

ON TRIPOLAR FUZZY SOFT INTERIOR IDEALS IN ORDERED SEMIGROUPS

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ABSTRACT. In this paper, we combine tripolar fuzzy sets with soft sets to introduce the concept of tripolar fuzzy soft interior ideals over ordered semigroups. This new concept generalizes both bipolar and intuitionistic fuzzy soft structures. We investigate the basic algebraic properties of these ideals. Main results show that in regular, intra-regular, and semisimple ordered semigroups, tripolar fuzzy soft interior ideals and tripolar fuzzy soft ideals are identical. Finally, we characterize simple ordered semigroups by using the concept of tripolar fuzzy soft simple ordered semigroups.

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1. INTRODUCTION

The theory of fuzzy sets was introduced by Zadeh [1] in 1965. Since then, fuzzy set theory has emerged as one of the most appropriate frameworks for modeling and dealing with uncertainty. Following Zadeh's pioneering work, numerous researchers have extensively investigated various generalizations of fuzzy sets. For instance, Atanassov [2] introduced intuitionistic fuzzy sets by adding a non-membership degree. Later, Lee [3] introduced bipolar-valued fuzzy sets to capture both positive and negative information. These concepts have found substantial applications across diverse fields, including computer science, artificial intelligence, control engineering, robotics, automata theory, decision theory, finite state machines, graph theory, mathematical logic, operations research, and multiple branches of pure and applied mathematics (see, for example, [4]).

In the theory of semigroups and ordered semigroups, ideals and interior ideals are the same in regular, intra-regular, and semisimple ordered semigroups. Many researchers have applied fuzzy set theory to study these structures. For instance, Kehayopulu and Tsingelis [5] showed that fuzzy interior ideals and fuzzy ideals are the same in regular, intra-regular, and semisimple ordered semigroups. In

2009, Khan and Shabir [6] introduced (α, β) -fuzzy interior ideals and proved that they are the same as (α, β) -fuzzy ideals in these semigroups. Similarly, Shabir and Khan [7] proved that intuitionistic fuzzy interior ideals and intuitionistic fuzzy ideals are the same. Ibrar et al. [8] extended these results to (α, β) -bipolar fuzzy structures. Recently, Linesawat et al. [9] introduced anti-hybrid interior ideals and showed that they also equal anti-hybrid ideals in these specific classes.

After Molodtsov [10] introduced soft sets, Maji et al. [11] combined this concept with fuzzy sets to define fuzzy soft sets. Later, Maji et al. [12] extended this idea to intuitionistic fuzzy soft sets. To include negative information in the soft set framework, Aslam and Abdullah [13] introduced bipolar fuzzy soft sets. These algebraic and soft frameworks have been widely used to study various structures in ordered semigroups.

The concept of tripolar fuzzy sets extends fuzzy sets, bipolar fuzzy sets, and intuitionistic fuzzy sets. Rao [14] introduced this concept and studied tripolar fuzzy interior ideals in Γ -semigroups. Later, Rao and Venkateswarlu [15, 16] studied tripolar fuzzy ideals and tripolar fuzzy soft interior ideals over Γ -semigroups and Γ -semirings. In 2020, Rao [17] studied these algebraic structures over semigroups and analyzed their properties. In 2022, Wattanasiripong et al. [18] proved that tripolar fuzzy ideals and tripolar fuzzy interior ideals are the same in certain classes of ordered semigroups. They also characterized simple and semisimple ordered semigroups by using tripolar fuzzy interior ideals.

Based on the framework in [18], this paper introduces the concept of tripolar fuzzy soft interior ideals over ordered semigroups. This new concept extends both bipolar and intuitionistic fuzzy soft structures. The paper is organized as follows. First, we define tripolar fuzzy soft interior ideals and study their basic algebraic properties. Second, we show that in regular, intra-regular, and semisimple ordered semigroups, tripolar fuzzy soft interior ideals and tripolar fuzzy soft ideals are the same. Using these results, we provide a complete characterization of semisimple ordered semigroups. Finally, we characterize simple ordered semigroups by using the concept of tripolar fuzzy soft simple ordered semigroups.

2. PRELIMINARY

In this section, we recall some fundamental definitions and results regarding ordered semigroups and tripolar fuzzy soft set theory that are used throughout this paper.

Let S be a nonempty set together with a binary operation denoted by $\cdot$. The structure $\langle S; \cdot \rangle$ is referred to as a *groupoid*. If the binary operation $\cdot$ satisfies the associative property, meaning that

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

holds for all $x, y, z \in S$, then the groupoid $\langle S; \cdot \rangle$ is called a *semigroup*.

The structure $\langle S; \preceq \rangle$ is called a *partially ordered set* (or simply a *poset*) if a binary relation $\preceq$ on a set S is a *partial order*.

Definition 2.1. [5] *The structure $\langle S; \cdot, \preceq \rangle$ is called an ordered semigroup if it satisfies the following conditions:*

- (1) $\langle S; \cdot \rangle$ is a semigroup.
- (2) $\langle S; \preceq \rangle$ is a partially ordered set.
- (3) For every $a, b, c \in S$ if $a \preceq b$, then $a \cdot c \preceq b \cdot c$ and $c \cdot a \preceq c \cdot b$.

For the sake of simplicity, we denote an ordered semigroup $\langle S; \cdot, \preceq \rangle$ simply by its carrier set S . For any $a, b \in S$, the product $a \cdot b$ is denoted by ab . Let A and B be nonempty subsets of S . The product AB is defined by

$$AB := \{ab : a \in A \text{ and } b \in B\}.$$

Let S be an ordered semigroup. A nonempty subset A of S is said to be a *subsemigroup* of S if $AA \subseteq A$.

Definition 2.2. [5] *Let S be an ordered semigroup. A nonempty subset A of S is called a left ideal (resp., right ideal) of S if it satisfies the following conditions:*

- (1) $SA \subseteq A$ (resp., $AS \subseteq A$).
- (2) For any $x, y \in S$, if $x \preceq y$ and $y \in A$, then $x \in A$.

A nonempty subset of S is called a *two-sided ideal*, or simply an *ideal*, of S if it is both a left ideal and a right ideal of S .

Let $\langle S; \preceq \rangle$ be an ordered set, and let $(\] : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ be a mapping define as follows,

$$(\] := \{x \in S : x \preceq a \text{ for some } a \in A\}.$$

Definition 2.3. [5] *Let S be an ordered semigroup. A subsemigroup A of S is called an interior ideal of S if it satisfies the following conditions:*

- (1) $SAS \subseteq A$.
- (2) For $x, y \in S$, if $x \preceq y$ and $y \in A$, then $x \in A$.

In classical set theory, the membership of an element in a set is binary: an element either belongs to the set or it does not. However, many concepts in real-world problems are vague and cannot be defined by precise criteria. To address this uncertainty, Zadeh [1] introduced the concept of fuzzy sets in 1965. Unlike classical sets, a fuzzy set allows elements to have partial membership degrees ranging between 0 and 1.

Definition 2.4. *Let X be a nonempty set. A fuzzy set A in X is defined as an object of the form:*

$$A = \{\langle x, \mu_A(x) \rangle : x \in X\}$$

where $\mu_A: X \rightarrow [0, 1]$ is a mapping called the membership function of A , and $\mu_A(x)$ represents the degree of membership of x in A .

Although fuzzy sets introduced by Zadeh provide a powerful tool for dealing with uncertainty, they assume that the degree of non-membership is simply the complement of the degree of membership (i.e., $1 - \mu_A(x)$). This assumption may not always accurately reflect real-world situations where there is a degree of hesitation or lack of knowledge. To overcome this limitation, Atanassov [2] introduced the concept of intuitionistic fuzzy sets (IFSs) in 1986. An IFS is characterized by two functions: a membership function μ_A and a non-membership function ν_A , with the condition that their sum does not exceed 1. The remaining part, $1 - (\mu_A(x) + \nu_A(x))$, represents the degree of hesitation or uncertainty.

Definition 2.5. Let X be a nonempty set. An intuitionistic fuzzy set A in X is defined as an object of the form:

$$A := \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in X \}$$

where the mappings $\mu_A: X \rightarrow [0, 1]$ and $\nu_A: X \rightarrow [0, 1]$ denote the degree of membership and the degree of non-membership of the element $x \in X$ to the set A , respectively, satisfying the condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

for all $x \in X$.

In 2000, Lee [3] introduced bipolar-valued fuzzy sets as an extension of traditional fuzzy sets. In a bipolar-valued fuzzy set, the range of the membership degree is enlarged from $[0, 1]$ to $[-1, 1]$. Here, a membership degree of 0 means the element is irrelevant to the property. A membership degree in the interval $(0, 1]$ means the element satisfies the property, while a degree in $[-1, 0)$ means the element satisfies an implicit counter-property [3].

Definition 2.6. A bipolar fuzzy set A in a universe U is an object having the form,

$$A := \{ \langle x, \mu_A^+(x), \mu_A^-(x) \rangle : x \in U \}$$

where $\mu_A^+: U \rightarrow [0, 1]$, $\mu_A^-: U \rightarrow [-1, 0]$. So μ_A^+ denote for positive function and μ_A^- denote for negative function.

In 1999, Molodtsov [10] introduced soft set theory as a general mathematical tool for dealing with uncertainty. However, the classical soft set theory is based on crisp sets. Recognizing that in many real-world problems, the parameters or the approximations themselves are fuzzy in nature, Maji et al. [11] combined the theories of fuzzy sets and soft sets. They introduced the concept of *fuzzy soft sets*, which allows for a gradation of membership in the approximation of parameters.

Definition 2.7. Let U be an initial universe set and E a set of parameters. Let $\mathcal{F}(U)$ denote the set of all fuzzy sets in U . A pair (F, E) is called a fuzzy soft set over U , where F is a mapping given by:

$$F: E \rightarrow \mathcal{F}(U).$$

For any parameter $e \in E$, $F(e)$ is a fuzzy set in U defined as an object of the form:

$$F(e) := \{\langle x, \mu_{F(e)}(x) \rangle : x \in U\}$$

where $\mu_{F(e)}: U \rightarrow [0, 1]$ is the membership function.

