

MODELING AND PREDICTIVE ANALYSIS OF ENVIRONMENTAL SUSTAINABILITY AND BASIC NEEDS INDICATORS IN SAUDI ARABIA UNDER KUMARASWAMY CURRENT RECORDS

ABDULAZIZ A. ALQARNI, RAMY A. ALDALLAL*

Department of Management, College of Business Administration in Hawtat Bani Tamim, Prince Sattam bin Abdulaziz University, Hawtat Bani Tamim, Saudi Arabia

*Corresponding author: r.eldallal@psau.edu.sa

Received Jul. 9, 2026

ABSTRACT. This paper develops a Kumaraswamy current-record framework to model and predict environmental sustainability indicators in Saudi Arabia. Using Gamma-based current-record representations, explicit forms of the probability density functions, cumulative distribution functions, joint and conditional densities, moments, entropy, and extropy are derived for upper and lower current records. Maximum likelihood and Bayesian estimation methods are then proposed for estimating the Kumaraswamy parameters from current-record data. Since the likelihood equations and posterior expectations are analytically intractable, numerical optimization and the Metropolis–Hastings algorithm are employed. A Monte Carlo simulation study evaluates the performance of the estimators using RMSE, MAD, standard deviation, and interval estimates. The results show that the Bayesian approach generally yields more stable and accurate estimates, especially for small and moderate sample sizes. The proposed methodology is applied to annual data on Saudi tree-covered area from 2001 to 2020. Goodness-of-fit results support the suitability of the Kumaraswamy distribution for the transformed data. Future current records are predicted using Bayesian and maximum likelihood approaches, indicating a continued increase in tree-covered area. The proposed framework provides a useful statistical tool for analyzing bounded environmental sustainability indicators and supporting evidence-based environmental planning in Saudi Arabia.

2020 Mathematics Subject Classification. 62G30; 62F15.

Key words and phrases. environmental sustainability; Bayesian inference; current records; Kumaraswamy distribution; Saudi Arabia tree-covered area data.

1. INTRODUCTION

Climate change poses substantial challenges to the global community, particularly to international food and nutritional security. Rising temperatures, fluctuations in precipitation patterns, and increasing greenhouse gas concentrations are among the key factors contributing to climate change and its impacts on natural and human systems [1]. Climate change, characterized by long-term shifts

in temperature and weather patterns, is primarily driven by human activities such as fossil fuel combustion, deforestation, land-use change, and industrial development. Since the Industrial Revolution, these activities have significantly increased atmospheric greenhouse gas concentrations, resulting in global warming and noticeable alterations in climate systems worldwide [2].

Saudi Arabia is particularly vulnerable to climate change due to its arid environment, limited renewable water resources, and economic dependence on fossil fuel exports. Climate change represents a major challenge with wide-ranging economic, environmental, and social implications for the Kingdom [2]. One of the most direct consequences is the increase in average temperatures. Saudi Arabia already experiences some of the highest temperatures globally, and climate projections suggest temperatures may rise by up to 5.6°C by the end of the century under high-emission scenarios. Such increases intensify existing challenges, including extreme summer heat that frequently exceeds 50°C, placing pressure on public health systems, increasing energy demand for cooling, and reducing labor productivity [2].

The impacts of climate change extend beyond environmental concerns to affect human health, food security, water resources, and economic development. As emphasized in the Climate Futures Report for Saudi Arabia, climate-related risks can affect virtually every aspect of life and development in the Kingdom [2]. Consequently, environmental sustainability has emerged as a central concern for governments, businesses, and societies worldwide. Growing awareness of climate change, environmental degradation, and resource depletion has accelerated the global transition toward sustainable development practices [3].

Many countries are implementing comprehensive strategies to reduce environmental impacts while maintaining economic growth and social development. These efforts often involve environmental innovation, green growth initiatives, and policies aligned with the Sustainable Development Goals (SDGs) [4]. In this context, Saudi Arabia has demonstrated a strong commitment to environmental sustainability through the Saudi Green Initiative (SGI), which serves as a cornerstone of the Kingdom's environmental transformation agenda [5]. Launched in 2021, the SGI supports Saudi Arabia's broader vision of economic diversification while addressing critical environmental challenges. The initiative focuses on large-scale afforestation programs, carbon emission reduction, renewable energy expansion, and sustainable urban development [6].

The Saudi Green Initiative reflects the Kingdom's recognition that environmental sustainability is an essential component of long-term national development. Its objectives align with global sustainability efforts and are supported by technological innovation, regulatory reforms, and sustainable resource management strategies [7]. To support these goals, the Saudi government has introduced policies and regulations to encourage the adoption of renewable energy, reduce greenhouse gas emissions, and promote environmental conservation. These measures create a regulatory framework that encourages both public- and private-sector participation in sustainability initiatives [8]. Furthermore, organizations

operating in Saudi Arabia are increasingly integrating environmental sustainability into their strategic planning and operations, reflecting the growing influence of the Saudi Green Initiative on corporate environmental commitments [9].

Motivated by the importance of environmental sustainability and the need for evidence-based planning, this paper investigates the evolution of tree cover in Saudi Arabia and develops predictive models for future tree-cover dynamics using current-record data under the Kumaraswamy distribution. Understanding future trends in tree cover is essential for assessing the effectiveness of environmental initiatives and supporting sustainable land management policies in the Kingdom.

The Kumaraswamy distribution is a flexible two-parameter continuous probability distribution defined on the bounded interval $(0, 1)$. It was introduced by Kumaraswamy (1980) for modeling hydrological phenomena and has since been widely applied in environmental sciences, reliability analysis, survival studies, and record-value theory. The distribution is characterized by two positive shape parameters, $\alpha > 0$ and $\beta > 0$, and possesses the cumulative distribution function

$$f(x; \alpha, \beta) = \alpha\beta x^{\alpha-1} (1 - x^\alpha)^{\beta-1}, \quad (1)$$

and

$$F(x; \alpha, \beta) = 1 - (1 - x^\alpha)^\beta, \quad 0 < x < 1. \quad (2)$$

One of the main advantages of the Kumaraswamy distribution is its simple closed-form expressions for the distribution, density, and quantile functions, which make it computationally more attractive than the Beta distribution. Furthermore, the distribution is highly flexible and can model a variety of shapes, including symmetric, left-skewed, right-skewed, J-shaped, and unimodal distributions. These properties make the Kumaraswamy distribution particularly suitable for modeling bounded environmental and hydrological data as well as for developing statistical inference procedures based on record and current-record observations.

Before fitting the Kumaraswamy distribution, the original observations are commonly transformed to the unit interval $(0, 1)$ using the min–max transformation

$$Y = \frac{X - \min(X)}{\max(X) - \min(X)}.$$

This transformation is applied because the Kumaraswamy distribution is defined only on the bounded support $(0, 1)$. It is important to note, however, that this rescaling does not imply that the transformed data follow a Kumaraswamy distribution. Rather, the transformation merely ensures that the data satisfy the model's support requirements, thereby allowing the Kumaraswamy distribution to be considered a candidate distribution.

The theory of current records stems from the study of stochastic extremes and ordered observations. Houchens [10] established their statistical properties and laid the foundation for inference based on

record data. Subsequent studies substantially advanced current record theory and its applications. Barakat et al. [11,12] investigated current records, record ranges, and exact prediction intervals, while Raqab [13–15] established several theoretical inequalities and distribution-free prediction methods. More recently, Aldallal and co-authors developed estimation, prediction, and Bayesian inference procedures for current records under various lifetime distributions, with applications to real datasets [16–21,24,25]. A comprehensive review of both classical and modern developments in current record theory is provided by Aldallal and Barakat [22].

2. PDF AND CDF OF CURRENT RECORDS

This section derives the probability density functions (PDFs) and cumulative distribution functions (CDFs) of the Kumaraswamy upper and lower current records. The derivations are based on the new Gamma-based current-record formulas established by [23], which are specialized here by substituting the Kumaraswamy distribution functions.

Theorem 1. From the new current records formulas founded by [23] and using equations (1) and (2), we get the PDF and CDF of Kumaraswamy upper and lower current records as follows

$$F_{L_{cr}^n}(x) = 1 + 2^n \left[1 - (1 - x^\alpha)^\beta \right] G \left(n, -\log \left[1 - (1 - x^\alpha)^\beta \right] \right) - G \left(n, -2 \log \left[1 - (1 - x^\alpha)^\beta \right] \right), \quad (3)$$

$$f_{L_{cr}^n}(x) = 2^n \alpha \beta x^{\alpha-1} (1 - x^\alpha)^{\beta-1} G \left(n, -\log \left[1 - (1 - x^\alpha)^\beta \right] \right), \quad (4)$$

$$F_{U_{cr}^n}(x) = G \left(n, -2 \log \left[(1 - x^\alpha)^\beta \right] \right) - 2^n (1 - x^\alpha)^\beta G \left(n, -\log \left[(1 - x^\alpha)^\beta \right] \right), \quad (5)$$

$$f_{U_{cr}^n}(x) = 2^n \alpha \beta x^{\alpha-1} (1 - x^\alpha)^{\beta-1} G \left(n, -\log \left[(1 - x^\alpha)^\beta \right] \right). \quad (6)$$

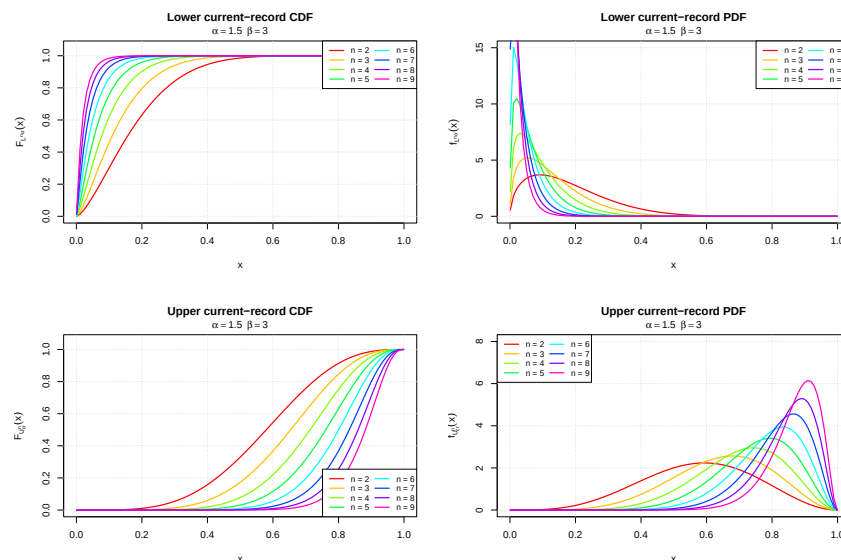


FIGURE 1. Plot of Kumaraswamy Current Records' PDF and CDF for different values of n , $\alpha = 1.5$, and $\beta = 3$.

