

# CLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH THE $q$ -BERNOULLI POLYNOMIALS

ABDULLAH ALSOBOH<sup>1</sup>, ALA AMOURAH<sup>2,3</sup>, YOUSEF ALQUDAH<sup>3</sup>, FAISAL AL-SHARQI<sup>4,\*</sup>,  
FAHAD ALABRI<sup>1,\*</sup>, SARAH JAWAD SHOJA<sup>5</sup>

<sup>1</sup>College of Applied and Health Sciences, A'Sharqiyah University, P.O. Box 42, Postal Code 400, Ibra, Oman

<sup>2</sup>Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar 311, Oman

<sup>3</sup>Department of Mathematics, Faculty of Arts and Science, Amman Arab University, Amman 11953, Jordan

<sup>4</sup>Department of Mathematics, Faculty of Education for Pure Sciences, University of Anbar, Ramadi, Anbar, Iraq

<sup>5</sup>Laser and Optoelectronics Department, College of Engineering, Al-Ayen Iraqi University (AUIQ), An Nasiriyah, Iraq

\*Corresponding authors: Faisal.ghazi@uoanbar.edu.iq, fahad.alabri@asu.edu.om

Received Jul. 13, 2026

**ABSTRACT.** In this work, we introduce and study new subclasses of bi-univalent functions generated by the  $q$ -Bernoulli polynomials within the framework of analytic subordination. The proposed families include generalized bi-starlike and bi-convex functions whose defining conditions incorporate the structure of the associated  $q$ -Bernoulli generating function. For these classes, explicit estimates are derived for the first few Taylor–Maclaurin coefficients. In addition, suitable bounds are established for the corresponding Fekete–Szegő functional. The results unify and extend several earlier findings in bi-univalent function theory and further demonstrate the usefulness of  $q$ -Bernoulli polynomials in the development of geometric function theory.

2020 Mathematics Subject Classification. 30C45.

Key words and phrases. analytic functions; convex functions; Fekete–Szegő functional;  $q$ -Bernoulli polynomials; starlike functions; univalent functions.

## 1. INTRODUCTION

Let  $\mathcal{A}$  denote the class of all functions analytic in the open unit disk

$$\mathcal{O} = \{z = a + ib \in \mathbb{C} : |z| < 1\}.$$

Throughout this paper, every function  $\mathfrak{F} \in \mathcal{A}$  is assumed to be normalized by

$$\mathfrak{F}(0) = 0 \quad \text{and} \quad \mathfrak{F}'(0) = 1.$$

Hence, each  $\mathfrak{F} \in \mathcal{A}$  has a Taylor–Maclaurin (T.M) series representation of the form

$$\mathfrak{S}(z) = z + \sum_{\iota=2}^{\infty} \delta_{\iota} z^{\iota}, \quad z \in \mathcal{O}. \quad (1.1)$$

The above normalization provides a unified setting in which the geometric behavior of analytic functions can be studied without the influence of arbitrary translations or dilations. It therefore serves as a natural starting point for investigating coefficient estimates, subordination properties, and various subclasses of univalent and bi-univalent functions.

Let  $\mathfrak{S}_1$  and  $\mathfrak{S}_2$  be analytic functions in  $\mathcal{O}$ . The function  $\mathfrak{S}_1$  is said to be subordinate to  $\mathfrak{S}_2$ , denoted by

$$\mathfrak{S}_1 \prec \mathfrak{S}_2,$$

if there exists a Schwarz function  $\eta$ , analytic in  $\mathcal{O}$ , such that

$$\eta(0) = 0 \quad \text{and} \quad |\eta(z)| < 1 \quad (z \in \mathcal{O}),$$

for which

$$\mathfrak{S}_1(z) = \mathfrak{S}_2(\eta(z)), \quad z \in \mathcal{O}.$$

In particular, when  $\mathfrak{S}_2$  is univalent in  $\mathcal{O}$ , the subordination relation  $\mathfrak{S}_1 \prec \mathfrak{S}_2$  is equivalent to

$$\mathfrak{S}_1(0) = \mathfrak{S}_2(0) \quad \text{and} \quad \mathfrak{S}_1(\mathcal{O}) \subseteq \mathfrak{S}_2(\mathcal{O}).$$

The symbol  $\mathcal{S}$  will be used for the family of all functions in  $\mathcal{A}$  that are univalent in the unit disk  $\mathcal{O}$ . Another fundamental family in geometric function theory (GFT) is the Carathéodory class  $\mathcal{P}$ , consisting of functions  $\mathcal{P}$  that are analytic in  $\mathcal{O}$  and satisfy

$$\mathcal{P}(0) = 1 \quad \text{and} \quad \operatorname{Re}\{\mathcal{P}(z)\} > 0, \quad z \in \mathcal{O}.$$

Each function  $\mathcal{P} \in \mathcal{P}$  possesses a T.M expansion of the form

$$\mathcal{P}(z) = 1 + \sum_{\iota=1}^{\infty} \mathcal{P}_{\iota} z^{\iota} = 1 + \mathcal{P}_1 z + \mathcal{P}_2 z^2 + \mathcal{P}_3 z^3 + \cdots, \quad z \in \mathcal{O}. \quad (1.2)$$

A classical result due to Carathéodory states that the coefficients of every function belonging to  $\mathcal{P}$  satisfy

$$|\mathcal{P}_{\iota}| \leq 2, \quad \iota \geq 1$$

(see, for instance, [1]). Equivalently, an analytic function  $\mathcal{P}$ , normalized by  $\mathcal{P}(0) = 1$ , belongs to  $\mathcal{P}$  if and only if

$$\mathcal{P}(z) \prec \frac{1+z}{1-z}, \quad z \in \mathcal{O}.$$

One of the central subclasses of  $\mathcal{S}$  is the family of starlike functions, customarily denoted by  $\mathcal{S}^*$ . A comprehensive generalization of this class was introduced by Ma and Minda [2] through the language of analytic subordination. They defined

$$\mathcal{S}^*(\Omega) = \left\{ \mathfrak{S} \in \mathcal{A} : \frac{z\mathfrak{S}'(z)}{\mathfrak{S}(z)} \prec \Omega(z), \quad z \in \mathcal{O} \right\},$$

where  $\Omega$  is analytic and univalent in  $\mathcal{O}$ , normalized by

$$\Omega(0) = 1, \quad \operatorname{Re}\{\Omega(z)\} > 0,$$

and maps  $\mathcal{O}$  onto a region that is symmetric with respect to the real axis and starlike with respect to the point 1.

The significance of this formulation lies in its ability to encompass a wide range of classical and recently introduced starlike families within a single framework. By choosing the comparison function  $\Omega$  appropriately, one recovers several familiar subclasses of analytic functions. Consequently, the Ma–Minda setting has become a powerful tool for studying coefficient bounds, geometric mapping properties, subordination relations, and various extremal problems in GFT.

The study of univalent functions, and starlike functions in particular, occupies a prominent position in geometric function theory because of its close relationship with complex analysis, conformal mapping, and the geometry of analytic functions. Within this broad setting, considerable attention has been devoted to constructing and analyzing new subclasses of starlike functions characterized by prescribed geometric or analytic behavior.

One especially effective approach is based on the incorporation of classical orthogonal polynomials, including the Legendre, Chebyshev, Hermite, and Gegenbauer polynomials, into the defining conditions of analytic function classes; see, for example, [1]. The rich algebraic structure and recurrence properties of these polynomial families provide a systematic mechanism for introducing additional parameters and geometric features into the associated function classes.

By embedding suitable orthogonal polynomial sequences into generating functions, coefficient relations, or subordination conditions, one can formulate subclasses exhibiting such properties as convexity, and starlikeness. This framework also facilitates the derivation of coefficient estimates and related functional inequalities. For completeness, we briefly recall below four classical families of orthogonal polynomials that are frequently employed in this context.

Orthogonal polynomials occupy an important position in GFT and in several related branches of mathematical analysis. Among the classical families most frequently encountered are the Legendre, Chebyshev, Hermite, and Gegenbauer polynomials. These polynomial sequences are distinguished by their well-established generating functions, recurrence formulas, and rich algebraic structure. Owing to these properties, they provide a convenient framework for introducing and studying new subclasses of analytic, univalent, and bi-univalent functions. In particular, their generating functions can be incorporated into subordination conditions, thereby allowing geometric properties and coefficient behavior to be investigated in a systematic manner.

