

A NEW FLEXIBLE XLINDLEY DISTRIBUTION WITH APPLICATIONS TO REAL DATA IN MEDICINE

MOHAMED KOUADRIA¹, HALIM ZEGHDOUDI^{2,*}, HASSAN ALSUHABI³, IBRAHIM HASSAN ALKHAIRY³, LAXMI PRASAD SAPKOTA⁴, ESLAM HUSSAM^{5,6}, AHMED M. GEMEAY⁷

¹Laboratory of Pure and Applied Mathematics (LMPA), Faculty of Sciences, Mouloud Mammeri University, B.P. N°17 RP, 15000 Tizi-Ouzou, Algeria

²LaPS Laboratory, Badji Mokhtar-Annaba University, Annaba 23000, Algeria

³Department of Mathematics, Al-Qunfudah University College, Umm Al-Qura University, Mecca, Saudi Arabia

⁴Department of Statistics, Tribhuvan University, Tribhuvan Multiple Campus, Palpa, Nepal

⁵Department of Accounting, College of Business Administration in Hawtat Bani Tamim, Prince Sattam bin Abdulaziz University, Hawtat Bani Tamim, Saudi Arabia

⁶Department of Mathematics, Faculty of Science, Capital University, Cairo, Egypt

⁷Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt

*Corresponding author: halim.zeghdoudi@univ-annaba.dz

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ABSTRACT. This paper presents the beta-new XLindley (BNXL) distribution, an innovative three-parameter extension of the XLindley model designed to enhance flexibility in the modeling of various lifetime and reliability data. The statistical properties of the BNXL, encompassing the moment-generating function, moments, entropy, stress-strength reliability, and order statistics, are derived and analyzed comprehensively. Parameters are estimated using various methods, including maximum likelihood, least squares, and the Anderson-Darling method. The model's efficacy is assessed using two disparate medical datasets: luteinizing hormone concentrations and Marburg virus recuperation durations. A comparative analysis of multiple established lifetime distributions reveals that the BNXL consistently offers a superior fit for the two real datasets across various measures, including the Akaike information criterion. The findings suggest that the BNXL distribution is a multifaceted instrument with prospective applications in medicine, biology, survival analysis, and related fields.

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1. INTRODUCTION

Modeling and analyzing survival data are crucial in applied fields such as medicine, engineering, and finance, where selecting an appropriate probability model is vital for reliable statistical inference. Over the years, numerous lifetime distributions have been proposed; however, empirical data often exhibit features such as heavy tails, skewness, or complex hazard rate shapes that classical models fail to adequately capture. This gap has motivated the development of flexible distribution families and robust statistical methodologies. For more details about new statistical models, see a new asymmetric modified Topp-Leone distribution [13], the new hybrid Weibull-inverse Weibull distribution [10], power Lambert uniform distribution [32], truncated inverse power Lindley distribution [14], odd reparameterized exponential transformed-X family of distributions [30], a new extended Fréchet model [15], the new version of Gompertz distribution [12], novel Kumaraswamy power Fréchet distribution [29], new heavy-tailed statistical model [40], an extended Rayleigh Weibull distribution [11], generalized linear model [27], hyperbolic sine-Rayleigh distribution [28], inverse power and truncated inverse power Ailamujia distributions [31], and heavy tailed power XLindley distribution [6]. Among these developments, beta-generalized distributions have emerged as a powerful and adaptable framework for enhancing baseline models. By embedding a given cumulative distribution function (CDF) into the beta distribution, one obtains models that can represent a wide variety of shapes for both the distribution and the hazard rate. Applications span reliability analysis, environmental science, and biomedical research. Examples include the beta-normal [8], beta-Fréchet [16], beta-Gumbel [17], beta-exponential [43], beta-generalized half-normal [20], beta-generalized Rayleigh [5], beta-generalized exponential [2], and beta Lindley [1]. These works demonstrate the beta family's ability to generalize standard models while retaining their tractable properties. Before introducing our proposed extension, it is essential to recall the *XLindley distribution* (XL), originally proposed by Chouia and Zeghdoudi [3] due to its properties and practical applications. The XLindley provides greater flexibility in hazard shapes compared to the Lindley. The density function of the XLindley is given by:

$$f_{XL}(x; \theta) = \begin{cases} \frac{\theta^2 (2 + \theta + x)}{(1 + \theta)^2} e^{-\theta x}, & x > 0, \theta > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Its recent extensions, including:

- a transmutation-based variant (e.g., the transmuted Lindley distribution [33]),
- odd log-logistic type extension [36],
- Marshall–Olkin family inspired variant [37].

The new XLindley distribution (NXLD) [18] refines the XLindley by modifying its CDF and PDF to better capture diverse hazard structures, and has demonstrated superior performance compared

to well-known lifetime models such as Weibull, gamma, exponential, Zeghdoudi [42], Akash [21], Lindley [9], Chris-Jerry [19], and Xgamma [44]. The CDF and PDF of the NXLD are:

$$F_{NXL}(x, \theta) = 1 - \left(\frac{x\theta}{2} + 1 \right) \exp(-x\theta), \quad f_{NXL}(x, \theta) = \frac{\theta}{2} (1 + x\theta) \exp(-x\theta), \quad x, \theta > 0. \quad (2)$$

with the PDF interpretable as an equal-weight mixture of $\text{Exp}(\theta)$ and $\text{Gamma}(2, \theta)$ distributions. In this paper, we propose the *beta-new XLindley* (BNXL) distribution, a three-parameter extension of the NXLD based on the beta-generalization scheme. Compared with other XLindley extensions, the BNXL uniquely combines:

- Enhanced flexibility: two additional shape parameters allow for modeling a wide spectrum of hazard rate behaviors, from increasing/decreasing to bathtub- and upside-down bathtub-shaped.
- Mathematical tractability: closed-form expressions for moments, entropy, and reliability measures remain available.
- Interpretability: the model preserves the intuitive structure of the NXLD while expanding its applicability.
- Practical performance: the linear-form representation of the PDF facilitates parameter estimation and application to both small and large datasets.

The remainder of this paper is organized as follows: Section 2 formally defines the BNXL distribution. Section 3 investigates its main distributional properties. Section 4 describes parameter estimation methods. Section 5 presents simulation results. Section 6 demonstrates applications to two real datasets. Section 7 concludes with remarks and future research directions.

2. BETA NEW XLINDLEY DISTRIBUTION (BNXL)

Let $F(x)$ be the CDF of a random variable X . Eugene et al. [8] introduced a generalized class of distributions by applying the inverse CDF transformation to a beta-distributed random variable. The resulting CDF is

$$G(x) = \frac{1}{\beta(a, b)} \int_0^{F(x)} t^{a-1} (1-t)^{b-1} dt, \quad 0 < a, b < +\infty. \quad (3)$$

The corresponding PDF is

$$g(x) = \frac{1}{\beta(a, b)} f(x) [F(x)]^{a-1} [1 - F(x)]^{b-1}. \quad (4)$$

We now define the three-parameter beta-new XLindley (BNXL) distribution by taking $G_{BNXL}(x)$ as the CDF of $F_{NXL}(x, \theta)$ (2). The BNXL CDF is

$$\begin{aligned} G_{BNXL}(x) &= I_{F_{NXL}(x, \theta)}(x) = \frac{1}{\beta(a, b)} \int_0^{F_{NXL}(x, \theta)} t^{a-1} (1-t)^{b-1} dt \\ &= \frac{1}{\beta(a, b)} \int_0^{1 - (\frac{x\theta}{2} + 1) \exp(-x\theta)} t^{a-1} (1-t)^{b-1} dt. \end{aligned} \quad (5)$$

This CDF can also be expressed in terms of the hypergeometric function (see Cordeiro and Nadarajah [4]) as

$$G_{BNXL}(x) = \frac{1}{\beta(a, b)} \frac{F_{NXL}^a(x, \theta)}{a} {}_2F_1(a, 1 - b, a + 1, F_{NXL}(x, \theta)), \quad (6)$$

where

$${}_2F_1(a, 1 - b, a + 1, F_{NXL}(x, \theta)) = \sum_{k=0}^{\infty} \frac{(a)_k (1 - b)_k (F_{NXL}(x, \theta))^k}{k! (a + 1)_k}, \quad (7)$$

and $(a)_k$ is the ascending factorial

$$(a)_k = a(a + 1)(a + 2) \cdots (a + k - 1), \quad (a)_0 = 1. \quad (8)$$

For b real and non-integer, the series expansion

$$(1 - z)^{b-1} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(b)}{k! \Gamma(b - k)} z^k, \quad |z| < 1, \quad b > 0, \quad (9)$$

leads to

$$G_{BNXL}(x) = \frac{\Gamma(a + b)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k [F_{NXL}(x, \theta)]^{a+k}}{k! (a + k) \Gamma(b - k)}. \quad (10)$$

Thus, for $b > k$,

$$\begin{aligned} G_{BNXL}(x) &= \frac{\Gamma(a + b)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k [1 - (\frac{x\theta}{2} + 1) \exp(-x\theta)]^{a+k}}{k! (a + k) \Gamma(b - k)} \\ &= \frac{\Gamma(a + b) [1 - (\frac{x\theta}{2} + 1) \exp(-x\theta)]^a}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k [1 - (\frac{x\theta}{2} + 1) \exp(-x\theta)]^k}{k! (a + k) \Gamma(b - k)}. \end{aligned} \quad (11)$$

The PDF of the BNXL distribution is

$$g_{BNXL}(x) = \frac{\left(\frac{1}{2}\right)^b \theta \Gamma(a + b)}{\Gamma(a) \Gamma(b)} (x\theta + 1) (x\theta + 2)^{b-1} \left[1 - \left(\frac{x\theta}{2} + 1\right) e^{-x\theta}\right]^{a-1} e^{-xb\theta}. \quad (12)$$

Remark 1. If $a = b = 1$, Eq. (12) reduces to the PDF of the new XLindley distribution with parameter θ .

Proposition 1. Let $g_{BNXL}(x)$ denote the PDF given in Eq. (12). Then:

- (1) If $0 < a \leq 1$ and $b \geq 1$, the PDF is monotonically decreasing.
- (2) If $a = 1$ and $0 < b < 1$, the PDF is unimodal, with mode

$$x_m = \frac{1/\sqrt{b} - 1}{\theta}.$$

- (3) If $a > 1$, the PDF is increasing near the origin and decreasing for sufficiently large x ; hence, it possesses at least one interior mode.

Proof. Differentiating Eq. (12) with respect to x yields

$$\begin{aligned} \frac{dg_{BNXL}(x)}{dx} &= 2 \left(\frac{1}{2}\right)^{a+b} e^{-bx\theta} \frac{\theta^2 \Gamma(a + b)}{\Gamma(a) \Gamma(b)} \frac{(x\theta + 2)^{b-2}}{(2e^{-x\theta} + x\theta e^{-x\theta} - 2)^2} \\ &\quad \times (2 - x\theta e^{-x\theta} - 2e^{-x\theta})^a L(x), \end{aligned} \quad (13)$$

where

$$L(x) = P_0(x) + P_1(x)e^{-x\theta}, \quad (14)$$

with

$$P_0(x) = 2 - 2b - 4bx\theta - 2bx^2\theta^2, \quad (15)$$

$$P_1(x) = (2a + 2b - 4) + (5a + 5b - 6)\theta x + (4a + 4b - 4)\theta^2 x^2 + (a + b - 1)\theta^3 x^3. \quad (16)$$

All factors in Eq. (13), except $L(x)$, are positive for $x > 0$; therefore, the sign of $g'_{BNXL}(x)$ is determined by $L(x)$.

Equivalently, logarithmic differentiation gives

$$\frac{g'_{BNXL}(x)}{g_{BNXL}(x)} = \theta \left[\frac{1}{1 + \theta x} + \frac{b - 1}{2 + \theta x} - b + (a - 1) \frac{(1 + \theta x)e^{-\theta x}}{2 \left[1 - \left(1 + \frac{\theta x}{2} \right) e^{-\theta x} \right]} \right].$$

If $0 < a \leq 1$ and $b \geq 1$, both the contribution associated with $a - 1$ and the remaining part are non-positive, so $g'_{BNXL}(x) < 0$ for $x > 0$.

For $a = 1$, the derivative simplifies to

$$\frac{g'_{BNXL}(x)}{g_{BNXL}(x)} = \theta \frac{1 - b(1 + \theta x)^2}{(1 + \theta x)(2 + \theta x)}.$$

Hence, for $0 < b < 1$, the derivative changes sign exactly once at

$$x_m = \frac{1/\sqrt{b} - 1}{\theta},$$

which proves unimodality.

Finally, if $a > 1$, then $g'_{BNXL}(x) > 0$ for sufficiently small $x > 0$, whereas

$$\frac{g'_{BNXL}(x)}{g_{BNXL}(x)} \longrightarrow -b\theta < 0, \quad x \rightarrow \infty.$$

Thus, the density has at least one interior mode. □

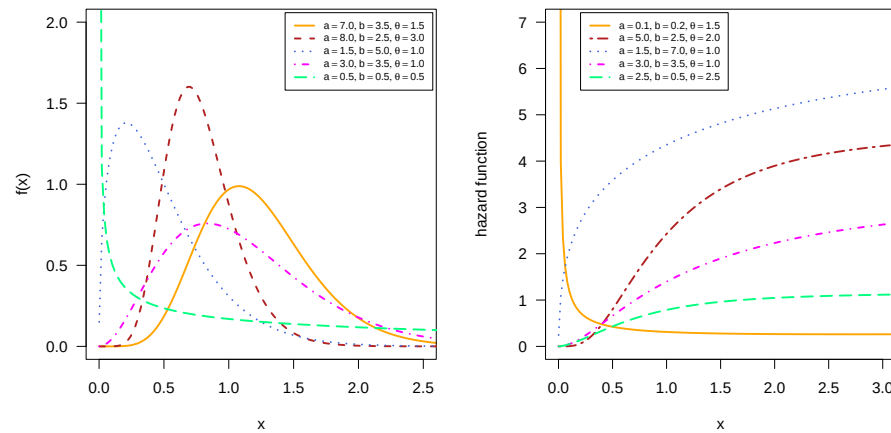


FIGURE 1. Shapes of the PDF and hazard rate function (hrf) of the BNXL distribution. The green PDF curve decreases for $a \leq 1$, whereas others are unimodal for $a > 1$.