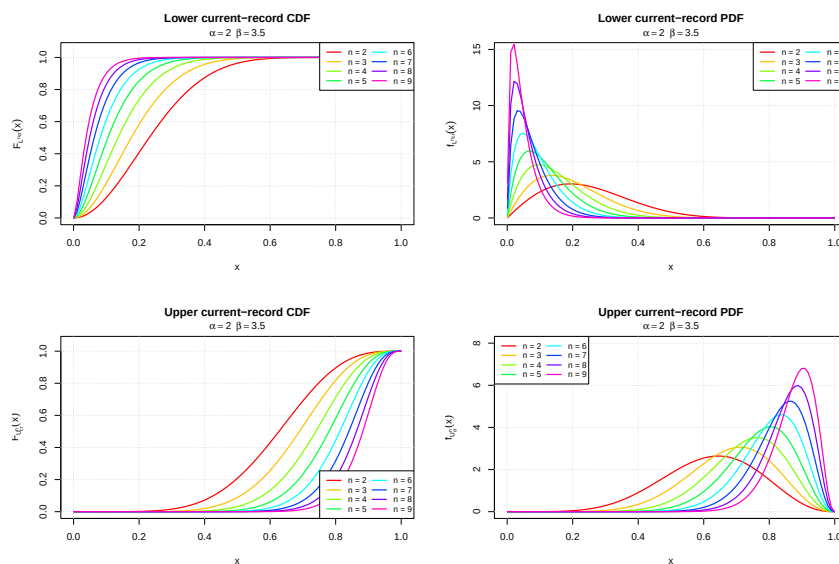


FIGURE 2. Plot of Kumaraswamy Current Records' PDF and CDF for different values of n , $\alpha = 2$, and $\beta = 3.5$.

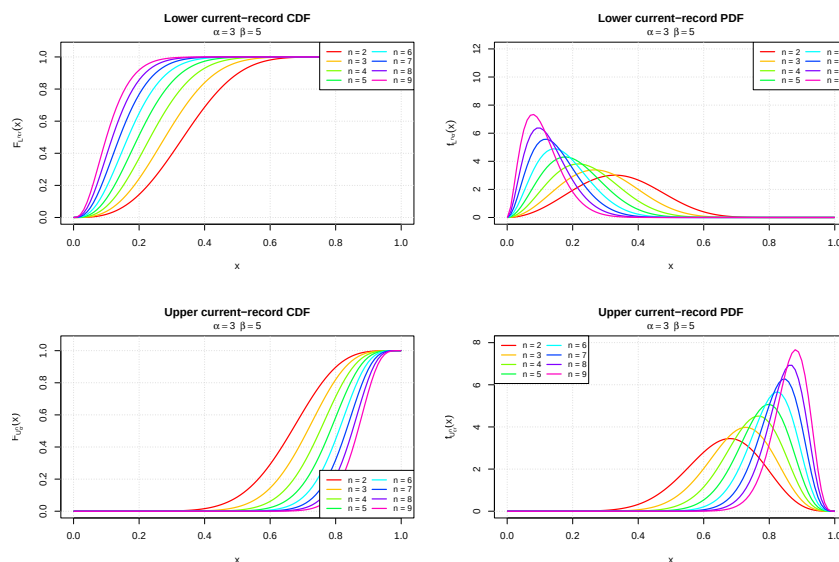


FIGURE 3. Plot of Kumaraswamy Current Records' PDF and CDF for different values of n , $\alpha = 3$, and $\beta = 5$.

Figures 1–3 illustrate the behavior of the proposed Kumaraswamy current-record distributions for different values of the record index n under three combinations of the shape parameters (α, β) .

Figure 1, corresponding to $\alpha = 1.5$ and $\beta = 3$, shows that increasing the record index n has a noticeable effect on both the lower and upper current-record distributions. The lower current-record CDF rises more rapidly as n increases, while the corresponding PDF becomes increasingly concentrated

near the left boundary of the support. In contrast, the upper current-record CDF shifts toward larger values of x , indicating that higher-order upper current records are increasingly likely to occur near the upper end of the support. The upper current-record PDF also shifts to the right and becomes more sharply peaked, reflecting reduced variability as successive upper records accumulate.

Figure 2 presents the same distributions for $\alpha = 2$ and $\beta = 3.5$. Compared with Figure 1, both the lower- and upper-current-record PDFs become more concentrated, while the CDFs exhibit steeper transitions. Increasing the shape parameters shifts the upper current-record distributions closer to one, whereas the lower current-record distributions become increasingly concentrated near smaller values of x . Consequently, higher-order current records become more predictable and less dispersed.

Figure 3 considers a larger parameter combination, $\alpha = 3$ and $\beta = 5$. The concentration effect becomes even more pronounced. The lower current-record PDF is sharply concentrated near the origin with very little dispersion, whereas the upper current-record PDF exhibits a high and narrow peak close to the upper boundary of the support. Likewise, both cumulative distribution functions become considerably steeper, indicating that most of the probability mass is concentrated within relatively narrow intervals. This behavior demonstrates that larger values of α and β substantially reduce the variability of both lower and upper current records.

Overall, Figures 1–3 demonstrate that both the record index n and the Kumaraswamy shape parameters (α, β) significantly influence the proposed current-record distributions. Increasing either n or the shape parameters produces more concentrated probability densities and steeper cumulative distribution functions, indicating that higher-order current records become progressively less variable and more concentrated near their respective boundaries. These characteristics highlight the flexibility of the proposed distributions for modeling a wide range of current-record behaviors.

3. JOINT AND CONDITIONAL DENSITY FUNCTIONS

The joint density of lower current records can be found by substituting the Kumaraswamy PDF (1) and CDF (2) in the following formula given by [10]

$$f_{L_{cr}^n, L_{cr}^s}(l_n, l_s) = \frac{f_p(l_n)f_p(l_s)}{F_p(l_n)(s-n-1)!} \left(H(l_n) - H(l_s) \right)^{s-n-1},$$

gives

$$\begin{aligned} f_{L_{cr}^n, L_{cr}^s}(l_n, l_s) &= \frac{\alpha^2 \beta^2 l_n^{\alpha-1} (1-l_n^\alpha)^{\beta-1} l_s^{\alpha-1} (1-l_s^\alpha)^{\beta-1}}{[1 - (1-l_n^\alpha)^\beta] (s-n-1)!} \\ &\quad \times \left(\log \left[1 - (1-l_n^\alpha)^\beta \right] - \log \left[1 - (1-l_s^\alpha)^\beta \right] \right)^{s-n-1}, \quad l_s < l_n. \end{aligned} \quad (7)$$

While to get the conditional density of it we substitute in

$$f_{L_{cr}^s | L_{cr}^n}(l_s | l_n) = \frac{f_p(l_s)}{F_p(l_n)(s-n-1)!} \left(H(l_n) - H(l_s) \right)^{s-n-1}.$$

Therefore,

$$f_{L_{cr}^s | L_{cr}^n}(l_s | l_n) = \frac{\alpha \beta l_s^{\alpha-1} (1 - l_s^\alpha)^{\beta-1}}{[1 - (1 - l_n^\alpha)^\beta] (s - n - 1)!} \times \left(\log \left[1 - (1 - l_n^\alpha)^\beta \right] - \log \left[1 - (1 - l_s^\alpha)^\beta \right] \right)^{s-n-1}, \quad l_s < l_n. \quad (8)$$

And the same goes for the upper current records

$$f_{U_{cr}^n, U_{cr}^s}(u_n, u_s) = \frac{f_p(u_n) f_p(u_s)}{\bar{F}_p(u_n) (s - n - 1)!} \left(R(u_s) - R(u_n) \right)^{s-n-1}.$$

Substituting the Kumaraswamy functions gives

$$f_{U_{cr}^n, U_{cr}^s}(u_n, u_s) = \frac{\alpha^2 \beta^2 u_n^{\alpha-1} (1 - u_n^\alpha)^{\beta-1} u_s^{\alpha-1} (1 - u_s^\alpha)^{\beta-1}}{(1 - u_n^\alpha)^\beta (s - n - 1)!} \left[\beta \log \left(\frac{1 - u_n^\alpha}{1 - u_s^\alpha} \right) \right]^{s-n-1}, \quad u_s > u_n. \quad (9)$$

And for its conditional density we use

$$f_{U_{cr}^s | U_{cr}^n}(u_s | u_n) = \frac{f_p(u_s)}{\bar{F}_p(u_n) (s - n - 1)!} \left(R(u_s) - R(u_n) \right)^{s-n-1}.$$

Thus,

$$f_{U_{cr}^s | U_{cr}^n}(u_s | u_n) = \frac{\alpha \beta u_s^{\alpha-1} (1 - u_s^\alpha)^{\beta-1}}{(1 - u_n^\alpha)^\beta (s - n - 1)!} \left[\beta \log \left(\frac{1 - u_n^\alpha}{1 - u_s^\alpha} \right) \right]^{s-n-1}, \quad u_s > u_n. \quad (10)$$

4. MOMENTS, ENTROPY AND EXTROPY

For the lower current record, the r^{th} moments will be

$$\delta_{L_{cr}^n}^r = 2^n \alpha \beta \int_0^1 x^{r+\alpha-1} (1 - x^\alpha)^{\beta-1} G \left(n, -\log \left[1 - (1 - x^\alpha)^\beta \right] \right) dx. \quad (11)$$

While for the upper current record will be

$$\delta_{U_{cr}^n}^r = 2^n \alpha \beta \int_0^1 x^{r+\alpha-1} (1 - x^\alpha)^{\beta-1} G \left(n, -\beta \log(1 - x^\alpha) \right) dx. \quad (12)$$

For the Entropy, we can use the following integrals

$$\begin{aligned} H(L_{cr}^n) &= -n \log 2 - 2^n \alpha \beta \int_0^1 x^{\alpha-1} (1 - x^\alpha)^{\beta-1} G \left(n, -\log \left[1 - (1 - x^\alpha)^\beta \right] \right) \\ &\quad \times \log \left[\alpha \beta x^{\alpha-1} (1 - x^\alpha)^{\beta-1} \right] dx \\ &\quad - 2^n \alpha \beta \int_0^1 x^{\alpha-1} (1 - x^\alpha)^{\beta-1} G \left(n, -\log \left[1 - (1 - x^\alpha)^\beta \right] \right) \\ &\quad \times \log G \left(n, -\log \left[1 - (1 - x^\alpha)^\beta \right] \right) dx. \end{aligned} \quad (13)$$

$$\begin{aligned} H(U_{cr}^n) &= -n \log 2 - 2^n \alpha \beta \int_0^1 x^{\alpha-1} (1 - x^\alpha)^{\beta-1} G \left(n, -\beta \log(1 - x^\alpha) \right) \\ &\quad \times \log \left[\alpha \beta x^{\alpha-1} (1 - x^\alpha)^{\beta-1} \right] dx \\ &\quad - 2^n \alpha \beta \int_0^1 x^{\alpha-1} (1 - x^\alpha)^{\beta-1} G \left(n, -\beta \log(1 - x^\alpha) \right) \end{aligned}$$

$$\times \log G(n, -\beta \log(1 - x^\alpha)) \, dx. \quad (14)$$

And the Extropy of them will be

$$J(L_{cr}^n) = -2^{2n-1} \alpha^2 \beta^2 \int_0^1 x^{2\alpha-2} (1 - x^\alpha)^{2\beta-2} G^2\left(n, -\log\left[1 - (1 - x^\alpha)^\beta\right]\right) \, dx. \quad (15)$$

$$J(U_{cr}^n) = -2^{2n-1} \alpha^2 \beta^2 \int_0^1 x^{2\alpha-2} (1 - x^\alpha)^{2\beta-2} G^2(n, -\beta \log(1 - x^\alpha)) \, dx. \quad (16)$$

5. ESTIMATION USING THE NEW PDFs

In this section, the unknown parameters α and β of the Kumaraswamy distribution are estimated directly from the newly derived current-record density functions. This approach uses the gamma-based PDFs of the lower and upper current records and thus incorporates the current-record-generating mechanism into the likelihood function.