Later, Maji et al. [12] generalized the concept of fuzzy soft sets to *intuitionistic fuzzy soft sets*. This extension is based on the combination of the intuitionistic fuzzy set theory and the soft set theory, which allows for dealing with both the degree of membership and the degree of non-membership of the parameters.

Definition 2.8. Let U be an initial universe set and let E be a set of parameters. Let $\mathcal{IF}(U)$ denote the set of all intuitionistic fuzzy sets in U . A pair (F, E) is called an intuitionistic fuzzy soft set over U , where F is a mapping given by:

$$F: E \rightarrow \mathcal{IF}(U).$$

For any parameter $e \in E$, $F(e)$ is an intuitionistic fuzzy set in U defined as an object of the form:

$$F(e) := \{\langle x, \mu_{F(e)}(x), \nu_{F(e)}(x) \rangle : x \in U\}$$

where $\mu_{F(e)}, \nu_{F(e)}: U \rightarrow [0, 1]$ are the membership and non-membership functions, respectively, satisfying the condition:

$$0 \leq \mu_{F(e)}(x) + \nu_{F(e)}(x) \leq 1$$

for all $x \in U$.

In 2013, M. Aslam and S. Abdullah [13] combined the concepts of bipolar fuzzy sets and soft sets. They introduced the notion of bipolar fuzzy soft sets and applied it to solve decision-making problems.

Definition 2.9. Let U be an initial universe set and E be a set of parameters. Let $\mathcal{BF}(U)$ denote the set of all bipolar fuzzy sets in U . A pair (F, E) is called a bipolar fuzzy soft set over U , where F is a mapping given by:

$$F: E \rightarrow \mathcal{BF}(U).$$

For each parameter $e \in E$, $F(e)$ is a bipolar fuzzy set in U given by:

$$F(e) := \{\langle x, \mu_{F(e)}^+(x), \mu_{F(e)}^-(x) \rangle : x \in U\}$$

where $\mu_{F(e)}^+: U \rightarrow [0, 1]$ and $\mu_{F(e)}^-: U \rightarrow [-1, 0]$ are the positive and negative membership functions, respectively.

The concept of tripolar fuzzy sets was originally introduced by M. Murali Krishna Rao [14] as a significant generalization of classical fuzzy sets, intuitionistic fuzzy sets, and bipolar fuzzy sets. This framework was specifically developed to capture and model tripolar decision-making information, as formally presented in the following definition.

Definition 2.10. Let X be a nonempty set. A tripolar fuzzy set A in X is an object of the form:

$$A := \{ \langle x, \mu_A(x), \nu_A(x), \delta_A(x) \rangle : x \in X \}$$

where $\mu_A, \nu_A : X \rightarrow [0, 1]$ and $\delta_A : X \rightarrow [-1, 0]$ are mappings such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

Motivated by these definitions, the integration of soft set and fuzzy set theories will be developed and applied to ordered semigroups in the main section.

3. MAIN RESULTS

In this section, we introduce the concept of tripolar fuzzy soft interior ideals over ordered semigroups and investigate their foundational algebraic properties. First, we demonstrate that the concepts of a tripolar fuzzy soft interior ideal and a tripolar fuzzy soft ideal coincide in regular, intra-regular, and semisimple ordered semigroups. Second, we utilize these interior ideals to provide a complete characterization of semisimple ordered semigroups. Finally, we introduce the notion of tripolar fuzzy soft simple ordered semigroups and establish its equivalence to the classical characterization of simple ordered semigroups.

Definition 3.1. Let U be an initial universe set and E be a set of parameters. Let $\mathcal{TF}(U)$ denote the collection of all tripolar fuzzy sets in U . A pair (F, E) is called a tripolar fuzzy soft set over U , where F is a mapping given by:

$$F : E \rightarrow \mathcal{TF}(U)$$

For each parameter $e \in E$, $F(e)$ is a tripolar fuzzy set in U defined as:

$$F(e) := \{ \langle x, \mu_{F(e)}(x), \nu_{F(e)}(x), \delta_{F(e)}(x) \rangle : x \in U \}$$

where $\mu_{F(e)}, \nu_{F(e)} : U \rightarrow [0, 1]$ and $\delta_{F(e)} : U \rightarrow [-1, 0]$ are the positive, neutral, and negative membership functions, respectively, satisfying the condition:

$$0 \leq \mu_{F(e)}(x) + \nu_{F(e)}(x) \leq 1$$

for all $x \in U$.

To better understand the concept of a tripolar fuzzy soft set, let us consider a practical application in decision-making problems. In the following example, we illustrate how a tripolar fuzzy soft set can represent uncertain data involving positive, neutral, and negative aspects simultaneously.

Example 3.1. Let $U = \{x_1, x_2\}$ be the set of two land plots and $E = \{e_1, e_2\}$ be the set of parameters representing nutrients Nitrogen and Potassium, respectively. A tripolar fuzzy soft set (F, E) describing the soil quality is given by:

$$F(e_1) = \{\langle x_1, 0.7, 0.2, -0.1 \rangle, \langle x_2, 0.4, 0.5, -0.3 \rangle\}$$

$$F(e_2) = \{\langle x_1, 0.6, 0.1, -0.2 \rangle, \langle x_2, 0.8, 0.1, -0.1 \rangle\}$$

The tabular representation of (F, E) is shown below:

$U \setminus E$	e_1	e_2
x_1	$(0.7, 0.2, -0.1)$	$(0.6, 0.1, -0.2)$
x_2	$(0.4, 0.5, -0.3)$	$(0.8, 0.1, -0.1)$

We denote by $\mathcal{TFSS}(U, E)$ the set of all tripolar fuzzy soft sets over U . In the following definition, we introduce the concepts of inclusion relation, intersection and union operations on $\mathcal{TFSS}(U, E)$.

Definition 3.2. Let (F, E) and (G, E) be two tripolar fuzzy soft sets over U .

- (1) We say that (F, E) is a tripolar fuzzy soft subset of (G, E) , denoted by $(F, E) \tilde{\subseteq} (G, E)$, if for all $e \in E$ and $x \in U$, the following conditions hold:

$$\mu_{F(e)}(x) \leq \mu_{G(e)}(x),$$

$$\nu_{F(e)}(x) \geq \nu_{G(e)}(x),$$

$$\delta_{F(e)}(x) \geq \delta_{G(e)}(x).$$

- (2) Two tripolar intuitionistic fuzzy soft sets (F, E) and (G, E) are said to be equal, denoted by $(F, E) = (G, E)$, if $(F, E) \tilde{\subseteq} (G, E)$ and $(G, E) \tilde{\subseteq} (F, E)$.
- (3) The intersection of (F, E) and (G, E) , denoted by $(F, E) \tilde{\cap} (G, E)$, is defined as a tripolar fuzzy soft set (H, E) over X , where for each $e \in E$ and $x \in X$:

$$\mu_{H(e)}(x) := \min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\},$$

$$\nu_{H(e)}(x) := \max\{\nu_{F(e)}(x), \nu_{G(e)}(x)\},$$

$$\delta_{H(e)}(x) := \max\{\delta_{F(e)}(x), \delta_{G(e)}(x)\}$$

for all $x \in U$.

(4) The union of (F, E) and (G, E) , denoted by $(F, E) \tilde{\cup} (G, E)$, is defined as a tripolar fuzzy soft set (H, E) over X , where for each $e \in E$ and $x \in X$:

$$\mu_{H(e)}(x) := \max\{\mu_{F(e)}(x), \mu_{G(e)}(x)\},$$

$$\nu_{H(e)}(x) := \min\{\nu_{F(e)}(x), \nu_{G(e)}(x)\},$$

$$\delta_{H(e)}(x) := \min\{\delta_{F(e)}(x), \delta_{G(e)}(x)\}$$

for all $x \in U$.

From the above definition, it is easy to verify that the relation $\tilde{\subseteq}$ is a partial order. Consequently, the structure $\langle \mathcal{TIFSS}(X, E), \tilde{\subseteq} \rangle$ is a partially ordered set.

Let $\mathbf{S}$ be an ordered semigroup and let $a \in S$. We define the set $\mathbf{S}_a$ by

$$\mathbf{S}_a := \{(x, y) \in S \times S : a \preceq xy\}.$$

We now introduce additional operations on $\mathcal{TFSS}(S, E)$ as follows.

Definition 3.3. Let $\mathbf{S}$ be an ordered semigroup. Let (F, E) and (G, E) be two tripolar fuzzy soft sets over S . The product of (F, E) and (G, E) , denoted by $(F, E) \tilde{\diamond} (G, E)$, is defined as a tripolar fuzzy soft set (H, E) over S , where for each $e \in E$ and $x \in S$:

$$\mu_{H(e)}(x) := \begin{cases} \bigvee_{(y,z) \in \mathbf{S}_x} \{\min\{\mu_{F(e)}(y), \mu_{G(e)}(z)\}\} & \text{if } \mathbf{S}_x \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu_{H(e)}(x) := \begin{cases} \bigwedge_{(y,z) \in \mathbf{S}_x} \{\max\{\nu_{F(e)}(y), \nu_{G(e)}(z)\}\} & \text{if } \mathbf{S}_x \neq \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

$$\delta_{H(e)}(x) := \begin{cases} \bigwedge_{(y,z) \in \mathbf{S}_x} \{\max\{\delta_{F(e)}(y), \delta_{G(e)}(z)\}\} & \text{if } \mathbf{S}_x \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

By Definitions 3.2(1) and 3.3, it is straightforward to verify that the operation $\tilde{\diamond}$ is compatible with the inclusion relation $\tilde{\subseteq}$. That is, for any $(A, E), (B, E), (C, E) \in \mathcal{TFSS}(S, E)$, if $(A, E) \tilde{\subseteq} (B, E)$, then

$$(A, E) \tilde{\diamond} (C, E) \tilde{\subseteq} (B, E) \tilde{\diamond} (C, E) \quad \text{and} \quad (C, E) \tilde{\diamond} (A, E) \tilde{\subseteq} (C, E) \tilde{\diamond} (B, E).$$

Consequently, the structure $\langle \mathcal{TFSS}(S, E), \tilde{\diamond}, \tilde{\subseteq} \rangle$ forms an ordered semigroup, which is called a *tripolar fuzzy soft ordered semigroup*.