Proposition 2. The BNXL PDF can be expressed in the linear form

$$g_{BNXL}(x) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{s=0}^r W_{j,i,r,s} (\theta x^{s+1} + x^s) \exp\{-x\theta(r+1)\}, \quad (17)$$

where

$$W_{j,i,r,s} = \frac{1}{B(a,b)} \left(\frac{\theta}{2}\right)^{s+1} \binom{r}{s} \binom{i}{r} \frac{(-1)^{j+i+r} \Gamma(b+j) \Gamma(a)}{i! \Gamma(b+j-i) j! \Gamma(a-j)}. \quad (18)$$

Proof. From Eq. (12), using the generalized binomial expansion

$$(1-z)^{a-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(a)}{j! \Gamma(a-j)} z^j, \quad |z| < 1, \quad (19)$$

and applying the corresponding binomial expansions to the remaining terms, the BNXL density can be written as

$$g_{BNXL}(x) = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{s=0}^r \left(\frac{\theta}{2}\right)^{s+1} \binom{r}{s} \binom{i}{r} \times \frac{(-1)^{j+i+r} \Gamma(b+j) \Gamma(a)}{i! \Gamma(b+j-i) j! \Gamma(a-j)} (\theta x^{s+1} + x^s) e^{-x\theta(r+1)}. \quad (20)$$

Defining $W_{j,i,r,s}$ as in Eq. (18) immediately gives Eq. (17).

3. STATISTICAL PROPERTIES

This section presents some important statistical properties of the BNXL distribution, including the reliability and hazard rate functions, moments, moment generating function, Rényi entropy, stress-strength reliability, and order statistics.

3.1. Reliability and hazard rate functions. The reliability function is defined by

$$R(t) = 1 - G_{BNXL}(t).$$

Let

$$z(t) = 1 - \left(\frac{t\theta}{2} + 1 \right) e^{-t\theta}.$$

Then

$$R(t) = 1 - \frac{\Gamma(a+b)z(t)^a}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k z(t)^k}{k!(a+k)\Gamma(b-k)}. \quad (21)$$

The hazard rate function is defined by

$$h(t) = \frac{g_{BNXL}(t)}{R(t)} = \frac{g_{BNXL}(t)}{1 - G_{BNXL}(t)}.$$

Therefore,

$$h(t) = \frac{\left(\frac{1}{2}\right)^b \theta \Gamma(a+b)(t\theta+1)(t\theta+2)^{b-1} z(t)^{a-1} e^{-tb\theta}}{\Gamma(a)\Gamma(b) R(t)}. \quad (22)$$

The hazard rate describes the instantaneous risk of failure at time t , conditional on survival up to that time. As illustrated in Figure 1, different hazard-rate shapes can be obtained for suitable parameter values.

3.2. Moments.

Theorem 1. If $X \sim BNXL(a, b, \theta)$, then the k th raw moment is given by

$$E(X^k) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{s=0}^r \frac{W_{j,i,r,s}}{\theta^{k+s+1}(r+1)^{k+s+1}} \left[\frac{\Gamma(k+s+2)}{r+1} + \Gamma(k+s+1) \right]. \quad (23)$$

Proof. Using Proposition 2, we obtain

$$\begin{aligned} E(X^k) &= \int_0^{\infty} x^k g_{BNXL}(x) dx \\ &= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{s=0}^r W_{j,i,r,s} \int_0^{\infty} \left(\theta x^{k+s+1} + x^{k+s} \right) e^{-x\theta(r+1)} dx. \end{aligned} \quad (24)$$

Using

$$\int_0^{\infty} x^{\nu} e^{-\lambda x} dx = \frac{\Gamma(\nu+1)}{\lambda^{\nu+1}}, \quad \lambda > 0,$$

gives

$$E(X^k) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{s=0}^r W_{j,i,r,s} \left[\frac{\theta \Gamma(k+s+2)}{[\theta(r+1)]^{k+s+2}} + \frac{\Gamma(k+s+1)}{[\theta(r+1)]^{k+s+1}} \right], \quad (25)$$

which reduces to Eq. (23).

Remark 2. The first four raw moments follow directly from Eq. (23) by taking $k = 1, 2, 3$, and 4. In particular,

$$\begin{aligned} E(X) &= \sum_{j,i,r,s} \frac{W_{j,i,r,s}}{\theta^{s+2}(r+1)^{s+2}} \left[\frac{\Gamma(s+3)}{r+1} + \Gamma(s+2) \right], \\ E(X^2) &= \sum_{j,i,r,s} \frac{W_{j,i,r,s}}{\theta^{s+3}(r+1)^{s+3}} \left[\frac{\Gamma(s+4)}{r+1} + \Gamma(s+3) \right], \\ E(X^3) &= \sum_{j,i,r,s} \frac{W_{j,i,r,s}}{\theta^{s+4}(r+1)^{s+4}} \left[\frac{\Gamma(s+5)}{r+1} + \Gamma(s+4) \right], \\ E(X^4) &= \sum_{j,i,r,s} \frac{W_{j,i,r,s}}{\theta^{s+5}(r+1)^{s+5}} \left[\frac{\Gamma(s+6)}{r+1} + \Gamma(s+5) \right], \end{aligned}$$

where

$$\sum_{j,i,r,s} = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{s=0}^r.$$

3.3. Moment generating function. The moment generating function of X is defined by

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} g_{BNXL}(x) dx.$$

Using Proposition 2, term-by-term integration gives

$$M_X(t) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{s=0}^r W_{j,i,r,s} \left[\frac{\theta \Gamma(s+2)}{[\theta(r+1) - t]^{s+2}} + \frac{\Gamma(s+1)}{[\theta(r+1) - t]^{s+1}} \right], \quad (26)$$

for values of t for which the above series converges.

3.4. Rényi entropy. The Rényi entropy of order $q > 0, q \neq 1$, is defined by

$$I_R(q) = \frac{1}{1-q} \log \left\{ \int_0^{\infty} g_{BNXL}^q(x) dx \right\}. \quad (27)$$

For a positive integer $q \geq 2$, the same expansion technique used in Proposition 2 gives the following representation.

Proposition 3. For an integer $q \geq 2$,

$$g_{BNXL}^q(x) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{l=0}^r \sum_{m=0}^q W_{j,i,r,l,m}^{(q)} (\theta x)^{m+l} e^{-x\theta(r+q)}, \quad (28)$$

where

$$\begin{aligned} W_{j,i,r,l,m}^{(q)} &= \frac{\theta^q}{2^q [B(a,b)]^q} \binom{r}{l} \binom{q}{m} \binom{i}{r} \left(\frac{1}{2} \right)^l \\ &\times \frac{(-1)^{j+i+r} \Gamma((b-1)q + j + 1) \Gamma((a-1)q + 1)}{i! \Gamma((b-1)q + j + 1 - i) j! \Gamma((a-1)q + 1 - j)}. \end{aligned} \quad (29)$$

Proof. The result follows by applying the generalized binomial expansion successively to the factors in $g_{BNXL}^q(x)$, in the same manner as in Proposition 2.

Theorem 2. For an integer $q \geq 2$, the Rényi entropy of the BNXL distribution is

$$I_R(q) = \frac{1}{1-q} \log \left\{ \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{l=0}^r \sum_{m=0}^q \frac{W_{j,i,r,l,m}^{(q)} \Gamma(m+l+1)}{\theta(r+q)^{m+l+1}} \right\}. \quad (30)$$

Proof. From Proposition 3,

$$\int_0^{\infty} g_{BNXL}^q(x) dx = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{l=0}^r \sum_{m=0}^q W_{j,i,r,l,m}^{(q)} \int_0^{\infty} (\theta x)^{m+l} e^{-x\theta(r+q)} dx. \quad (31)$$

Taking $v = x\theta(r+q)$ gives

$$\int_0^{\infty} (\theta x)^{m+l} e^{-x\theta(r+q)} dx = \frac{\Gamma(m+l+1)}{\theta(r+q)^{m+l+1}}.$$

Substitution into Eq. (27) yields Eq. (30).

For noninteger values of q , the defining integral in Eq. (27) can be evaluated numerically.

3.5. Stress–strength reliability. The stress–strength reliability measures the probability that a component with random strength X withstands a random stress Y . It is defined by

$$R_{SS} = P(Y < X).$$

If X and Y are independent BNXL random variables with parameters (a_1, b_1, θ_1) and (a_2, b_2, θ_2) , respectively, then

$$R_{SS} = \int_0^{\infty} g_{BNXL}(x; a_1, b_1, \theta_1) G_{BNXL}(x; a_2, b_2, \theta_2) dx. \quad (32)$$

3.6. Order statistics. Let $X_1, \dots, X_n$ be a random sample from the BNXL distribution, and let

$$X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$$

denote the corresponding order statistics. The PDF of the j th order statistic is

$$g_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} g_{BNXL}(x) [G_{BNXL}(x)]^{j-1} [1 - G_{BNXL}(x)]^{n-j}, \quad j = 1, \dots, n. \quad (33)$$

Using the BNXL PDF and CDF, we obtain

$$\begin{aligned} g_{X_{(j)}}(x) &= \frac{n!}{(j-1)!(n-j)!} \left[\frac{\left(\frac{1}{2}\right)^b \theta}{B(a, b)} (x\theta + 1)(x\theta + 2)^{b-1} z(x)^{a-1} e^{-xb\theta} \right] \\ &\times \left[\frac{\Gamma(a+b)z(x)^a}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k z(x)^k}{k!(a+k)\Gamma(b-k)} \right]^{j-1} \\ &\times \left[1 - \frac{\Gamma(a+b)z(x)^a}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k z(x)^k}{k!(a+k)\Gamma(b-k)} \right]^{n-j}, \end{aligned} \quad (34)$$

where

$$z(x) = 1 - \left(\frac{x\theta}{2} + 1 \right) e^{-x\theta}.$$

4. ESTIMATION OF THE BNXL DISTRIBUTION PARAMETERS

Several estimation techniques for the Beta New XLindley (BNXL) distribution's parameters are presented in this section. We start with the maximum likelihood estimation (MLE), which under regularity conditions offers effective and asymptotically unbiased estimators. Alternative methods, including least squares, Anderson–Darling, Cramér–von Mises, maximum product spacings, and other variants, are then covered. These approaches help when MLE is sensitive to data features or difficult to estimate. Every method is developed in terms of its goal function, which helps to estimate parameters by numerical optimization. For more details, see [34,35].

4.1. Maximum likelihood estimation (MLE). This section will focus on estimating the proposed model parameter using the maximum likelihood method, MLEs, of the parameters that are inherent within the beta-new XLindley distribution function is obtained as follows. The likelihood function of the observed sample $\underline{x} = \{x_1, \dots, x_n\}$ of size n , is defined as

$$\begin{aligned} L(\underline{x}, a, b, \theta) &= \prod_{i=1}^n g(x_i) \\ &= \prod_{i=1}^n \left[\frac{\left(\frac{1}{2}\right)^b \theta}{B(a, b)} (x_i \theta + 1) (x_i \theta + 2)^{b-1} \left[1 - \left(\frac{x_i \theta}{2} + 1\right) \exp(-x_i \theta)\right]^{a-1} \exp(-x_i b \theta) \right] \\ &= \frac{\left(\frac{1}{2}\right)^{bn} \theta^n}{[B(a, b)]^n} \exp(-b \theta \sum_{i=1}^n x_i) \prod_{i=1}^n (x_i \theta + 1) (x_i \theta + 2)^{b-1} \left[1 - \left(\frac{x_i \theta}{2} + 1\right) \exp(-x_i \theta)\right]^{a-1} \end{aligned}$$

The corresponding log-likelihood function is given by

$$\begin{aligned} l(\underline{x}, a, b, \theta) &= \log \left\{ \frac{\left(\frac{1}{2}\right)^{bn} \theta^n}{[B(a, b)]^n} \exp(-b \theta \sum_{i=1}^n x_i) \prod_{i=1}^n (x_i \theta + 1) (x_i \theta + 2)^{b-1} \left[1 - \left(\frac{x_i \theta}{2} + 1\right) \exp(-x_i \theta)\right]^{a-1} \right\} \\ &= -bn \log(2) + n \log(\theta) - n \log[B(a, b)] - b \theta \sum_{i=1}^n x_i + \sum_{i=1}^n \log(x_i \theta + 1) \\ &\quad + (b-1) \log(x_i \theta + 2) + (a-1) \sum_{i=1}^n \log\left[1 - \left(\frac{x_i \theta}{2} + 1\right) \exp(-x_i \theta)\right] \\ &= n[-b \log(2) + \log(\theta) + \log \Gamma(a+b) - \log \Gamma(a) - \log \Gamma(b)] - b \theta \sum_{i=1}^n x_i \\ &\quad + \sum_{i=1}^n \log(x_i \theta + 1) + (b-1) \log(x_i \theta + 2) + (a-1) \sum_{i=1}^n \log\left[1 - \left(\frac{x_i \theta}{2} + 1\right) \exp(-x_i \theta)\right] \end{aligned}$$

Now, setting

$$\frac{\partial l(\underline{x}, a, b, \theta)}{\partial a}, \quad \frac{\partial l(\underline{x}, a, b, \theta)}{\partial b}, \quad \text{and} \quad \frac{\partial l(\underline{x}, a, b, \theta)}{\partial \theta},$$

we get the following nonlinear system of equations

$$\begin{cases} \frac{\partial l(x, a, b, \theta)}{\partial a} = 0 & \Leftrightarrow n\psi(a+b) - n\psi(a) + \sum_{i=1}^n \log[1 - (\frac{x_i\theta}{2} + 1) \exp(-x_i\theta)] = 0 \\ \frac{\partial l(x, a, b, \theta)}{\partial b} = 0 & \Leftrightarrow -n \log(2) + n\psi(a+b) - n\psi(b) - \theta \sum_{i=1}^n x_i + \sum_{i=1}^n \log(x_i\theta + 2) = 0 \\ \frac{\partial l(x, a, b, \theta)}{\partial \theta} = 0 & \Leftrightarrow \frac{n}{\theta} - b \sum_{i=1}^n x_i + \sum_{i=1}^n \frac{x_i}{x_i\theta+1} + (b-1) \frac{x_i}{x_i\theta+2} + (a-1) \sum_{i=1}^n \frac{\exp(-x_i\theta)(\frac{x_i}{2}(\theta-1)+1)}{[1-(\frac{x_i\theta}{2}+1)\exp(-x_i\theta)]} = 0 \end{cases}$$

where $\psi(\cdot)$ is digamma function. The MLEs $\hat{\lambda} = (\hat{a}, \hat{b}, \hat{\theta})$ of $\lambda = (a, b, \theta)$, respectively, are obtained by solving this nonlinear system of equations. To numerically maximize the sample likelihood function given in the likelihood function, it is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton algorithm.