5.1. MLE under new PDF. Let $U_{cr}^1, U_{cr}^2, \dots, U_{cr}^n$ denote the observed upper current records and let $L_{cr}^1, L_{cr}^2, \dots, L_{cr}^n$ denote the observed lower current records. Based on (6), the likelihood function for the upper current records is

$$L_{U_{cr}^n}(\alpha, \beta) = \prod_{i=1}^n 2^i \alpha \beta u_i^{\alpha-1} (1 - u_i^\alpha)^{\beta-1} G(i, -\beta \log(1 - u_i^\alpha)), \quad 0 < u_i < 1. \quad (17)$$

Thus, the corresponding log-likelihood function is

$$\begin{aligned} \ell_{U_{cr}^n}(\alpha, \beta) = \sum_{i=1}^n \left[i \log 2 + \log \alpha + \log \beta + (\alpha - 1) \log u_i + (\beta - 1) \log(1 - u_i^\alpha) \right. \\ \left. + \log G(i, -\beta \log(1 - u_i^\alpha)) \right]. \end{aligned} \quad (18)$$

Similarly, based on (4), the likelihood function for the lower current records is

$$L_{L_{cr}^n}(\alpha, \beta) = \prod_{i=1}^n 2^i \alpha \beta l_i^{\alpha-1} (1 - l_i^\alpha)^{\beta-1} G\left(i, -\log\left[1 - (1 - l_i^\alpha)^\beta\right]\right), \quad 0 < l_i < 1. \quad (19)$$

Therefore, the log-likelihood function is

$$\begin{aligned} \ell_{L_{cr}^n}(\alpha, \beta) = \sum_{i=1}^n \left[i \log 2 + \log \alpha + \log \beta + (\alpha - 1) \log l_i + (\beta - 1) \log(1 - l_i^\alpha) \right. \\ \left. + \log G\left(i, -\log\left[1 - (1 - l_i^\alpha)^\beta\right]\right) \right]. \end{aligned} \quad (20)$$

The likelihood equations for the upper current records are obtained by differentiating $\ell_{U_{cr}^n}(\alpha, \beta)$ with respect to α and β . Thus,

$$\frac{\partial \ell_{U_{cr}^n}}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log u_i - (\beta - 1) \sum_{i=1}^n \frac{u_i^\alpha \log u_i}{1 - u_i^\alpha} + \sum_{i=1}^n \frac{G'_i}{G_i} \frac{\beta u_i^\alpha \log u_i}{1 - u_i^\alpha} = 0, \quad (21)$$

$$\frac{\partial \ell_{U_{cr}^n}}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^n \log(1 - u_i^\alpha) - \sum_{i=1}^n \frac{G'_i}{G_i} \log(1 - u_i^\alpha) = 0. \quad (22)$$

Similarly, the likelihood equations for the lower current records are

$$\frac{\partial \ell_{L_{cr}^n}}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log l_i - (\beta - 1) \sum_{i=1}^n \frac{l_i^\alpha \log l_i}{1 - l_i^\alpha} - \sum_{i=1}^n \frac{G'_i}{G_i} \frac{\beta(1 - l_i^\alpha)^{\beta-1} l_i^\alpha \log l_i}{1 - (1 - l_i^\alpha)^\beta} = 0, \quad (23)$$

$$\frac{\partial \ell_{L_{cr}^n}}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^n \log(1 - l_i^\alpha) + \sum_{i=1}^n \frac{G'_i}{G_i} \frac{(1 - l_i^\alpha)^\beta \log(1 - l_i^\alpha)}{1 - (1 - l_i^\alpha)^\beta} = 0. \quad (24)$$

where $G'_i = \frac{t_i^{i-1} e^{-t_i}}{\Gamma(i)}$ for the upper case, and $G'_i = \frac{s_i^{i-1} e^{-s_i}}{\Gamma(i)}$ for the lower case.

Since the likelihood equations contain nonlinear terms involving the regularized incomplete gamma function, closed-form solutions are not available. Hence, numerical optimization methods are used to obtain the MLEs.

5.2. BE under new PDF. For Bayesian estimation, we assume that α and β are independent gamma random variables with prior densities

$$\pi(\alpha) \propto \alpha^{a_1-1} \exp(-b_1\alpha), \quad \alpha > 0, \quad \pi(\beta) \propto \beta^{a_2-1} \exp(-b_2\beta), \quad \beta > 0. \quad (25)$$

Thus, the joint prior density is

$$\pi(\alpha, \beta) \propto \alpha^{a_1-1} \beta^{a_2-1} \exp(-b_1\alpha - b_2\beta). \quad (26)$$

Using the upper current-record likelihood (17), the posterior density is

$$\pi_{U_{cr}^n}(\alpha, \beta | \mathbf{u}) \propto L_{U_{cr}^n}(\alpha, \beta) \pi(\alpha, \beta). \quad (27)$$

Similarly, using the lower current-record likelihood (19), the posterior density is

$$\pi_{L_{cr}^n}(\alpha, \beta | \mathbf{l}) \propto L_{L_{cr}^n}(\alpha, \beta) \pi(\alpha, \beta). \quad (28)$$

Under the squared error loss function, the Bayes estimators are the posterior means:

$$\hat{\alpha}_{U_{cr}}^{BE} = E(\alpha | \mathbf{u}), \hat{\beta}_{U_{cr}}^{BE} = E(\beta | \mathbf{u}), \quad \hat{\alpha}_{L_{cr}}^{BE} = E(\alpha | \mathbf{l}), \hat{\beta}_{L_{cr}}^{BE} = E(\beta | \mathbf{l}). \quad (29)$$

Since these posterior expectations cannot be obtained in closed form, Markov Chain Monte Carlo methods may be used to approximate them. If $(\alpha^{(t)}, \beta^{(t)})$, $t = 1, \dots, T$, are posterior samples after burn-in, then

$$\hat{\alpha}^{BE} = \frac{1}{T} \sum_{t=1}^T \alpha^{(t)}, \quad \hat{\beta}^{BE} = \frac{1}{T} \sum_{t=1}^T \beta^{(t)}. \quad (30)$$

The corresponding credible intervals are obtained from the empirical posterior quantiles.

6. SIMULATION

To evaluate the performance of the proposed estimation procedures, a Monte Carlo simulation study was conducted. Following the structure of the simulation study in the related current-record paper, 1000 random samples of size 100 were generated from the Kumaraswamy distribution for selected values of the parameters α and β . For each generated sample, the upper and lower current records were extracted. Then, the unknown parameters were estimated using maximum likelihood estimation and Bayesian estimation based on the newly derived current-record density functions.

The likelihood functions used in the simulation were those obtained from the proposed gamma-based PDFs of the Kumaraswamy upper and lower current records. Since the likelihood equations contain the regularized incomplete gamma function, the MLEs were obtained numerically using constrained optimization. For Bayesian estimation, independent gamma priors were assumed for α and β , and posterior estimates were obtained using the Metropolis–Hastings algorithm. Under the squared error loss function, the Bayesian estimates were computed as posterior means.

For each record size, the mean estimate and standard deviation were computed. In addition, the root mean square error (RMSE) and mean absolute deviation (MAD) were calculated to assess the accuracy of the estimators. Bootstrap confidence intervals were also obtained from the empirical quantiles of the simulated estimates. The simulation results are reported in Tables 1–3.

TABLE 1. Simulation results when $\alpha = 1.5$ and $\beta = 3$.

n	Parameter	MLE					BE				
		Mean	SD	RMSE	MAD	95% BCI	Mean	SD	RMSE	MAD	95% BCI
4 ($n=18$)	$\alpha_{U_{cr}}$	1.3076	0.7520	0.7531	0.6849	[0.4377,2.5411]	1.4849	0.7769	0.7523	0.6465	[0.5374,2.8752]
	$\beta_{U_{cr}}$	1.6142	0.4264	1.4460	1.3858	[0.9488,2.4325]	1.7310	0.4247	1.3340	1.2690	[1.0707,2.4244]
	$\alpha_{L_{cr}}$	0.8510	0.2135	0.6811	0.6490	[0.4274,1.1812]	0.8898	0.2061	0.6420	0.6102	[0.4680,1.2226]
	$\beta_{L_{cr}}$	1.4083	0.6170	1.7001	1.5917	[0.6171,2.2812]	1.4465	0.5796	1.6517	1.5535	[0.6954,2.4770]
5 ($n=36$)	$\alpha_{U_{cr}}$	1.5035	0.8358	0.8241	0.7262	[0.4589,2.7665]	1.6608	0.8762	0.8788	0.7193	[0.5601,3.3058]
	$\beta_{U_{cr}}$	2.0295	0.5168	1.0961	0.9705	[1.1713,2.8475]	2.1471	0.5360	1.0033	0.8544	[1.2482,2.9639]
	$\alpha_{L_{cr}}$	0.9693	0.1892	0.5625	0.5307	[0.5946,1.2656]	1.0069	0.2100	0.5348	0.4931	[0.6257,1.3537]
	$\beta_{L_{cr}}$	1.4600	0.6794	1.6794	1.5400	[0.6049,2.4786]	1.4605	0.6096	1.6527	1.5395	[0.6931,2.4293]
6 ($n=84$)	$\alpha_{U_{cr}}$	1.4390	0.8543	0.8513	0.7386	[0.4833,3.1680]	1.5555	0.8664	0.8630	0.7027	[0.5544,3.4448]
	$\beta_{U_{cr}}$	2.1474	0.6156	1.0494	0.9008	[1.1661,3.3915]	2.2264	0.5892	0.9703	0.8180	[1.3259,3.3204]
	$\alpha_{L_{cr}}$	1.1925	0.2439	0.3916	0.3340	[0.8560,1.6607]	1.1938	0.2420	0.3894	0.3326	[0.8071,1.6757]
	$\beta_{L_{cr}}$	1.7821	0.7182	1.4117	1.2179	[0.6055,2.6673]	1.7109	0.6357	1.4356	1.2891	[0.6223,2.5954]