Definition 3.4. Let $\mathbf{S}$ be an ordered semigroup and let (F, E) be a tripolar fuzzy soft set over S . Then (F, E) is called a *tripolar fuzzy soft subsemigroup over $\mathbf{S}$* if for all $e \in E$ and $x, y \in S$, the following conditions hold:

$$\mu_{F(e)}(xy) \geq \min\{\mu_{F(e)}(x), \mu_{F(e)}(y)\},$$

$$\nu_{F(e)}(xy) \leq \max\{\nu_{F(e)}(x), \nu_{F(e)}(y)\},$$

$$\delta_{F(e)}(xy) \leq \max\{\delta_{F(e)}(x), \delta_{F(e)}(y)\}.$$

Example 3.2. Let $S = \{a, b, c\}$ be a semigroup with the multiplication table defined as follows:

$*$	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

We define an order relation $\preceq$ on S by:

$$\preceq := \{(a, b), (a, c), (b, c)\} \cup \Delta_S,$$

where Δ_S is the identity relation on S . It is easy to verify that $\mathbf{S} := \langle S, *, \preceq \rangle$ is an ordered semigroup. Let $E = \{e\}$ be a set of parameters. We define a tripolar fuzzy soft set (F, E) over S by defining the values for $x \in S$ as follows:

$$F(e) = \{\langle a, 0.8, 0.1, -0.9 \rangle, \langle b, 0.6, 0.3, -0.4 \rangle, \langle c, 0.4, 0.5, -0.2 \rangle\}.$$

It is easy to verify that, (F, E) is a tripolar fuzzy soft subsemigroup over $\mathbf{S}$.

Definition 3.5. Let $\mathbf{S}$ be an ordered semigroup and let (F, E) be a tripolar fuzzy soft set over S . Then (F, E) is called a tripolar fuzzy soft left ideal over $\mathbf{S}$ if for all $e \in E$ and $x, y \in S$, the following conditions hold:

(1) The left ideal condition:

$$\mu_{F(e)}(xy) \geq \mu_{F(e)}(y),$$

$$\nu_{F(e)}(xy) \leq \nu_{F(e)}(y),$$

$$\delta_{F(e)}(xy) \leq \delta_{F(e)}(y).$$

(2) The order condition: If $x \preceq y$, then

$$\mu_{F(e)}(x) \geq \mu_{F(e)}(y),$$

$$\nu_{F(e)}(x) \leq \nu_{F(e)}(y),$$

$$\delta_{F(e)}(x) \leq \delta_{F(e)}(y).$$

Example 3.3. Consider an ordered semigroup $S = \{a, b, c\}$ with the multiplication table defined by:

$*$	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

and the order relation $\preceq$ defined by:

$$\preceq := \{(a, b), (b, c), (a, c)\} \cup \Delta_S,$$

where Δ_S is an identity relation on S . It is easy to calculate that $\mathbf{S} := \langle S; *, \preceq \rangle$ is an ordered semigroup. Let $E = \{e\}$ be a set of parameter. We define a tripolar fuzzy soft set (F, E) over S as follows:

$$F(e) = \{\langle a, 0.9, 0.0, -0.8 \rangle, \langle b, 0.5, 0.4, -0.5 \rangle, \langle c, 0.2, 0.7, -0.1 \rangle\}.$$

Hence, (F, E) is a tripolar fuzzy soft left ideal over $\mathbf{S}$.

Definition 3.6. Let $\mathbf{S}$ be an ordered semigroup and let (F, E) be a tripolar fuzzy soft set over S . Then (F, E) is called a tripolar fuzzy soft right ideal over $\mathbf{S}$ if for all $e \in E$ and $x, y \in S$, the following conditions hold:

(1) The right ideal condition:

$$\mu_{F(e)}(xy) \geq \mu_{F(e)}(x),$$

$$\nu_{F(e)}(xy) \leq \nu_{F(e)}(x),$$

$$\delta_{F(e)}(xy) \leq \delta_{F(e)}(x).$$

(2) The order condition: If $x \preceq y$, then

$$\mu_{F(e)}(x) \geq \mu_{F(e)}(y),$$

$$\nu_{F(e)}(x) \leq \nu_{F(e)}(y),$$

$$\delta_{F(e)}(x) \leq \delta_{F(e)}(y).$$

Example 3.4. Let $S = \{a, b, c\}$ be an ordered semigroup with the multiplication table defined by:

$*$	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

and the order relation $\preceq$ defined by:

$$\preceq := \{(a, b), (b, c), (a, c)\} \cup \Delta_S,$$

where $\Delta_S := \{(x, x) : x \in S\}$. Therefore $\mathbf{S} = \langle S; *, \preceq \rangle$ is an ordered semigroup. Let $E = \{e\}$ be a set of parameters. We define a tripolar fuzzy soft set (F, E) over S as follows:

$$F(e) = \{\langle a, 0.7, 0.2, -0.9 \rangle, \langle b, 0.5, 0.4, -0.6 \rangle, \langle c, 0.3, 0.6, -0.2 \rangle\}.$$

Thus, (F, E) is a tripolar fuzzy soft right ideal over $\mathbf{S}$.

A tripolar fuzzy soft set (F, E) over S is called a *tripolar fuzzy soft two-sided ideal* (or simply, a *tripolar fuzzy soft ideal*) over S if it is both a tripolar fuzzy soft left ideal and a tripolar fuzzy soft right ideal over S .

Definition 3.7. Let S be an ordered semigroup, and let (F, E) be a tripolar fuzzy soft subsemigroup over S . Then (F, E) is called *tripolar fuzzy soft interior ideal* over S if for all $e \in E$ and $x, y, z \in S$, the following conditions hold:

(1) *The interior ideal condition:*

$$\mu_{F(e)}(xyz) \geq \mu_{F(e)}(y),$$

$$\nu_{F(e)}(xyz) \leq \nu_{F(e)}(y),$$

$$\delta_{F(e)}(xyz) \leq \delta_{F(e)}(y).$$

(2) *The order condition: If $x \preceq y$, then*

$$\mu_{F(e)}(x) \geq \mu_{F(e)}(y),$$

$$\nu_{F(e)}(x) \leq \nu_{F(e)}(y),$$

$$\delta_{F(e)}(x) \leq \delta_{F(e)}(y).$$

Example 3.5. Let $S = \{a, b, c\}$ be a set with the multiplication defined by the following Cayley table:

$*$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	b

We define an order relation $\preceq$ on S as follows:

$$\preceq := \{(a, a), (b, b), (c, c), (a, b)\}.$$

Then $S := \langle S, *, \preceq \rangle$ is an ordered semigroup. Let $E = \{e\}$ be a parameter set.

We construct a tripolar fuzzy soft set (F, E) over S by defining the component functions piecewise as follows:

$$\mu_{F(e)}(x) := \begin{cases} 0.8 & \text{if } x = a \\ 0.5 & \text{if } x = b \\ 0.5 & \text{if } x = c, \end{cases} \quad \nu_{F(e)}(x) := \begin{cases} 0.1 & \text{if } x = a \\ 0.3 & \text{if } x = b \\ 0.4 & \text{if } x = c, \end{cases} \quad \text{and} \quad \delta_{F(e)}(x) := \begin{cases} -0.6 & \text{if } x = a \\ -0.4 & \text{if } x = b \\ -0.2 & \text{if } x = c. \end{cases}$$

It is easy to verify that (F, E) is a tripolar fuzzy soft interior ideal over S .

We now investigate some algebraic properties of tripolar fuzzy soft subsemigroups and tripolar fuzzy soft interior ideals.

Proposition 3.1. Let $\mathbf{S}$ be an ordered semigroup. If $\{(F_i, E) : i \in I\}$ is a family of tripolar fuzzy soft subsemigroups over $\mathbf{S}$, then their intersection

$$(M, E) := \bigcap_{i \in I} (F_i, E)$$

is a tripolar fuzzy soft subsemigroup over $\mathbf{S}$, where for each $e \in E$ and $x \in S$, the components are defined as follows:

$$\bigcap_{i \in I} (F_i, E)(e)(x) := \left\langle \bigwedge_{i \in I} \mu_{F_i(e)}(x), \bigvee_{i \in I} \nu_{F_i(e)}(x), \bigvee_{i \in I} \delta_{F_i(e)}(x) \right\rangle.$$

Proof. Let $\{(F_i, E) : i \in I\}$ be a family of tripolar fuzzy soft subsemigroups over $\mathbf{S}$ and let $(M, E) := \bigcap_{i \in I} (F_i, E)$. To prove that (M, E) is a tripolar fuzzy soft subsemigroups over $\mathbf{S}$, we let $e \in E$ and $x, y \in S$. Since each (F_i, E) is a tripolar fuzzy soft subsemigroup over $\mathbf{S}$, it satisfies the given conditions for all $i \in I$. Now, we evaluate each component function of the intersection (M, E) at the product $xy \in S$ as follows:

(i) For the positive membership function $\mu_{M(e)}$, we have:

$$\begin{aligned} \mu_{M(e)}(xy) &= \bigwedge_{i \in I} \mu_{F_i(e)}(xy) \\ &\geq \bigwedge_{i \in I} \min\{\mu_{F_i(e)}(x), \mu_{F_i(e)}(y)\} \\ &= \min \left\{ \bigwedge_{i \in I} \mu_{F_i(e)}(x), \bigwedge_{i \in I} \mu_{F_i(e)}(y) \right\} \\ &= \min\{\mu_{M(e)}(x), \mu_{M(e)}(y)\}. \end{aligned}$$