For interval estimation and tests of hypotheses on the parameters in λ , we require the 3×3 unit expected information matrix, that is

$$I(\lambda) = \begin{pmatrix} \frac{\partial^2 l(x, a, b, \theta)}{\partial a^2} & \frac{\partial^2 l(x, a, b, \theta)}{\partial a \partial b} & \frac{\partial^2 l(x, a, b, \theta)}{\partial a \partial \theta} \\ \frac{\partial^2 l(x, a, b, \theta)}{\partial b \partial a} & \frac{\partial^2 l(x, a, b, \theta)}{\partial b^2} & \frac{\partial^2 l(x, a, b, \theta)}{\partial b \partial \theta} \\ \frac{\partial^2 l(x, a, b, \theta)}{\partial \theta \partial a} & \frac{\partial^2 l(x, a, b, \theta)}{\partial \theta \partial b} & \frac{\partial^2 l(x, a, b, \theta)}{\partial \theta^2} \end{pmatrix},$$

where

$$\begin{aligned} \frac{\partial^2 l(x, a, b, \theta)}{\partial a^2} &= n\psi'(a+b) - n\psi'(a) \\ \frac{\partial^2 l(x, a, b, \theta)}{\partial a \partial b} &= \frac{\partial^2 l(x, a, b, \theta)}{\partial b \partial a} = n\psi'(a+b) \\ \frac{\partial^2 l(x, a, b, \theta)}{\partial a \partial \theta} &= \frac{\partial^2 l(x, a, b, \theta)}{\partial \theta \partial a} = \sum_{i=1}^n \frac{\exp(-x_i\theta)(\frac{x_i}{2}(\theta-1)+1)}{[1-(\frac{x_i\theta}{2}+1)\exp(-x_i\theta)]} \\ \frac{\partial^2 l(x, a, b, \theta)}{\partial b^2} &= n\psi'(a+b) - n\psi'(b) \\ \frac{\partial^2 l(x, a, b, \theta)}{\partial b \partial \theta} &= \frac{\partial^2 l(x, a, b, \theta)}{\partial \theta \partial b} = -\sum_{i=1}^n x_i + \sum_{i=1}^n \frac{x_i}{x_i\theta+2} \\ \frac{\partial^2 l(x, a, b, \theta)}{\partial \theta^2} &= -\frac{n}{\theta^2} - \sum_{i=1}^n \frac{x_i^2}{(x_i\theta+1)^2} + (b-1) \frac{x_i^2}{(x_i\theta+2)^2} + (a-1) \\ &\quad \times \sum_{i=1}^n \left\{ \frac{\exp(-x_i\theta)(x_i - \frac{x_i}{2}\theta - 1) \times [1 - (\frac{x_i\theta}{2} + 1) \exp(-x_i\theta)] - (\exp(-x_i\theta)(\frac{x_i}{2}(\theta-1)+1))^2}{[1 - (\frac{x_i\theta}{2} + 1) \exp(-x_i\theta)]} \right\} \end{aligned}$$

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\lambda} - \lambda)$ is $N(0, I^{-1}(\lambda))$. The estimated asymptotic multivariate normal $N(0, \frac{1}{n}I^{-1}(\hat{\lambda}))$ distribution of $\hat{\lambda}$ can be used to construct approximate confidence intervals for the parameters and the hazard rate and survival functions. An asymptotic confidence interval with significance level α for each parameter λ_k for $k = 1, 2, 3$ is given by:

$$ACI_{100(1-\alpha)} = \left[\hat{\lambda}_k \pm z_{\frac{\alpha}{2}} \sqrt{\hat{I}_{kk}(\lambda)} \right]$$

Where $\hat{I}_{kk}(\lambda)$ is the k th diagonal element of $\frac{1}{n}I^{-1}(\hat{\lambda})$ estimated at $\hat{\lambda}$ for $k = 1, 2, 3$, and $z_{\frac{\alpha}{2}}$ is the quantile $1 - \frac{\alpha}{2}$ of the standard normal distribution. Using R software, we can easily compute the

Hessian matrix (information matrix) and its inverse and hence the standard errors and asymptotic confidence intervals.

To test some BNXL sub-models, we can compute the likelihood ratio (LR) test statistic by maximizing unrestricted and restricted log-likelihood functions. For example, we can use the LR test statistic to check whether the BNXL distribution is statistically superior to the new XLindley distribution for a given data set.

4.2. Least square estimation (LSE). The least-square estimation is an alternative to maximum likelihood estimation. For the BNXL model, the least square estimates (LSEs), $\hat{a}$, $\hat{b}$ and $\hat{\theta}$ of a , b and θ are defined as those arguments that minimize the objective function:

$$Q(a, b, \theta) = \sum_{i=1}^n \left\{ \frac{\Gamma(a+b)[1 - (\frac{x_{(i)}\theta}{2} + 1) \exp(-x_{(i)}\theta)]^a}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k [1 - (\frac{x_{(i)}\theta}{2} + 1) \exp(-x_{(i)}\theta)]^k}{k!(a+k)\Gamma(b-k)} - \frac{i}{n+1} \right\}^2,$$

where $x_{(i)}$ is a possible outcome of the i th order statistic based on a n -points random sample obtained from $X \sim \text{BNXL}(a, b, \theta)$.

The minimum point $\hat{a}, \hat{b}, \hat{\theta}$ can also be given as a solution of the following system of non-linear equations:

$$\frac{dQ(a, b, \theta)}{da} = \frac{dQ(a, b, \theta)}{db} = \frac{dQ(a, b, \theta)}{d\theta} = 0.$$

4.3. Anderson-Darling estimation (ADE). Suppose we have an ordered random sample of size n given by $X_1, X_2, \dots, X_n$, where each X_i is distributed according to a BNXL distribution with parameters a, b, θ . To estimate the BNXL distribution coefficients using the Anderson-Darling estimation approach, we must calculate the minimization of the following equation

$$A = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) [\log(G_{\text{BNXL}}(x_i)) + \log(R(x_i))].$$

4.4. Cramér-Von Mises estimation (CVME). To estimate the BNXL distribution coefficients using the Cramér-von Mises estimation approach, we must calculate the minimization of the following equation

$$C = -\frac{1}{12n} + \sum_{i=1}^n \left[G_{\text{BNXL}}(x_i) - \frac{2i-1}{2n} \right]^2.$$

4.5. Spacings estimation (MPSE). In order to estimate the BNXL distribution coefficients using the spacings estimation approach, we must calculate the maximum of the following equation

$$T = \frac{1}{1+n} \sum_{i=1}^{n+1} \log M_i \quad \text{where} \quad M_i = G_{\text{BNXL}}(x_{(i)}) - G_{\text{BNXL}}(x_{(i-1)}).$$

4.6. Right tail Anderson-Darling estimation (RTADE). In order to estimate the BNXL distribution coefficients using the right-tail Anderson-Darling estimation approach, we must calculate the minimization of the following equation

$$RL = \frac{n}{2} - 2 \sum_{i=1}^n G_{\text{BNXL}}(x_i) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log(R(x_{n+1-i})).$$

4.7. Weighted least-squares estimation (WLSE). To estimate the BNXL distribution coefficients using the weighted least-squares estimation approach, we must calculate the minimization of the following equation

$$W = \sum_{i=1}^n \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[G_{BNXL}(x_i) - \frac{i}{n+1} \right]^2.$$

4.8. The minimum spacing absolute distance estimation (MSADE). In order to estimate the BNXL distribution coefficients using the minimum spacing absolute distance estimation approach, we must calculate the minimization of the following equation

$$\omega_1(x_i) = \sum_{i=1}^{n+1} \left| M_i - \frac{1}{n+1} \right|.$$

4.9. The minimum spacing absolute-log distance estimation (MSALDE). To estimate the BNXL distribution coefficients using the minimum spacing absolute-log distance estimation approach, we must calculate the minimization of the following equation

$$\omega_2(x_i) = \sum_{i=1}^{n+1} \left| \log M_i - \log \frac{1}{n+1} \right|.$$

Remark 3. *The adaptability of the BNXL distribution is a strength; however, it may result in overfitting when utilized with small datasets. This study primarily focused on model formulation and initial empirical evaluation; however, the incorporation of cross-validation and robustness checks is an essential subsequent measure to evaluate and mitigate over fitting.*

5. SIMULATION STUDY

The goal of the simulation experiment is to see how well different estimation methods work at finding the parameters of the BNXL distribution. We performed a comprehensive simulation experiment in this section to assess different estimation methods, such as MLE, ADE, CVME, MPSE, LSE, RTADE, WLSE, MSADE, and MSALDE. We created sample sizes of 35, 100, 250, 400, and 600 from a pool of 1000 Monte Carlo samples for this experiment. Tables 1 to 6 show the bias (BIAS), mean square error (MSE), mean residual error (MRE), and rank values for each method of estimation. These calculations were made using the following formulas

- BIAS: The bias is calculated as the average difference between the estimated parameter ($\hat{\theta}_i$) and the true parameter (θ). $B = \frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta)$.
- MSE: The MSE is the average of the squared differences between the estimated parameter ($\hat{\theta}_i$) and the true parameter (θ). $MSE = \frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta)^2$.
- MRE: The MRE is the average of the absolute differences between the estimated parameter ($\hat{\theta}_i$) and the true parameter (θ). $MRE = \frac{1}{N} \sum_{i=1}^N |\hat{\theta}_i - \theta|$,

where N is the number of simulated data sets, $\hat{\theta}_i$ is the estimated parameter in the i^{th} data set, and θ is the actual value of the parameter. A noticeable pattern can be seen in these tables, showing that Bias,

MSE, and MRE all slowly move closer to zero as the sample size gets bigger. This result confirms the consistent estimability of parameters defining the BNXL distribution.

The simulation results were obtained by using R Programming software, and these results are presented in Tables 1–6, which indicate that all estimators exhibit consistency, with BIAS, MSE, and MRE diminishing as n increases. The overall rankings (Table 6) demonstrate that CVME achieves superior performance on average, closely followed by MSALDE and WLSE. In small samples ($n \leq 100$), WLSE and MSALDE typically produce the lowest errors, while CVME remains competitive. For moderate sample sizes ($100 < n < 400$), CVME and WLSE are excellent options, with MSALDE frequently close behind. In large samples ($n \geq 400$), CVME prevails, and MLE approaches optimal efficiency. Overall, WLSE and MSALDE are advantageous for small datasets, CVME for large datasets, and MLE is efficient when n is substantial. This provides a practical guideline for analysts: for small-scale biomedical studies ($n < 100$), the WLSE method is preferable, whereas for large epidemiological datasets, CVME should be the estimator of choice.

6. APPLICATIONS

In this section, the practical performance of the beta-new XLindley (BNXL) distribution is examined using two real data sets. The BNXL model is compared with several well-known lifetime distributions, namely the exponential, Lindley [41], XLindley [3], new XLindley [18], Xgamma [44], Zeghdoudi [42], two-parameter Lindley I [25], two-parameter Lindley II [26], gamma Lindley [23], new quasi-Lindley [24], beta-Lindley [1], beta-exponential [43], and Chen [22] distributions.

The competing models are fitted by maximum likelihood estimation. Model comparison is based on the Akaike information criterion (AIC), Bayesian information criterion (BIC), minus twice the maximized log-likelihood (-2ℓ), and the corrected Akaike information criterion (AICc). These criteria are defined as

$$\text{AIC} = -2\ell + 2k, \quad \text{BIC} = -2\ell + k \log n,$$

and

$$\text{AICc} = \text{AIC} + \frac{2k(k+1)}{n-k-1},$$

where k denotes the number of estimated parameters and n is the sample size. Smaller values indicate a better trade-off between goodness of fit and model complexity.

Data set 1. The first data set consists of 48 measurements of luteinizing hormone (LH) concentrations obtained from blood samples collected at 10-minute intervals from a healthy adult woman. The data were originally reported by Diggle [7] and have been widely used in biomedical statistical studies. Although the observations were collected sequentially in time, the present analysis concerns the marginal distribution of the observed LH concentrations rather than their temporal dependence or autocorrelation structure.