The Legendre polynomials  $P_\ell(\vartheta)$  are defined through the generating function

$$\frac{1}{\sqrt{1-2\vartheta t+t^2}} = \sum_{\iota=0}^{\infty} P_{\iota}(\vartheta)t^{\iota}. \quad (1.3)$$

They also obey the three-term recurrence formula

$$(\iota+1)P_{\iota+1}(\vartheta) = (2\iota+1)\vartheta P_{\iota}(\vartheta) - \iota P_{\iota-1}(\vartheta). \quad (1.4)$$

Using this polynomial sequence, one may construct an associated analytic function by incorporating the Legendre coefficients into its Taylor–Maclaurin expansion. More precisely, we consider

$$\mathfrak{S}(z) = z + \sum_{\iota=2}^{\infty} \delta_{\iota} P_{\iota-1}(\lambda) z^{\iota}. \quad (1.5)$$

This representation provides a convenient mechanism for transferring the algebraic structure of the Legendre polynomials to the coefficient framework of analytic functions.

In a similar manner, the Chebyshev polynomials of the first kind, denoted by  $T_{\iota}(\vartheta)$ , are characterized by the generating function

$$\frac{1-\vartheta t}{1-2\vartheta t+t^2} = \sum_{\iota=0}^{\infty} T_{\iota}(\vartheta)t^{\iota}. \quad (1.6)$$

These polynomials satisfy the three-term recurrence relation

$$T_{\iota+1}(\vartheta) = 2\vartheta T_{\iota}(\vartheta) - T_{\iota-1}(\vartheta), \quad \iota \geq 1. \quad (1.7)$$

By incorporating the Chebyshev polynomial coefficients into the Taylor–Maclaurin expansion, one obtains the associated analytic function

$$\mathfrak{S}(z) = z + \sum_{\iota=2}^{\infty} \delta_{\iota} T_{\iota-1}(\lambda) z^{\iota}. \quad (1.8)$$

This construction transfers the structural and recurrence properties of the Chebyshev polynomials to the coefficient behavior of the corresponding analytic function.

The Hermite polynomials  $H_{\iota}(\vartheta)$  are determined by the exponential generating function

$$e^{2\vartheta t-t^2} = \sum_{\iota=0}^{\infty} \frac{H_{\iota}(\vartheta)}{\iota!} t^{\iota}. \quad (1.9)$$

They obey the recurrence formula

$$H_{\iota+1}(\vartheta) = 2\vartheta H_{\iota}(\vartheta) - 2\iota H_{\iota-1}(\vartheta), \quad \iota \geq 1. \quad (1.10)$$

Accordingly, the analytic function associated with the Hermite polynomial sequence may be written as

$$\mathfrak{S}(z) = z + \sum_{\iota=2}^{\infty} \delta_{\iota} H_{\iota-1}(\lambda) z^{\iota}. \quad (1.11)$$

Thus, the algebraic structure of the Hermite polynomials is naturally embedded in the Taylor coefficients of  $\mathfrak{S}$ .

Finally, the Gegenbauer polynomials  $C_\iota^{(\delta)}(\vartheta)$  are introduced through the generating function

$$\frac{1}{(1 - 2\vartheta t + t^2)^\delta} = \sum_{\iota=0}^{\infty} C_\iota^{(\delta)}(\vartheta) t^\iota. \quad (1.12)$$

The corresponding three-term recurrence relation is given by

$$(\iota + 1)C_{\iota+1}^{(\delta)}(\vartheta) = 2(\iota + \delta)\vartheta C_\iota^{(\delta)}(\vartheta) - (\iota + 2\delta - 1)C_{\iota-1}^{(\delta)}(\vartheta), \quad \iota \geq 1. \quad (1.13)$$

Using the Gegenbauer polynomial sequence, the related analytic function takes the form

$$\mathfrak{S}(z) = z + \sum_{\iota=2}^{\infty} \delta_\iota C_{\iota-1}^{(\delta)}(\lambda) z^\iota. \quad (1.14)$$

This representation allows the parameters and structural features of the Gegenbauer polynomials to be incorporated directly into the coefficient expansion of  $\mathfrak{S}$ .

Taken together, these classical orthogonal polynomial families furnish a broad and coherent framework for defining and investigating subclasses of analytic, univalent, and bi-univalent functions. Their generating functions and recurrence relations make it possible to encode special algebraic structures into analytic expansions, thereby facilitating the study of coefficient estimates, geometric properties, subordination conditions, and related extremal problems.

Each function  $\mathfrak{S}$  introduced through the preceding constructions belongs to a particular subclass of analytic functions whose geometric properties depend essentially on the selected family of orthogonal polynomials and the value of the parameter  $\lambda$ . Such polynomial-based representations have proved highly effective in the investigation of coefficient estimates, Hankel determinants, Fekete–Szegő-type functionals, and other extremal problems; see, for example, [9–11].

We next recall some basic facts concerning bi-univalent functions. For every function  $\mathfrak{S} \in \mathcal{S} \subset \mathcal{A}$ , there exists an inverse function  $\mathfrak{S}^{-1}$  satisfying

$$\mathfrak{S}^{-1}(\mathfrak{S}(z)) = z \quad \text{and} \quad \mathfrak{S}(\mathfrak{S}^{-1}(\xi)) = \xi, \quad (1.15)$$

where

$$(z \in \mathcal{O}, |\xi| < r_0(\mathfrak{S})),$$

and  $r_0(\mathfrak{S})$  denotes a positive radius within which the inverse function is well defined and analytic.

Suppose that  $\mathfrak{S} \in \mathcal{S}$  has the Taylor–Maclaurin expansion given in (1.1). Then the local inverse

$$\chi(\xi) := \mathfrak{S}^{-1}(\xi)$$

admits the series representation

$$\chi(\xi) = \mathfrak{S}^{-1}(\xi) = \xi - \delta_2 \xi^2 + (2\delta_2^2 - \delta_3) \xi^3 - (5\delta_2^3 - 5\delta_2\delta_3 + \delta_4) \xi^4 + \cdots, \quad (1.16)$$

which is valid for  $|\xi| < r_0(\mathfrak{S})$ . The coefficients appearing in this inverse expansion provide an essential link between the geometry of  $\mathfrak{S}$  and that of its inverse. Consequently, (1.16) plays a central role in

inverse coefficient problems, Hankel determinant estimates, and related extremal questions in geometric function theory.

A function  $\mathfrak{S} \in \mathcal{S}$  is called *bi-univalent* in  $\mathcal{O}$  whenever both  $\mathfrak{S}$  and its inverse  $\mathfrak{S}^{-1}$  are univalent in the unit disk. The family of all such functions is conventionally denoted by  $\Sigma$ . Thus,

$$\mathfrak{S} \in \Sigma \iff \mathfrak{S} \in \mathcal{S} \text{ and } \mathfrak{S}^{-1} \in \mathcal{S}.$$

Quantum calculus (or  $q$ -calculus), introduced by Jackson [16, 17], offers a discrete analogue of classical calculus and has gained considerable attention in GFT. Aral and Gupta [18] further advanced the subject by studying  $q$ -analogues of various operators. In this framework, the parameter  $q \in (0, 1)$  ensures convergence and analytical stability.

A fundamental object in quantum calculus is the  $q$ -analogue of the classical exponential function. Following [27], it is defined by

$$e_q(z) = \sum_{\iota=0}^{\infty} \frac{z^{\iota}}{[\iota; q]!}, \quad 0 < q < 1, \quad |z| < \frac{1}{1-q}, \quad (1.17)$$

where the  $q$ -factorial is given by

$$[\iota; q]! = [\iota; q][\iota - 1; q] \cdots [3; q][2; q][1; q],$$

with  $[0; q]! = 1$  and  $[\iota; q]$  denotes the corresponding quantum integer; see [21].