n	Parameter	MLE					BE				
		Mean	SD	RMSE	MAD	95% BCI	Mean	SD	RMSE	MAD	95% BCI
7 ($n=119$)	$\alpha_{U_{cr}}$	1.5206	0.7741	0.7711	0.6496	[0.4396,3.2313]	1.6014	0.8224	0.8252	0.6474	[0.5175,3.1679]
	$\beta_{U_{cr}}$	2.5720	0.7441	0.8556	0.7210	[1.3883,4.0218]	2.5853	0.6098	0.7353	0.5860	[1.4457,3.5599]
	$\alpha_{L_{cr}}$	1.2934	0.2468	0.3211	0.2651	[0.8795,1.7396]	1.2850	0.2518	0.3303	0.2742	[0.8709,1.8736]
	$\beta_{L_{cr}}$	1.8270	0.7300	1.3799	1.1730	[0.6098,2.8442]	1.7180	0.6071	1.4174	1.2820	[0.6426,2.6071]
8 ($n=130$)	$\alpha_{U_{cr}}$	1.5343	0.7011	0.6992	0.5700	[0.5875,3.0417]	1.5063	0.6218	0.6195	0.4928	[0.6127,2.8692]
	$\beta_{U_{cr}}$	3.1878	0.7993	0.8181	0.6790	[1.7763,4.5000]	3.0496	0.5503	0.5504	0.4653	[1.8636,3.7864]
	$\alpha_{L_{cr}}$	1.4265	0.2744	0.2830	0.2297	[0.9839,1.9820]	1.4151	0.2650	0.2773	0.2233	[0.9895,1.9886]
	$\beta_{L_{cr}}$	1.9978	0.6972	1.2194	1.0045	[0.6868,2.9921]	1.8930	0.5726	1.2453	1.1070	[0.8044,2.6945]
9 ($n=163$)	$\alpha_{U_{cr}}$	1.4345	0.6203	0.6219	0.5168	[0.5429,2.6815]	1.3877	0.5198	0.5302	0.4447	[0.5902,2.4813]
	$\beta_{U_{cr}}$	3.3455	0.8445	0.9100	0.7660	[1.7506,4.5000]	3.1694	0.5797	0.6023	0.5160	[1.8419,3.8810]
	$\alpha_{L_{cr}}$	1.5261	0.2586	0.2592	0.2028	[1.0825,2.1549]	1.5117	0.2566	0.2561	0.2010	[1.0906,2.1268]
	$\beta_{L_{cr}}$	2.1398	0.6924	1.1029	0.8851	[0.9179,3.1527]	2.0276	0.5438	1.1133	0.9724	[0.9103,2.7964]
10 ($n=138$)	$\alpha_{U_{cr}}$	1.4736	0.5371	0.5358	0.4300	[0.6353,2.6742]	1.4104	0.4409	0.4484	0.3643	[0.6893,2.4376]
	$\beta_{U_{cr}}$	3.6354	0.7071	0.9487	0.8046	[2.1864,4.5000]	3.4342	0.4590	0.6306	0.5570	[2.2906,3.9952]
	$\alpha_{L_{cr}}$	1.7308	0.2721	0.3560	0.2922	[1.1565,2.2120]	1.7012	0.2666	0.3332	0.2703	[1.1458,2.2207]
	$\beta_{L_{cr}}$	2.2817	0.7447	1.0327	0.7957	[0.9326,3.3128]	2.0937	0.5703	1.0697	0.9066	[0.9903,2.8886]
11 ($n=113$)	$\alpha_{U_{cr}}$	1.5291	0.5213	0.5198	0.4179	[0.6896,2.5505]	1.4456	0.4625	0.4637	0.3729	[0.7003,2.3247]
	$\beta_{U_{cr}}$	3.9000	0.6934	1.1343	1.0078	[2.4464,4.5000]	3.6232	0.4761	0.7830	0.7131	[2.5069,4.1629]
	$\alpha_{L_{cr}}$	1.7945	0.2884	0.4113	0.3367	[1.2808,2.3239]	1.7685	0.2808	0.3876	0.3113	[1.2816,2.3116]
	$\beta_{L_{cr}}$	2.2084	0.7795	1.1086	0.9123	[0.7911,3.4683]	2.0426	0.5841	1.1202	0.9594	[0.8141,2.9105]
12 ($n=89$)	$\alpha_{U_{cr}}$	1.3076	0.4709	0.5063	0.4006	[0.6138,2.4752]	1.2378	0.4347	0.5056	0.4135	[0.6643,2.2371]
	$\beta_{U_{cr}}$	3.9202	0.6637	1.1324	1.0097	[2.3917,4.5000]	3.6715	0.4677	0.8169	0.7465	[2.4916,4.2027]
	$\alpha_{L_{cr}}$	1.9985	0.3287	0.5961	0.5212	[1.3752,2.6122]	1.9576	0.3143	0.5541	0.4783	[1.3378,2.5535]
	$\beta_{L_{cr}}$	2.5866	0.7793	0.8783	0.6711	[0.9291,3.6352]	2.3179	0.5670	0.8850	0.6889	[0.8829,2.9811]
13 ($n=50$)	$\alpha_{U_{cr}}$	1.2328	0.3172	0.4122	0.3589	[0.7754,2.0106]	1.1594	0.2629	0.4286	0.3895	[0.7802,1.7909]
	$\beta_{U_{cr}}$	4.1309	0.5374	1.2498	1.1691	[2.7314,4.5000]	3.8692	0.3714	0.9438	0.8992	[2.8051,4.2120]
	$\alpha_{L_{cr}}$	2.0845	0.3414	0.6752	0.5945	[1.4373,2.7017]	2.0413	0.3266	0.6306	0.5523	[1.4304,2.6630]
	$\beta_{L_{cr}}$	2.6827	0.8642	0.9124	0.7200	[0.9853,3.8299]	2.3810	0.5996	0.8576	0.6224	[1.0752,2.9769]
14 ($n=32$)	$\alpha_{U_{cr}}$	1.1951	0.3176	0.4366	0.3812	[0.8004,2.0851]	1.1236	0.2817	0.4675	0.4318	[0.7821,1.8837]
	$\beta_{U_{cr}}$	4.3042	0.3641	1.3526	1.3042	[3.4448,4.5000]	4.0486	0.2592	1.0791	1.0486	[3.4016,4.2886]
	$\alpha_{L_{cr}}$	2.2837	0.4208	0.8864	0.7881	[1.5280,2.9893]	2.2373	0.4070	0.8391	0.7420	[1.5178,2.9278]
	$\beta_{L_{cr}}$	2.9149	0.7778	0.7703	0.6516	[1.5711,3.8868]	2.5683	0.5026	0.6566	0.4732	[1.5844,3.1443]

n	Parameter	MLE					BE				
		Mean	SD	RMSE	MAD	95% BCI	Mean	SD	RMSE	MAD	95% BCI
15 ($n=16$)	$\alpha_{U_{c_r}}$	1.2247	0.3501	0.4367	0.3684	[0.7123,1.9186]	1.1678	0.3210	0.4549	0.3939	[0.7153,1.8007]
	$\beta_{U_{c_r}}$	4.2864	0.5133	1.3791	1.3517	[3.0317,4.5000]	4.0671	0.4323	1.1462	1.1232	[3.0491,4.3425]
	$\alpha_{L_{c_r}}$	2.4280	0.4766	1.0364	0.9584	[1.4727,3.1499]	2.3745	0.4639	0.9831	0.9025	[1.4471,3.0701]
	$\beta_{L_{c_r}}$	3.0408	0.8109	0.7862	0.6416	[1.5038,4.1462]	2.6041	0.4554	0.5925	0.4113	[1.5843,3.0767]
16 ($n=12$)	$\alpha_{U_{c_r}}$	1.0451	0.1938	0.4881	0.4549	[0.8048,1.2593]	0.9851	0.1722	0.5384	0.5149	[0.7844,1.1687]
	$\beta_{U_{c_r}}$	4.5000	0.0000	1.5000	1.5000	[4.5000,4.5000]	4.2432	0.0908	1.2460	1.2432	[4.1272,4.3506]
	$\alpha_{L_{c_r}}$	2.3774	0.3337	0.9288	0.8774	[1.9466,2.8381]	2.2870	0.2932	0.8313	0.7870	[1.8889,2.6853]
	$\beta_{L_{c_r}}$	3.2997	0.7973	0.7871	0.7620	[2.2250,4.0324]	2.6755	0.4330	0.5114	0.3894	[2.1109,3.1605]

TABLE 2. Simulation results when $\alpha = 2$ and $\beta = 3.5$.

n	Parameter	MLE					BE				
		Mean	SD	RMSE	MAD	95% BCI	Mean	SD	RMSE	MAD	95% BCI
4 ($n=12$)	$\alpha_{U_{c_r}}$	1.4449	0.7028	0.8782	0.7856	[0.5602,2.5634]	1.6722	0.7395	0.7875	0.6879	[0.6940,2.9883]
	$\beta_{U_{c_r}}$	1.6600	0.3893	1.8782	1.8400	[1.0273,2.4080]	1.8141	0.4097	1.7319	1.6859	[1.1491,2.4541]
	$\alpha_{L_{c_r}}$	1.0826	0.2583	0.9509	0.9174	[0.5613,1.4769]	1.1380	0.2498	0.8953	0.8620	[0.6211,1.5311]
	$\beta_{L_{c_r}}$	1.4088	0.5781	2.1648	2.0912	[0.6586,2.2253]	1.5040	0.6017	2.0793	1.9960	[0.7379,2.6031]
5 ($n=30$)	$\alpha_{U_{c_r}}$	1.6531	0.7863	0.8493	0.7810	[0.5965,2.7793]	1.8693	0.8851	0.8825	0.7649	[0.7044,3.6278]
	$\beta_{U_{c_r}}$	2.0767	0.4659	1.4956	1.4233	[1.2709,2.8304]	2.2511	0.5532	1.3628	1.2489	[1.3529,3.1570]
	$\alpha_{L_{c_r}}$	1.2400	0.2309	0.7933	0.7600	[0.7786,1.5957]	1.2898	0.2574	0.7542	0.7102	[0.8082,1.6911]
	$\beta_{L_{c_r}}$	1.4698	0.6409	2.1263	2.0302	[0.6501,2.4178]	1.5058	0.6074	2.0822	1.9942	[0.7471,2.5616]
6 ($n=79$)	$\alpha_{U_{c_r}}$	1.6119	0.8201	0.9029	0.8095	[0.6289,3.1688]	1.7882	0.8724	0.8927	0.7749	[0.7012,3.5953]
	$\beta_{U_{c_r}}$	2.2232	0.5735	1.3983	1.2769	[1.2804,3.3506]	2.3653	0.6043	1.2839	1.1458	[1.4421,3.5055]
	$\alpha_{L_{c_r}}$	1.5199	0.2936	0.5619	0.4969	[1.1028,2.0759]	1.5276	0.2944	0.5557	0.4893	[1.0477,2.1062]
	$\beta_{L_{c_r}}$	1.7802	0.6735	1.8455	1.7198	[0.6548,2.6042]	1.7712	0.6497	1.8455	1.7288	[0.7201,2.7331]
7 ($n=107$)	$\alpha_{U_{c_r}}$	1.7474	0.7706	0.8079	0.6957	[0.5768,3.2985]	1.8763	0.8508	0.8562	0.7052	[0.6761,3.4849]
	$\beta_{U_{c_r}}$	2.6916	0.6980	1.0661	0.8893	[1.4989,4.0455]	2.7830	0.6488	0.9652	0.7872	[1.6054,3.8269]
	$\alpha_{L_{c_r}}$	1.6519	0.3048	0.4618	0.3862	[1.1483,2.1960]	1.6550	0.3128	0.4648	0.3924	[1.1311,2.3610]
	$\beta_{L_{c_r}}$	1.8447	0.6811	1.7889	1.6553	[0.6636,2.7856]	1.8098	0.6379	1.8056	1.6902	[0.6837,2.8155]
8 ($n=125$)	$\alpha_{U_{c_r}}$	1.7909	0.7109	0.7384	0.6350	[0.7651,3.1521]	1.8136	0.6773	0.7000	0.5686	[0.8086,3.3423]
	$\beta_{U_{c_r}}$	3.3543	0.7901	0.8004	0.6517	[1.9426,4.9042]	3.3174	0.6112	0.6356	0.5027	[2.0640,4.1786]
	$\alpha_{L_{c_r}}$	1.8255	0.3329	0.3747	0.3085	[1.2744,2.4910]	1.8191	0.3250	0.3709	0.3006	[1.2898,2.5050]
	$\beta_{L_{c_r}}$	2.0139	0.6513	1.6216	1.4861	[0.7433,2.9448]	1.9786	0.5875	1.6301	1.5214	[0.8656,2.8646]