Table 1: Simulation values of BIAS, MSE and MRE for ($a = 1.5$, $b = 0.5$, $\theta = 0.25$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	MSADE	MSALDE
35	BIAS	$\hat{a}$	0.38907 ^{7}	0.38869 ^{6}	0.40419 ^{9}	0.39846 ^{8}	0.38188 ^{3}	0.3844 ^{4}	0.38111 ^{2}	0.38495 ^{5}	0.36899 ^{1}
		$\hat{b}$	0.17434 ^{5}	0.17496 ^{6}	0.17124 ^{3}	0.17247 ^{4}	0.17819 ^{8}	0.16978 ^{2}	0.16929 ^{1}	0.18243 ^{9}	0.17517 ^{7}
		$\hat{\theta}$	0.09065 ^{9}	0.08752 ^{4}	0.08674 ^{3}	0.08841 ^{6}	0.08914 ^{7}	0.08669 ^{2}	0.08785 ^{5}	0.09004 ^{8}	0.08591 ^{1}
	MSE	$\hat{a}$	0.26235 ^{7}	0.25696 ^{5}	0.28022 ^{9}	0.2751 ^{8}	0.25209 ^{4}	0.25197 ^{3}	0.24924 ^{2}	0.25908 ^{6}	0.23804 ^{1}
		$\hat{b}$	0.04296 ^{5}	0.04435 ^{7}	0.042 ^{3}	0.04206 ^{4}	0.04516 ^{8}	0.04135 ^{2}	0.04097 ^{1}	0.04681 ^{9}	0.04379 ^{6}
		$\hat{\theta}$	0.01258 ^{8}	0.01199 ^{5}	0.01172 ^{2}	0.01211 ^{6}	0.01246 ^{7}	0.01179 ^{3}	0.01194 ^{4}	0.01261 ^{9}	0.01152 ^{1}
	MRE	$\hat{a}$	0.25938 ^{7}	0.25913 ^{6}	0.26946 ^{9}	0.26564 ^{8}	0.25459 ^{3}	0.25627 ^{4}	0.25407 ^{2}	0.25664 ^{5}	0.24599 ^{1}
		$\hat{b}$	0.34867 ^{5}	0.34992 ^{6}	0.34248 ^{3}	0.34494 ^{4}	0.35639 ^{8}	0.33957 ^{2}	0.33858 ^{1}	0.36487 ^{9}	0.35033 ^{7}
		$\hat{\theta}$	0.36261 ^{9}	0.35008 ^{4}	0.34694 ^{3}	0.35366 ^{6}	0.35657 ^{7}	0.34675 ^{2}	0.35138 ^{5}	0.36015 ^{8}	0.34365 ^{1}
	$\sum Ranks$		62 ^{8}	49 ^{5}	44 ^{4}	54 ^{6}	55 ^{7}	24 ^{2}	23 ^{1}	68 ^{9}	26 ^{3}
100	BIAS	$\hat{a}$	0.23603 ^{5}	0.24925 ^{8}	0.23013 ^{1}	0.25015 ^{9}	0.23532 ^{3}	0.24489 ^{6}	0.24638 ^{7}	0.23525 ^{2}	0.23591 ^{4}
		$\hat{b}$	0.17064 ^{5}	0.17119 ^{6}	0.16954 ^{3}	0.17018 ^{4}	0.17338 ^{8}	0.16793 ^{1}	0.16804 ^{2}	0.17716 ^{9}	0.17192 ^{7}
		$\hat{\theta}$	0.08665 ^{8}	0.08494 ^{4}	0.084 ^{2}	0.0855 ^{5}	0.08375 ^{1}	0.08572 ^{6}	0.08613 ^{7}	0.08734 ^{9}	0.08472 ^{3}
	MSE	$\hat{a}$	0.10417 ^{5}	0.1121 ^{7}	0.09423 ^{2}	0.12072 ^{9}	0.0939 ^{1}	0.11008 ^{6}	0.11709 ^{8}	0.10152 ^{3}	0.10368 ^{4}
		$\hat{b}$	0.04066 ^{3}	0.04153 ^{5}	0.04088 ^{4}	0.04157 ^{6}	0.04258 ^{8}	0.04049 ^{2}	0.03956 ^{1}	0.04395 ^{9}	0.04225 ^{7}
		$\hat{\theta}$	0.01198 ^{8}	0.01126 ^{3}	0.01115 ^{2}	0.01151 ^{6}	0.01086 ^{1}	0.01174 ^{7}	0.01131 ^{4.5}	0.012 ^{9}	0.01131 ^{4.5}
	MRE	$\hat{a}$	0.15736 ^{5}	0.16616 ^{8}	0.15342 ^{1}	0.16676 ^{9}	0.15688 ^{3}	0.16326 ^{6}	0.16426 ^{7}	0.15683 ^{2}	0.15727 ^{4}
		$\hat{b}$	0.34127 ^{5}	0.34238 ^{6}	0.33908 ^{3}	0.34036 ^{4}	0.34676 ^{8}	0.33586 ^{1}	0.33607 ^{2}	0.35433 ^{9}	0.34384 ^{7}
		$\hat{\theta}$	0.34661 ^{8}	0.33976 ^{4}	0.33599 ^{2}	0.34199 ^{5}	0.33501 ^{1}	0.34289 ^{6}	0.34453 ^{7}	0.34936 ^{9}	0.3389 ^{3}
	$\sum Ranks$		52 ^{7}	51 ^{6}	20 ^{1}	57 ^{8}	34 ^{2}	41 ^{3}	45.5 ^{5}	61 ^{9}	43.5 ^{4}
250	BIAS	$\hat{a}$	0.14873 ^{6}	0.14257 ^{3}	0.14935 ^{7}	0.14763 ^{5}	0.15409 ^{9}	0.14308 ^{4}	0.13606 ^{1}	0.15046 ^{8}	0.13672 ^{2}
		$\hat{b}$	0.1692 ^{7}	0.16925 ^{8}	0.16786 ^{5}	0.16733 ^{3}	0.17181 ^{9}	0.16242 ^{2}	0.16168 ^{1}	0.16818 ^{6}	0.16734 ^{4}
		$\hat{\theta}$	0.0828 ^{5}	0.08378 ^{9}	0.08121 ^{3}	0.08293 ^{6}	0.08311 ^{7}	0.07999 ^{2}	0.08128 ^{4}	0.08341 ^{8}	0.07887 ^{1}
	MSE	$\hat{a}$	0.03809 ^{5}	0.03538 ^{3}	0.03982 ^{7}	0.03891 ^{6}	0.04138 ^{9}	0.03545 ^{4}	0.03151 ^{1}	0.03997 ^{8}	0.03212 ^{2}
		$\hat{b}$	0.04046 ^{6}	0.04094 ^{8}	0.03967 ^{3}	0.04014 ^{4}	0.04235 ^{9}	0.03806 ^{2}	0.03745 ^{1}	0.04045 ^{5}	0.04082 ^{7}
		$\hat{\theta}$	0.01076 ^{6}	0.01101 ^{8}	0.01032 ^{2}	0.0108 ^{7}	0.01072 ^{4}	0.01033 ^{3}	0.01075 ^{5}	0.01113 ^{9}	0.01013 ^{1}
	MRE	$\hat{a}$	0.09915 ^{6}	0.09505 ^{3}	0.09956 ^{7}	0.09842 ^{5}	0.10272 ^{9}	0.09538 ^{4}	0.09071 ^{1}	0.10031 ^{8}	0.09115 ^{2}
		$\hat{b}$	0.33839 ^{7}	0.33849 ^{8}	0.33572 ^{5}	0.33465 ^{3}	0.34363 ^{9}	0.32484 ^{2}	0.32335 ^{1}	0.33637 ^{6}	0.33468 ^{4}
		$\hat{\theta}$	0.33119 ^{5}	0.33511 ^{9}	0.32484 ^{3}	0.33172 ^{6}	0.33245 ^{7}	0.31996 ^{2}	0.32512 ^{4}	0.33364 ^{8}	0.31547 ^{1}
	$\sum Ranks$		53 ^{6}	59 ^{7}	42 ^{4}	45 ^{5}	72 ^{9}	25 ^{3}	19 ^{1}	66 ^{8}	24 ^{2}
400	BIAS	$\hat{a}$	0.11617 ^{3}	0.11789 ^{6}	0.11335 ^{1}	0.11635 ^{4}	0.1191 ^{8}	0.11909 ^{7}	0.11553 ^{2}	0.11722 ^{5}	0.11991 ^{9}
		$\hat{b}$	0.15942 ^{1}	0.1643 ^{7}	0.16361 ^{6}	0.15974 ^{2}	0.16824 ^{9}	0.16164 ^{4}	0.1606 ^{3}	0.16255 ^{5}	0.16646 ^{8}
		$\hat{\theta}$	0.07623 ^{2}	0.07959 ^{8}	0.07689 ^{3}	0.07458 ^{1}	0.08148 ^{9}	0.07946 ^{6}	0.07778 ^{4}	0.07955 ^{7}	0.07842 ^{5}
	MSE	$\hat{a}$	0.02357 ^{3}	0.02392 ^{5}	0.02171 ^{1}	0.02484 ^{7}	0.02491 ^{8}	0.02466 ^{6}	0.02273 ^{2}	0.02371 ^{4}	0.02576 ^{9}
		$\hat{b}$	0.03709 ^{1}	0.03955 ^{8}	0.03878 ^{6}	0.0377 ^{5}	0.03988 ^{9}	0.03767 ^{4}	0.03712 ^{2}	0.03732 ^{3}	0.0393 ^{7}
		$\hat{\theta}$	0.00925 ^{2}	0.01 ^{6}	0.00941 ^{3}	0.00902 ^{1}	0.01033 ^{9}	0.01008 ^{7}	0.00972 ^{5}	0.01009 ^{8}	0.00957 ^{4}
	MRE	$\hat{a}$	0.07744 ^{3}	0.07859 ^{6}	0.07556 ^{1}	0.07756 ^{4}	0.0794 ^{7.5}	0.0794 ^{7.5}	0.07702 ^{2}	0.07815 ^{5}	0.07994 ^{9}
		$\hat{b}$	0.31885 ^{1}	0.32861 ^{7}	0.32722 ^{6}	0.31947 ^{2}	0.33648 ^{9}	0.32328 ^{4}	0.32119 ^{3}	0.3251 ^{5}	0.33292 ^{8}
		$\hat{\theta}$	0.30493 ^{2}	0.31838 ^{8}	0.30758 ^{3}	0.29833 ^{1}	0.32592 ^{9}	0.31782 ^{6}	0.3111 ^{4}	0.31819 ^{7}	0.31367 ^{5}
	$\sum Ranks$		18 ^{1}	61 ^{7}	30 ^{4}	27 ^{2.5}	77.5 ^{9}	51.5 ^{6}	27 ^{2.5}	49 ^{5}	64 ^{8}
600	BIAS	$\hat{a}$	0.09438 ^{4}	0.09615 ^{7}	0.095 ^{5}	0.08866 ^{1}	0.09705 ^{8}	0.09405 ^{3}	0.09355 ^{2}	0.09754 ^{9}	0.0958 ^{6}
		$\hat{b}$	0.15184 ^{1}	0.1562 ^{6}	0.15355 ^{3}	0.15428 ^{4}	0.15885 ^{8}	0.15216 ^{2}	0.1544 ^{5}	0.15812 ^{7}	0.16051 ^{9}
		$\hat{\theta}$	0.07078 ^{1}	0.07612 ^{9}	0.07233 ^{4}	0.07129 ^{2}	0.0752 ^{7}	0.07199 ^{3}	0.07275 ^{5}	0.07605 ^{8}	0.07341 ^{6}
	MSE	$\hat{a}$	0.01509 ^{2}	0.01557 ^{6}	0.01549 ^{5}	0.0135 ^{1}	0.01701 ^{9}	0.01524 ^{4}	0.01523 ^{3}	0.01605 ^{8}	0.01579 ^{7}
		$\hat{b}$	0.03426 ^{2}	0.03556 ^{5}	0.03505 ^{3}	0.03529 ^{4}	0.03649 ^{8}	0.03397 ^{1}	0.03565 ^{6}	0.03623 ^{7}	0.03727 ^{9}
		$\hat{\theta}$	0.00834 ^{2}	0.00929 ^{9}	0.00855 ^{5}	0.00814 ^{1}	0.00899 ^{7}	0.00835 ^{3}	0.00846 ^{4}	0.00912 ^{8}	0.00859 ^{6}
	MRE	$\hat{a}$	0.06292 ^{4}	0.0641 ^{7}	0.06334 ^{5}	0.05911 ^{1}	0.0647 ^{8}	0.0627 ^{3}	0.06237 ^{2}	0.06503 ^{9}	0.06387 ^{6}
		$\hat{b}$	0.30368 ^{1}	0.31241 ^{6}	0.3071 ^{3}	0.30857 ^{4}	0.31769 ^{8}	0.30432 ^{2}	0.3088 ^{5}	0.31624 ^{7}	0.32103 ^{9}
		$\hat{\theta}$	0.28314 ^{1}	0.30447 ^{9}	0.28931 ^{4}	0.28515 ^{2}	0.3008 ^{7}	0.28798 ^{3}	0.29099 ^{5}	0.3042 ^{8}	0.29365 ^{6}
	$\sum Ranks$		18 ^{1}	64 ^{6.5}	37 ^{4.5}	20 ^{2}	70 ^{8}	24 ^{3}	37 ^{4.5}	71 ^{9}	64 ^{6.5}