(ii) For the neutral membership function $\nu_{M(e)}$, we have:

$$\begin{aligned} \nu_{M(e)}(xy) &= \bigvee_{i \in I} \nu_{F_i(e)}(xy) \\ &\leq \bigvee_{i \in I} \max\{\nu_{F_i(e)}(x), \nu_{F_i(e)}(y)\} \\ &= \max \left\{ \bigvee_{i \in I} \nu_{F_i(e)}(x), \bigvee_{i \in I} \nu_{F_i(e)}(y) \right\} \\ &= \max\{\nu_{M(e)}(x), \nu_{M(e)}(y)\}. \end{aligned}$$

(iii) For the negative membership function $\delta_{M(e)}$, we have:

$$\begin{aligned} \delta_{M(e)}(xy) &= \bigvee_{i \in I} \delta_{F_i(e)}(xy) \\ &\leq \bigvee_{i \in I} \max\{\delta_{F_i(e)}(x), \delta_{F_i(e)}(y)\} \end{aligned}$$

$$\begin{aligned}
&= \max \left\{ \bigvee_{i \in I} \delta_{F_i(e)}(x), \bigvee_{i \in I} \delta_{F_i(e)}(y) \right\} \\
&= \max \{ \delta_{M(e)}(x), \delta_{M(e)}(y) \}.
\end{aligned}$$

From the evaluations above, all the required inequalities for a subsemigroup are satisfied. Consequently,

$(M, E) = \bigcap_{i \in I} (F_i, E)$ is a tripolar fuzzy soft subsemigroup over $\mathbf{S}$. $\square$

Proposition 3.2. *Let $\mathbf{S}$ be an ordered semigroup. If $\{(F_i, E) : i \in I\}$ is a family of tripolar fuzzy soft interior ideals over $\mathbf{S}$, then their intersection*

$$(M, E) := \bigcap_{i \in I} (F_i, E)$$

is a tripolar fuzzy soft interior ideal over $\mathbf{S}$, where for each $e \in E$ and $x \in S$, the components are defined as follows:

$$\bigcap_{i \in I} (F_i, E)(e)(x) := \left\langle \bigwedge_{i \in I} \mu_{F_i(e)}(x), \bigvee_{i \in I} \nu_{F_i(e)}(x), \bigvee_{i \in I} \delta_{F_i(e)}(x) \right\rangle.$$

Proof. Let $\{(F_i, E) : i \in I\}$ be a family of tripolar fuzzy soft interior ideals over $\mathbf{S}$ and let $(M, E) := \bigcap_{i \in I} (F_i, E)$. To demonstrate that (M, E) is a tripolar fuzzy soft interior ideal over $\mathbf{S}$, we must verify that it satisfies the subsemigroup condition, the interior ideal condition, and the order condition. Let $e \in E$ and $x, y, z \in S$.

(1) The tripolar fuzzy soft subsemigroup condition: Since each (F_i, E) is a tripolar fuzzy soft subsemigroup over $\mathbf{S}$ for all $i \in I$, we evaluate the components of the intersection at the product $xy \in S$ as follows:

$$\begin{aligned}
\mu_{M(e)}(xy) &= \bigwedge_{i \in I} \mu_{F_i(e)}(xy) \\
&\geq \bigwedge_{i \in I} \{ \min \{ \mu_{F_i(e)}(x), \mu_{F_i(e)}(y) \} \} \\
&= \left\{ \min \left\{ \bigwedge_{i \in I} \mu_{F_i(e)}(x), \bigwedge_{i \in I} \mu_{F_i(e)}(y) \right\} \right\} \\
&= \min \{ \mu_{M(e)}(x), \mu_{M(e)}(y) \}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nu_{M(e)}(xy) &= \bigvee_{i \in I} \nu_{F_i(e)}(xy) \\
&\leq \bigvee_{i \in I} \{ \max \{ \nu_{F_i(e)}(x), \nu_{F_i(e)}(y) \} \} \\
&= \max \left\{ \bigvee_{i \in I} \nu_{F_i(e)}(x), \bigvee_{i \in I} \nu_{F_i(e)}(y) \right\} \\
&= \max \{ \nu_{M(e)}(x), \nu_{M(e)}(y) \},
\end{aligned}$$

and

$$\begin{aligned}
 \delta_{M(e)}(xy) &= \bigvee_{i \in I} \delta_{F_i(e)}(xy) \\
 &\leq \bigvee_{i \in I} \{\max\{\delta_{F_i(e)}(x), \delta_{F_i(e)}(y)\}\} \\
 &= \max \left\{ \bigvee_{i \in I} \delta_{F_i(e)}(x), \bigvee_{i \in I} \delta_{F_i(e)}(y) \right\} \\
 &= \max\{\delta_{M(e)}(x), \delta_{M(e)}(y)\}.
 \end{aligned}$$

Thus, (M, E) satisfies the tripolar fuzzy soft subsemigroup condition over $\mathbf{S}$.

(2) The interior ideal condition: Consider the product element $xyz \in S$. Since each (F_i, E) is a tripolar fuzzy soft interior ideal over $\mathbf{S}$, it satisfies $\mu_{F_i(e)}(xyz) \geq \mu_{F_i(e)}(y)$, $\nu_{F_i(e)}(xyz) \leq \nu_{F_i(e)}(y)$, and $\delta_{F_i(e)}(xyz) \leq \delta_{F_i(e)}(y)$ for all $i \in I$. We have that,

$$\begin{aligned}
 \mu_{M(e)}(xyz) &= \bigwedge_{i \in I} \mu_{F_i(e)}(xyz) \\
 &\geq \bigwedge_{i \in I} \mu_{F_i(e)}(y) \\
 &= \mu_{M(e)}(y).
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 \nu_{M(e)}(xyz) &= \bigvee_{i \in I} \nu_{F_i(e)}(xyz) \\
 &\leq \bigvee_{i \in I} \nu_{F_i(e)}(y) \\
 &= \nu_{M(e)}(y),
 \end{aligned}$$

and

$$\begin{aligned}
 \delta_{M(e)}(xyz) &= \bigvee_{i \in I} \delta_{F_i(e)}(xyz) \\
 &\leq \bigvee_{i \in I} \delta_{F_i(e)}(y) \\
 &= \delta_{M(e)}(y).
 \end{aligned}$$

Hence, (M, E) satisfies the interior ideal condition over $\mathbf{S}$.

(3) The order condition: Let $x, y \in S$ such that $x \preceq y$. Since each (F_i, E) satisfies the order condition, we have $\mu_{F_i(e)}(x) \geq \mu_{F_i(e)}(y)$, $\nu_{F_i(e)}(x) \leq \nu_{F_i(e)}(y)$, and $\delta_{F_i(e)}(x) \leq \delta_{F_i(e)}(y)$ for all $i \in I$. It follows that:

$$\mu_{M(e)}(x) = \bigwedge_{i \in I} \mu_{F_i(e)}(x)$$

$$\begin{aligned}
&\geq \bigwedge_{i \in I} \mu_{F_i(e)}(y) \\
&= \mu_{M(e)}(y).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nu_{M(e)}(x) &= \bigvee_{i \in I} \nu_{F_i(e)}(x) \\
&\leq \bigvee_{i \in I} \nu_{F_i(e)}(y) \\
&= \nu_{M(e)}(y),
\end{aligned}$$

and

$$\begin{aligned}
\delta_{M(e)}(x) &= \bigvee_{i \in I} \delta_{F_i(e)}(x) \\
&\leq \bigvee_{i \in I} \delta_{F_i(e)}(y) \\
&= \delta_{M(e)}(y).
\end{aligned}$$

Thus, (M, E) satisfies the order condition over $\mathbf{S}$. Consequently, $(M, E) = \widetilde{\bigcap}_{i \in I} (F_i, E)$ is a tripolar fuzzy soft interior ideal over $\mathbf{S}$. $\square$

The following result illustrates a relationship between tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals in ordered semigroups.

Proposition 3.3. *Let $\mathbf{S}$ be an ordered semigroup. Then every tripolar fuzzy soft ideal over $\mathbf{S}$ is a tripolar fuzzy soft interior ideal over $\mathbf{S}$.*

Proof. Let (F, E) be a tripolar fuzzy soft ideal over $\mathbf{S}$. Then, by definition, (F, E) is both a tripolar fuzzy soft left ideal and a tripolar fuzzy soft right ideal over $\mathbf{S}$. Consequently, for the order condition, (F, E) satisfies condition (2) of Definition 3.7. To verify the interior ideal condition, let $e \in E$ and $x, y, z \in S$. Since (F, E) is a tripolar fuzzy soft left and right ideal over $\mathbf{S}$, we have:

$$\begin{aligned}
\mu_{F(e)}(xyz) &= \mu_{F(e)}((xy)z) \\
&\geq \mu_{F(e)}(xy) \\
&\geq \mu_{F(e)}(y).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nu_{F(e)}(xyz) &= \nu_{F(e)}((xy)z) \\
&\leq \nu_{F(e)}(xy)
\end{aligned}$$

$$\leq \nu_{F(e)}(y),$$

and

$$\begin{aligned}\delta_{F(e)}(xyz) &= \delta_{F(e)}((xy)z) \\ &\leq \delta_{F(e)}(xy) \\ &\leq \delta_{F(e)}(y).\end{aligned}$$

Therefore, (F, E) satisfies the interior ideal condition. Hence, every tripolar fuzzy soft ideal over $\mathbf{S}$ is a tripolar fuzzy soft interior ideal over $\mathbf{S}$. $\square$

The converse of Proposition 3.3, in general, does not hold, as the following counterexample.

Example 3.6. Let $S = \{a, b, c\}$ be a set with the multiplication defined by the following multiplication table:

$*$	a	b	c
a	a	a	a
b	a	a	b
c	a	a	a

We define a partial order relation $\preceq$ on S as the identity relation:

$$\preceq := \{(a, a), (b, b), (c, c)\}.$$

Then $\mathbf{S} := \langle S, *, \preceq \rangle$ is an ordered semigroup. Let $E = \{e\}$ be a parameter set.