The function  $e_q$  may be viewed as the natural counterpart of the ordinary exponential function within the framework of  $q$ -calculus. In fact, it is characterized as the unique analytic function satisfying

$$\partial_q \langle e_q(z) \rangle = e_q(z) \quad \text{and} \quad e_q(0) = 1.$$

Indeed, using

$$\partial_q \langle z^{\iota} \rangle = [\iota; q] z^{\iota-1},$$

we obtain

$$\begin{aligned} \partial_q \langle e_q(z) \rangle &= \sum_{\iota=0}^{\infty} \frac{\partial_q \langle z^{\iota} \rangle}{[\iota; q]!} \\ &= \sum_{\iota=1}^{\infty} \frac{[\iota; q] z^{\iota-1}}{[\iota; q]!} = \sum_{\iota=1}^{\infty} \frac{z^{\iota-1}}{[\iota-1; q]!} \\ &= \sum_{\iota=0}^{\infty} \frac{z^{\iota}}{[\iota; q]!} = e_q(z), \quad z \in \mathcal{O}. \end{aligned} \quad (1.18)$$

Thus, the  $q$ -exponential function retains the familiar eigenfunction property of the classical exponential, with the ordinary derivative replaced by the  $q$ -derivative.

The  $q$ -Bernoulli polynomials  $\mathcal{B}_{q,\iota}(\vartheta)$  which arise in the context of GFT in complex analysis, satisfy the following linear homogeneous recurrence relation (see [3]):

$$\mathcal{B}_{q,\iota}(q\vartheta) = q^\iota \left( \vartheta - \frac{1}{q[2;q]} \right) \mathcal{B}_{q,\iota-1}(\vartheta) - \frac{1}{[ \iota; q]} \sum_{j=0}^{\iota-2} \binom{\iota}{j}_q q^{j-1} \mathcal{B}_{q,\iota-j}(\vartheta). \quad (1.19)$$

The generating function of the  $q$ -Bernoulli polynomials  $\mathcal{B}_{q,\iota}(\vartheta)$  is given as follows (see [4]):

$$\mathcal{B}_{q,\iota}(\vartheta, h) = \frac{h}{e_q(h) - 1} e_q(h\vartheta) = \sum_{\iota=0}^{\infty} \mathcal{B}_{q,\iota}(\vartheta) \frac{h^\iota}{[\iota; q]!}, \quad |h| < 2\pi. \quad (1.20)$$

The first few  $q$ -Bernoulli polynomials are given by

$$\begin{aligned} \mathcal{B}_{q,0}(\vartheta) &= 1, \\ \mathcal{B}_{q,1}(\vartheta) &= \vartheta - \frac{q}{[2; q]}, \\ \mathcal{B}_{q,2}(\vartheta) &= \vartheta(\vartheta - 1) + \frac{q}{[2; q][3; q]}. \end{aligned} \quad (1.21)$$

The development of  $q$ -calculus has significantly broadened the scope of analytic function theory by providing an effective framework for constructing and studying new subclasses with rich geometric and algebraic structures. As a natural extension of classical calculus, this theory introduces additional flexibility into the formulation of analytic operators, subordination conditions, and coefficient problems, thereby revealing phenomena that are not readily accessible through traditional methods. Its influence is not confined to purely theoretical investigations, since  $q$ -calculus also arises in several areas of applied mathematics and mathematical physics. Consequently, it continues to provide a fertile foundation for the advancement of geometric function theory and for the exploration of new analytic models and related extremal problems; see, for example, [6, 22, 23, 28–34, 36–40].

## 2. THE NOVEL BI-UNIVALENT FUNCTION CLASSE

The framework of  $q$ -calculus, together with the algebraic structure of the  $q$ -Bernoulli polynomials, provides a natural and versatile setting for introducing new families of analytic and bi-univalent functions. Motivated by the generating function of these polynomials, we define new subclass of the bi-univalent family  $\Sigma$ , which represent generalized bi-starlike and bi-convex function classes.

**Definition 2.1.** Let  $0 \leq \beta \leq 1$ ,  $0 < q < 1$ , and  $\vartheta \in [-\pi, \pi]$ . A function  $\mathfrak{F} \in \Sigma$ , having the TM expansion given by (1.1), is said to belong to the class  $\mathcal{BS}_\Sigma^*(\beta; q, \vartheta)$  if the following subordination conditions hold:

$$(1 - \beta) \frac{z \partial_q \langle \mathfrak{F}(z) \rangle}{\mathfrak{F}(z)} + \beta \frac{\partial_q \langle z \partial_q \langle \mathfrak{F}(z) \rangle \rangle}{\partial_q \langle \mathfrak{F}(z) \rangle} \prec \mathcal{B}_q(\vartheta, z), \quad (2.1)$$

and

$$(1 - \beta) \frac{\xi \partial_q \langle \chi(\xi) \rangle}{\chi(\xi)} + \beta \frac{\partial_q \langle \xi \partial_q \langle \chi(\xi) \rangle \rangle}{\partial_q \langle \chi(\xi) \rangle} \prec \mathcal{B}_q(\vartheta, \xi), \quad (2.2)$$

where  $z, \xi \in \mathcal{O}$  and

$$\chi(\xi) = \mathfrak{S}^{-1}(\xi)$$

is the inverse function of  $\mathfrak{S}$ , whose expansion is given in (1.16).

The following examples illustrate several important specializations of the class introduced in Definition 2.1.

**Example 2.1.** Taking  $\beta = 0$  in Definition 2.1, we obtain the  $q$ -bi-starlike class

$$\mathcal{BS}_{\Sigma}^*(q, \vartheta) := \mathcal{BS}_{\Sigma}^*(0; q, \vartheta).$$

Thus, a function  $\mathfrak{S} \in \Sigma$ , of the form (1.1), belongs to  $\mathcal{BS}_{\Sigma}^*(q, \vartheta)$  if and only if

$$\frac{z \mathfrak{D}_q \langle \mathfrak{S}(z) \rangle}{\mathfrak{S}(z)} \prec \mathcal{B}_q(\vartheta, z), \quad z \in \mathcal{O}, \quad (2.3)$$

and

$$\frac{\xi \mathfrak{D}_q \langle \chi(\xi) \rangle}{\chi(\xi)} \prec \mathcal{B}_q(\vartheta, \xi), \quad \xi \in \mathcal{O}, \quad (2.4)$$

where  $\chi = \mathfrak{S}^{-1}$  is given by (1.16). Hence, this specialization describes the bi-starlike family generated by the  $q$ -Bernoulli comparison function.

**Example 2.2.** For  $\beta = 1$ , Definition 2.1 gives the  $q$ -bi-convex class

$$\mathcal{BC}_{\Sigma}(q, \vartheta) := \mathcal{BS}_{\Sigma}^*(1; q, \vartheta).$$

Accordingly, a function  $\mathfrak{S} \in \Sigma$  belongs to  $\mathcal{BC}_{\Sigma}(q, \vartheta)$  precisely when

$$\frac{\mathfrak{D}_q \langle z \mathfrak{D}_q \langle \mathfrak{S}(z) \rangle \rangle}{\mathfrak{D}_q \langle \mathfrak{S}(z) \rangle} \prec \mathcal{B}_q(\vartheta, z), \quad z \in \mathcal{O}, \quad (2.5)$$

and

$$\frac{\mathfrak{D}_q \langle \xi \mathfrak{D}_q \langle \chi(\xi) \rangle \rangle}{\mathfrak{D}_q \langle \chi(\xi) \rangle} \prec \mathcal{B}_q(\vartheta, \xi), \quad \xi \in \mathcal{O}, \quad (2.6)$$

where  $\chi = \mathfrak{S}^{-1}$  has the expansion stated in (1.16).

Using the Jackson  $q$ -derivative identity

$$\mathfrak{D}_q \langle z \mathfrak{D}_q \langle \mathfrak{S}(z) \rangle \rangle = \mathfrak{D}_q \langle \mathfrak{S}(z) \rangle + qz \mathfrak{D}_q^2 \langle \mathfrak{S}(z) \rangle,$$

the condition in (2.5) may equivalently be written as

$$1 + \frac{qz \mathfrak{D}_q^2 \langle \mathfrak{S}(z) \rangle}{\mathfrak{D}_q \langle \mathfrak{S}(z) \rangle} \prec \mathcal{B}_q(\vartheta, z).$$

A corresponding identity holds for the inverse function  $\chi$ .