TABLE 2. Simulation values of BIAS, MSE and MRE for ($a = 2.5$, $b = 0.75$, $\theta = 1.5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	MSADE	MSALDE
35	BIAS	$\hat{a}$	0.71836 ^{7}	0.6894 ^{2.5}	0.70916 ^{6}	0.7015 ^{5}	0.6894 ^{2.5}	0.72325 ^{9}	0.71929 ^{8}	0.69872 ^{4}	0.6817 ^{1}
		$\hat{b}$	0.26312 ^{2}	0.27062 ^{7}	0.26981 ^{5}	0.26689 ^{4}	0.27005 ^{6}	0.27093 ^{8}	0.2759 ^{9}	0.26641 ^{3}	0.26285 ^{1}
		$\hat{\theta}$	0.54919 ^{4}	0.57489 ^{9}	0.56226 ^{7}	0.55146 ^{6}	0.53477 ^{2}	0.57454 ^{8}	0.55145 ^{5}	0.53078 ^{1}	0.53609 ^{3}
	MSE	$\hat{a}$	0.872 ^{7}	0.7912 ^{2}	0.8434 ^{6}	0.83293 ^{5}	0.7975 ^{3}	0.90819 ^{9}	0.87631 ^{8}	0.83168 ^{4}	0.78701 ^{1}
		$\hat{b}$	0.09486 ^{2}	0.09939 ^{7}	0.09725 ^{3}	0.09773 ^{4}	0.09833 ^{5}	0.1002 ^{8}	0.10267 ^{9}	0.0984 ^{6}	0.09441 ^{1}
		$\hat{\theta}$	0.47229 ^{6}	0.50846 ^{9}	0.48132 ^{7}	0.46675 ^{5}	0.44206 ^{2}	0.50497 ^{8}	0.46201 ^{4}	0.43668 ^{1}	0.4464 ^{3}
	MRE	$\hat{a}$	0.28734 ^{7}	0.27576 ^{2.5}	0.28366 ^{6}	0.2806 ^{5}	0.27576 ^{2.5}	0.2893 ^{9}	0.28772 ^{8}	0.27949 ^{4}	0.27268 ^{1}
		$\hat{b}$	0.35083 ^{2}	0.36082 ^{7}	0.35975 ^{5}	0.35586 ^{4}	0.36007 ^{6}	0.36124 ^{8}	0.36787 ^{9}	0.35522 ^{3}	0.35046 ^{1}
		$\hat{\theta}$	0.36613 ^{4}	0.38326 ^{9}	0.37484 ^{7}	0.36764 ^{6}	0.35651 ^{2}	0.38302 ^{8}	0.36763 ^{5}	0.35385 ^{1}	0.35739 ^{3}
	$\sum Ranks$		41 ^{4}	55 ^{7}	52 ^{6}	44 ^{5}	31 ^{3}	75 ^{9}	65 ^{8}	27 ^{2}	15 ^{1}
100	BIAS	$\hat{a}$	0.47922 ^{4}	0.47765 ^{2}	0.48235 ^{6}	0.48368 ^{7}	0.47994 ^{5}	0.49385 ^{8}	0.47772 ^{3}	0.50474 ^{9}	0.47304 ^{1}
		$\hat{b}$	0.26097 ^{4}	0.26335 ^{6}	0.26552 ^{8}	0.26134 ^{5}	0.25625 ^{2}	0.26618 ^{9}	0.26439 ^{7}	0.25872 ^{3}	0.25467 ^{1}
		$\hat{\theta}$	0.523 ^{6}	0.53008 ^{7}	0.51742 ^{3}	0.51902 ^{4}	0.50309 ^{1}	0.53454 ^{8}	0.53465 ^{9}	0.51944 ^{5}	0.50557 ^{2}
	MSE	$\hat{a}$	0.43547 ^{7}	0.42411 ^{2}	0.43255 ^{4}	0.43503 ^{6}	0.43377 ^{5}	0.44956 ^{8}	0.41801 ^{11}	0.47164 ^{9}	0.42616 ^{3}
		$\hat{b}$	0.09225 ^{4}	0.09466 ^{7}	0.09665 ^{8}	0.09419 ^{6}	0.08997 ^{1}	0.0972 ^{9}	0.09316 ^{5}	0.09043 ^{3}	0.09019 ^{2}
		$\hat{\theta}$	0.42671 ^{6}	0.4442 ^{8}	0.41432 ^{3}	0.42425 ^{5}	0.40661 ^{2}	0.44551 ^{9}	0.44065 ^{7}	0.41949 ^{4}	0.39631 ^{1}
	MRE	$\hat{a}$	0.19169 ^{4}	0.19106 ^{2}	0.19294 ^{6}	0.19347 ^{7}	0.19198 ^{5}	0.19754 ^{8}	0.19109 ^{3}	0.20189 ^{9}	0.18922 ^{1}
		$\hat{b}$	0.34796 ^{4}	0.35114 ^{6}	0.35403 ^{8}	0.34845 ^{5}	0.34166 ^{2}	0.3549 ^{9}	0.35252 ^{7}	0.34497 ^{3}	0.33956 ^{1}
		$\hat{\theta}$	0.34867 ^{6}	0.35338 ^{7}	0.34495 ^{3}	0.34601 ^{4}	0.33539 ^{11}	0.35636 ^{8}	0.35643 ^{9}	0.34629 ^{5}	0.33705 ^{2}
	$\sum Ranks$		45 ^{3}	47 ^{4}	49 ^{5.5}	49 ^{5.5}	24 ^{2}	76 ^{9}	51 ^{8}	50 ^{7}	14 ^{1}
250	BIAS	$\hat{a}$	0.331 ^{9}	0.30906 ^{3}	0.31852 ^{6}	0.308 ^{2}	0.31277 ^{4}	0.32367 ^{8}	0.30532 ^{11}	0.32252 ^{7}	0.31452 ^{5}
		$\hat{b}$	0.24507 ^{6}	0.24212 ^{2}	0.24959 ^{9}	0.24379 ^{5}	0.23689 ^{11}	0.24924 ^{7.5}	0.24289 ^{3}	0.24344 ^{4}	0.24924 ^{7.5}
		$\hat{\theta}$	0.45478 ^{5}	0.44322 ^{11}	0.45954 ^{8}	0.447 ^{4}	0.44519 ^{2}	0.45905 ^{7}	0.44583 ^{3}	0.45604 ^{6}	0.46229 ^{9}
	MSE	$\hat{a}$	0.20458 ^{9}	0.18174 ^{5}	0.1892 ^{6}	0.18108 ^{4}	0.17948 ^{3}	0.2011 ^{8}	0.17259 ^{11}	0.19118 ^{7}	0.17752 ^{2}
		$\hat{b}$	0.08736 ^{7}	0.08369 ^{3}	0.08801 ^{8}	0.0853 ^{5}	0.07804 ^{11}	0.08608 ^{6}	0.08264 ^{2}	0.08485 ^{4}	0.08856 ^{9}
		$\hat{\theta}$	0.33734 ^{8}	0.31381 ^{11}	0.33359 ^{5}	0.3242 ^{4}	0.31773 ^{2}	0.33605 ^{7}	0.32157 ^{3}	0.33386 ^{6}	0.34121 ^{9}
	MRE	$\hat{a}$	0.1324 ^{9}	0.12363 ^{3}	0.12741 ^{6}	0.1232 ^{2}	0.12511 ^{4}	0.12947 ^{8}	0.12213 ^{11}	0.12901 ^{7}	0.12581 ^{5}
		$\hat{b}$	0.32675 ^{6}	0.32283 ^{2}	0.33279 ^{9}	0.32506 ^{5}	0.31585 ^{11}	0.33231 ^{7}	0.32386 ^{3}	0.32459 ^{4}	0.33232 ^{8}
		$\hat{\theta}$	0.30318 ^{5}	0.29548 ^{11}	0.30636 ^{8}	0.298 ^{4}	0.29679 ^{2}	0.30604 ^{7}	0.29722 ^{3}	0.30403 ^{6}	0.3082 ^{9}
	$\sum Ranks$		64 ^{7}	21 ^{3}	65 ^{8}	35 ^{4}	20 ^{1.5}	65.5 ^{9}	20 ^{1.5}	51 ^{5}	63.5 ^{6}
400	BIAS	$\hat{a}$	0.2318 ^{11}	0.24528 ^{5}	0.24575 ^{7}	0.24802 ^{8}	0.23968 ^{3}	0.24926 ^{9}	0.2385 ^{2}	0.24549 ^{6}	0.24525 ^{4}
		$\hat{b}$	0.224 ^{4}	0.23807 ^{8}	0.22398 ^{3}	0.23248 ^{7}	0.22162 ^{11}	0.23842 ^{9}	0.2232 ^{2}	0.23112 ^{6}	0.22725 ^{5}
		$\hat{\theta}$	0.38425 ^{5}	0.40013 ^{8}	0.37585 ^{3}	0.3942 ^{7}	0.3738 ^{2}	0.41042 ^{9}	0.37253 ^{11}	0.38983 ^{6}	0.37917 ^{4}
	MSE	$\hat{a}$	0.10112 ^{11}	0.1089 ^{5}	0.11022 ^{6}	0.10761 ^{4}	0.10718 ^{3}	0.11384 ^{8}	0.10652 ^{2}	0.11144 ^{7}	0.11483 ^{9}
		$\hat{b}$	0.07488 ^{2}	0.08355 ^{9}	0.07595 ^{4}	0.07915 ^{6}	0.07523 ^{3}	0.08237 ^{8}	0.07401 ^{11}	0.07965 ^{7}	0.07785 ^{5}
		$\hat{\theta}$	0.2419 ^{5}	0.2548 ^{8}	0.23013 ^{2}	0.24619 ^{6}	0.23484 ^{3}	0.26794 ^{9}	0.22588 ^{11}	0.24709 ^{7}	0.23965 ^{4}
	MRE	$\hat{a}$	0.09272 ^{11}	0.09811 ^{5}	0.0983 ^{7}	0.09921 ^{8}	0.09587 ^{3}	0.09971 ^{9}	0.0954 ^{2}	0.0982 ^{6}	0.0981 ^{4}
		$\hat{b}$	0.29867 ^{4}	0.31742 ^{8}	0.29864 ^{3}	0.30998 ^{7}	0.29549 ^{11}	0.3179 ^{9}	0.2976 ^{2}	0.30816 ^{6}	0.303 ^{5}
		$\hat{\theta}$	0.25617 ^{5}	0.26675 ^{8}	0.25057 ^{3}	0.2628 ^{7}	0.2492 ^{2}	0.27362 ^{9}	0.24835 ^{11}	0.25989 ^{6}	0.25278 ^{4}
	$\sum Ranks$		28 ^{3}	64 ^{8}	38 ^{4}	60 ^{7}	21 ^{2}	79 ^{9}	14 ^{1}	57 ^{6}	44 ^{5}
600	BIAS	$\hat{a}$	0.19689 ^{4}	0.19976 ^{5}	0.19339 ^{2}	0.20318 ^{8}	0.20538 ^{9}	0.19365 ^{3}	0.20026 ^{6}	0.18748 ^{11}	0.20144 ^{7}
		$\hat{b}$	0.21033 ^{4}	0.21286 ^{8}	0.20614 ^{2}	0.21246 ^{7}	0.21493 ^{9}	0.20999 ^{3}	0.20072 ^{11}	0.21109 ^{5}	0.21229 ^{6}
		$\hat{\theta}$	0.33473 ^{6}	0.33223 ^{5}	0.3308 ^{4}	0.34344 ^{8}	0.355 ^{9}	0.32483 ^{2}	0.31764 ^{11}	0.32914 ^{3}	0.33812 ^{7}
	MSE	$\hat{a}$	0.07173 ^{5}	0.07335 ^{6}	0.07017 ^{4}	0.07688 ^{8}	0.07878 ^{9}	0.06657 ^{2}	0.06937 ^{3}	0.06402 ^{11}	0.07595 ^{7}
		$\hat{b}$	0.0689 ^{5}	0.06856 ^{4}	0.06588 ^{2}	0.07092 ^{9}	0.07075 ^{8}	0.0684 ^{3}	0.06245 ^{11}	0.06953 ^{6}	0.06972 ^{7}
		$\hat{\theta}$	0.18464 ^{6}	0.17392 ^{3}	0.18068 ^{5}	0.19215 ^{8}	0.2042 ^{9}	0.16652 ^{2}	0.16386 ^{11}	0.17799 ^{4}	0.187 ^{7}
	MRE	$\hat{a}$	0.07876 ^{4}	0.0799 ^{5}	0.07736 ^{2}	0.08127 ^{8}	0.08215 ^{9}	0.07746 ^{3}	0.0801 ^{6}	0.07499 ^{11}	0.08058 ^{7}
		$\hat{b}$	0.28044 ^{4}	0.28382 ^{8}	0.27486 ^{2}	0.28328 ^{7}	0.28657 ^{9}	0.27999 ^{3}	0.26763 ^{11}	0.28145 ^{5}	0.28306 ^{6}
		$\hat{\theta}$	0.22315 ^{6}	0.22149 ^{5}	0.22053 ^{4}	0.22896 ^{8}	0.23666 ^{9}	0.21655 ^{2}	0.21176 ^{11}	0.21943 ^{3}	0.22541 ^{7}
	$\sum Ranks$		44 ^{5}	49 ^{6}	27 ^{3}	71 ^{8}	80 ^{9}	23 ^{2}	21 ^{1}	29 ^{4}	61 ^{7}