We construct a tripolar fuzzy soft set (F, E) over S by defining the component functions as follows:

$$\mu_{F(e)}(x) := \begin{cases} 0.6 & \text{if } x = a \\ 0.2 & \text{if } x = b \\ 0.5 & \text{if } x = c, \end{cases} \quad \nu_{F(e)}(x) := \begin{cases} 0.1 & \text{if } x = a \\ 0.4 & \text{if } x = b \\ 0.3 & \text{if } x = c, \end{cases} \quad \text{and} \quad \delta_{F(e)}(x) := \begin{cases} -0.7 & \text{if } x = a \\ -0.2 & \text{if } x = b \\ -0.4 & \text{if } x = c. \end{cases}$$

Thus, (F, E) is a tripolar fuzzy soft interior ideal over $\mathbf{S}$. However, (F, E) is not a tripolar fuzzy soft right ideal over $\mathbf{S}$, because of

$$\mu_{F(e)}(b * c) = \mu_{F(e)}(b) = 0.2 \not\geq 0.5 = \mu_{F(e)}(c).$$

Consequently, (F, E) is not a tripolar fuzzy soft ideal over $\mathbf{S}$.

However, within certain classes of ordered semigroups, the concepts of tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals coincide.

An ordered semigroup $\mathbf{S}$ is called *regular* if for each $a \in S$, there exists an element $x \in S$ such that $a \preceq axa$. First, we investigate the converse of Proposition 3.3 within the framework of regular ordered semigroups.

Theorem 3.1. *Let $\mathbf{S}$ be a regular ordered semigroup. Then every tripolar fuzzy soft interior ideal over $\mathbf{S}$ is a tripolar fuzzy soft ideal over $\mathbf{S}$.*

Proof. Let $\mathbf{S}$ be a regular ordered semigroup and let (F, E) be a tripolar fuzzy soft interior ideal over $\mathbf{S}$. To prove that (F, E) is a tripolar fuzzy soft ideal over $\mathbf{S}$, we need to show that it is both a tripolar fuzzy soft left ideal and a tripolar fuzzy soft right ideal over $\mathbf{S}$. Let $e \in E$ and $a, b \in S$. Since $\mathbf{S}$ is regular, for the element $a \in S$, there exists $x \in S$ such that $a \preceq axa$. This implies that the product ab satisfies the following relation based on the compatibility of the partial order $\preceq$ with the semigroup operation:

$$ab \preceq (axa)b = a(xa)b.$$

Since (F, E) satisfies the order condition, we obtain the following inequalities for each component:

$$\begin{aligned}\mu_{F(e)}(ab) &\geq \mu_{F(e)}((ax)ab) \\ &\geq \mu_{F(e)}(a).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nu_{F(e)}(ab) &\leq \nu_{F(e)}((ax)ab) \\ &\leq \nu_{F(e)}(a),\end{aligned}$$

and

$$\begin{aligned}\delta_{F(e)}(ab) &\leq \delta_{F(e)}((ax)ab) \\ &\leq \delta_{F(e)}(a).\end{aligned}$$

Thus, (F, E) is a tripolar fuzzy soft right ideal over $\mathbf{S}$. Similarly, we can show that (F, E) is a tripolar fuzzy soft left ideal over $\mathbf{S}$. Therefore, (F, E) is a tripolar fuzzy soft ideal over $\mathbf{S}$. $\square$

By combining Proposition 3.3 and Theorem 3.1, we obtain the following corollary.

Corollary 3.1. *Let $\mathbf{S}$ be a regular ordered semigroup and let (F, E) be a tripolar fuzzy soft set over S . Then the following conditions are equivalent:*

- (1) (F, E) is a tripolar fuzzy soft ideal over $\mathbf{S}$.
- (2) (F, E) is a tripolar fuzzy soft interior ideal over $\mathbf{S}$.

Proof. (1) $\Rightarrow$ (2). By Proposition 3.3. (2) $\Rightarrow$ (1). By Theorem 3.1. $\square$

An ordered semigroup $\mathbf{S}$ is called intra-regular if for each element $a \in S$ there exist elements $x, y \in S$ such that $a \preceq xa^2y$. For intra-regular ordered semigroups, the other direction of Proposition 3.3 is provided below.

Theorem 3.2. *Let $\mathbf{S}$ be an intra-regular ordered semigroup. Then every tripolar fuzzy soft interior ideal over $\mathbf{S}$ is a tripolar fuzzy soft ideal over $\mathbf{S}$.*

Proof. Let $\mathbf{S}$ be an intra-regular ordered semigroup and let (F, E) be a tripolar fuzzy soft interior ideal over $\mathbf{S}$. To show that (F, E) is a tripolar fuzzy soft ideal over $\mathbf{S}$, we must prove that it is both a tripolar fuzzy soft left ideal and a tripolar fuzzy soft right ideal over $\mathbf{S}$. Let $e \in E$ and $a, b \in S$. Since $\mathbf{S}$ is intra-regular, for the element $a \in S$, there exist elements $x, y \in S$ such that $a \preceq xa^2y$. By the compatibility of the partial order $\preceq$ with the semigroup operation, we can analyze the product ba as follows:

$$ba \preceq b(xa^2y) = b(xaa)y = (bx)a(ay).$$

Since (F, E) satisfies the order condition, we obtain the following inequalities for each component:

$$\begin{aligned}\mu_{F(e)}(ba) &\geq \mu_{F(e)}((bx)a(ay)) \\ &\geq \mu_{F(e)}(a).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nu_{F(e)}(ba) &\leq \nu_{F(e)}((bx)a(ay)) \\ &\leq \nu_{F(e)}(a),\end{aligned}$$

and

$$\begin{aligned}\delta_{F(e)}(ba) &\leq \delta_{F(e)}((bx)a(ay)) \\ &\leq \delta_{F(e)}(a).\end{aligned}$$

Thus, (F, E) satisfies the tripolar fuzzy soft left ideal over $\mathbf{S}$. By a similar argument, (F, E) satisfies the tripolar fuzzy soft right ideal condition over $\mathbf{S}$. Since (F, E) is both a tripolar fuzzy soft left ideal and a tripolar fuzzy soft right ideal over $\mathbf{S}$, we conclude that (F, E) is a tripolar fuzzy soft ideal over $\mathbf{S}$. $\square$

We obtain the coincidence of tripolar fuzzy ideals and tripolar fuzzy interior ideals by combining Proposition 3.2 and Theorem 3.2.

Corollary 3.2. *Let $\mathbf{S}$ be an intra-regular ordered semigroup, and let (F, E) be a tripolar fuzzy soft set over S . Then the following conditions are equivalent:*

- (1) (F, E) is a tripolar fuzzy soft ideal over $\mathbf{S}$.
- (2) (F, E) is a tripolar fuzzy soft interior ideal over $\mathbf{S}$.

Proof. (1) $\Rightarrow$ (2). By Proposition 3.2. (2) $\Rightarrow$ (1). By Theorem 3.2. $\square$

An ordered semigroup $\mathbf{S}$ is called *semisimple* if for each element $a \in S$, there exist elements $x, y, z \in S$ such that $a \preceq xayaz$. Within the framework of semisimple ordered semigroups, the converse of Proposition 3.2 holds, as established in the following theorem.

Theorem 3.3. *Let $\mathbf{S}$ be a semisimple ordered semigroup. Then every tripolar fuzzy soft interior ideal over $\mathbf{S}$ is a tripolar fuzzy soft ideal over $\mathbf{S}$.*

Proof. Let $\mathbf{S}$ be a semisimple ordered semigroup, and let (F, E) be a tripolar fuzzy soft interior ideal over $\mathbf{S}$. To demonstrate that (F, E) is a tripolar fuzzy soft ideal over $\mathbf{S}$, it suffices to show that it is both a tripolar fuzzy soft left ideal and a tripolar fuzzy soft right ideal over $\mathbf{S}$. Let $e \in E$ and $a, b \in S$. We prove that (F, E) satisfies the tripolar fuzzy soft left ideal condition over $\mathbf{S}$. Since $\mathbf{S}$ is semisimple, for the element $a \in S$, there exist elements $x, y, z \in S$ such that $a \preceq xayaz$. By the compatibility of the partial order $\preceq$ with the semigroup operation, we can express the product ba under the order relation as follows:

$$ba \preceq b(xayaz) = (bxaya)z = (bxay)a(z).$$

Since (F, E) satisfies the order condition, we obtain:

$$\begin{aligned} \mu_{F(e)}(ba) &\geq \mu_{F(e)}((bxay)a(z)) \\ &\geq \mu_{F(e)}(a). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \nu_{F(e)}(ba) &\leq \nu_{F(e)}((bxay)a(z)) \\ &\leq \nu_{F(e)}(a), \end{aligned}$$

and

$$\begin{aligned} \delta_{F(e)}(ba) &\leq \delta_{F(e)}((bxay)a(z)) \\ &\leq \delta_{F(e)}(a). \end{aligned}$$

This demonstrates that (F, E) is a tripolar fuzzy soft left ideal over $\mathbf{S}$. Similarly, we can prove that (F, E) satisfies the tripolar fuzzy soft right ideal condition over $\mathbf{S}$. Consequently, since (F, E) is both a tripolar fuzzy soft left ideal and a tripolar fuzzy soft right ideal over $\mathbf{S}$, we conclude that (F, E) is a tripolar fuzzy soft ideal over $\mathbf{S}$. $\square$

Corollary 3.3. *Let $\mathbf{S}$ be a semisimple ordered semigroup and let (F, E) be a tripolar fuzzy soft set over S . Then the following conditions are equivalent:*

- (1) (F, E) is a tripolar fuzzy soft ideal over $\mathbf{S}$.
- (2) (F, E) is a tripolar fuzzy soft interior ideal over $\mathbf{S}$.

Proof. (1) $\Rightarrow$ (2). By Proposition 3.2. (2) $\Rightarrow$ (1). By Theorem 3.3. $\square$

Next, we study the relationship between original ideals and tripolar fuzzy soft ideals. To do this, we need a special type of tripolar fuzzy soft set, which is introduced in the following definition.