n	Parameter	MLE					BE				
		Mean	SD	RMSE	MAD	95% BCI	Mean	SD	RMSE	MAD	95% BCI
9 ($n=175$)	$\alpha_{U_{cr}}$	1.7292	0.6773	0.7275	0.6308	[0.7159,3.0533]	1.7080	0.5918	0.6583	0.5666	[0.7709,2.8684]
	$\beta_{U_{cr}}$	3.6045	0.9185	0.9216	0.7669	[1.9424,5.2500]	3.4862	0.6592	0.6573	0.5461	[2.0177,4.3213]
	$\alpha_{L_{cr}}$	1.9544	0.3153	0.3176	0.2556	[1.4080,2.7135]	1.9469	0.3157	0.3192	0.2575	[1.4245,2.6653]
	$\beta_{L_{cr}}$	2.1660	0.6414	1.4793	1.3340	[0.9834,3.0861]	2.1340	0.5736	1.4809	1.3660	[1.0303,3.0171]
10 ($n=152$)	$\alpha_{U_{cr}}$	1.7987	0.6102	0.6404	0.5350	[0.8287,3.2399]	1.7571	0.5054	0.5590	0.4656	[0.9022,2.8677]
	$\beta_{U_{cr}}$	3.9647	0.8223	0.9419	0.7592	[2.4086,5.2500]	3.8136	0.5393	0.6221	0.5369	[2.5446,4.5361]
	$\alpha_{L_{cr}}$	2.2155	0.3328	0.3955	0.3191	[1.5099,2.7959]	2.1876	0.3286	0.3773	0.3012	[1.5210,2.8120]
	$\beta_{L_{cr}}$	2.3053	0.6928	1.3798	1.1947	[1.0141,3.2324]	2.2075	0.5922	1.4208	1.2925	[1.0499,3.0558]
11 ($n=101$)	$\alpha_{U_{cr}}$	1.9155	0.6179	0.6209	0.5073	[0.9024,3.1320]	1.8108	0.5347	0.5649	0.4610	[0.9181,2.8317]
	$\beta_{U_{cr}}$	4.3501	0.8424	1.1942	1.0372	[2.6797,5.2500]	4.0574	0.5782	0.8013	0.7186	[2.7934,4.7759]
	$\alpha_{L_{cr}}$	2.3047	0.3535	0.4655	0.3727	[1.6759,2.9510]	2.2789	0.3483	0.4450	0.3516	[1.6578,2.9882]
	$\beta_{L_{cr}}$	2.2549	0.7283	1.4409	1.2451	[0.8748,3.4051]	2.1695	0.6153	1.4648	1.3305	[0.9149,3.1446]
12 ($n=91$)	$\alpha_{U_{cr}}$	1.6637	0.5595	0.6501	0.5403	[0.8140,3.0856]	1.5794	0.5110	0.6596	0.5630	[0.8803,2.7421]
	$\beta_{U_{cr}}$	4.3923	0.8168	1.2066	1.0378	[2.6476,5.2500]	4.1200	0.5627	0.8351	0.7473	[2.7506,4.7944]
	$\alpha_{L_{cr}}$	2.5579	0.4014	0.6860	0.5928	[1.7820,3.3043]	2.5218	0.3834	0.6463	0.5545	[1.7461,3.2245]
	$\beta_{L_{cr}}$	2.6128	0.7248	1.1430	0.8937	[1.0051,3.5502]	2.4676	0.5983	1.1915	1.0327	[1.0023,3.2322]
13 ($n=58$)	$\alpha_{U_{cr}}$	1.5994	0.3952	0.5599	0.4907	[1.0101,2.5590]	1.5048	0.3170	0.5862	0.5317	[1.0257,2.2579]
	$\beta_{U_{cr}}$	4.6768	0.6922	1.3618	1.2367	[3.0251,5.2500]	4.3758	0.4693	0.9914	0.9237	[3.0865,4.8565]
	$\alpha_{L_{cr}}$	2.6719	0.4165	0.7883	0.6913	[1.8824,3.4153]	2.6299	0.4017	0.7449	0.6497	[1.8752,3.3947]
	$\beta_{L_{cr}}$	2.7193	0.8058	1.1161	0.8348	[1.0830,3.7604]	2.5431	0.6392	1.1472	0.9569	[1.1699,3.2388]
14 ($n=44$)	$\alpha_{U_{cr}}$	1.5754	0.3861	0.5698	0.5011	[1.0692,2.5970]	1.4764	0.3397	0.6213	0.5766	[1.0486,2.3372]
	$\beta_{U_{cr}}$	4.9352	0.5124	1.5212	1.4352	[3.8147,5.2500]	4.6130	0.3456	1.1638	1.1130	[3.7718,4.9318]
	$\alpha_{L_{cr}}$	2.9210	0.5106	1.0492	0.9291	[1.9973,3.7737]	2.8816	0.5038	1.0115	0.8924	[1.9927,3.7413]
	$\beta_{L_{cr}}$	2.9541	0.7182	0.8931	0.6580	[1.6827,3.8134]	2.7585	0.5381	0.9112	0.7472	[1.6974,3.4341]
15 ($n=26$)	$\alpha_{U_{cr}}$	1.6161	0.4292	0.5657	0.4728	[0.9421,2.4319]	1.5386	0.3912	0.5970	0.5200	[0.9532,2.2920]
	$\beta_{U_{cr}}$	4.9308	0.6533	1.5644	1.5228	[3.3672,5.2500]	4.6621	0.5245	1.2682	1.2433	[3.4124,5.0105]
	$\alpha_{L_{cr}}$	3.1056	0.5859	1.2426	1.1490	[1.9184,3.9783]	3.0571	0.5821	1.1979	1.1004	[1.8817,3.9198]
	$\beta_{L_{cr}}$	3.0842	0.7402	0.8285	0.6052	[1.6191,4.0592]	2.8050	0.5158	0.8558	0.6950	[1.6547,3.3531]

TABLE 3. Simulation results when $\alpha = 3$ and $\beta = 5$.