TABLE 3. Simulation values of BIAS, MSE and MRE for ($a = 2.0$, $b = 1.5$, $\theta = 0.5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	MSADE	MSALDE
35	BIAS	$\hat{a}$	0.40401 ^{3}	0.41613 ^{8}	0.40746 ^{4}	0.42101 ^{9}	0.41133 ^{6}	0.4095 ^{5}	0.41215 ^{7}	0.38922 ^{1}	0.40114 ^{2}
		$\hat{b}$	0.65918 ^{8}	0.63204 ^{4}	0.62999 ^{3}	0.6424 ^{6}	0.6269 ^{1}	0.67124 ^{9}	0.6397 ^{5}	0.64671 ^{7}	0.62763 ^{2}
		$\hat{\theta}$	0.19084 ^{6}	0.18732 ^{2}	0.18861 ^{4}	0.19464 ^{9}	0.19329 ^{8}	0.19308 ^{7}	0.19066 ^{5}	0.18753 ^{3}	0.18608 ^{1}
	MSE	$\hat{a}$	0.29097 ^{7}	0.28955 ^{6}	0.28345 ^{4}	0.294 ^{8}	0.28909 ^{5}	0.2823 ^{3}	0.29702 ^{9}	0.25057 ^{1}	0.27034 ^{2}
		$\hat{b}$	0.60432 ^{8}	0.55715 ^{4}	0.55075 ^{3}	0.56722 ^{6}	0.54999 ^{2}	0.61616 ^{9}	0.56437 ^{5}	0.57933 ^{7}	0.54763 ^{1}
		$\hat{\theta}$	0.0528 ^{5}	0.05124 ^{3}	0.05153 ^{4}	0.05463 ^{9}	0.05433 ^{8}	0.05285 ^{6}	0.0531 ^{7}	0.05011 ^{1}	0.05025 ^{2}
	MRE	$\hat{a}$	0.20201 ^{3}	0.20807 ^{8}	0.20373 ^{4}	0.21051 ^{9}	0.20567 ^{6}	0.20475 ^{5}	0.20608 ^{7}	0.19461 ^{1}	0.20057 ^{2}
		$\hat{b}$	0.43945 ^{8}	0.42136 ^{4}	0.42 ^{3}	0.42826 ^{6}	0.41794 ^{1}	0.44749 ^{9}	0.42647 ^{5}	0.43114 ^{7}	0.41842 ^{2}
		$\hat{\theta}$	0.38169 ^{6}	0.37464 ^{2}	0.37722 ^{4}	0.38928 ^{9}	0.38657 ^{8}	0.38616 ^{7}	0.38132 ^{5}	0.37506 ^{3}	0.37215 ^{1}
	$\sum Ranks$		54 ^{6}	41 ^{4}	33 ^{3}	71 ^{9}	45 ^{5}	60 ^{8}	55 ^{7}	31 ^{2}	15 ^{1}
100	BIAS	$\hat{a}$	0.25095 ^{6}	0.25828 ^{8}	0.24918 ^{5}	0.2521 ^{7}	0.246 ^{3}	0.24686 ^{4}	0.26275 ^{9}	0.24382 ^{1}	0.24454 ^{2}
		$\hat{b}$	0.60033 ^{9}	0.57941 ^{1}	0.58165 ^{2}	0.5861 ^{3}	0.59578 ^{6}	0.58784 ^{5}	0.59725 ^{7}	0.5998 ^{8}	0.58759 ^{4}
		$\hat{\theta}$	0.18372 ^{6}	0.18057 ^{3}	0.1794 ^{1}	0.18832 ^{9}	0.18821 ^{8}	0.18384 ^{7}	0.18342 ^{5}	0.17991 ^{2}	0.18088 ^{4}
	MSE	$\hat{a}$	0.10854 ^{6}	0.11429 ^{8}	0.10736 ^{5}	0.11076 ^{7}	0.10101 ^{1}	0.10376 ^{3}	0.11643 ^{9}	0.10298 ^{2}	0.10398 ^{4}
		$\hat{b}$	0.49408 ^{7}	0.47296 ^{2}	0.47462 ^{3}	0.47488 ^{4}	0.48069 ^{5}	0.48248 ^{6}	0.50212 ^{9}	0.50111 ^{8}	0.47245 ^{1}
		$\hat{\theta}$	0.05074 ^{5}	0.05 ^{4}	0.04976 ^{3}	0.05346 ^{9}	0.05228 ^{8}	0.05127 ^{7}	0.05085 ^{6}	0.04861 ^{1}	0.04914 ^{2}
	MRE	$\hat{a}$	0.12548 ^{6}	0.12914 ^{8}	0.12459 ^{5}	0.12605 ^{7}	0.123 ^{3}	0.12343 ^{4}	0.13138 ^{9}	0.12191 ^{1}	0.12227 ^{2}
		$\hat{b}$	0.40022 ^{9}	0.38627 ^{1}	0.38777 ^{2}	0.39073 ^{3}	0.39719 ^{6}	0.39189 ^{5}	0.39817 ^{7}	0.39986 ^{8}	0.39172 ^{4}
		$\hat{\theta}$	0.36744 ^{6}	0.36113 ^{3}	0.3588 ^{1}	0.37664 ^{9}	0.37643 ^{8}	0.36769 ^{7}	0.36683 ^{5}	0.35981 ^{2}	0.36177 ^{4}
	$\sum Ranks$		60 ^{8}	38 ^{4}	27 ^{1.5}	58 ^{7}	48 ^{5.5}	48 ^{5.5}	66 ^{9}	33 ^{3}	27 ^{1.5}
250	BIAS	$\hat{a}$	0.15273 ^{2}	0.16105 ^{8}	0.15399 ^{5}	0.15523 ^{6}	0.15792 ^{7}	0.16365 ^{9}	0.15279 ^{3}	0.14782 ^{1}	0.15376 ^{4}
		$\hat{b}$	0.5915 ^{8}	0.56567 ^{1}	0.57433 ^{4}	0.57525 ^{5}	0.57335 ^{2}	0.58608 ^{7}	0.59646 ^{9}	0.57361 ^{3}	0.58507 ^{6}
		$\hat{\theta}$	0.18086 ^{6}	0.17935 ^{5}	0.17854 ^{3}	0.17898 ^{4}	0.18404 ^{9}	0.18146 ^{7}	0.18311 ^{8}	0.17773 ^{2}	0.17734 ^{1}
	MSE	$\hat{a}$	0.03914 ^{3}	0.04329 ^{8}	0.03964 ^{4}	0.04114 ^{6}	0.04135 ^{7}	0.04485 ^{9}	0.03781 ^{2}	0.03601 ^{1}	0.03991 ^{5}
		$\hat{b}$	0.48555 ^{9}	0.44768 ^{1}	0.46063 ^{5}	0.45917 ^{4}	0.44807 ^{2}	0.46848 ^{6}	0.48047 ^{8}	0.45126 ^{3}	0.47114 ^{7}
		$\hat{\theta}$	0.04999 ^{7}	0.04914 ^{5}	0.04912 ^{4}	0.0486 ^{3}	0.0517 ^{9}	0.04956 ^{6}	0.05014 ^{8}	0.04832 ^{2}	0.04763 ^{1}
	MRE	$\hat{a}$	0.07637 ^{2}	0.08053 ^{8}	0.077 ^{5}	0.07761 ^{6}	0.07896 ^{7}	0.08183 ^{9}	0.07639 ^{3}	0.07391 ^{1}	0.07688 ^{4}
		$\hat{b}$	0.39433 ^{8}	0.37711 ^{1}	0.38289 ^{4}	0.3835 ^{5}	0.38223 ^{2}	0.39072 ^{7}	0.39764 ^{9}	0.3824 ^{3}	0.39005 ^{6}
		$\hat{\theta}$	0.36172 ^{6}	0.35871 ^{5}	0.35709 ^{3}	0.35797 ^{4}	0.36807 ^{9}	0.36293 ^{7}	0.36622 ^{8}	0.35546 ^{2}	0.35468 ^{1}
	$\sum Ranks$		51 ^{6}	42 ^{4}	37 ^{3}	43 ^{5}	54 ^{7}	67 ^{9}	58 ^{8}	18 ^{1}	35 ^{2}
400	BIAS	$\hat{a}$	0.12418 ^{3}	0.12428 ^{4}	0.13181 ^{9}	0.12607 ^{7}	0.12603 ^{6}	0.1249 ^{5}	0.12286 ^{2}	0.11795 ^{1}	0.13113 ^{8}
		$\hat{b}$	0.55058 ^{4}	0.54638 ^{1}	0.54999 ^{3}	0.5624 ^{8}	0.56793 ^{9}	0.54835 ^{2}	0.55737 ^{7}	0.55088 ^{5}	0.5565 ^{6}
		$\hat{\theta}$	0.17159 ^{1}	0.172 ^{2}	0.17247 ^{3}	0.17767 ^{8}	0.17932 ^{9}	0.17362 ^{6}	0.17301 ^{4}	0.17611 ^{7}	0.17317 ^{5}
	MSE	$\hat{a}$	0.02463 ^{2}	0.02572 ^{4}	0.02864 ^{8.5}	0.02578 ^{5}	0.02714 ^{7}	0.02603 ^{6}	0.02473 ^{3}	0.02355 ^{1}	0.02864 ^{8.5}
		$\hat{b}$	0.41886 ^{3}	0.41329 ^{1}	0.42355 ^{5}	0.43577 ^{8}	0.44131 ^{9}	0.42103 ^{4}	0.42946 ^{6}	0.41516 ^{2}	0.43299 ^{7}
		$\hat{\theta}$	0.04454 ^{1}	0.04587 ^{3}	0.04641 ^{5}	0.04777 ^{8}	0.04908 ^{9}	0.04739 ^{6}	0.04597 ^{4}	0.04751 ^{7}	0.04583 ^{2}
	MRE	$\hat{a}$	0.06209 ^{3}	0.06214 ^{4}	0.0659 ^{9}	0.06303 ^{7}	0.06301 ^{6}	0.06245 ^{5}	0.06143 ^{2}	0.05898 ^{1}	0.06557 ^{8}
		$\hat{b}$	0.36705 ^{4}	0.36426 ^{1}	0.36666 ^{3}	0.37493 ^{8}	0.37862 ^{9}	0.36557 ^{2}	0.37158 ^{7}	0.36725 ^{5}	0.371 ^{6}
		$\hat{\theta}$	0.34319 ^{1}	0.34399 ^{2}	0.34493 ^{3}	0.35534 ^{8}	0.35865 ^{9}	0.34724 ^{6}	0.34603 ^{4}	0.35223 ^{7}	0.34634 ^{5}
	$\sum Ranks$		22 ^{1.5}	22 ^{1.5}	48.5 ^{6}	67 ^{8}	73 ^{9}	42 ^{5}	39 ^{4}	36 ^{3}	55.5 ^{7}
600	BIAS	$\hat{a}$	0.10053 ^{2}	0.10264 ^{5}	0.10334 ^{6}	0.10019 ^{1}	0.10076 ^{4}	0.10418 ^{8}	0.10055 ^{3}	0.10592 ^{9}	0.10414 ^{7}
		$\hat{b}$	0.5424 ^{8}	0.52844 ^{3}	0.53022 ^{4}	0.5372 ^{6}	0.53479 ^{5}	0.5236 ^{2}	0.55149 ^{9}	0.5413 ^{7}	0.52248 ^{1}
		$\hat{\theta}$	0.16705 ^{7}	0.16212 ^{3}	0.16899 ^{8}	0.16357 ^{5}	0.1639 ^{6}	0.15908 ^{1}	0.17071 ^{9}	0.16339 ^{4}	0.16174 ^{2}
	MSE	$\hat{a}$	0.0169 ^{4}	0.01693 ^{5}	0.0176 ^{6}	0.01682 ^{3}	0.01664 ^{2}	0.01766 ^{7}	0.01657 ^{1}	0.01808 ^{9}	0.01789 ^{8}
		$\hat{b}$	0.41807 ^{8}	0.39602 ^{4}	0.39226 ^{2}	0.40775 ^{6}	0.40041 ^{5}	0.39476 ^{3}	0.42006 ^{9}	0.40783 ^{7}	0.38721 ^{1}
		$\hat{\theta}$	0.04359 ^{7}	0.04127 ^{3}	0.04424 ^{8}	0.04155 ^{4}	0.04177 ^{6}	0.03953 ^{1}	0.04448 ^{9}	0.04163 ^{5}	0.04121 ^{2}
	MRE	$\hat{a}$	0.05026 ^{2}	0.05132 ^{5}	0.05167 ^{6}	0.0501 ^{1}	0.05038 ^{4}	0.05209 ^{8}	0.05028 ^{3}	0.05296 ^{9}	0.05207 ^{7}
		$\hat{b}$	0.3616 ^{8}	0.35229 ^{3}	0.35348 ^{4}	0.35813 ^{6}	0.35653 ^{5}	0.34906 ^{2}	0.36766 ^{9}	0.36087 ^{7}	0.34832 ^{1}
		$\hat{\theta}$	0.33409 ^{7}	0.32424 ^{3}	0.33798 ^{8}	0.32713 ^{5}	0.3278 ^{6}	0.31816 ^{1}	0.34141 ^{9}	0.32678 ^{4}	0.32348 ^{2}
	$\sum Ranks$		53 ^{7}	34 ^{3}	52 ^{6}	37 ^{4}	43 ^{5}	33 ^{2}	61 ^{8.5}	61 ^{8.5}	31 ^{1}

TABLE 4. Simulation values of BIAS, MSE and MRE for $(a = 3.5, b = 0.2, \theta = 0.9)$.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	MSADE	MSALDE
35	BIAS	$\hat{a}$	1.11231 ^{7}	1.08021 ^{2}	1.0966 ^{3}	1.1179 ^{8}	1.11148 ^{5}	1.11157 ^{6}	1.14508 ^{9}	1.02864 ^{1}	1.10167 ^{4}
		$\hat{b}$	0.0706 ^{8}	0.06965 ^{4}	0.06985 ^{5.5}	0.06985 ^{5.5}	0.06788 ^{1}	0.07205 ^{9}	0.06942 ^{3}	0.07013 ^{7}	0.06839 ^{2}
		$\hat{\theta}$	0.23338 ^{4}	0.24155 ^{7}	0.23727 ^{5}	0.24464 ^{8}	0.22909 ^{2}	0.24518 ^{9}	0.22664 ^{1}	0.23854 ^{6}	0.22994 ^{3}
	MSE	$\hat{a}$	1.97278 ^{7}	1.86501 ^{2}	1.92368 ^{5}	1.96395 ^{6}	1.92296 ^{4}	1.99038 ^{8}	2.02977 ^{9}	1.74002 ^{1}	1.87597 ^{3}
		$\hat{b}$	0.0075 ^{8}	0.00738 ^{5}	0.00723 ^{3}	0.00737 ^{4}	0.00702 ^{1}	0.00771 ^{9}	0.0074 ^{6}	0.00748 ^{7}	0.00711 ^{2}
		$\hat{\theta}$	0.08699 ^{4}	0.09258 ^{7}	0.08516 ^{3}	0.0944 ^{9}	0.08447 ^{2}	0.09312 ^{8}	0.08228 ^{1}	0.09249 ^{6}	0.08719 ^{5}
	MRE	$\hat{a}$	0.3178 ^{7}	0.30863 ^{2}	0.31331 ^{3}	0.3194 ^{8}	0.31757 ^{5}	0.31759 ^{6}	0.32717 ^{9}	0.2939 ^{1}	0.31476 ^{4}
		$\hat{b}$	0.353 ^{8}	0.34825 ^{4}	0.34924 ^{5.5}	0.34924 ^{5.5}	0.33942 ^{1}	0.36025 ^{9}	0.34709 ^{3}	0.35066 ^{7}	0.34194 ^{2}
		$\hat{\theta}$	0.25931 ^{4}	0.26839 ^{7}	0.26363 ^{5}	0.27182 ^{8}	0.25455 ^{2}	0.27242 ^{9}	0.25182 ^{1}	0.26504 ^{6}	0.25549 ^{3}
	$\sum Ranks$		57 ^{7}	40 ^{4}	38 ^{3}	62 ^{8}	23 ^{1}	73 ^{9}	42 ^{5.5}	42 ^{5.5}	28 ^{2}
100	BIAS	$\hat{a}$	0.95109 ^{3}	0.98818 ^{9}	0.95322 ^{4}	0.94664 ^{2}	0.97491 ^{5}	0.9226 ^{1}	0.98181 ^{6}	0.98367 ^{7}	0.98633 ^{8}
		$\hat{b}$	0.06252 ^{5}	0.063 ^{7}	0.066 ^{9}	0.06383 ^{8}	0.06146 ^{2}	0.06275 ^{6}	0.06215 ^{3}	0.06081 ^{1}	0.06231 ^{4}
		$\hat{\theta}$	0.22458 ^{3}	0.23845 ^{9}	0.23456 ^{8}	0.22529 ^{5}	0.226 ^{6}	0.22296 ^{1}	0.22498 ^{4}	0.22868 ^{7}	0.22409 ^{2}
	MSE	$\hat{a}$	1.57368 ^{4}	1.6596 ^{9}	1.54491 ^{3}	1.52937 ^{2}	1.60339 ^{5}	1.44943 ^{1}	1.64625 ^{8}	1.63772 ^{7}	1.61158 ^{6}
		$\hat{b}$	0.00604 ^{4.5}	0.00605 ^{6}	0.0067 ^{9}	0.00632 ^{8}	0.00596 ^{2}	0.00611 ^{7}	0.00601 ^{3}	0.0057 ^{1}	0.00604 ^{4.5}
		$\hat{\theta}$	0.07959 ^{5}	0.08894 ^{9}	0.08302 ^{6}	0.07743 ^{1}	0.08395 ^{7}	0.079 ^{3}	0.07924 ^{4}	0.08412 ^{8}	0.07817 ^{2}
	MRE	$\hat{a}$	0.27174 ^{3}	0.28234 ^{9}	0.27235 ^{4}	0.27047 ^{2}	0.27855 ^{5}	0.2636 ^{1}	0.28052 ^{6}	0.28105 ^{7}	0.28181 ^{8}
		$\hat{b}$	0.31262 ^{5}	0.31501 ^{7}	0.33002 ^{9}	0.31913 ^{8}	0.3073 ^{2}	0.31375 ^{6}	0.31077 ^{3}	0.30407 ^{1}	0.31153 ^{4}
		$\hat{\theta}$	0.24953 ^{3}	0.26494 ^{9}	0.26063 ^{8}	0.25032 ^{5}	0.25111 ^{6}	0.24773 ^{1}	0.24998 ^{4}	0.25409 ^{7}	0.24899 ^{2}
	$\sum Ranks$		35.5 ^{2}	74 ^{9}	60 ^{8}	41 ^{5.5}	40 ^{3}	27 ^{1}	41 ^{5.5}	46 ^{7}	40.5 ^{4}
250	BIAS	$\hat{a}$	0.78611 ^{2}	0.80316 ^{3}	0.76803 ^{1}	0.82119 ^{5}	0.82709 ^{7}	0.82313 ^{6}	0.86141 ^{9}	0.80404 ^{4}	0.82822 ^{8}
		$\hat{b}$	0.05488 ^{6}	0.0553 ^{8}	0.05237 ^{1}	0.0549 ^{7}	0.05434 ^{5}	0.05403 ^{4}	0.05328 ^{3}	0.05624 ^{9}	0.05282 ^{2}
		$\hat{\theta}$	0.20126 ^{7}	0.19982 ^{4}	0.18797 ^{1}	0.20054 ^{6}	0.19727 ^{2}	0.20026 ^{5}	0.19946 ^{3}	0.20129 ^{8}	0.20318 ^{9}
	MSE	$\hat{a}$	1.14053 ^{2}	1.14561 ^{3}	1.05426 ^{1}	1.2305 ^{6}	1.22893 ^{5}	1.24908 ^{7}	1.33801 ^{9}	1.14669 ^{4}	1.26124 ^{8}
		$\hat{b}$	0.00471 ^{4}	0.00492 ^{8}	0.00454 ^{3}	0.00479 ^{7}	0.00475 ^{5}	0.00476 ^{6}	0.00444 ^{1}	0.00504 ^{9}	0.00449 ^{2}
		$\hat{\theta}$	0.06512 ^{8}	0.06174 ^{2}	0.05579 ^{1}	0.06371 ^{7}	0.06249 ^{3}	0.06344 ^{5}	0.06302 ^{4}	0.06347 ^{6}	0.0667 ^{9}
	MRE	$\hat{a}$	0.2246 ^{2}	0.22947 ^{3}	0.21944 ^{1}	0.23462 ^{5}	0.23631 ^{7}	0.23518 ^{6}	0.24612 ^{9}	0.22973 ^{4}	0.23664 ^{8}
		$\hat{b}$	0.27438 ^{6}	0.2765 ^{8}	0.26183 ^{1}	0.27449 ^{7}	0.2717 ^{5}	0.27017 ^{4}	0.26642 ^{3}	0.28122 ^{9}	0.26412 ^{2}
		$\hat{\theta}$	0.22362 ^{7}	0.22203 ^{4}	0.20885 ^{1}	0.22282 ^{6}	0.21919 ^{2}	0.22251 ^{5}	0.22162 ^{3}	0.22365 ^{8}	0.22575 ^{9}
	$\sum Ranks$		44 ^{4.5}	43 ^{3}	11 ^{1}	56 ^{7}	41 ^{2}	48 ^{6}	44 ^{4.5}	61 ^{9}	57 ^{8}
400	BIAS	$\hat{a}$	0.72511 ^{8}	0.69183 ^{5}	0.72313 ^{7}	0.68504 ^{2}	0.72918 ^{9}	0.71778 ^{6}	0.66407 ^{1}	0.68603 ^{3}	0.68725 ^{4}
		$\hat{b}$	0.05005 ^{8}	0.05027 ^{9}	0.0486 ^{4}	0.04828 ^{3}	0.04967 ^{6}	0.0495 ^{5}	0.05001 ^{7}	0.04812 ^{2}	0.0468 ^{1}
		$\hat{\theta}$	0.18303 ^{9}	0.17924 ^{6}	0.17796 ^{5}	0.17493 ^{4}	0.17927 ^{7}	0.17934 ^{8}	0.17413 ^{3}	0.17355 ^{2}	0.17003 ^{1}
	MSE	$\hat{a}$	0.93647 ^{6}	0.89678 ^{5}	0.9776 ^{8}	0.86046 ^{2}	0.98931 ^{9}	0.97578 ^{7}	0.78785 ^{1}	0.8792 ^{4}	0.87616 ^{3}
		$\hat{b}$	0.00386 ^{5}	0.00419 ^{9}	0.00385 ^{4}	0.00377 ^{2}	0.004 ^{6}	0.00403 ^{8}	0.00401 ^{7}	0.00384 ^{3}	0.00356 ^{1}
		$\hat{\theta}$	0.05084 ^{8}	0.05081 ^{7}	0.04947 ^{5}	0.04914 ^{4}	0.05027 ^{6}	0.05245 ^{9}	0.0473 ^{2}	0.04817 ^{3}	0.04501 ^{1}
	MRE	$\hat{a}$	0.20717 ^{8}	0.19766 ^{5}	0.20661 ^{7}	0.19573 ^{2}	0.20834 ^{9}	0.20508 ^{6}	0.18973 ^{1}	0.19601 ^{3}	0.19636 ^{4}
		$\hat{b}$	0.25025 ^{8}	0.25137 ^{9}	0.24302 ^{4}	0.2414 ^{3}	0.24833 ^{6}	0.24748 ^{5}	0.25003 ^{7}	0.2406 ^{2}	0.234 ^{1}
		$\hat{\theta}$	0.20336 ^{9}	0.19915 ^{6}	0.19773 ^{5}	0.19437 ^{4}	0.19919 ^{7}	0.19926 ^{8}	0.19347 ^{3}	0.19284 ^{2}	0.18892 ^{1}
	$\sum Ranks$		69 ^{9}	61 ^{6}	49 ^{5}	26 ^{3}	65 ^{8}	62 ^{7}	32 ^{4}	24 ^{2}	17 ^{1}
600	BIAS	$\hat{a}$	0.59176 ^{7}	0.58586 ^{5}	0.58491 ^{4}	0.60071 ^{8}	0.55951 ^{1}	0.58991 ^{6}	0.60198 ^{9}	0.57781 ^{2}	0.58391 ^{3}
		$\hat{b}$	0.04164 ^{8}	0.04162 ^{7}	0.04131 ^{6}	0.04292 ^{9}	0.0403 ^{3}	0.04096 ^{4}	0.04025 ^{2}	0.03995 ^{1}	0.04126 ^{5}
		$\hat{\theta}$	0.15248 ^{8}	0.15082 ^{6}	0.14843 ^{4}	0.15406 ^{9}	0.14766 ^{2}	0.15121 ^{7}	0.14786 ^{3}	0.14765 ^{1}	0.14985 ^{5}
	MSE	$\hat{a}$	0.65994 ^{6}	0.63873 ^{4}	0.64113 ^{5}	0.67208 ^{8}	0.58479 ^{1}	0.69661 ^{9}	0.67053 ^{7}	0.61673 ^{2}	0.62322 ^{3}
		$\hat{b}$	0.00288 ^{8}	0.00286 ^{6.5}	0.00282 ^{5}	0.00306 ^{9}	0.00266 ^{2}	0.00272 ^{3.5}	0.00272 ^{3.5}	0.00263 ^{1}	0.00286 ^{6.5}
		$\hat{\theta}$	0.03752 ^{8}	0.03636 ^{4}	0.0354 ^{3}	0.03817 ^{9}	0.03495 ^{2}	0.03706 ^{7}	0.03646 ^{5.5}	0.03476 ^{1}	0.03646 ^{5.5}
	MRE	$\hat{a}$	0.16907 ^{7}	0.16739 ^{5}	0.16712 ^{4}	0.17163 ^{8}	0.15986 ^{1}	0.16855 ^{6}	0.17199 ^{9}	0.16509 ^{2}	0.16683 ^{3}
		$\hat{b}$	0.20821 ^{8}	0.20811 ^{7}	0.20656 ^{6}	0.21459 ^{9}	0.20148 ^{3}	0.20479 ^{4}	0.20125 ^{2}	0.19973 ^{1}	0.20631 ^{5}
		$\hat{\theta}$	0.16942 ^{8}	0.16758 ^{6}	0.16492 ^{4}	0.17118 ^{9}	0.16407 ^{2}	0.16801 ^{7}	0.16429 ^{3}	0.16405 ^{1}	0.1665 ^{5}
	$\sum Ranks$		68 ^{8}	50.5 ^{6}	41 ^{3.5}	78 ^{9}	17 ^{2}	53.5 ^{7}	44 ^{5}	12 ^{1}	41 ^{3.5}