Definition 3.8. Let A be a subset of S and let E be a set of parameters. The characteristic tripolar fuzzy soft set over S , denoted by (χ_A, E) , is defined by the mapping $\chi_A : E \rightarrow \mathcal{TF}(S)$ where for each $e \in E$, the tripolar fuzzy soft set $\chi_A(e)$ is given by:

$$\chi_A(e) = \{\langle x, \mu_{\chi_A(e)}(x), \nu_{\chi_A(e)}(x), \delta_{\chi_A(e)}(x) \rangle : x \in S\}$$

The positive, neutral, and negative membership functions are defined for each $x \in S$ and $e \in E$ as follows:

$$\begin{aligned} (1) \text{ Positive membership function: } \mu_{\chi_A(e)}(x) &:= \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases} \\ (2) \text{ Neutral membership function: } \nu_{\chi_A(e)}(x) &:= \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{otherwise.} \end{cases} \\ (3) \text{ Negative membership function: } \delta_{\chi_A(e)}(x) &:= \begin{cases} -1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If $A = S$, then the characteristic tripolar fuzzy soft set (χ_A, E) is denoted by $(\mathbf{1}, E)$. For each $e \in E$, the tripolar fuzzy set $\mathbf{1}(e)$ is defined by:

$$\mu_{\mathbf{1}(e)}(x) = 1, \quad \nu_{\mathbf{1}(e)}(x) = 0, \quad \text{and} \quad \delta_{\mathbf{1}(e)}(x) = -1$$

for all $x \in S$.

In terms of tripolar fuzzy soft ideals, the following remark provides a characterization of left (right, and interior) ideals in ordered semigroups using characteristic tripolar fuzzy soft sets.

Remark 3.1. Let A be a nonempty subset of S . The characteristic tripolar fuzzy soft set (χ_A, E) over S . For each $e \in E$ and $x \in S$, the following conditions are equivalent:

- (1) $x \in A$;
- (2) $\mu_{\chi_A(e)}(x) = 1, \nu_{\chi_A(e)}(x) = 0, \text{ and } \delta_{\chi_A(e)}(x) = -1.$

The following lemma illustrates a characterization of left ideals in ordered semigroups using characteristic tripolar fuzzy soft sets.

Lemma 3.1. Let S be an ordered semigroup, and let L be a nonempty subset of S . Then the following statements are equivalent:

- (1) L is a left ideal of S ;
- (2) The characteristic tripolar fuzzy soft set (χ_L, E) is a tripolar fuzzy soft left ideal over S .

Proof. Let $\mathbf{S}$ be an ordered semigroup and let L be a nonempty subset of S .

(1) $\Rightarrow$ (2). Assume that L is a left ideal of $\mathbf{S}$. Let $e \in E$ and $x, y \in S$. To show that (χ_L, E) is a tripolar fuzzy soft left ideal over $\mathbf{S}$, we condition two cases as follows:

Case 1: If $y \notin L$, we have $\mu_{\chi_L(e)}(xy) \geq 0 = \mu_{\chi_L(e)}(xy)$, $\nu_{\chi_L(e)}(xy) \leq 1 = \nu_{\chi_L(e)}(y)$, and $\delta_{\chi_L(e)}(xy) \leq 0 = \delta_{\chi_L(e)}(y)$.

Case 2: If $y \in L$, since L is a left ideal of $\mathbf{S}$, it follows that $xy \in SL \subseteq L$, which means $xy \in L$. Thus, we obtain $\mu_{\chi_L(e)}(xy) = 1 = \mu_{\chi_L(e)}(y)$, $\nu_{\chi_L(e)}(xy) = 0 = \nu_{\chi_L(e)}(y)$, and $\delta_{\chi_L(e)}(xy) = -1 = \delta_{\chi_L(e)}(y)$. Hence, the left ideal condition holds in all cases. Next, we check the order condition. Let $x, y \in S$ such that $x \preceq y$. We consider two cases as follows: If $y \notin L$, the inequalities hold trivially as shown in **case 1**. If $y \in L$, since L is an ordered left ideal, $x \preceq y \in L$ implies $x \in L$. Thus, $\mu_{\chi_L(e)}(x) = 1 = \mu_{\chi_L(e)}(y)$, $\nu_{\chi_L(e)}(x) = 0 = \nu_{\chi_L(e)}(y)$, and $\delta_{\chi_L(e)}(x) = -1 = \delta_{\chi_L(e)}(y)$. For any two cases, we have that (χ_L, E) satisfied the order condition. Therefore, (χ_L, E) is a tripolar fuzzy soft left ideal over $\mathbf{S}$.

(2) $\Rightarrow$ (1). Assume that (2) holds. Let $e \in E$ and $x \in S$, $y \in L$. Since $y \in L$, we have $\mu_{\chi_L(e)}(y) = 1$, $\nu_{\chi_L(e)}(y) = 0$, and $\delta_{\chi_L(e)}(y) = -1$. Since (χ_L, E) is a tripolar fuzzy soft left ideal over $\mathbf{S}$, it satisfies $1 \geq \mu_{\chi_L(e)}(xy) \geq \mu_{\chi_L(e)}(y) = 1$, this means that $\mu_{\chi_L(e)}(xy) = 1$. By Remark 3.1, this directly implies that $xy \in L$. Thus, $SL \subseteq L$. Now, let $x \in S$ and $y \in L$ such that $x \preceq y$. Since $y \in L$, we have $\mu_{\chi_L(e)}(y) = 1$. Utilizing the order condition of (χ_L, E) , we obtain that $1 \geq \mu_{\chi_L(e)}(x) \geq \mu_{\chi_L(e)}(y) = 1$. This means that $\mu_{\chi_L(e)}(x) = 1$, which implies $x \in L$. Hence, L satisfies the order condition. Consequently, L is a left ideal of $\mathbf{S}$. $\square$

Using arguments similar to those employed in Lemma 3.1, we can establish the following lemma for right ideals.

Lemma 3.2. *Let $\mathbf{S}$ be an ordered semigroup, and let R be a nonempty subset of S . Then the following statements are equivalent:*

- (1) R is a right ideal of $\mathbf{S}$;
- (2) The characteristic tripolar fuzzy soft set (χ_R, E) is a tripolar fuzzy soft right ideal over $\mathbf{S}$.

By combining Lemmas 3.1 and 3.2, we directly obtain the following corollary characterizing two-sided ideals.

Corollary 3.4. *Let $\mathbf{S}$ be an ordered semigroup, and let D be a nonempty subset of S . Then the following statements are equivalent:*

- (1) D is an ideal of $\mathbf{S}$;
- (2) The characteristic tripolar fuzzy soft set (χ_D, E) is a tripolar fuzzy soft ideal over $\mathbf{S}$.

The following lemma provides a characterization of interior ideals in ordered semigroups by utilizing characteristic tripolar fuzzy soft sets.

Lemma 3.3. *Let $\mathbf{S}$ be an ordered semigroup, and let I be a nonempty subset of S . Then the following statements are equivalent:*

- (1) I is an interior ideal of $\mathbf{S}$;
- (2) The characteristic tripolar fuzzy soft set (χ_I, E) is a tripolar fuzzy soft interior ideal over $\mathbf{S}$.

Proof. Let $\mathbf{S}$ be an ordered semigroup and let I be a nonempty subset of S .

(1) $\Rightarrow$ (2). Suppose that (1) holds. Let $e \in E$ and $x, y, z \in S$. To show that (χ_I, E) is a tripolar fuzzy soft interior ideal over $\mathbf{S}$, we first verify the interior ideal condition. We consider two cases as follows:

Case 1. If $y \notin I$, then we have $\mu_{\chi_I(e)}(xyz) \geq 0 = \mu_{\chi_I(e)}(y)$, $\nu_{\chi_I(e)}(xyz) \leq 1 = \nu_{\chi_I(e)}(y)$, and $\delta_{\chi_I(e)}(xyz) \leq 0 = \delta_{\chi_I(e)}(y)$.

Case 2. If $y \in I$, since I is an interior ideal of $\mathbf{S}$, it follows that $xyz \in SIS \subseteq I$, which implies $xyz \in I$. We obtain: $\mu_{\chi_I(e)}(xyz) = 1 = \mu_{\chi_I(e)}(y)$, $\nu_{\chi_I(e)}(xyz) = 0 = \nu_{\chi_I(e)}(y)$, and $\delta_{\chi_I(e)}(xyz) = -1 = \delta_{\chi_I(e)}(y)$. Hence, the interior ideal condition holds for all cases.

Next, we verify the order condition. Let $x, y \in S$ such that $x \preceq y$. We consider two cases

Case 1. If $y \notin I$, then we have $\mu_{\chi_I(e)}(x) \geq 0 = \mu_{\chi_I(e)}(y)$, $\nu_{\chi_I(e)}(x) \leq 1 = \nu_{\chi_I(e)}(y)$, and $\delta_{\chi_I(e)}(x) \leq 0 = \delta_{\chi_I(e)}(y)$.

Case 2. If $y \in I$, since I is an ordered interior ideal, $x \preceq y \in I$ implies $x \in I$. Thus, we have $\mu_{\chi_I(e)}(x) = 1 = \mu_{\chi_I(e)}(y)$, $\nu_{\chi_I(e)}(x) = 0 = \nu_{\chi_I(e)}(y)$, and $\delta_{\chi_I(e)}(x) = -1 = \delta_{\chi_I(e)}(y)$.

Furthermore, (χ_I, E) is a tripolar fuzzy soft subsemigroup over $\mathbf{S}$ since any interior ideal is a subsemigroup. Therefore, (χ_I, E) is a tripolar fuzzy soft interior ideal over $\mathbf{S}$.