n	Parameter	MLE					BE				
		Mean	SD	RMSE	MAD	95% BCI	Mean	SD	RMSE	MAD	95% BCI
5 ($n=45$)	$\alpha_{U_{cr}}$	1.8353	0.6595	1.3339	1.1647	[0.8527,2.7370]	2.0922	0.7833	1.1920	1.0345	[1.0003,3.7270]
	$\beta_{U_{cr}}$	2.2562	0.4044	2.7726	2.7438	[1.5147,2.8333]	2.4896	0.5537	2.5690	2.5104	[1.6303,3.4588]
	$\alpha_{L_{cr}}$	1.7115	0.2899	1.3198	1.2885	[1.1258,2.1392]	1.7782	0.3276	1.2637	1.2218	[1.1701,2.2800]
	$\beta_{L_{cr}}$	1.5036	0.5622	3.5400	3.4964	[0.7623,2.3091]	1.5910	0.5944	3.4590	3.4090	[0.8392,2.7005]
6 ($n=76$)	$\alpha_{U_{cr}}$	1.8527	0.7012	1.3424	1.1610	[0.8947,3.0960]	2.0650	0.7782	1.2135	1.0440	[0.9962,3.6895]
	$\beta_{U_{cr}}$	2.4798	0.5130	2.5712	2.5202	[1.5814,3.4041]	2.6863	0.6085	2.3915	2.3137	[1.6970,3.9601]
	$\alpha_{L_{cr}}$	2.0803	0.3569	0.9857	0.9204	[1.5386,2.7335]	2.0946	0.3606	0.9737	0.9055	[1.4743,2.7728]
	$\beta_{L_{cr}}$	1.7926	0.5832	3.2594	3.2074	[0.7785,2.5024]	1.8513	0.6258	3.2095	3.1487	[0.8522,2.7855]
7 ($n=123$)	$\alpha_{U_{cr}}$	2.0334	0.6886	1.1851	0.9989	[0.8358,3.2622]	2.2096	0.8082	1.1280	0.9460	[0.9420,3.7204]
	$\beta_{U_{cr}}$	3.0113	0.6199	2.0823	1.9887	[1.7899,4.1467]	3.2077	0.6840	1.9174	1.7923	[2.0083,4.4736]
	$\alpha_{L_{cr}}$	2.2725	0.3766	0.8185	0.7306	[1.6448,2.9590]	2.2822	0.3971	0.8195	0.7341	[1.6176,3.1904]
	$\beta_{L_{cr}}$	1.8611	0.5971	3.1947	3.1389	[0.7915,2.6718]	1.9079	0.6236	3.1538	3.0921	[0.7993,2.9407]
8 ($n=128$)	$\alpha_{U_{cr}}$	2.0935	0.6253	1.0999	0.9410	[1.0851,3.1982]	2.2086	0.6835	1.0440	0.8983	[1.0950,3.6485]
	$\beta_{U_{cr}}$	3.7087	0.6771	1.4569	1.2913	[2.3511,4.8788]	3.8805	0.7037	1.3209	1.1217	[2.4639,4.9532]
	$\alpha_{L_{cr}}$	2.5138	0.4103	0.6352	0.5418	[1.8184,3.3242]	2.5116	0.4015	0.6313	0.5408	[1.8287,3.3226]
	$\beta_{L_{cr}}$	2.0610	0.5582	2.9911	2.9390	[0.8839,2.8576]	2.1094	0.5882	2.9494	2.8906	[1.0455,3.1420]
9 ($n=161$)	$\alpha_{U_{cr}}$	2.0798	0.6209	1.1090	0.9486	[1.0448,3.1929]	2.1601	0.6330	1.0505	0.8974	[1.0864,3.3928]
	$\beta_{U_{cr}}$	4.0620	0.8378	1.2559	1.0247	[2.4427,5.5027]	4.1886	0.8153	1.1485	0.9175	[2.5635,5.4547]
	$\alpha_{L_{cr}}$	2.6991	0.3902	0.4917	0.4204	[2.0050,3.6093]	2.7001	0.3965	0.4962	0.4132	[2.0587,3.5774]
	$\beta_{L_{cr}}$	2.2142	0.5465	2.8386	2.7858	[1.1301,2.9679]	2.3003	0.5725	2.7593	2.6997	[1.2129,3.2052]
10 ($n=148$)	$\alpha_{U_{cr}}$	2.2112	0.5762	0.9756	0.8485	[1.1885,3.4082]	2.2537	0.5428	0.9217	0.8076	[1.2764,3.4077]
	$\beta_{U_{cr}}$	4.5798	0.8121	0.9117	0.7531	[2.9776,6.0516]	4.6530	0.7009	0.7798	0.6300	[3.1688,5.7333]
	$\alpha_{L_{cr}}$	3.0490	0.4169	0.4182	0.3225	[2.1620,3.8033]	3.0293	0.4109	0.4105	0.3201	[2.2074,3.7880]
	$\beta_{L_{cr}}$	2.3672	0.5850	2.6966	2.6328	[1.1931,3.0960]	2.4098	0.6148	2.6617	2.5902	[1.2391,3.3393]
11 ($n=124$)	$\alpha_{U_{cr}}$	2.3850	0.6319	0.8798	0.7265	[1.2821,3.6113]	2.3673	0.5940	0.8660	0.7142	[1.3039,3.4335]
	$\beta_{U_{cr}}$	5.1582	0.9833	0.9916	0.8232	[3.2511,6.6976]	5.0839	0.8188	0.8194	0.6753	[3.3483,6.2801]
	$\alpha_{L_{cr}}$	3.1912	0.4435	0.4812	0.3770	[2.4086,3.9921]	3.1676	0.4435	0.4723	0.3685	[2.3481,3.9863]
	$\beta_{L_{cr}}$	2.3635	0.6199	2.7077	2.6365	[1.0871,3.2970]	2.3863	0.6352	2.6892	2.6137	[1.0694,3.4567]
12 ($n=69$)	$\alpha_{U_{cr}}$	2.1748	0.6122	1.0254	0.9042	[1.2003,3.6642]	2.1329	0.5777	1.0401	0.9315	[1.2638,3.3554]
	$\beta_{U_{cr}}$	5.3531	1.0645	1.1158	0.9074	[3.3001,7.3743]	5.1997	0.8005	0.8207	0.6812	[3.3923,6.3228]
	$\alpha_{L_{cr}}$	3.5173	0.5000	0.7175	0.5991	[2.5120,4.4347]	3.4900	0.4784	0.6829	0.5700	[2.4905,4.2756]
	$\beta_{L_{cr}}$	2.6778	0.6136	2.4010	2.3222	[1.1717,3.3980]	2.7087	0.6445	2.3793	2.2913	[1.2012,3.6413]

n	Parameter	MLE					BE				
		Mean	SD	RMSE	MAD	95% BCI	Mean	SD	RMSE	MAD	95% BCI
13 ($n=47$)	$\alpha_{U_{cr}}$	2.2003	0.5035	0.9423	0.8388	[1.4313,3.3293]	2.1060	0.3923	0.9747	0.8976	[1.4602,2.9995]
	$\beta_{U_{cr}}$	5.9519	1.1221	1.4629	1.1984	[3.7777,7.5000]	5.6448	0.7294	0.9681	0.8361	[3.9468,6.5675]
	$\alpha_{L_{cr}}$	3.6835	0.5160	0.8533	0.7352	[2.7080,4.5982]	3.6474	0.5105	0.8213	0.7024	[2.6772,4.5992]
	$\beta_{L_{cr}}$	2.8035	0.6834	2.2983	2.1965	[1.3406,3.6084]	2.8048	0.6700	2.2932	2.1952	[1.4266,3.5562]
14 ($n=39$)	$\alpha_{U_{cr}}$	2.2594	0.5036	0.8911	0.7921	[1.5224,3.2618]	2.1143	0.4251	0.9795	0.9175	[1.5626,3.0299]
	$\beta_{U_{cr}}$	6.5528	1.0172	1.8476	1.6109	[4.7433,7.5000]	6.0718	0.6204	1.2335	1.1255	[4.7026,6.7436]
	$\alpha_{L_{cr}}$	4.0091	0.6262	1.1825	1.0338	[2.8635,5.0462]	3.9925	0.6258	1.1681	1.0179	[2.8751,5.0892]
	$\beta_{L_{cr}}$	3.0391	0.5939	2.0462	1.9609	[1.9325,3.6875]	3.0959	0.5777	1.9872	1.9041	[2.0516,3.9495]
15 ($n=26$)	$\alpha_{U_{cr}}$	2.3321	0.5261	0.8400	0.7254	[1.3832,3.1775]	2.2283	0.4965	0.9092	0.8198	[1.4177,3.1481]
	$\beta_{U_{cr}}$	6.6455	1.1394	1.9811	1.8309	[4.2262,7.5000]	6.2515	0.8156	1.4798	1.4233	[4.3425,6.9238]
	$\alpha_{L_{cr}}$	4.2604	0.7384	1.4491	1.3354	[2.7347,5.3240]	4.2306	0.7492	1.4285	1.3066	[2.6960,5.3220]
	$\beta_{L_{cr}}$	3.1755	0.6006	1.9149	1.8245	[1.8855,3.9066]	3.1611	0.6015	1.9290	1.8389	[1.9330,4.0093]
16 ($n=20$)	$\alpha_{U_{cr}}$	2.1245	0.3024	0.9180	0.8755	[1.8017,2.5097]	1.9883	0.2510	1.0373	1.0117	[1.7382,2.2925]
	$\beta_{U_{cr}}$	7.0876	0.5205	2.1410	2.0876	[6.3037,7.5000]	6.5952	0.3317	1.6237	1.5952	[6.1779,6.9922]
	$\alpha_{L_{cr}}$	4.1953	0.5020	1.2801	1.1953	[3.5307,4.8756]	4.0818	0.4422	1.1546	1.0818	[3.4400,4.6352]
	$\beta_{L_{cr}}$	3.3510	0.6364	1.7484	1.6490	[2.4955,3.9832]	3.0633	0.5956	2.0116	1.9367	[2.3746,3.8068]
17 ($n=4$)	$\alpha_{U_{cr}}$	1.9549	0.1921	1.0583	1.0451	[1.7731,2.1815]	1.8383	0.1976	1.1742	1.1617	[1.6711,2.0893]
	$\beta_{U_{cr}}$	7.1828	0.6343	2.2509	2.1828	[6.3265,7.5000]	6.7195	0.5115	1.7756	1.7195	[6.0327,7.0810]
	$\alpha_{L_{cr}}$	4.6896	0.7404	1.8072	1.6896	[3.9298,5.4921]	4.5784	0.6672	1.6809	1.5784	[3.9215,5.2259]
	$\beta_{L_{cr}}$	3.8692	0.1770	1.1412	1.1308	[3.7232,4.0811]	3.5844	0.4302	1.4638	1.4156	[3.1062,4.0344]

6.1. Discussion of the simulation results.

Tables 1–3 summarize the finite-sample performance of the proposed maximum likelihood estimator (MLE) and Bayesian estimator (BE) based on the newly derived current-record density functions for three different parameter settings. The estimators were compared using the mean estimate, standard deviation (SD), root mean square error (RMSE), mean absolute deviation (MAD), and the corresponding 95% Bayesian credible intervals.

Overall, both estimation procedures successfully recover the true parameter values from current-record data. As the number of observed current records (n) increases, the sampling variability of both estimators generally decreases, while the estimated values become progressively closer to the true parameters. This behaviour is reflected by the overall reduction in SD, RMSE, and MAD, demonstrating the consistency of the proposed estimation procedures.

For the upper current records, the Bayesian estimator generally produces more stable estimates than the MLE. In particular, the Bayesian estimates usually exhibit smaller standard deviations and mean

absolute deviations. Furthermore, for the parameter β , the Bayesian estimator achieves smaller RMSE values in most record sizes, especially when the number of observed current records is moderate or large. This improvement is accompanied by narrower credible intervals, indicating greater estimation precision.

A similar pattern is observed for the lower current records. Although the MLE and Bayesian estimator provide comparable estimates for the parameter α , the Bayesian estimator frequently exhibits reduced variability and slightly improved estimation accuracy. For the parameter β , the Bayesian estimator again provides more stable estimates with smaller SD and MAD values across most simulation scenarios.

The simulation results also indicate that estimation becomes more challenging as the true parameter values increase from $(\alpha, \beta) = (1.5, 3)$ to $(2, 3.5)$ and $(3, 5)$. This is reflected by the larger RMSE values observed for both estimation methods. Nevertheless, the Bayesian procedure maintains its stability and continues to produce competitive estimates together with relatively narrow credible intervals, demonstrating its robustness under different parameter configurations.

Overall, the simulation study confirms that the proposed gamma-based current-record likelihood functions provide an effective framework for parameter estimation under the Kumaraswamy distribution. Both estimation methods perform satisfactorily and become increasingly accurate as more current-record observations become available. Although neither estimator uniformly dominates the other for every parameter and every record size, the Bayesian estimator consistently exhibits lower estimator variability and generally smaller RMSE and MAD values for the parameter β . Consequently, Bayesian estimation may be regarded as the preferred estimation approach for the proposed current-record models because it provides the most reliable overall performance while naturally accounting for parameter uncertainty.

7. BAYESIAN PREDICTION (BP)

In this subsection, Bayesian prediction of future current records is developed using the posterior predictive approach based on the proposed current-record distributions. The predictive distribution is obtained by integrating the conditional distribution of a future current record with respect to the posterior distribution of the unknown parameters derived in Section 5. Since the posterior densities involve the regularized incomplete gamma function, closed-form Bayesian predictors are not available, and Monte Carlo simulation is employed for numerical implementation.