TABLE 5. Simulation values of BIAS, MSE and MRE for ($a = 3.0$, $b = 0.9$, $\theta = 0.2$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	MSADE	MSALDE
35	BIAS	$\hat{a}$	0.84106 ^{4}	0.86579 ^{8}	0.83381 ^{3}	0.82683 ^{2}	0.82575 ^{1}	0.84176 ^{5}	0.89041 ^{9}	0.86449 ^{7}	0.85383 ^{6}
		$\hat{b}$	0.32947 ^{4}	0.33016 ^{5}	0.32317 ^{2}	0.32657 ^{3}	0.33596 ^{6}	0.34315 ^{9}	0.3215 ^{1}	0.34172 ^{8}	0.33788 ^{7}
		$\hat{\theta}$	0.7355 ^{3}	0.75642 ^{7}	0.75302 ^{5}	0.73018 ^{2}	0.73659 ^{4}	0.79221 ^{9}	0.72242 ^{1}	0.78203 ^{8}	0.75503 ^{6}
	MSE	$\hat{a}$	1.18909 ^{3}	1.25316 ^{8}	1.1971 ^{4}	1.16063 ^{1}	1.16591 ^{2}	1.22341 ^{6}	1.29588 ^{9}	1.21723 ^{5}	1.2239 ^{7}
		$\hat{b}$	0.14457 ^{3}	0.14553 ^{5}	0.13804 ^{1}	0.14467 ^{4}	0.14844 ^{6}	0.15517 ^{9}	0.1395 ^{2}	0.15272 ^{8}	0.15207 ^{7}
		$\hat{\theta}$	0.83556 ^{3}	0.8659 ^{6}	0.86107 ^{5}	0.84557 ^{4}	0.83488 ^{2}	0.92984 ^{9}	0.80528 ^{1}	0.91477 ^{8}	0.86743 ^{7}
	MRE	$\hat{a}$	0.28035 ^{4}	0.2886 ^{8}	0.27794 ^{3}	0.27561 ^{2}	0.27525 ^{1}	0.28059 ^{5}	0.2968 ^{9}	0.28816 ^{7}	0.28461 ^{6}
		$\hat{b}$	0.36608 ^{4}	0.36684 ^{5}	0.35908 ^{2}	0.36285 ^{3}	0.37329 ^{6}	0.38128 ^{9}	0.35722 ^{1}	0.37969 ^{8}	0.37542 ^{7}
		$\hat{\theta}$	0.36775 ^{3}	0.37821 ^{7}	0.37651 ^{5}	0.36509 ^{2}	0.36829 ^{4}	0.3961 ^{9}	0.36121 ^{1}	0.39101 ^{8}	0.37751 ^{6}
	$\sum Ranks$		31 ^{3}	59 ^{6,5}	30 ^{2}	23 ^{1}	32 ^{4}	70 ^{9}	34 ^{5}	67 ^{8}	59 ^{6,5}
100	BIAS	$\hat{a}$	0.59279 ^{5}	0.61825 ^{8}	0.59097 ^{2}	0.60289 ^{7}	0.58279 ^{1}	0.62448 ^{9}	0.59122 ^{3}	0.60016 ^{6}	0.59234 ^{4}
		$\hat{b}$	0.3133 ^{2}	0.31989 ^{6}	0.30745 ^{1}	0.32136 ^{7}	0.32607 ^{9}	0.3246 ^{8}	0.316 ^{3}	0.31751 ^{5}	0.31681 ^{4}
		$\hat{\theta}$	0.67705 ^{2}	0.71732 ^{9}	0.67169 ^{1}	0.69994 ^{5}	0.70508 ^{6}	0.71555 ^{8}	0.71066 ^{7}	0.68141 ^{3}	0.68434 ^{4}
	MSE	$\hat{a}$	0.65735 ^{4}	0.72247 ^{9}	0.66182 ^{6}	0.69203 ^{7}	0.62598 ^{1}	0.71433 ^{8}	0.64943 ^{3}	0.64631 ^{2}	0.65928 ^{5}
		$\hat{b}$	0.13265 ^{2}	0.14146 ^{7}	0.13124 ^{1}	0.14095 ^{6}	0.14536 ^{9}	0.14172 ^{8}	0.13432 ^{3}	0.13654 ^{5}	0.13457 ^{4}
		$\hat{\theta}$	0.71118 ^{1}	0.80857 ^{9}	0.71727 ^{2}	0.75761 ^{5}	0.77219 ^{6}	0.80166 ^{8}	0.78422 ^{7}	0.7304 ^{3}	0.73835 ^{4}
	MRE	$\hat{a}$	0.1976 ^{5}	0.20608 ^{8}	0.19699 ^{2}	0.20096 ^{7}	0.19426 ^{1}	0.20816 ^{9}	0.19707 ^{3}	0.20005 ^{6}	0.19745 ^{4}
		$\hat{b}$	0.34811 ^{2}	0.35543 ^{6}	0.34161 ^{1}	0.35707 ^{7}	0.3623 ^{9}	0.36067 ^{8}	0.35111 ^{3}	0.35279 ^{5}	0.35201 ^{4}
		$\hat{\theta}$	0.33853 ^{2}	0.35866 ^{9}	0.33585 ^{1}	0.34997 ^{5}	0.35254 ^{6}	0.35777 ^{8}	0.35533 ^{7}	0.3407 ^{3}	0.34217 ^{4}
	$\sum Ranks$		25 ^{2}	71 ^{8}	17 ^{1}	56 ^{7}	48 ^{6}	74 ^{9}	39 ^{5}	38 ^{4}	37 ^{3}
250	BIAS	$\hat{a}$	0.39662 ^{2}	0.4051 ^{6}	0.39933 ^{3}	0.42327 ^{8}	0.40451 ^{5}	0.39937 ^{4}	0.41941 ^{7}	0.38515 ^{1}	0.42853 ^{9}
		$\hat{b}$	0.29421 ^{8}	0.29096 ^{5}	0.29122 ^{6}	0.29928 ^{9}	0.28822 ^{4}	0.28672 ^{3}	0.28541 ^{2}	0.27759 ^{1}	0.29164 ^{7}
		$\hat{\theta}$	0.57047 ^{3}	0.5736 ^{4}	0.57915 ^{6}	0.59318 ^{8}	0.57607 ^{5}	0.56211 ^{2}	0.58368 ^{7}	0.55989 ^{1}	0.59556 ^{9}
	MSE	$\hat{a}$	0.30221 ^{2}	0.30551 ^{4}	0.30311 ^{3}	0.35154 ^{9}	0.31591 ^{6}	0.30607 ^{5}	0.33139 ^{7}	0.27854 ^{1}	0.34993 ^{8}
		$\hat{b}$	0.12537 ^{8}	0.12205 ^{7}	0.12146 ^{6}	0.1263 ^{9}	0.11979 ^{5}	0.11895 ^{4}	0.11488 ^{2}	0.11025 ^{1}	0.11808 ^{3}
		$\hat{\theta}$	0.53497 ^{3}	0.53952 ^{4}	0.54302 ^{5}	0.56407 ^{9}	0.55325 ^{7}	0.51686 ^{1}	0.55198 ^{6}	0.51757 ^{2}	0.56229 ^{8}
	MRE	$\hat{a}$	0.13221 ^{2}	0.13503 ^{6}	0.13311 ^{3}	0.14109 ^{8}	0.13484 ^{5}	0.13312 ^{4}	0.1398 ^{7}	0.12838 ^{1}	0.14284 ^{9}
		$\hat{b}$	0.3269 ^{8}	0.32329 ^{5}	0.32357 ^{6}	0.33253 ^{9}	0.32025 ^{4}	0.31858 ^{3}	0.31712 ^{2}	0.30844 ^{1}	0.32405 ^{7}
		$\hat{\theta}$	0.28524 ^{3}	0.2868 ^{4}	0.28958 ^{6}	0.29659 ^{8}	0.28803 ^{5}	0.28105 ^{2}	0.29184 ^{7}	0.27995 ^{1}	0.29778 ^{9}
	$\sum Ranks$		39 ^{3}	45 ^{5}	44 ^{4}	77 ^{9}	46 ^{6}	28 ^{2}	47 ^{7}	10 ^{1}	69 ^{8}
400	BIAS	$\hat{a}$	0.30222 ^{2}	0.33022 ^{9}	0.31215 ^{6}	0.30516 ^{3}	0.31041 ^{5}	0.30617 ^{4}	0.31526 ^{7}	0.29711 ^{1}	0.32329 ^{8}
		$\hat{b}$	0.27488 ^{7}	0.26769 ^{4}	0.26906 ^{5}	0.26205 ^{1}	0.27651 ^{9}	0.27337 ^{6}	0.26712 ^{3}	0.2639 ^{2}	0.27532 ^{8}
		$\hat{\theta}$	0.49448 ^{4}	0.50706 ^{8}	0.4965 ^{5}	0.48847 ^{3}	0.49784 ^{6}	0.48449 ^{2}	0.5015 ^{7}	0.46853 ^{1}	0.51461 ^{9}
	MSE	$\hat{a}$	0.17002 ^{2}	0.20461 ^{9}	0.18494 ^{6}	0.1718 ^{3}	0.18208 ^{5}	0.17697 ^{4}	0.19295 ^{8}	0.16879 ^{1}	0.19085 ^{7}
		$\hat{b}$	0.1126 ^{8}	0.1031 ^{2}	0.10766 ^{5}	0.10205 ^{1}	0.11365 ^{9}	0.11236 ^{7}	0.10415 ^{3}	0.10554 ^{4}	0.1092 ^{6}
		$\hat{\theta}$	0.39733 ^{4}	0.40919 ^{7}	0.40055 ^{6}	0.38915 ^{3}	0.39882 ^{5}	0.38375 ^{2}	0.41294 ^{8}	0.36142 ^{1}	0.42418 ^{9}
	MRE	$\hat{a}$	0.10074 ^{2}	0.11007 ^{9}	0.10405 ^{6}	0.10172 ^{3}	0.10347 ^{5}	0.10206 ^{4}	0.10509 ^{7}	0.09904 ^{1}	0.10776 ^{8}
		$\hat{b}$	0.30542 ^{7}	0.29743 ^{4}	0.29895 ^{5}	0.29116 ^{1}	0.30724 ^{9}	0.30374 ^{6}	0.29679 ^{3}	0.29322 ^{2}	0.30592 ^{8}
		$\hat{\theta}$	0.24724 ^{4}	0.25353 ^{8}	0.24825 ^{5}	0.24423 ^{3}	0.24892 ^{6}	0.24224 ^{2}	0.25075 ^{7}	0.23426 ^{1}	0.2573 ^{9}
	$\sum Ranks$		40 ^{4}	60 ^{8}	49 ^{5}	21 ^{2}	59 ^{7}	37 ^{3}	53 ^{6}	14 ^{1}	72 ^{9}
600	BIAS	$\hat{a}$	0.24666 ^{2}	0.25303 ^{9}	0.24706 ^{3}	0.24525 ^{1}	0.25105 ^{8}	0.25007 ^{5}	0.25015 ^{6}	0.2471 ^{4}	0.25027 ^{7}
		$\hat{b}$	0.25303 ^{8}	0.2493 ^{6}	0.25202 ^{7}	0.24822 ^{3}	0.2477 ^{2}	0.24892 ^{5}	0.24884 ^{4}	0.2476 ^{1}	0.25639 ^{9}
		$\hat{\theta}$	0.42389 ^{6}	0.42183 ^{4}	0.42726 ^{8}	0.41651 ^{1}	0.4262 ^{7}	0.41882 ^{2}	0.42371 ^{5}	0.42089 ^{3}	0.43371 ^{9}
	MSE	$\hat{a}$	0.11338 ^{7}	0.12115 ^{9}	0.10938 ^{1}	0.10992 ^{2}	0.11256 ^{4}	0.11593 ^{8}	0.11162 ^{3}	0.11312 ^{6}	0.1129 ^{5}
		$\hat{b}$	0.1021 ^{8}	0.09816 ^{7}	0.09746 ^{6}	0.09544 ^{4}	0.09329 ^{1}	0.09714 ^{5}	0.0946 ^{3}	0.09418 ^{2}	0.10292 ^{9}
		$\hat{\theta}$	0.29258 ^{8}	0.28816 ^{5}	0.28676 ^{4}	0.27799 ^{1}	0.2905 ^{7}	0.28128 ^{2}	0.2892 ^{6}	0.28308 ^{3}	0.29795 ^{9}
	MRE	$\hat{a}$	0.08222 ^{2}	0.08434 ^{9}	0.08235 ^{3}	0.08175 ^{1}	0.08368 ^{8}	0.08336 ^{5}	0.08338 ^{6}	0.08237 ^{4}	0.08342 ^{7}
		$\hat{b}$	0.28114 ^{8}	0.277 ^{6}	0.28002 ^{7}	0.2758 ^{3}	0.27522 ^{2}	0.27658 ^{5}	0.27649 ^{4}	0.27511 ^{1}	0.28488 ^{9}
		$\hat{\theta}$	0.21194 ^{6}	0.21091 ^{4}	0.21363 ^{8}	0.20826 ^{1}	0.2131 ^{7}	0.20941 ^{2}	0.21185 ^{5}	0.21044 ^{3}	0.21685 ^{9}
	$\sum Ranks$		55 ^{7}	59 ^{8}	47 ^{6}	17 ^{1}	46 ^{5}	39 ^{3}	42 ^{4}	27 ^{2}	73 ^{9}