(2) $\Rightarrow$ (1). Assume that (2) holds. Let $e \in E$, $x, z \in S$ and $y \in I$. Since $y \in I$, we have $\mu_{\chi_I(e)}(y) = 1$. Using the interior ideal condition of (χ_I, E) , we obtain that

$$\begin{aligned} 1 &\geq \mu_{\chi_I(e)}(xyz) \\ &\geq \mu_{\chi_I(e)}(y) \\ &= 1, \end{aligned}$$

it follows that $\mu_{\chi_I(e)}(xyz) = 1$, which implies that $xyz \in I$ by Remark 3.1. Thus, $SIS \subseteq I$. Now, let $x \in S$ and $y \in I$ such that $x \preceq y$. Since $y \in I$, we have $\mu_{\chi_I(e)}(y) = 1$. By applying the order condition of (χ_I, E) , we obtain that

$$\begin{aligned} 1 &\geq \mu_{\chi_I(e)}(x) \\ &\geq \mu_{\chi_I(e)}(y) \\ &= 1, \end{aligned}$$

it follows that $\mu_{\chi_I(e)}(x) = 1$, which implies $x \in I$. Consequently, I is an interior ideal of $\mathbf{S}$. $\square$

The algebraic connections between classical ideals and tripolar fuzzy soft ideals are successfully established in Lemmas 3.1, 3.2, 3.3, and Corollary 3.4. According to these results, any crisp ideal in an ordered semigroup can be uniquely characterized by its corresponding characteristic tripolar fuzzy soft set. To further describe the structures of semisimple ordered semigroups, a series of fundamental lemmas regarding characteristic tripolar fuzzy soft operations is presented below.

Lemma 3.4. *Let $\mathbf{S}$ be an ordered semigroup, and let A and B be nonempty subsets of S . Then for each $e \in E$, the following statements hold:*

- (1) $(\chi_A, E) = (\chi_B, E)$ if and only if $A = B$.
- (2) $(\chi_A, E) \tilde{\circ} (\chi_B, E) = (\chi_{(AB]}, E)$.
- (3) $(\chi_A, E) \tilde{\cap} (\chi_B, E) = (\chi_{A \cap B}, E)$.

Proof. We give only the proof of (2) since the rests are not difficult to verify. Let $\mathbf{S}$ be an ordered semigroup and let A, B be nonempty subsets of S . Let $e \in E$ and $x \in S$. We now consider the following two mutually exclusive cases:

Case 1: Let $x \in (AB]$. By the definition of the ordered closure, there exist elements $a \in A$ and $b \in B$ such that $x \preceq ab$. This implies that the pair $(a, b) \in \mathbf{S}_x$, so $\mathbf{S}_x \neq \emptyset$. Since $a \in A$ and $b \in B$, we have $\mu_{\chi_A(e)}(a) = 1$, $\nu_{\chi_A(e)}(a) = 0$, $\delta_{\chi_A(e)}(a) = -1$, and $\mu_{\chi_B(e)}(b) = 1$, $\nu_{\chi_B(e)}(b) = 0$, $\delta_{\chi_B(e)}(b) = -1$. Evaluating the soft product components for x , we obtain:

$$\begin{aligned}
 1 &\geq \mu_{(\chi_A \tilde{\circ} \chi_B)(e)}(x) \\
 &= \bigvee_{(u,v) \in \mathbf{S}_x} \{ \min\{ \mu_{\chi_A(e)}(u), \mu_{\chi_B(e)}(v) \} \} \\
 &\geq \min\{ \mu_{\chi_A(e)}(a), \mu_{\chi_B(e)}(b) \} \\
 &= \min\{ 1, 1 \} \\
 &= 1.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 0 &\leq \nu_{(\chi_A \tilde{\circ} \chi_B)(e)}(x) \\
 &= \bigwedge_{(u,v) \in \mathbf{S}_x} \{ \max\{ \nu_{\chi_A(e)}(u), \nu_{\chi_B(e)}(v) \} \} \\
 &\leq \max\{ \nu_{\chi_A(e)}(a), \nu_{\chi_B(e)}(b) \} \\
 &= \max\{ 0, 0 \} \\
 &= 0,
 \end{aligned}$$

and

$$\begin{aligned}
 -1 &\leq \delta_{(\chi_A \tilde{\circ} \chi_B)(e)}(x) \\
 &= \bigwedge_{(u,v) \in \mathbf{S}_x} \{\max\{\delta_{\chi_A(e)}(u), \delta_{\chi_B(e)}(v)\}\} \\
 &\leq \max\{\delta_{\chi_A(e)}(a), \delta_{\chi_B(e)}(b)\} \\
 &= \max\{-1, -1\} \\
 &= -1.
 \end{aligned}$$

It follows that $\mu_{(\chi_A \tilde{\circ} \chi_B)(e)}(x) = 1$, $\nu_{(\chi_A \tilde{\circ} \chi_B)(e)}(x) = 0$, and $\delta_{(\chi_A \tilde{\circ} \chi_B)(e)}(x) = -1$. On the other hand, since $x \in (AB]$, the characteristic function gives $\mu_{\chi_{(AB)](e)}(x) = 1$, $\nu_{\chi_{(AB)](e)}(x) = 0$, and $\delta_{\chi_{(AB)](e)}(x) = -1$. Thus, both sides match perfectly.

Case 2: Let $x \notin (AB]$. This means that for any elements $a, b \in S$ such that $x \preceq ab$, it is impossible for both $a \in A$ and $b \in B$ to hold simultaneously. That is, for every $(a, b) \in \mathbf{S}_x$, either $a \notin A$ or $b \notin B$. If $\mathbf{S}_x = \emptyset$, then by the definition of the operation, we immediately have $\mu_{\chi_A(e)}(x) = 0$, $\nu_{\chi_A(e)}(x) = 1$, and $\delta_{\chi_A(e)}(x) = 0$. Since $x \notin (AB]$, these values exactly match $\chi_{(AB)](e)}(x)$. If $\mathbf{S}_x \neq \emptyset$, then for any $(a, b) \in \mathbf{S}_x$, since either $a \notin A$ or $b \notin B$, we obtain that

$$\begin{aligned}
 \mu_{(\chi_A \tilde{\circ} \chi_B)(e)}(x) &= \bigvee_{(a,b) \in \mathbf{S}_x} \{\min\{\mu_{\chi_A(e)}(a), \mu_{\chi_B(e)}(b)\}\} \\
 &= \bigvee_{(a,b) \in \mathbf{S}_x} \{\min\{0, 0\}\} \\
 &= 0.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 \nu_{(\chi_A \tilde{\circ} \chi_B)(e)}(x) &= \bigwedge_{(a,b) \in \mathbf{S}_x} \{\max\{\nu_{\chi_A(e)}(a), \nu_{\chi_B(e)}(b)\}\} \\
 &= \bigwedge_{(a,b) \in \mathbf{S}_x} \{\max\{1, 1\}\} \\
 &= 1,
 \end{aligned}$$

and

$$\begin{aligned}
 \delta_{(\chi_A \tilde{\circ} \chi_B)(e)}(x) &= \bigwedge_{(a,b) \in \mathbf{S}_x} \{\max\{\delta_{\chi_A(e)}(a), \delta_{\chi_B(e)}(b)\}\} \\
 &= \bigwedge_{(a,b) \in \mathbf{S}_x} \{\max\{0, 0\}\} \\
 &= 0.
 \end{aligned}$$

Since $x \notin (AB]$, we also have $\mu_{\chi_{(AB]}(e)}(x) = 0$, $\nu_{\chi_{(AB]}(e)}(x) = 1$, and $\delta_{\chi_{(AB]}(e)}(x) = 0$. Hence, the equations hold for all cases. $\square$

The following result demonstrates that the soft product of any two tripolar fuzzy soft interior ideals is contained in their soft intersection.

Lemma 3.5. *Let $\mathbf{S}$ be a semisimple ordered semigroup, and let (F, E) and (G, E) be tripolar fuzzy soft interior ideals over $\mathbf{S}$. Then we have*

$$(F, E) \tilde{\circ} (G, E) \tilde{\subseteq} (F, E) \tilde{\cap} (G, E).$$

Proof. Let (F, E) and (G, E) be tripolar fuzzy soft interior ideals over $\mathbf{S}$. Since $\mathbf{S}$ is semisimple, it follows from Theorem 3.3 that (F, E) and (G, E) are tripolar fuzzy soft ideals over $\mathbf{S}$. Let $e \in E$ and $a \in S$. Since $\mathbf{S}$ is semisimple, there exist elements $x, y, z \in S$ such that $a \preceq (xa)y(az) = xayaz$. This implies that the set $\mathbf{S}_a \neq \emptyset$. Utilizing the definition of the soft product along with the ideal properties of F and G , we obtain:

$$\begin{aligned} \mu_{(F \tilde{\circ} G)(e)}(a) &= \bigvee_{(u,v) \in \mathbf{S}_a} \{ \min\{\mu_{F(e)}(u), \mu_{G(e)}(v)\} \} \\ &\leq \bigvee_{(u,v) \in \mathbf{S}_a} \{ \min\{\mu_{F(e)}(uv), \mu_{G(e)}(uv)\} \} \\ &\leq \bigvee_{(u,v) \in \mathbf{S}_a} \{ \min\{\mu_{F(e)}(a), \mu_{G(e)}(a)\} \} \\ &= \min\{\mu_{F(e)}(a), \mu_{G(e)}(a)\} \\ &= \mu_{(F \tilde{\cap} G)(e)}(a). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \nu_{(F \tilde{\circ} G)(e)}(a) &= \bigwedge_{(u,v) \in \mathbf{S}_a} \{ \max\{\nu_{F(e)}(u), \nu_{G(e)}(v)\} \} \\ &\geq \bigwedge_{(u,v) \in \mathbf{S}_a} \{ \max\{\nu_{F(e)}(uv), \nu_{G(e)}(uv)\} \} \\ &\geq \bigwedge_{(u,v) \in \mathbf{S}_a} \{ \max\{\nu_{F(e)}(a), \nu_{G(e)}(a)\} \} \\ &= \max\{\nu_{F(e)}(a), \nu_{G(e)}(a)\} \\ &= \nu_{(F \tilde{\cap} G)(e)}(a), \end{aligned}$$

and finally,

$$\delta_{(F \tilde{\circ} G)(e)}(a) = \bigwedge_{(u,v) \in \mathbf{S}_a} \{ \max\{\delta_{F(e)}(u), \delta_{G(e)}(v)\} \}$$

$$\begin{aligned}
&\geq \bigwedge_{(u,v) \in \mathbf{S}_a} \{\max\{\delta_{F(e)}(uv), \delta_{G(e)}(uv)\}\} \\
&\geq \bigwedge_{(u,v) \in \mathbf{S}_a} \{\max\{\delta_{F(e)}(a), \delta_{G(e)}(a)\}\} \\
&= \max\{\delta_{F(e)}(a), \delta_{G(e)}(a)\} \\
&= \delta_{(F \tilde{\cap} G)(e)}(a).
\end{aligned}$$