**Example 2.3.** Letting  $q \rightarrow 1^-$  and choosing  $\beta = 0$ , the  $q$ -derivative reduces to the ordinary derivative. Therefore, the class in Example 2.1 approaches the bi-starlike family determined by

$$\frac{z \mathfrak{S}'(z)}{\mathfrak{S}(z)} \prec \mathcal{B}_1(\vartheta, z), \quad z \in \mathcal{O}, \quad (2.7)$$

and

$$\frac{\xi \chi'(\xi)}{\chi(\xi)} \prec \mathcal{B}_1(\vartheta, \xi), \quad \xi \in \mathcal{O}. \quad (2.8)$$

This limiting case recovers the corresponding classical bi-starlike setting associated with the Bernoulli generating function.

**Example 2.4.** If  $q \rightarrow 1^-$  and  $\beta = 1$ , then the conditions defining the class in Example 2.2 reduce to

$$1 + \frac{z \mathfrak{S}''(z)}{\mathfrak{S}'(z)} \prec \mathcal{B}_1(\vartheta, z), \quad z \in \mathcal{O}, \quad (2.9)$$

and

$$1 + \frac{\xi \chi''(\xi)}{\chi'(\xi)} \prec \mathcal{B}_1(\vartheta, \xi), \quad \xi \in \mathcal{O}. \quad (2.10)$$

Hence, the limiting form yields the classical bi-convex class governed by the Bernoulli comparison function.

These specializations demonstrate that the parameter  $\beta$  continuously connects the  $q$ -bi-starlike and  $q$ -bi-convex conditions, whereas the limiting process  $q \rightarrow 1^-$  recovers their ordinary-calculus counterparts. Consequently, the class  $\mathcal{BS}_{\Sigma}^*(\beta; q, \vartheta)$  provides a unified framework that simultaneously contains quantum and classical families of bi-univalent functions.

### 3. MAIN RESULTS

In this section, we establish estimates for the initial T.M coefficients  $|\delta_2|$  and  $|\delta_3|$  for functions belonging to the generalized bi-univalent class  $\mathcal{BS}_{\Sigma}^*(\beta; q, \vartheta)$ . As immediate consequences, the corresponding coefficient bounds for the bi-starlike and bi-convex subclasses

$$\mathcal{BS}_{\Sigma}^*(q, \vartheta) \quad \text{and} \quad \mathcal{BC}_{\Sigma}(q, \vartheta),$$

respectively, are obtained by taking  $\beta = 0$  and  $\beta = 1$ . We further derive the associated Fekete–Szegő inequality for the functional

$$|\delta_3 - \mu \delta_2^2|,$$

where  $\mu \in \mathbb{R}$ .

Firstly, let  $\mathcal{P}(z) = 1 + \mathcal{P}_1 z + \mathcal{P}_2 z^2 + \mathcal{P}_3 z^3 + \dots$ , and  $\mathcal{P}(z) \prec \mathcal{B}_q(\vartheta, z)$ . Then there exist  $\varphi \in \mathcal{P}$  such that  $|\varphi(z)| < 1$  in  $\mathcal{O}$  and  $\mathcal{P}(z) = \mathcal{B}_q(\vartheta, \varphi(z))$ , we have

$$\hbar(z) = (1 + \varphi(z))(1 - \varphi(z))^{-1} = 1 + \ell_1 z + \ell_2 z^2 + \dots \in \mathcal{P}. \quad (3.1)$$

It follows that

$$\varphi(z) = \frac{\ell_1 z}{2} + \left( \ell_2 - \frac{\ell_1^2}{2} \right) \frac{z^2}{2} + \left( \ell_3 - \ell_1 \ell_2 - \frac{\ell_1^3}{4} \right) \frac{z^3}{2} + \dots, \quad (3.2)$$

and

$$\begin{aligned}\mathcal{B}_q(\vartheta, \varphi(z)) &= 1 + \mathcal{B}_{q,1}(\vartheta) \varphi(z) + \frac{1}{[2; q]} \mathcal{B}_{q,2}(\vartheta) (\varphi(z))^2 + \dots \\ &= 1 + \mathcal{B}_{q,1}(\vartheta) \ell_1 z + \left[ \mathcal{B}_{q,1}(\vartheta) \ell_2 + \frac{1}{[2; q]} \mathcal{B}_{q,2}(\vartheta) \ell_1^2 \right] z^2 + \dots.\end{aligned}\quad (3.3)$$

It is a classical result that whenever  $|\varphi(z)| < 1$  holds for all  $z \in \mathcal{O}$ , the coefficients  $\ell_\iota$  in its expansion satisfy the uniform bound

$$|\ell_\iota| \leq 1, \quad \forall \iota \in \mathbb{N}. \quad (3.4)$$

In a similar vein, one can find an analytic function  $\nu$  defined on  $\mathcal{O}$  such that  $|\nu(\xi)| < 1$  and

$$\varphi(\xi) = \Upsilon(\nu(\xi); q).$$

Accordingly, the function

$$\kappa(\xi) = (1 + \nu(\xi))(1 - \nu(\xi))^{-1} = 1 + \tau_1 \xi + \tau_2 \xi^2 + \dots \quad (3.5)$$

belongs to the class  $\mathcal{P}$ .

It follows that

$$\nu(\xi) = \frac{\tau_1 \xi}{2} + \left( \tau_2 - \frac{\tau_1^2}{2} \right) \frac{\xi^2}{2} + \left( \tau_3 - \tau_1 \tau_2 - \frac{\tau_1^3}{4} \right) \frac{\xi^3}{2} + \dots, \quad (3.6)$$

and

$$\mathcal{B}_q(\vartheta, \nu(\xi)) = 1 + \mathcal{B}_{q,1}(\vartheta) \tau_1 \xi + \left[ \mathcal{B}_{q,1}(\vartheta) \tau_2 + \frac{1}{[2; q]} \mathcal{B}_{q,2}(\vartheta) \tau_1^2 \right] \xi^2 + \dots. \quad (3.7)$$

In the following theorem we determine the initial Taylor coefficients  $|\delta_2|$  and  $|\delta_3|$  for the class  $\mathcal{BS}_\Sigma^*(\beta; q, \vartheta)$ .

**Theorem 3.1.** *Let  $\mathfrak{S}$ , given by (1.1), belong to the class  $\mathcal{BS}_\Sigma^*(\beta; q, \vartheta)$ . Then*

$$|\delta_2| \leq \min \left\{ \frac{|[2; q]\vartheta - q|}{[2; q]\sqrt{C}}, M \right\},$$

and

$$|\delta_3| \leq \min \left\{ \frac{([2; q]\vartheta - q)^2}{[2; q]^2 C} + \frac{|[2; q]\vartheta - q|}{[2; q]K}, N \right\},$$

where

$$\begin{aligned}M &= \frac{\frac{1}{[2; q]} |[2; q]\vartheta - q|^{\frac{3}{2}}}{\sqrt{\left| ([2; q](K - L) - C)\vartheta^2 + (C - 2q(K - L))\vartheta + \frac{q(q[3; q](K - L) - C)}{[2; q][3; q]} \right|}} \\ N &= \frac{|[2; q]\vartheta - q|}{[2; q]|K - L|} + \frac{|[2; q]\vartheta - q|}{[2; q]K} + \frac{|\vartheta(\vartheta - 1)|}{[2; q]|K - L|} + \frac{q}{[2; q]^2[3; q]|K - L|},\end{aligned}$$

and

$$K = q [2; q] (1 + q [2; q] \beta), \quad (3.8)$$

$$L = q \left[ 1 + \beta \left( [2; q]^2 - 1 \right) \right], \quad (3.9)$$

$$C = q^2 \left( 1 + q \beta \right)^2. \quad (3.10)$$

*Proof.* Let  $\mathfrak{S} \in \mathcal{BS}_{\Sigma}^*(\beta; q, \vartheta)$  and  $\xi = \mathfrak{S}^{-1}$ . Considering (2.1) and (2.2) we have