Bayesian prediction for upper current records: Let U_{cr}^{n+s} denote the $(n + s)$ -th upper current record, given the observed upper current records

$$\underline{U}_{cr} = (U_{cr}^1, U_{cr}^2, \dots, U_{cr}^n).$$

The posterior predictive density is defined by

$$f_{U_{cr}^{n+s}|\underline{U}_{cr}}^{BP}(u) = \int f_{U_{cr}^{n+s}|U_{cr}^n}(u | u_n, \alpha, \beta) \pi(\alpha, \beta | \underline{U}_{cr}) d\alpha d\beta, \quad (31)$$

where $f_{U_{cr}^{n+s}|U_{cr}^n}(\cdot)$ denotes the conditional density of the future upper current record and $\pi(\alpha, \beta | \underline{U}_{cr})$ is the posterior distribution obtained in Section 5.

Substituting the conditional density of the proposed upper current-record model yields

$$\begin{aligned} f_{U_{cr}^{n+s}|\underline{U}_{cr}}^{BP}(u) &\propto \int f_{U_{cr}^{n+s}|U_{cr}^n}(u | u_n, \alpha, \beta) \\ &\times \pi(\alpha, \beta | \underline{U}_{cr}) d\alpha d\beta, \quad u \geq u_n. \end{aligned} \quad (32)$$

Under the squared error loss function, the Bayesian point predictor is

$$\widehat{U_{cr}^{n+s}}^{BP} = E(U_{cr}^{n+s} | \underline{U}_{cr}) = \int_{u_n}^1 u f_{U_{cr}^{n+s}|\underline{U}_{cr}}^{BP}(u) du. \quad (33)$$

Since the posterior predictive density has no closed-form expression, the Bayesian predictor is approximated using Monte Carlo simulation,

$$\widehat{U_{cr}^{n+s}}^{BP} \approx \frac{1}{T} \sum_{t=1}^T U_{cr}^{n+s,(t)},$$

where $\{U_{cr}^{n+s,(t)}\}_{t=1}^T$ are generated from the posterior predictive distribution. A $100(1 - \gamma)\%$ Bayesian prediction interval is obtained from the posterior predictive distribution,

$$P(L_\gamma \leq U_{cr}^{n+s} \leq U_\gamma | \underline{U}_{cr}) = 1 - \gamma,$$

whose limits are estimated using the empirical posterior predictive quantiles,

$$(L_\gamma, U_\gamma) = (Q_{\gamma/2}, Q_{1-\gamma/2}).$$

Bayesian prediction for lower current records: Exactly the same work can be done as the upper current record but using relevant formulas.

The Bayesian predictive approach naturally incorporates parameter uncertainty through the posterior distribution while preserving the stochastic structure of the proposed current-record model. Consequently, both the Bayesian point predictors and prediction intervals fully reflect the uncertainty associated with future current records. Since the predictive integrals involve the regularized incomplete gamma function and do not admit closed-form solutions, posterior predictive quantities are evaluated numerically using the Metropolis–Hastings algorithm described in Section 5.

8. REAL SAUDI ENVIRONMENTAL DATA

To illustrate the practical applicability of the proposed methodology, a real environmental dataset from Saudi Arabia is analyzed. The data consist of annual tree-covered area (measured in thousand hectares) obtained from the Climate Change Initiative Land Cover (CCI_LC) database for the period 2001–2020. Tree-covered area is an important environmental sustainability indicator because it reflects changes in vegetation cover, ecosystem health, and the effectiveness of national afforestation and conservation initiatives. During the study period, Saudi Arabia implemented several environmental programs aimed at increasing vegetation cover, culminating in the Saudi Green Initiative. The observed annual measurements are presented in Table 4. Since the Kumaraswamy distribution is defined on the unit interval $(0, 1)$, the original observations are transformed using the min–max transformation where $\min(x) = 112.79$ and $\max(x) = 143.91$. Table 4 presents both the original observations and their corresponding transformed values. The transformed data are subsequently used to extract the upper and lower current records and to implement the estimation and Bayesian prediction procedures developed in the previous sections.

TABLE 4. Annual tree-covered area in Saudi Arabia and the corresponding min–max transformed observations used in the analysis.

Year	Tree-Covered Area (1000 ha)	Transformed Value
2001	113.68	0.0286
2002	113.65	0.0276
2003	113.21	0.0135
2004	113.88	0.0350
2005	112.79	0.0000
2006	113.20	0.0132
2007	115.51	0.0874
2008	116.56	0.1211
2009	117.47	0.1504
2010	118.58	0.1861
2011	119.61	0.2192
2012	120.82	0.2580
2013	126.19	0.4306
2014	130.41	0.5662
2015	131.60	0.6044
2016	134.73	0.7050
2017	135.96	0.7445
2018	136.75	0.7699
2019	141.66	0.9271
2020	143.91	1.0000

The data were obtained from the KAPSARC Data Portal (CCI_LC dataset). <https://datasource.kapsarc.org/explore/assets/land-cover/view/?page=1&search=sau&refine=item%3ATree-cover>

ed+areas&refine=element%3AArea+from+CCI_LC The suitability of the Kumaraswamy distribution for modeling the transformed tree-covered area data was assessed using the Kolmogorov–Smirnov (KS), Cramér–von Mises (CvM), and Anderson–Darling (AD) goodness-of-fit tests. The maximum likelihood estimates of the shape parameters were obtained as $\hat{\alpha} = 0.2595$ and $\hat{\beta} = 0.4889$. The KS test produced a statistic of $D = 0.1640$ with a p -value of 0.5982, while the CvM and AD tests yielded statistics of 0.1146 and 0.7845 with corresponding p -values of 0.5218 and 0.4906, respectively. Since all three p -values are substantially greater than the 5% significance level, there is insufficient evidence to reject the null hypothesis that the transformed data follow a Kumaraswamy distribution. Furthermore, the histogram with the fitted density, the empirical and fitted CDFs, and the Q–Q and P–P plots in Figure 4 demonstrate good agreement between the observed data and the fitted model. These results confirm that the Kumaraswamy distribution provides an adequate fit to the transformed Saudi tree-covered area data and is therefore appropriate for the subsequent current-record estimation and Bayesian prediction analyses.

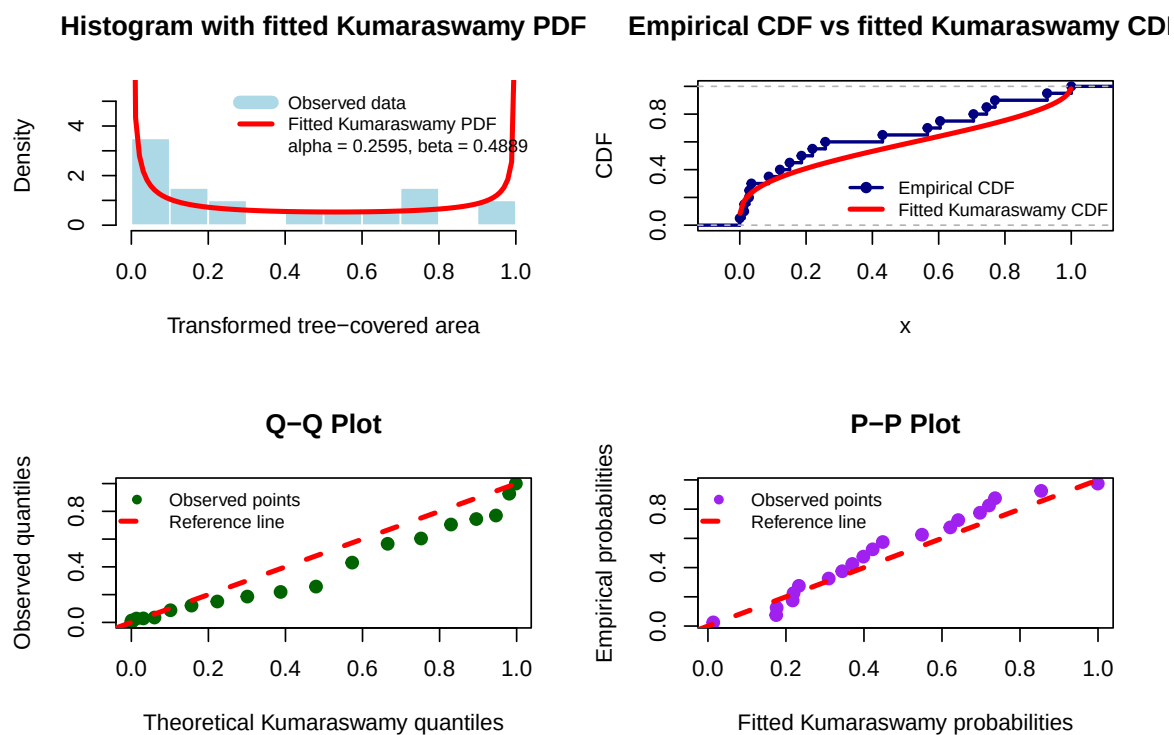


FIGURE 4. Goodness-of-fit diagnostics for the fitted Kumaraswamy distribution with $\hat{\alpha} = 0.2595$ and $\hat{\beta} = 0.4889$.

The following are the upper and lower current records with the record range

n	1	2	3	4	5	6	7	8	9	10
$U_{c_r}^n$	0.0286	0.0286	0.0286	0.0350	0.0350	0.0874	0.1211	0.1504	0.1861	0.2192
$L_{c_r}^n$	0.0286	0.0276	0.0135	0.0135	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$R_{c_r}^n$	0.0000	0.0010	0.0151	0.0215	0.0350	0.0874	0.1211	0.1504	0.1861	0.2192

n	11	12	13	14	15	16	17	18	19
$U_{c_r}^n$	0.2580	0.4306	0.5662	0.6044	0.7050	0.7445	0.7699	0.9271	1.0000
$L_{c_r}^n$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$R_{c_r}^n$	0.2580	0.4306	0.5662	0.6044	0.7050	0.7445	0.7699	0.9271	1.0000

TABLE 5. $L_{c_r}^n$, $U_{c_r}^n$, and $R_{c_r}^n$ records obtained from transformed values.

The transformed observations in Table 4 yield the upper and lower current record sequences presented in Table 5. These record values form the basis for all subsequent statistical analyses. The maximum-likelihood and Bayesian estimation procedures developed in Sections 4 and 5 are applied separately to the upper and lower current records to estimate the unknown Kumaraswamy parameters. The resulting parameter estimates, together with their associated uncertainty measures, are reported in Table 6. These estimates are subsequently employed to construct Bayesian point predictions (BPV) and Bayesian prediction intervals (BPI) for future current records.

TABLE 6. Parameter estimates for the transformed Saudi tree-covered area current records.

Method	$\hat{\alpha}$	$\hat{\beta}$
Upper MLE	0.0536	1.7200
Lower MLE	0.5000	2.7150
Upper BE	0.0732	1.8573
Lower BE	0.5971	2.9164

Using the Bayesian prediction methodology developed in Section 6, future upper and lower current records are predicted based on the transformed Saudi tree-covered area data. The posterior predictive distributions are obtained by integrating the proposed current-record densities with respect to the posterior distribution of the unknown parameters. Since the resulting predictive integrals do not admit closed-form expressions, the Bayesian point predictors and prediction intervals are computed using the Metropolis–Hastings algorithm described previously.