Consequently, $(F, E) \tilde{\succ} (G, E) \tilde{\subseteq} (F, E) \tilde{\cap} (G, E)$ holds. $\square$

To characterize simple ordered semigroups, we introduce a useful notion of level subsets within the tripolar fuzzy soft framework. Let (F, E) be a tripolar fuzzy soft set over an ordered semigroup $\mathbf{S}$. For each $e \in E$ and $a \in S$, we define the subset $(F, E)_e(a)$ of S as follows:

$$(F, E)_e(a) := \{b \in S : \mu_{F(e)}(b) \geq \mu_{F(e)}(a), \nu_{F(e)}(b) \leq \nu_{F(e)}(a) \text{ and } \delta_{F(e)}(b) \leq \delta_{F(e)}(a)\}.$$

Lemma 3.6. *Let $\mathbf{S}$ be an ordered semigroup and let (F, E) be a tripolar fuzzy soft set over S . If (F, E) is a tripolar fuzzy soft left ideal over $\mathbf{S}$, then for each $e \in E$ and $a \in S$, the subset $(F, E)_e(a)$ is a left ideal of $\mathbf{S}$.*

Proof. Let $e \in E$ and $a \in S$. To show that $S(F, E)_e(a) \subseteq (F, E)_e(a)$, let $x \in S$ and $b \in (F, E)_e(a)$. By the definition of $(F, E)_e(a)$, we have $\mu_{F(e)}(b) \geq \mu_{F(e)}(a)$, $\nu_{F(e)}(b) \leq \nu_{F(e)}(a)$, and $\delta_{F(e)}(b) \leq \delta_{F(e)}(a)$. Since (F, E) is a tripolar fuzzy soft left ideal over $\mathbf{S}$, it follows from the left ideal condition that:

$$\begin{aligned}
\mu_{F(e)}(xb) &\geq \mu_{F(e)}(b) \geq \mu_{F(e)}(a), \\
\nu_{F(e)}(xb) &\leq \nu_{F(e)}(b) \leq \nu_{F(e)}(a), \\
\delta_{F(e)}(xb) &\leq \delta_{F(e)}(b) \leq \delta_{F(e)}(a).
\end{aligned}$$

This implies that $xb \in (F, E)_e(a)$, which yields $S(F, E)_e(a) \subseteq (F, E)_e(a)$.

Next, we verify the order condition. Let $x, y \in S$ be such that $x \preceq y$ and $y \in (F, E)_e(a)$. By the definition of $(F, E)_e(a)$, we have $\mu_{F(e)}(y) \geq \mu_{F(e)}(a)$, $\nu_{F(e)}(y) \leq \nu_{F(e)}(a)$, and $\delta_{F(e)}(y) \leq \delta_{F(e)}(a)$. Since (F, E) satisfies the order condition of a tripolar fuzzy soft left ideal, we obtain:

$$\begin{aligned}
\mu_{F(e)}(x) &\geq \mu_{F(e)}(y) \geq \mu_{F(e)}(a), \\
\nu_{F(e)}(x) &\leq \nu_{F(e)}(y) \leq \nu_{F(e)}(a), \\
\delta_{F(e)}(x) &\leq \delta_{F(e)}(y) \leq \delta_{F(e)}(a).
\end{aligned}$$

Consequently, $x \in (F, E)_e(a)$. Therefore, $(F, E)_e(a)$ is a left ideal of $\mathbf{S}$. $\square$

By employing arguments similar to those used in Lemma 3.6, we can establish the following result for right ideals.

Lemma 3.7. Let $\mathbf{S}$ be an ordered semigroup and let (F, E) be a tripolar fuzzy soft set over S . If (F, E) is a tripolar fuzzy soft right ideal over $\mathbf{S}$, then for each $e \in E$ and $a \in S$, the subset $(F, E)_e(a)$ is a right ideal of $\mathbf{S}$.

As an immediate consequence of Lemmas 3.6 and 3.7, we establish the following lemma.

Lemma 3.8. Let $\mathbf{S}$ be an ordered semigroup and let (F, E) be a tripolar fuzzy soft set over S . If (F, E) is a tripolar fuzzy soft ideal over $\mathbf{S}$, then for each $e \in E$ and $a \in S$, the subset $(F, E)_e(a)$ is an ideal of $\mathbf{S}$.

We recall the concept of simple ordered semigroups. An ordered semigroup $\mathbf{S}$ is called *simple* [3] if $\mathbf{S}$ contains no proper ideals; that is, for any ideal D of $\mathbf{S}$, we have $D = S$.

Definition 3.9. Let $\mathbf{S}$ be an ordered semigroup. Then $\mathbf{S}$ is called tripolar fuzzy soft simple if every tripolar fuzzy soft ideal (F, E) over $\mathbf{S}$ is a constant function; that is, for any tripolar fuzzy soft ideal (F, E) over $\mathbf{S}$, each $e \in E$, and any $a, b \in S$, we have:

$$\mu_{F(e)}(a) = \mu_{F(e)}(b), \quad \nu_{F(e)}(a) = \nu_{F(e)}(b), \quad \text{and} \quad \delta_{F(e)}(a) = \delta_{F(e)}(b).$$

The following theorem establishes a characterization of simple ordered semigroups via tripolar fuzzy soft simplicity.

Theorem 3.4. Let $\mathbf{S}$ be an ordered semigroup. Then the following statements are equivalent:

- (1) $\mathbf{S}$ is simple;
- (2) $\mathbf{S}$ is tripolar fuzzy soft simple.

Proof. (1) $\Rightarrow$ (2). Suppose that (1) holds. Let (F, E) be a tripolar fuzzy soft ideal over $\mathbf{S}$. To show that $\mathbf{S}$ is tripolar fuzzy soft simple, we must verify that (F, E) is a constant function. Let $e \in E$ and $a, b \in S$. We define a subset D of S by:

$$D = \{x \in S : \mu_{F(e)}(x) \geq \mu_{F(e)}(a), \nu_{F(e)}(x) \leq \nu_{F(e)}(a), \text{ and } \delta_{F(e)}(x) \leq \delta_{F(e)}(a)\}.$$

Since (F, E) is a tripolar fuzzy soft ideal over $\mathbf{S}$, it follows that D is an ideal of $\mathbf{S}$. Moreover, since $a \in D$, we have $D \neq \emptyset$. By hypothesis, $\mathbf{S}$ is simple, which means $\mathbf{S}$ contains no proper ideals. Therefore, we must have $D = S$. Since $b \in S = D$, by the definition of the set D , we obtain:

$$\mu_{F(e)}(b) \geq \mu_{F(e)}(a), \quad \nu_{F(e)}(b) \leq \nu_{F(e)}(a), \quad \text{and} \quad \delta_{F(e)}(b) \leq \delta_{F(e)}(a).$$

By symmetry, by interchanging the roles of a and b , we also obtain:

$$\mu_{F(e)}(a) \geq \mu_{F(e)}(b), \quad \nu_{F(e)}(a) \leq \nu_{F(e)}(b), \quad \text{and} \quad \delta_{F(e)}(a) \leq \delta_{F(e)}(b).$$

Consequently, we have $\mu_{F(e)}(a) = \mu_{F(e)}(b)$, $\nu_{F(e)}(a) = \nu_{F(e)}(b)$, and $\delta_{F(e)}(a) = \delta_{F(e)}(b)$. Hence, (F, E) is a constant function, and then $\mathbf{S}$ is a tripolar fuzzy soft simple ordered semigroup.

(2) $\Rightarrow$ (1). Assume that (2) holds. Let I be an ideal of $\mathbf{S}$. We need to show that $I = S$. Clearly, $I \subseteq S$. Let us consider the characteristic tripolar fuzzy soft set (χ_I, E) over $\mathbf{S}$. Then, since I is an ideal of $\mathbf{S}$, we obtain (χ_I, E) is a tripolar fuzzy soft ideal over $\mathbf{S}$. By our assumption, $\mathbf{S}$ is tripolar fuzzy soft simple, which implies that (χ_I, E) must be a constant function. Thus, for any $a \in I$ and $b \in S$, we have that $\mu_{\chi_I(e)}(a) = \mu_{\chi_I(e)}(b)$, $\nu_{\chi_I(e)}(a) = \nu_{\chi_I(e)}(b)$, and $\delta_{\chi_I(e)}(a) = \delta_{\chi_I(e)}(b)$. Since $a \in I$, we have $\mu_{\chi_I(e)}(a) = 1$, $\nu_{\chi_I(e)}(a) = 0$, and $\delta_{\chi_I(e)}(a) = -1$. This forces $\mu_{\chi_I(e)}(b) = 1$, $\nu_{\chi_I(e)}(b) = 0$, and $\delta_{\chi_I(e)}(b) = -1$, which implies that $b \in I$. Therefore, $S \subseteq I$. It follows that $I = S$. Hence, S contains no proper ideals, which means $\mathbf{S}$ is simple. $\square$

4. CONCLUSION

In this paper, we studied tripolar fuzzy soft ideals in ordered semigroups. We introduced different types of these ideals and studied their properties. Specifically, we proved that in regular, intra-regular, and semisimple ordered semigroups, tripolar fuzzy soft interior ideals and tripolar fuzzy soft ideals are the same.

Additionally, we connected classical ideal theory with the tripolar fuzzy soft framework. In this structure, every tripolar fuzzy soft ideal is a constant function. We then proved that an ordered semigroup $\mathbf{S}$ is simple if and only if it is tripolar fuzzy soft simple.

In the future, we plan to extend these concepts and results to the framework of algebraic hyperstructures.

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