$$(1 - \beta) \frac{z \partial_q \langle \mathfrak{S}(z) \rangle}{\mathfrak{S}(z)} + \beta \frac{\partial_q (z \partial_q \langle \mathfrak{S}(z) \rangle)}{\partial_q \langle \mathfrak{S}(z) \rangle} = \mathcal{B}_q(\vartheta, \varphi(z)), \quad (3.11)$$

and

$$(1 - \beta) \frac{\xi \partial_q \langle \chi(\xi) \rangle}{\chi(\xi)} + \beta \frac{\partial_q (\xi \partial_q \langle \chi(\xi) \rangle)}{\partial_q \langle \chi(\xi) \rangle} = \mathcal{B}_q(\vartheta, \nu(\xi)), \quad (3.12)$$

where  $(z, \xi \in \mathcal{O})$ . Since

$$\begin{aligned} (1 - \beta) \frac{z \partial_q \langle \mathfrak{S}(z) \rangle}{\mathfrak{S}(z)} + \beta \frac{\partial_q (z \partial_q \langle \mathfrak{S}(z) \rangle)}{\partial_q \langle \mathfrak{S}(z) \rangle} \\ = q(1 + q\beta)\delta_2 z + \left[ q[2; q](1 + q[2; q]\beta)\delta_3 - q(1 + \beta([2; q]^2 - 1))\delta_2^2 \right] z^2 + \dots \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} (1 - \beta) \frac{\xi \partial_q \langle \chi(\xi) \rangle}{\chi(\xi)} + \beta \frac{\partial_q (\xi \partial_q \langle \chi(\xi) \rangle)}{\partial_q \langle \chi(\xi) \rangle} = -q(1 + q\beta)\delta_2 \xi \\ + \left[ 2q[2; q](1 + q[2; q]\beta)\delta_2^2 - q(1 + \beta([2; q]^2 - 1))\delta_2^2 - q[2; q](1 + q[2; q]\beta)\delta_3 \right] \xi^2 + \dots \end{aligned} \quad (3.14)$$

By comparing (3.11) and (3.13), along (3.3), yields

$$\begin{aligned} q(1 + q\beta)\delta_2 z + \left[ q[2; q](1 + q[2; q]\beta)\delta_3 - q(1 + \beta([2; q]^2 - 1))\delta_2^2 \right] z^2 + \dots \\ = \mathcal{B}_{q,1}(\vartheta)\ell_1 z + \left[ \mathcal{B}_{q,1}(\vartheta)\ell_2 + \frac{1}{[2; q]}\mathcal{B}_{q,2}(\vartheta)\ell_1^2 \right] z^2 + \dots \end{aligned} \quad (3.15)$$

Besied that, by comparing (3.12) and (3.14), along (3.7), yields

$$\begin{aligned} -q(1 + q\beta)\delta_2 \xi + \left[ 2q[2; q](1 + q[2; q]\beta)\delta_2^2 - q(1 + \beta([2; q]^2 - 1))\delta_2^2 \right. \\ \left. - q[2; q](1 + q[2; q]\beta)\delta_3 \right] \xi^2 + \dots \\ = \mathcal{B}_{q,1}(\vartheta)\tau_1 \xi + \left[ \mathcal{B}_{q,1}(\vartheta)\tau_2 + \frac{1}{[2; q]}\mathcal{B}_{q,2}(\vartheta)\tau_1^2 \right] \xi^2 + \dots \end{aligned} \quad (3.16)$$

Equating the pertinent coefficient in (3.15) and (3.16), we obtain

$$q(1 + q\beta)\delta_2 = \mathcal{B}_{q,1}(\vartheta) \ell_1 \quad (3.17)$$

$$-q(1 + q\beta)\delta_2 = \mathcal{B}_{q,1}(\vartheta) \tau_1 \quad (3.18)$$

$$K\delta_3 - L\delta_2^2 = \mathcal{B}_{q,1}(\vartheta) \ell_2 + \frac{\mathcal{B}_{q,2}(\vartheta)\ell_1^2}{[2; q]} \quad (3.19)$$

$$(2K - L)\delta_2^2 - K\delta_3 = \mathcal{B}_{q,1}(\vartheta) \tau_2 + \frac{\mathcal{B}_{q,2}(\vartheta)\tau_1^2}{[2; q]} \quad (3.20)$$

From (3.17) and (3.18), we have

$$\ell_1 = -\tau_1 \iff \ell_1^2 = \tau_1^2, \quad (3.21)$$

and

$$2C\delta_2^2 = \mathcal{B}_{q,1}^2(\vartheta)(\ell_1^2 + \tau_1^2) \iff \ell_1^2 + \tau_1^2 = \frac{2C}{\mathcal{B}_{q,1}^2(\vartheta)}\delta_2^2. \quad (3.22)$$

Now, by summing (3.19) and (3.20), we obtain

$$2(K - L)\delta_2^2 = \mathcal{B}_{q,1}(\vartheta)(\ell_2 + \tau_2) + \frac{\mathcal{B}_{q,2}(\vartheta)}{[2; q]}(\ell_1^2 + \tau_1^2). \quad (3.23)$$

By putting (3.22) in (3.23), we obtain

$$\delta_2^2 = \frac{[2; q]\mathcal{B}_{q,1}^3(\vartheta)(\ell_2 + \tau_2)}{2([2; q](K - L)\mathcal{B}_{q,1}^2(\vartheta) - C\mathcal{B}_{q,2}(\vartheta))}. \quad (3.24)$$

By using  $\mathcal{B}_{q,1}(\vartheta)$  and  $\mathcal{B}_{q,2}(\vartheta)$  where given by (1.21), we deduce that

$$\delta_2^2 = \frac{\frac{([2; q]\vartheta - q)^3}{2[2; q]^2}(\ell_2 + \tau_2)}{\left( ([2; q](K - L) - C)\vartheta^2 + (C - 2q(K - L))\vartheta + \frac{q(q[3; q](K - L) - C)}{[2; q][3; q]} \right)}. \quad (3.25)$$

By applying (3.4) for (3.22) and (3.25), we have

$$|\delta_2| \leq \min \left\{ \frac{|[2; q]\vartheta - q|}{[2; q]\sqrt{C}}, M \right\}, \quad (3.26)$$

where

$$M = \frac{\frac{1}{[2; q]}|[2; q]\vartheta - q|^{\frac{3}{2}}}{\sqrt{\left| ([2; q](K - L) - C)\vartheta^2 + (C - 2q(K - L))\vartheta + \frac{q(q[3; q](K - L) - C)}{[2; q][3; q]} \right|}}$$

Now, so as to find the bound on  $|\delta_3|$ , let's subtract from (3.19) and (3.20) along (3.22), we obtain

$$\delta_3 - \delta_2^2 = \frac{\mathcal{B}_{q,1}(\vartheta)}{2K}(\ell_2 - \tau_2). \quad (3.27)$$

In view of (3.21) and using (3.22), we obtain

$$\delta_3 = \frac{\mathcal{B}_{q,1}^2(\vartheta)}{2C}(\ell_1^2 + \tau_1^2) + \frac{\mathcal{B}_{q,1}(\vartheta)}{2K}(\ell_2 - \tau_2). \quad (3.28)$$

By applying (3.4), we have

$$|\delta_3| \leq \frac{([2; q]\vartheta - q)^2}{[2; q]^2C} + \frac{|[2; q]\vartheta - q|}{[2; q]K}. \quad (3.29)$$

In addition, substituting the value of  $\delta_2^2$  from (3.23) into (3.27), we find

$$\begin{aligned} \delta_3 &= \frac{\mathcal{B}_{q,1}(\vartheta)}{2(K - L)}(\ell_2 + \tau_2) + \frac{\mathcal{B}_{q,2}(\vartheta)}{2[2; q](K - L)}(\ell_1^2 + \tau_1^2) \\ &\quad + \frac{\mathcal{B}_{q,1}(\vartheta)}{2K}(\ell_2 - \tau_2). \end{aligned}$$

Then, we have

$$\begin{aligned} \delta_3 &\leq \frac{|[2; q]\vartheta - q|}{[2; q]|K - L|} + \frac{|[2; q]\vartheta - q|}{[2; q]K} \\ &\quad + \frac{|\vartheta(\vartheta - 1)|}{[2; q]|K - L|} + \frac{q}{[2; q]^2[3; q]|K - L|}. \end{aligned}$$

Hence, the assertion of Theorem 3.1 is established.  $\square$

The next theorem derives the Fekete–Szegő functional for functions  $\mathfrak{F}$  belonging to the class  $\mathcal{BS}_{\Sigma}^*(\beta; q, \vartheta)$ .