The Bayesian point values (BPV), together with their corresponding 95% Bayesian prediction intervals (BPI), for the next three future current records ($s = 1, 2$, and 3) are presented in Table 7 after

transforming them back using the following equation

$$X = x^* (\max(X) - \min(X)) + \min(X), \quad (34)$$

For comparison, the corresponding predictions obtained by substituting the maximum likelihood estimates into the predictive distributions are reported in Table 8.

TABLE 7. BPV and BPI for future current records under BE on the original scale.

s	$\hat{U}_{cr}^{n+s}$	$\hat{L}_{cr}^{n+s}$	$\hat{R}_{cr}^{n+s}$
1	146.3532 (143.9694, 152.9417)	111.2353 (106.3232, 112.7557)	35.1179 (31.5644, 43.1885)
2	148.8105 (144.5050, 157.8617)	109.6505 (102.2646, 112.4809)	39.1601 (33.1196, 50.4096)
3	151.3058 (145.3786, 162.3788)	108.0971 (98.2647, 112.0440)	43.2087 (34.9372, 57.2267)

TABLE 8. BPV and BPI for future current records under MLE on the original scale.

s	$\hat{U}_{cr}^{n+s}$	$\hat{L}_{cr}^{n+s}$	$\hat{R}_{cr}^{n+s}$
1	146.5011 (143.9756, 153.5091)	108.2139 (96.1408, 112.6794)	38.2871 (31.9282, 51.6884)
2	149.1592 (144.5249, 158.4519)	103.4300 (86.6769, 111.6606)	45.7292 (34.9618, 64.0781)
3	151.7906 (145.4967, 162.9701)	98.7455 (78.9702, 109.9169)	53.0451 (38.7967, 75.1079)

The prediction results presented in Tables 7 and 8 indicate a continued increase in the annual tree-covered area in Saudi Arabia over the next three future current records. Under both the Bayesian and maximum-likelihood approaches, the predicted upper current records increase steadily from one prediction horizon to the next, exceeding the largest observed value of 143.91 thousand hectares. This finding is consistent with the recent upward trend in the data and reflects the expected expansion of tree-covered areas associated with ongoing afforestation and environmental sustainability initiatives.

Similarly, the predicted lower current records decrease gradually as the prediction horizon increases. Since the proposed prediction procedure permits extrapolation beyond the observed data, the predicted lower values fall below the historical minimum of 112.79 thousand hectares. Moreover, the prediction intervals widen as s increases under both estimation methods, indicating greater uncertainty in long-term predictions.

Comparing the two estimation approaches, the Bayesian method yields more stable point predictions and narrower prediction intervals than the maximum-likelihood approach. This is particularly evident in the lower-current records, where the maximum-likelihood intervals are considerably wider, reflecting greater variability in the predictions. Overall, the Bayesian approach provides more reliable predictions

for the annual tree-covered area by incorporating parameter uncertainty into the predictive distribution, making it the preferred method for forecasting future environmental extremes.

9. CONCLUSION

This paper developed a comprehensive framework for modeling upper and lower current records under the Kumaraswamy distribution. Using the recently proposed Gamma-based current-record representation of [23], explicit expressions for the PDFs, CDFs, joint and conditional distributions, moments, entropy, and extropy were derived. Based on the newly obtained current-record density functions, maximum likelihood and Bayesian estimation procedures were proposed, with Bayesian estimation implemented through the Metropolis–Hastings algorithm.

The proposed estimators were evaluated through an extensive Monte Carlo simulation study. The results showed that both estimation methods perform well and become more accurate as the number of current records increases. Overall, the Bayesian approach generally produced smaller RMSE and MAD values and lower variability than the maximum likelihood estimator, particularly for moderate and small record samples.

The methodology was further applied to annual tree-covered area data for Saudi Arabia after transforming the observations to the unit interval, demonstrating the practical usefulness of the proposed model for environmental sustainability analysis. The prediction results indicate a continued increase in future upper current records and a wider gap between future upper and lower records, suggesting a sustained expansion of tree-covered areas. These findings provide quantitative evidence to monitor afforestation progress, evaluate the effectiveness of the Saudi Green Initiative, and support environmental planning and policy decisions related to sustainable land management and natural resource conservation.

The proposed framework provides an effective statistical tool for analyzing bounded environmental and sustainability indicators using current-record data. Future research may extend the proposed methodology to generalized Kumaraswamy models, progressive current records, multivariate settings, and other environmental and climate-related sustainability indicators.

Acknowledgment: The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2026/02/44661).

Authors' Contributions: All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

- [1] A.A. Rashed, Climate Change Challenges in Saudi Arabia: Strategies for Mitigation, *J. Umm Al-Qura Univ. Appl. Sci.* (2025), 1–15. <https://doi.org/10.1007/s43994-025-00278-7>.
- [2] M. McCabe, M. AlShalan, M. Hejazi, H. Beck, F.T. Maestre, et al., Climate Futures Report: Saudi Arabia in a 3 Degrees Warmer World, KAUST, AEON Collective, KAPSARC, 2023. <https://doi.org/10.25781/KAUST-8XY63>.
- [3] D.V. Ogunkan, Achieving Sustainable Environmental Governance in Nigeria: A Review for Policy Consideration, *Urban Gov.* 2 (2022), 212–220. <https://doi.org/10.1016/j.ugj.2022.04.004>.
- [4] N. Saqib, M. Usman, I. Ozturk, A. Sharif, Harnessing the Synergistic Impacts of Environmental Innovations, Financial Development, Green Growth, and Ecological Footprint through the Lens of SDGs Policies for Countries Exhibiting High Ecological Footprints, *Energy Policy* 184 (2024), 113863. <https://doi.org/10.1016/j.enpol.2023.113863>.
- [5] A.M. Ghanem, Y.A. Alamri, The Impact of the Green Middle East Initiative on Sustainable Development in the Kingdom of Saudi Arabia, *J. Saudi Soc. Agric. Sci.* 22 (2023), 35–46. <https://doi.org/10.1016/j.jssas.2022.06.001>.
- [6] A.K. Nassar, Strategic Energy Transition in the Gulf Cooperation Council: Balancing Economic, Social, Political, and Environmental Dynamics for Sustainable Development, *Int. J. Green Energy* 22 (2025), 1570–1586. <https://doi.org/10.1080/15435075.2024.2435434>.
- [7] M.K. Nematchoua, M. Sadeghi, S. Reiter, Strategies and Scenarios to Reduce Energy Consumption and CO₂ Emission in the Urban, Rural and Sustainable Neighbourhoods, *Sustain. Cities Soc.* 72 (2021), 103053. <https://doi.org/10.1016/j.scs.2021.103053>.
- [8] S. AlQuhtani, From Oil to Electrification: A Qualitative Assessment of E-Mobility Policy in Saudi Arabia, *Sustainability* 17 (2025), 10915. <https://doi.org/10.3390/su172410915>.
- [9] A.B.H. Alkahtani, Saudi Companies' Approaches to Environmental Sustainability in the Context of the Saudi Green Initiative: An Assessment of Organizational Commitment, *Int. J. Adv. Appl. Sci.* 13 (2026), 145–156. <https://doi.org/10.21833/ijaas.2026.05.014>.
- [10] R.L. Houchens, Record Value Theory and Inference, Ph.D. Thesis, University of California, Riverside, 1984. <https://www.proquest.com/openview/e8ed089e30157f985e6e202525caa98b/1>.
- [11] H.M. Barakat, E.M. Nigm, R.A. Aldallal, Current Records and Record Range with Some Applications, *J. Korean Stat. Soc.* 43 (2014), 263–273. <https://doi.org/10.1016/j.jkss.2013.09.004>.
- [12] H.M. Barakat, E.M. Nigm, R.A. Aldallal, Exact Prediction Intervals for Future Current Records and Record Range from Any Continuous Distribution, *SORT* 38 (2014), 251–270.
- [13] M.Z. Raqab, Bounds on Expectations of Record Range and Record Increment from Distributions with Bounded Support, *J. Inequal. Pure Appl. Math.* 8 (2007), 21. <https://eudml.org/doc/128223>.
- [14] M.Z. Raqab, Inequalities for Expected Current Record Statistics, *Commun. Stat. Theory Methods* 36 (2007), 1367–1380. <https://doi.org/10.1080/03610920601077097>.
- [15] M.Z. Raqab, Distribution-Free Prediction Intervals for the Future Current Record Statistics, *Stat. Pap.* 50 (2009), 429–439. <https://doi.org/10.1007/s00362-007-0082-3>.
- [16] R.A. Aldallal, Best Linear Unbiased Estimation and Prediction of Record Values Based on Kumaraswamy Distributed Data, *Stat. Optim. Inf. Comput.* 10 (2022), 1250–1266. <https://doi.org/10.19139/soic-2310-5070-1397>.
- [17] R.A. Aldallal, Moment Generating Function of Current Records Based on Generalized Exponential Distribution with Some Recurrence Relations, *Eur. J. Pure Appl. Math.* 16 (2023), 97–111. <https://doi.org/10.29020/nybg.ejpam.v16i1.4641>.

- [18] R.A. Aldallal, Bayesian Estimation and Prediction of Weibull Current Records with Application to Saudi Industrial Data, *AIMS Math.* 11 (2026), 4369–4394. <https://doi.org/10.3934/math.2026175>.
- [19] R.A. Aldallal, Fréchet Current Records: Bayesian and Non-Bayesian Estimation and Prediction with Application to Saudi Industrial Data, *Lett. Biomath.* 13 (2026), 194–213.
- [20] R.A. Aldallal, Rayleigh Current Record Distributed Data: Estimation and Prediction with Application on Saudi Arabia Industrial Data, *J. Stat. Theory Appl.* 25 (2026), 2. <https://doi.org/10.1007/s44199-025-00150-x>.
- [21] R.A. Aldallal, Current Record Range of Laplace Distributed Data with Application on Saudi Arabia Industrial Data, *J. Stat. Appl. Probab.* 14 (2025), 723–729. <https://doi.org/10.18576/jsap/140504>.
- [22] R.A. Aldallal, H.M. Barakat, A Comprehensive Review: Current Record Values and Their Applications, *Appl. Math. Inf. Sci.* 20 (2026), 401–423. <https://doi.org/10.18576/amis/200208>.
- [23] R.A. Aldallal, M.R. Alseedy, Bayesian Prediction of Future Network Latency Times Under Two-Parameter Exponential Current Records, *Math. Comput. Appl.* 31 (2026), 130. <https://doi.org/10.3390/mca31040130>.
- [24] M. Barbary, R.A. Aldallal, Statistical Analysis of Saudi Arabia's Environmental Sustainability Indicators Using Log-Logistic Current Records, *Stat. Optim. Inf. Comput.* in Press.
- [25] R.A. Aldallal, I. Sadok, Bayesian Current Record Analysis for Mixed Rayleigh–Weibull Distributions with Application to Saudi Humidity Data, *Stat. Optim. Inf. Comput.* in Press.