TABLE 6. Partial and overall ranks of all estimation methods of PXL distribution.

Parameter	n	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	MSADE	MSALDE
$a = 1.5, b = 0.5, \theta = 0.25$	35	8.0	5.0	4.0	6.0	7.0	2.0	1.0	9.0	3.0
	100	7.0	6.0	1.0	8.0	2.0	3.0	5.0	9.0	4.0
	250	6.0	7.0	4.0	5.0	9.0	3.0	1.0	8.0	2.0
	400	1.0	7.0	4.0	2.5	9.0	6.0	2.5	5.0	8.0
	600	1.0	6.5	4.5	2.0	8.0	3.0	4.5	9.0	6.5
$a = 2.5, b = 0.75, \theta = 1.5$	35	4.0	7.0	6.0	5.0	3.0	9.0	8.0	2.0	1.0
	100	3.0	4.0	5.5	5.5	2.0	9.0	8.0	7.0	1.0
	250	7.0	3.0	8.0	4.0	1.5	9.0	1.5	5.0	6.0
	400	3.0	8.0	4.0	7.0	2.0	9.0	1.0	6.0	5.0
	600	5.0	6.0	3.0	8.0	9.0	2.0	1.0	4.0	7.0
$a = 2.0, b = 1.5, \theta = 0.5$	35	6.0	4.0	3.0	9.0	5.0	8.0	7.0	2.0	1.0
	100	8.0	4.0	1.5	7.0	5.5	5.5	9.0	3.0	1.5
	250	6.0	4.0	3.0	5.0	7.0	9.0	8.0	1.0	2.0
	400	1.5	1.5	6.0	8.0	9.0	5.0	4.0	3.0	7.0
	600	7.0	3.0	6.0	4.0	5.0	2.0	8.5	8.5	1.0
$a = 3.5, b = 0.2, \theta = 0.9$	35	7.0	4.0	3.0	8.0	1.0	9.0	5.5	5.5	2.0
	100	2.0	9.0	8.0	5.5	3.0	1.0	5.5	7.0	4.0
	250	4.5	3.0	1.0	7.0	2.0	6.0	4.5	9.0	8.0
	400	9.0	6.0	5.0	3.0	8.0	7.0	4.0	2.0	1.0
	600	8.0	6.0	3.5	9.0	2.0	7.0	5.0	1.0	3.5
$a = 3.0, b = 0.9, \theta = 0.2$	35	3.0	6.5	2.0	1.0	4.0	9.0	5.0	8.0	6.5
	100	2.0	8.0	1.0	7.0	6.0	9.0	5.0	4.0	3.0
	250	3.0	5.0	4.0	9.0	6.0	2.0	7.0	1.0	8.0
	400	4.0	8.0	5.0	2.0	7.0	3.0	6.0	1.0	9.0
	600	7.0	8.0	6.0	1.0	5.0	3.0	4.0	2.0	9.0
$\sum$ Ranks		123.0	139.5	102.0	138.5	128.0	140.5	121.5	122.0	110.0
Overall Rank		5	8	1	7	6	9	3	4	2

For the BNXL model, the maximum likelihood estimates are

$$\hat{a} = 16.68304, \quad \hat{b} = 7.036943, \quad \hat{\theta} = 0.8074539.$$

The estimated variance–covariance matrix is

$$\widehat{\text{Var}}(\hat{\boldsymbol{\lambda}}) = \begin{pmatrix} 141.64163 & -399.45164 & 27.74016 \\ -399.45164 & 1256.84388 & -86.16419 \\ 27.74016 & -86.16419 & 5.916374 \end{pmatrix}.$$

The corresponding estimated standard errors are

$$SE(\hat{a}) = 11.9013, \quad SE(\hat{b}) = 35.4520, \quad SE(\hat{\theta}) = 2.4324.$$

The relatively large standard errors, particularly for a and b , suggest that the individual shape parameters are estimated with considerable uncertainty for this data set.

Table 7 presents the maximum likelihood estimates and goodness-of-fit criteria for the competing distributions. The BNXL distribution attains the smallest AIC, BIC, -2ℓ , and AICc values. However, its values are very close to those of the beta-exponential and beta-Lindley models. Thus, the results indicate that the BNXL distribution provides the best fit according to the reported criteria, while these beta-generated alternatives remain highly competitive.

TABLE 7. Maximum likelihood estimates and goodness-of-fit criteria for data set 1.

Model	a	b	θ	γ	AIC	BIC	-2ℓ	AICc
Exponential	–	–	0.4166675	–	182.045	183.9162	180.045	182.132
Lindley	–	–	0.6666664	–	166.2406	168.1118	164.2406	166.3275
XLindley	–	–	0.5772491	–	174.4943	176.3655	172.4943	174.5812
New XLindley	–	–	0.670488	–	170.271	172.1422	168.271	170.3579
Xgamma	–	–	0.9279678	–	169.8614	171.7326	167.8614	169.9484
Zeghdoudi	–	–	1.101979	–	138.8313	140.7025	136.8313	138.9182
Two-parameter Lindley I	–	–	0.8546036	8.38814e-05	149.545	153.2874	145.545	149.8117
Gamma Lindley	–	–	0.8314911	42.62868	149.9641	153.7065	145.9641	150.2308
New quasi-Lindley	–	–	0.8248596	34.76987	150.3554	154.0978	146.3554	150.6221
Two-parameter Lindley II	–	–	0.8281973	42.72697	150.345	154.0874	146.345	150.6116
Power XLindley	–	–	1.599186	0.4308425	291.5391	295.2815	287.5391	291.8058
Beta-Lindley	18.89298	4.256092	1.08148	–	82.8295	88.44311	76.8295	83.37496
Beta-exponential	25.13779	5.168618	0.7716722	–	82.82384	88.43744	76.82384	83.3693
Chen	–	–	0.02940777	1.288413	91.30852	95.05093	87.30852	91.57519
Beta-New XLindley	16.68304	7.036943	0.8074539	–	82.77138	88.38499	76.77138	83.31684

Data set 2. The second data set consists of the recovery durations, measured in weeks, of 74 individuals affected by Marburg virus disease in Angola. The data provide an example of positive and right-skewed duration measurements and are therefore useful for assessing the ability of lifetime distributions to model heterogeneous recovery times. Additional epidemiological information on Marburg virus disease is available from the World Health Organization (<https://www.who.int/>).

For the BNXL distribution, the maximum likelihood estimates are

$$\hat{a} = 99.46829, \quad \hat{b} = 0.08597327, \quad \hat{\theta} = 3.779259.$$

The estimated variance–covariance matrix is

$$\widehat{\text{Var}}(\hat{\boldsymbol{\lambda}}) = \begin{pmatrix} 6426.206462 & -1.010469 & 42.351445 \\ -1.010469 & 0.0003792497 & -0.0101268704 \\ 42.351445 & -0.0101268704 & 0.37801906 \end{pmatrix}.$$

The corresponding estimated standard errors are

$$SE(\hat{a}) = 80.1636, \quad SE(\hat{b}) = 0.01947, \quad SE(\hat{\theta}) = 0.61483.$$

Table 8 shows that the BNXL distribution again produces the smallest AIC, BIC, -2ℓ , and AICc values among the fitted models. In particular, its AIC value is 340.5521, compared with 341.4334 for the beta-exponential distribution and 342.4860 for the beta-Lindley distribution. These relatively small differences indicate that the three beta-generated models provide competitive fits, with the BNXL distribution having the lowest values of the reported criteria.

TABLE 8. Maximum likelihood estimates and goodness-of-fit criteria for data set 2.

Model	a	b	θ	γ	AIC	BIC	-2ℓ	AICc
Exponential	–	–	0.1989245	–	388.9947	391.2987	386.9947	389.0502
Lindley	–	–	0.3466344	–	367.9865	370.2906	365.9865	368.0421
XLindley	–	–	0.3128214	–	374.8971	377.2012	372.8971	374.9527
New XLindley	–	–	0.311163	–	377.825	380.129	375.825	377.8805
Xgamma	–	–	0.4833675	–	370.3286	372.6327	368.3286	370.3841
Zeghdoudi	–	–	0.5536467	–	351.3284	353.6325	349.3284	351.384
Two-parameter Lindley I	–	–	0.3961074	0.000256688	358.5446	363.1527	354.5446	358.7136
Gamma Lindley	–	–	0.3965907	58.91614	358.7517	363.3598	354.7517	358.9207
New quasi-Lindley	–	–	0.3966442	46.35307	358.688	363.2962	354.688	358.8571
Two-parameter Lindley II	–	–	0.3963045	67.04979	358.7988	363.4069	354.7988	358.9678
Power XLindley	–	–	1.413932	0.3184585	576.7272	581.3353	572.7272	576.8962
Beta-Lindley	55.89799	0.1041794	3.051239	–	342.486	349.3982	336.486	342.8289
Beta-exponential	121.4675	0.101828	3.009173	–	341.4334	348.3456	335.4334	341.7763
Chen	–	–	0.0439729	0.6392914	357.2962	361.9043	353.2962	357.4652
Beta-New XLindley	99.46829	0.08597327	3.779259	–	340.5521	347.4643	334.5521	340.895

Likelihood ratio test. Since the new XLindley distribution is obtained from the BNXL model by setting $a = b = 1$, a likelihood ratio (LR) test can be used to assess whether the two additional beta-generated shape parameters provide a significant improvement in fit. We consider

$$H_0 : a = b = 1 \quad \text{versus} \quad H_1 : (a, b) \neq (1, 1).$$

The LR statistic is

$$\Lambda = 2 \left\{ \ell(\hat{\boldsymbol{\Theta}}) - \ell(\hat{\boldsymbol{\Theta}}_0) \right\}, \quad (35)$$

where $\hat{\Theta}$ and $\hat{\Theta}_0$ denote the MLEs under the unrestricted BNXL model and the restricted new XLindley model, respectively. Under H_0 , Λ is asymptotically distributed as a chi-square random variable with two degrees of freedom.

For data sets 1 and 2, the LR statistics are 91.49962 and 41.27290, respectively, with corresponding p -values

$$1.35 \times 10^{-20} \quad \text{and} \quad 1.09 \times 10^{-9}.$$

Thus, H_0 is strongly rejected for both data sets, providing statistical evidence that the additional shape parameters of the BNXL distribution significantly improve the fit relative to the new XLindley distribution.

7. CONCLUSION

We have introduced a more versatile three-parameter extension of the XLindley distribution, termed the beta-new XLindley (BNXL). The BNXL model encompasses several significant mathematical attributes, including reliability and hazard rate functions, moments, the moment-generating function, entropy, stress–strength reliability, and order statistics. We obtained the anticipated information matrix via comprehensive parameter analysis employing various established estimation methods. Two disparate real data sets were employed with the BNXL distribution to evaluate its efficacy. Even though the BNXL often fits perfectly, it is important to remember that this can lead to unstable parameter estimates and lower performance. When there aren't many samples, the model's flexibility could lead to unstable parameter estimates and less reliable predictions. When datasets have very skewed or heavy tails that go beyond what BNXL can represent, other models may work better. So, it is best to do thorough preliminary model diagnostics and goodness-of-fit tests before putting them into action.

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