**Theorem 3.2.** Let  $\mathfrak{F}$ , given by (1.1), belong to the class  $\mathcal{BS}_{\Sigma}^*(\beta; q, \vartheta)$ , and let  $\varepsilon_1 \in \mathbb{R}$ . Furthermore, define

$$\Lambda = \left| 1 - \frac{L}{K} - \frac{C([2; q][3; q]\vartheta(\vartheta - 1) + q)}{K[3; q]([2; q]\vartheta - q)^2} \right|.$$

Then

$$|\delta_3 - \varepsilon_1 \delta_2^2| \leq \frac{|[2; q]\vartheta - q|}{[2; q]K},$$

whenever  $0 \leq |1 - \varepsilon_1| \leq \Lambda$ , while

$$|\delta_3 - \varepsilon_1 \delta_2^2| \leq \frac{\frac{1}{[2; q]^2} |[2; q]\vartheta - q|^3 |1 - \varepsilon_1|}{\left| ([2; q](K - L) - C)\vartheta^2 + (C - 2q(K - L))\vartheta + q \left( \frac{q[3; q](K - L) - C}{[2; q][3; q]} \right) \right|},$$

whenever  $|1 - \varepsilon_1| \geq \Lambda$ .

*Proof.* Let  $\mathfrak{F} \in \mathcal{BS}_{\Sigma}^*(\beta; q, \vartheta)$ , from (3.24) and (3.27) we have

$$\begin{aligned} \delta_3 - \varepsilon_1 \delta_2^2 &= \frac{(1 - \varepsilon_1)[2; q]\mathcal{B}_{q,1}^3(\vartheta)(\ell_2 + \tau_2)}{2([2; q](K - L)\mathcal{B}_{q,1}^2(\vartheta) - C\mathcal{B}_{q,2}(\vartheta))} + \frac{\mathcal{B}_{q,1}(\vartheta)}{2K} (\ell_2 - \tau_2) \\ &= \frac{\mathcal{B}_{q,1}}{2} \left[ \left( \mathcal{N}(\varepsilon_1) + \frac{1}{K} \right) \ell_2 + \left( \mathcal{N}(\varepsilon_1) - \frac{1}{K} \right) \tau_2 \right], \end{aligned} \quad (3.30)$$

where

$$\mathcal{N}(\varepsilon_1) = \frac{(1 - \varepsilon_1)[2; q]\mathcal{B}_{q,1}^2(\vartheta)}{[2; q](K - L)\mathcal{B}_{q,1}^2(\vartheta) - C\mathcal{B}_{q,2}(\vartheta)}. \quad (3.31)$$

Then, by taking modulus of (3.30), we conclude that

$$|\delta_3 - \varepsilon_1 \delta_2^2| \leq \begin{cases} \frac{|\mathcal{B}_{q,1}(\vartheta)|}{K}, & 0 \leq |\mathcal{N}(\varepsilon_1)| \leq \frac{1}{K} \\ |\mathcal{B}_{q,1}(\vartheta)| |\mathcal{N}(\varepsilon_1)|, & |\mathcal{N}(\varepsilon_1)| \geq \frac{1}{K} \end{cases}$$

$\square$

**Corollary 3.1.** Let  $\mathfrak{F}$ , given by (1.1), belong to the class  $\mathcal{BS}_{\Sigma}^*(q, \vartheta)$ . Then

$$|\delta_2| \leq \min \left\{ \frac{|[2; q]\vartheta - q|}{q[2; q]}, M_1 \right\},$$

and

$$|\delta_3| \leq \min \left\{ \frac{([2; q]\vartheta - q)^2}{q^2[2; q]^2} + \frac{|[2; q]\vartheta - q|}{q[2; q]^2}, N_1 \right\},$$

where

$$M_1 = \frac{|[2; q]\vartheta - q|^{3/2}}{q[2; q] \sqrt{\left| ([2; q] - 1)\vartheta^2 + (1 - 2q)\vartheta + \frac{q(q[3; q] - 1)}{[2; q][3; q]} \right|}},$$

and

$$N_1 = \left( \frac{([2; q] + q)|[2; q]\vartheta - q|}{q^2[2; q]^2} + \frac{|\vartheta(\vartheta - 1)|}{q^2[2; q]} + \frac{1}{q[2; q]^2[3; q]} \right).$$

Furthermore, define

$$\Lambda_1 = \frac{\left| ([2; q] - 1)\vartheta^2 + (1 - 2q)\vartheta + \frac{q(q[3; q] - 1)}{[2; q][3; q]} \right|}{([2; q]\vartheta - q)^2}.$$

Then

$$|\delta_3 - \varepsilon_1 \delta_2^2| \leq \frac{|[2; q]\vartheta - q|}{q[2; q]^2},$$

whenever  $|1 - \varepsilon_1| \leq \Lambda_1$ , while

$$|\delta_3 - \varepsilon_1 \delta_2^2| \leq \frac{|[2; q]\vartheta - q|^3 |1 - \varepsilon_1|}{q[2; q]^2 \left| ([2; q] - 1)\vartheta^2 + (1 - 2q)\vartheta + \frac{q(q[3; q] - 1)}{[2; q][3; q]} \right|},$$

whenever  $|1 - \varepsilon_1| \geq \Lambda_1$ .

**Corollary 3.2.** Let  $\mathfrak{S}$ , given by (1.1), belong to the class  $\mathcal{BC}_\Sigma(q, \vartheta)$ . Then

$$|\delta_2| \leq \min \left\{ \frac{|[2; q]\vartheta - q|}{[2; q]^2}, M_2 \right\},$$

and

$$|\delta_3| \leq \min \left\{ \frac{([2; q]\vartheta - q)^2}{[2; q]^4} + \frac{|[2; q]\vartheta - q|}{[2; q]^2[3; q]}, N_2 \right\},$$

where

$$M_2 = \frac{|[2; q]\vartheta - q|^{3/2}}{[2; q]^2 \sqrt{\left| (q^2 - 1)\vartheta^2 - \left( \frac{[2; q] - 2q^3}{[2; q]} \right) \vartheta + \frac{q^4[3; q] - q[2; q]}{[2; q]^2[3; q]} \right|}},$$

and

$$N_2 = \left( \frac{|[2; q]\vartheta - q|(q^2 + [3; q])}{[2; q]^2[3; q]} + \frac{|\vartheta(\vartheta - 1)|}{q^2[2; q]^2} + \frac{1}{q[2; q]^3[3; q]} \right).$$

Furthermore, define

$$\Lambda_2 = \frac{[2; q]}{[3; q]} \left| \frac{q^2}{[2; q]} - \frac{[2; q][3; q]\vartheta(\vartheta - 1) + q}{[3; q]([2; q]\vartheta - q)^2} \right|.$$

Then

$$|\delta_3 - \varepsilon_2 \delta_2^2| \leq \frac{|[2; q]\vartheta - q|}{[2; q]^2[3; q]},$$

whenever  $|1 - \varepsilon_2| \leq \Lambda_2$ , while

$$|\delta_3 - \varepsilon_2 \delta_2^2| \leq \frac{|[2; q]\vartheta - q|^3 |1 - \varepsilon_2|}{[2; q]^4 \left| (q^2 - 1)\vartheta^2 + \left( 1 - \frac{2q^3}{[2; q]} \right) \vartheta + \frac{q(q^3[3; q] - [2; q])}{[2; q]^2[3; q]} \right|},$$

whenever  $|1 - \varepsilon_2| \geq \Lambda_2$ .

