrak{h}_{q+s-2\ell}, \quad (74)$$

where $\mathfrak{h}_p$ is the quantity defined in (73).

Proof. Integrate the finite expansion from formula (33); each Mott term is then evaluated using (68). $\square$

We conclude the section with products of a Mott polynomial and each Chebyshev kind.

6.2. Mixed Mott–Chebyshev integral formulas. The elementary Chebyshev integrals needed for these product evaluations are collected first.

Lemma 2. *For every $v \in \mathbb{Z}$, the following four integral identities hold:*

$$\int_{\alpha_0}^{\beta_0} T_v(\xi) d\xi = \mathcal{J}_v^{(T)} = \begin{cases} \beta_0 - \alpha_0, & v = 0, \\ \frac{\beta_0^2 - \alpha_0^2}{2}, & v \in \{-1, 1\}, \\ \frac{\alpha_0(v+1)T_v(\alpha_0) - \beta_0(v+1)T_v(\beta_0) + v(T_{v+1}(\beta_0) - T_{v+1}(\alpha_0))}{v^2 - 1}, & v \in \mathbb{Z} \setminus \{-1, 0, 1\}, \end{cases} \quad (75)$$

$$\int_{\alpha_0}^{\beta_0} U_v(\xi) d\xi = \mathcal{J}_v^{(U)} = \begin{cases} 0, & v = -1, \\ \frac{T_{v+1}(\beta_0) - T_{v+1}(\alpha_0)}{v+1}, & v \in \mathbb{Z} \setminus \{-1\}, \end{cases} \quad (76)$$

$$\int_{\alpha_0}^{\beta_0} V_v(\xi) d\xi = \mathcal{J}_v^{(V)} = \begin{cases} \beta_0 - \alpha_0, & v \in \{-1, 0\}, \\ \frac{T_v(\alpha_0) - T_v(\beta_0)}{v} + \frac{T_{v+1}(\beta_0) - T_{v+1}(\alpha_0)}{v+1}, & v \in \mathbb{Z} \setminus \{-1, 0\}, \end{cases} \quad (77)$$

$$\int_{\alpha_0}^{\beta_0} W_v(\xi) d\xi = \mathcal{J}_v^{(W)}$$

$$= \begin{cases} \beta_0 - \alpha_0, & v = 0, \\ \alpha_0 - \beta_0, & v = -1, \\ \frac{T_v(\beta_0) - T_v(\alpha_0)}{v} + \frac{T_{v+1}(\beta_0) - T_{v+1}(\alpha_0)}{v+1}, & v \in \mathbb{Z} \setminus \{-1, 0\}. \end{cases} \quad (78)$$

Proof. Apply the structure relation of the relevant Chebyshev kind and evaluate it at the two endpoints. $\square$

Corollary 5. For the first-kind polynomial $T_v(\xi)$,

$$\int_{\alpha_0}^{\beta_0} \mathcal{Q}_r(\xi) T_v(\xi) d\xi = (-1)^r r!(r-1)! \times \sum_{q=0}^r \frac{4^{-r} (-8qr(r-q) {}_3F_0(1-q, 1-r, 1+q-r; -4) + r {}_3F_0(-q, q-r, -r; -4))}{q! r! (r-q)!} \mathcal{J}_{r+v-2q}^{(T)}, \quad (79)$$

where $\mathcal{J}_v^{(T)}$ is specified in (75).

Proof. Integrate (53) termwise and evaluate the resulting factors through (75). $\square$

Corollary 6. For the second-kind polynomial $U_v(\xi)$, the result is

$$\int_{\alpha_0}^{\beta_0} \mathcal{Q}_r(\xi) U_v(\xi) d\xi = (-1)^r r!(r-1)! \times \sum_{q=0}^r \frac{4^{-r} (-8qr(r-q) {}_3F_0(1-q, 1-r, 1+q-r; -4) + r {}_3F_0(-q, q-r, -r; -4))}{q! r! (r-q)!} \mathcal{J}_{r+v-2q}^{(U)}, \quad (80)$$

where $\mathcal{J}_v^{(U)}$ is defined in (76).

Proof. Use the finite expansion (53), integrate, and substitute (76). $\square$

Corollary 7. The third-kind polynomial $V_v(\xi)$ satisfies

$$\int_{\alpha_0}^{\beta_0} \mathcal{Q}_r(\xi) V_v(\xi) d\xi = (-1)^r r!(r-1)! \times \sum_{q=0}^r \frac{4^{-r} (-8qr(r-q) {}_3F_0(1-q, 1-r, 1+q-r; -4) + r {}_3F_0(-q, q-r, -r; -4))}{q! r! (r-q)!} \mathcal{J}_{r+v-2q}^{(V)}, \quad (81)$$

where $\mathcal{J}_v^{(V)}$ is the value in (77).

Proof. After integrating (53), evaluate every term by (77). $\square$

Corollary 8. For the fourth-kind polynomial $W_v(\xi)$,

$$\int_{\alpha_0}^{\beta_0} Q_r(\xi) W_v(\xi) d\xi = (-1)^r r! (r-1)! \times \sum_{q=0}^r \frac{4^{-r} (-8qr(r-q) {}_3F_0(1-q, 1-r, 1+q-r; -4) + r {}_3F_0(-q, q-r, -r; -4))}{q! r! (r-q)!} \mathcal{J}_{r+v-2q}^{(W)}, \quad (82)$$

where $\mathcal{J}_v^{(W)}$ is defined in (78).

Proof. Integrate (53) and invoke (78). □

7. CONCLUSIONS AND PROSPECTIVE APPLICATIONS

This paper investigated the Mott polynomials from a theoretical point of view. First, an inverse monomial formula was developed and used to establish, along with the series form, many new formulas regarding these polynomials. Closed expressions were obtained for moments, derivatives, iterated integrations, and both standard and mixed product expansions. Derivative connections with six familiar polynomial classes were derived in both directions. Finite hypergeometric coefficients were simplified through recurrences generated by symbolic summation, and the resulting identities yielded explicit integral evaluations. A natural continuation would be to construct operational matrices from these relations and assess the Mott basis in spectral discretizations of differential and integral equations.

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