**Corollary 3.3** ([5]). Let  $\mathfrak{S}$ , given by (1.1), belong to the class  $\mathcal{S}_{\Sigma}^*(q, \vartheta)$ . Then

$$|\delta_2| \leq \min \left\{ \frac{|\lceil 2; q \rceil \vartheta - q|}{\lceil 2; q \rceil}, M_3 \right\},$$

and

$$|\delta_3| \leq \min \left\{ \frac{(\lceil 2; q \rceil \vartheta - q)^2}{2\lceil 2; q \rceil} + \frac{|\lceil 2; q \rceil \vartheta - q|}{\lceil 2; q \rceil^2}, N_3 \right\},$$

where

$$M_3 = \frac{|\lceil 2; q \rceil \vartheta - q|^{3/2}}{\lceil 2; q \rceil \sqrt{\left| (\lceil 2; q \rceil - 1)\vartheta^2 + (1 - 2q)\vartheta + \frac{q(q\lceil 3; q \rceil - 1)}{\lceil 2; q \rceil \lceil 3; q \rceil} \right|}},$$

and

$$N_3 = \left( \frac{3|\lceil 2; q \rceil \vartheta - q|}{2\lceil 2; q \rceil} + \frac{|\vartheta(\vartheta - 1)|}{\lceil 2; q \rceil} + \frac{q}{\lceil 2; q \rceil^2 \lceil 3; q \rceil} \right).$$

Define

$$\Lambda_3 = \frac{\lceil 2; q \rceil^2 \left| (\lceil 2; q \rceil - 1)\vartheta^2 + (1 - 2q)\vartheta + \frac{q(q\lceil 3; q \rceil + 1)}{\lceil 2; q \rceil \lceil 3; q \rceil} \right|}{2(\lceil 2; q \rceil \vartheta - q)^2}.$$

Then

$$|\delta_3 - \varepsilon_3 \delta_2^2| \leq \frac{|\lceil 2; q \rceil \vartheta - q|}{2\lceil 2; q \rceil},$$

whenever  $|1 - \varepsilon_3| \leq \Lambda_3$ , while

$$|\delta_3 - \varepsilon_3 \delta_2^2| \leq \frac{|\lceil 2; q \rceil \vartheta - q|^3 |1 - \varepsilon_3|}{\lceil 2; q \rceil^3 \left| (\lceil 2; q \rceil - 1)\vartheta^2 + (1 - 2q)\vartheta + \frac{q(q\lceil 3; q \rceil - 1)}{\lceil 2; q \rceil \lceil 3; q \rceil} \right|}},$$

whenever  $|1 - \varepsilon_3| \geq \Lambda_3$ .

**Corollary 3.4** ([5]). Let  $\mathfrak{S}$ , given by (1.1), belong to the class  $\mathcal{C}_{\Sigma}(q, \vartheta)$ . Then

$$|\delta_2| \leq \min \left\{ \frac{|\lceil 2; q \rceil \vartheta - q|}{\lceil 2; q \rceil^2}, M_4 \right\},$$

and

$$|\delta_3| \leq \min \left\{ \frac{(\lceil 2; q \rceil \vartheta - q)^2}{4\lceil 2; q \rceil^2} + \frac{|\lceil 2; q \rceil \vartheta - q|}{6\lceil 2; q \rceil}, N_4 \right\},$$

where

$$M_4 = \frac{|\lceil 2; q \rceil \vartheta - q|^{3/2}}{\lceil 2; q \rceil \sqrt{2 \left| (\lceil 2; q \rceil - 2)\vartheta^2 - 2(1 - q)\vartheta + \frac{q(q\lceil 3; q \rceil - 2)}{\lceil 2; q \rceil \lceil 3; q \rceil} \right|}},$$

and

$$N_4 = \left( \frac{2|\lceil 2; q \rceil \vartheta - q|}{3\lceil 2; q \rceil} + \frac{|\vartheta(\vartheta - 1)|}{2\lceil 2; q \rceil} + \frac{q}{2\lceil 2; q \rceil^2 \lceil 3; q \rceil} \right).$$

Furthermore, define

$$\Lambda_4 = \frac{[2; q]^2 \left| ([2; q] - 2)\vartheta^2 + 2(1 - 2q)\vartheta + \frac{q(q[3; q] - 2)}{[2; q][3; q]} \right|}{3([2; q]\vartheta - q)^2}.$$

Then

$$|\delta_3 - \varepsilon_4 \delta_2^2| \leq \frac{|[2; q]\vartheta - q|}{6[2; q]},$$

whenever  $|1 - \varepsilon_4| \leq \Lambda_4$ , while

$$|\delta_3 - \varepsilon_4 \delta_2^2| \leq \frac{|[2; q]\vartheta - q|^3 |1 - \varepsilon_4|}{2[2; q]^3 \left| ([2; q] - 2)\vartheta^2 + 2(1 - q)\vartheta + \frac{q(q[3; q] - 2)}{[2; q][3; q]} \right|},$$

whenever  $|1 - \varepsilon_4| \geq \Lambda_4$ .

#### 4. CONCLUSION

In this work, we investigated two subclasses of bi-univalent functions associated with  $q$ -Bernoulli Polynomials, namely the starlike and convex classes. Utilizing the subordination principle, we established coefficient bounds for the initial terms of these function classes and derived the corresponding Fekete–Szegő inequalities. These results enhance the theoretical framework of bi-univalent function theory and elucidate its deeper connections with special function spaces.

Future research could extend these findings by exploring higher-order coefficient estimates, refining the structural characteristics of these subclasses, and examining their geometric properties. Moreover, investigating upper bounds related to the Zalcman conjecture and analyzing Hankel determinants of orders two and three within these subclasses could provide new insights and open further avenues in the study of analytic and bi-univalent function theory.

**Authors' Contributions:** All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

#### REFERENCES

- [1] P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften 259, Springer, New York, 1983.
- [2] W.C. Ma, D. Minda, A Unified Treatment of Some Special Classes of Univalent Functions, in: *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, Conf. Proc. Lect. Notes Anal. 1, International Press, Cambridge, MA, 1994, 157–169.
- [3] N.I. Mahmudov, Difference Equations of  $q$ -Appell Polynomials, *Appl. Math. Comput.* 245 (2014), 539–543. <https://doi.org/10.1016/j.amc.2014.07.107>.
- [4] W.A. Al-Salam,  $q$ -Bernoulli Numbers and Polynomials, *Math. Nachr.* 17 (1958), 239–260. <https://doi.org/10.1002/mana.19580170311>.

- [5] A.K. Wanas, S.C. Khachi, Coefficient Bounds and Fekete-Szegő Inequalities for New Families of Bi-Starlike and Bi-Convex Functions Associated with the  $q$ -Bernoulli Polynomial, *Appl. Math. E-Notes* 25 (2025), 105–117. <https://www.math.nthu.edu.tw/~amen/2025/AMEN-A240105.pdf>.
- [6] A. Alsoboh, A. Amourah, O. Alnajar, M. Ahmed, T.M. Seoudy, Exploring  $q$ -Fibonacci Numbers in Geometric Function Theory: Univalence and Shell-like Starlike Curves, *Mathematics* 13 (2025), 1294. <https://doi.org/10.3390/math13081294>.
- [7] J. Sokół, On Starlike Functions Connected with Fibonacci Numbers, *Zesz. Nauk. Politech. Rzesz. Mat.* 23 (1999), 111–116.
- [8] J. Sokół, A Certain Class of Starlike Functions, *Comput. Math. Appl.* 62 (2011), 611–619. <https://doi.org/10.1016/j.camwa.2011.05.041>.
- [9] A.W. Goodman, *Univalent Functions*, Vols. I–II, Mariner Publishing Co., Tampa, FL, 1983.
- [10] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, *Encyclopedia of Mathematics and Its Applications* 98, Cambridge University Press, Cambridge, 2005.
- [11] G. Szegő, *Orthogonal Polynomials*, *Amer. Math. Soc. Colloq. Publ.* 23, American Mathematical Society, New York, 1939.
- [12] V.S. Masih, A. Ebadian, S. Yalçın, Some Properties Associated to a Certain Class of Starlike Functions, *Math. Slovaca* 69 (2019), 1329–1340. <https://doi.org/10.1515/ms-2017-0311>.
- [13] M.S. Robertson, Certain Classes of Starlike Functions, *Mich. Math. J.* 32 (1985), 135–140. <https://doi.org/10.1307/mmj/1029003181>.
- [14] H.E. Özkan Uçar, Coefficient Inequality for  $q$ -Starlike Functions, *Appl. Math. Comput.* 276 (2016), 122–126. <https://doi.org/10.1016/j.amc.2015.12.008>.
- [15] A. Alsoboh, A. Malkawi, A. Rabaiah, A. Amourah, A. Al-Maqbali, et al., Neutrosophic MR-Metric Spaces: Theory, Compactness, and Applications in Machine Learning, *Nonlinear Funct. Anal. Appl.* 31 (2026), 487–500. <https://doi.org/10.22771/nfaa.2026.31.02.11>.
- [16] F.H. Jackson, On  $q$ -Functions and a Certain Difference Operator, *Trans. R. Soc. Edinb.* 46 (1909), 253–281. <https://doi.org/10.1017/S0080456800002751>.
- [17] F.H. Jackson, On  $q$ -Definite Integrals, *Quart. J. Pure Appl. Math.* 41 (1910), 193–203.
- [18] A. Aral, V. Gupta, Generalized  $q$ -Baskakov Operators, *Math. Slovaca* 61 (2011), 619–634. <https://doi.org/10.2478/s12175-011-0032-3>.
- [19] A. Aral, V. Gupta, On the Durrmeyer Type Modification of the  $q$ -Baskakov Type Operators, *Nonlinear Anal. Theory Methods Appl.* 72 (2010), 1171–1180. <https://doi.org/10.1016/j.na.2009.07.052>.
- [20] A. Aral, V. Gupta, R.P. Agarwal, *Applications of  $q$ -Calculus in Operator Theory*, Springer New York, 2013. <https://doi.org/10.1007/978-1-4614-6946-9>.
- [21] A. Almalkawi, A. Alsoboh, A. Amourah, T. Sasa, Estimates for the Coefficients of Subclasses Defined by the  $q$ -Babalola Convolution Operator of Bi-Univalent Functions Subordinate to the  $q$ -Fibonacci Analogue, *Eur. J. Pure Appl. Math.* 18 (2025), 6499. <https://doi.org/10.29020/nybg.ejpam.v18i3.6499>.
- [22] A. Amourah, A. Alsoboh, D. Breaz, S.M. El-Deeb, A Bi-Starlike Class in a Leaf-Like Domain Defined through Subordination via  $q$ -Calculus, *Mathematics* 12 (2024), 1735. <https://doi.org/10.3390/math12111735>.
- [23] A. Alsoboh, M. Çağlar, M. Buyankara, Fekete-Szegő Inequality for a Subclass of Bi-Univalent Functions Linked to  $q$ -Ultraspherical Polynomials, *Contemp. Math.* 5 (2024), 2366–2380. <https://doi.org/10.37256/cm.5220243737>.
- [24] S. Elhaddad, H. Aldweby, M. Darus, Some Properties on a Class of Harmonic Univalent Functions Defined by  $q$ -Analogue of Ruscheweyh Operator, *J. Math. Anal.* 9 (2018), 28–35.

- [25] M. Çağlar, H. Orhan, N. Yağmur, Coefficient Bounds for New Subclasses of Bi-Univalent Functions, *Filomat* 27 (2013), 1165–1171. <https://doi.org/10.2298/FIL1307165C>.
- [26] G.E. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge University Press, 1999. <https://doi.org/10.1017/CB09781107325937>.
- [27] D.S. McAnally,  $q$ -Exponential and  $q$ -Gamma Functions. II.  $q$ -Gamma Functions, *J. Math. Phys.* 36 (1995), 574–595. <https://doi.org/10.1063/1.531323>.
- [28] M. El-Ityan, A. Amourah, S. Hammad, R. Buti, A. Alsoboh, New Subclass of Bi-Univalent Functions Involving the Wright Function Associated with the Jung-Kim-Srivastav Operator, *Gulf J. Math.* 19 (2025), 451–462. <https://doi.org/10.56947/gjom.v19i2.2817>.
- [29] S.H. Hadi, M. Darus, B. Alamri, Ş. Altınkaya, A. Alatawi, On Classes of  $\zeta$ -Uniformly  $q$ -Analogue of Analytic Functions With Some Subordination Results, *Appl. Math. Sci. Eng.* 32 (2024), 2312803. <https://doi.org/10.1080/27690911.2024.2312803>.
- [30] M. Ahmed, A. Alsoboh, A. Amourah, J. Salah, On the Fractional  $q$ -Differintegral Operator for Subclasses of Bi-Univalent Functions Subordinate to  $q$ -Ultraspherical Polynomials, *Eur. J. Pure Appl. Math.* 18 (2025), 6586. <https://doi.org/10.29020/nybg.ejpam.v18i3.6586>.
- [31] M. Rasheed, A.H. Majeed, E.N. Abdulwahab, D.S. Ali, Z. Mohammed, Certain Results of Riemann-Liouville Fractional Calculus Involving Seven-Parametric Mittag-Leffler Function, *Gulf J. Math.* 21 (2025), 625–632. <https://doi.org/10.56947/gjom.v21i1.3585>.
- [32] A. Alsoboh, A. Amourah, K. Al Mashrafi, T. Sasa, Bi-Starlike and Bi-Convex Function Classes Connected to Shell-Like Curves and the  $q$ -Analogue of Fibonacci Numbers, *Int. J. Anal. Appl.* 23 (2025), 201. <https://doi.org/10.28924/2291-8639-23-2025-201>.
- [33] S. Mahmood, Q.Z. Ahmad, H.M. Srivastava, N. Khan, B. Khan, et al., A Certain Subclass of Meromorphically  $q$ -Starlike Functions Associated with the Janowski Functions, *J. Inequal. Appl.* 2019 (2019), 88. <https://doi.org/10.1186/s13660-019-2020-z>.
- [34] S. Mahmood, H.M. Srivastava, N. Khan, Q.Z. Ahmad, B. Khan, et al., Upper Bound of the Third Hankel Determinant for a Subclass of  $q$ -Starlike Functions, *Symmetry* 11 (2019), 347. <https://doi.org/10.3390/sym11030347>.
- [35] H.M. Srivastava, B. Khan, N. Khan, M. Tahir, S. Ahmad, et al., Upper Bound of the Third Hankel Determinant for a Subclass of  $q$ -Starlike Functions Associated with the  $q$ -Exponential Function, *Bull. Sci. Math.* 167 (2021), 102942. <https://doi.org/10.1016/j.bulsci.2020.102942>.
- [36] A.M. Malkawi, A.M. Rabaiah, *Fuzzy Metric Spaces: Theory and Applications in Medical Imaging, Transportation, and Social Networks*, *Nonlinear Funct. Anal. Appl.* 30 (2025), 1255–1272. <https://doi.org/10.22771/nfaa.2025.30.04.17>.
- [37] Y. Al-Qudah, A. O. Hamadameen, F. Al-Sharqi, A new generalization of interval-valued  $Q$ -neutrosophic soft matrix and its applications, *Int. J. Neutrosophic Sci.* 25 (2025), 242–257.
- [38] Y. Al-Qudah, F. Al-Sharqi, M. Mishlish, M.M. Rasheed, Hybrid Integrated Decision-Making Algorithm Based on AO of Possibility Interval-Valued Neutrosophic Soft Settings, *Int. J. Neutrosophic Sci.* 22 (2023), 84–98. <https://doi.org/10.54216/IJNS.220306>.
- [39] A. Hazaymeh, Y. Al-Qudah, F. Al-Sharqi, A. Bataihah, A Novel  $Q$ -Neutrosophic Soft under Interval Matrix Setting and Its Applications, *Int. J. Neutrosophic Sci.* 25 (2025), 156–168. <https://doi.org/10.54216/IJNS.250413>.
- [40] Y. Al-Qudah, N. Hassan, Complex Multi-Fuzzy Relation for Decision Making Using Uncertain Periodic Data, *Int. J. Eng. Technol.* 7 (2018), 2437–2445. <https://doi.org/10.14419/ijet.v7i4.16976>.