

ALMOST IDEAL STRUCTURES IN ORDERED BISEMIRINGS

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ABSTRACT. In this paper, we introduce and investigate several classes of almost ideals in ordered bisemirings. An ordered bisemiring has two ordered semiring reducts, called the Type 1 and Type 2 semiring reducts. For each reduct, we study right almost ideals, left almost ideals, bi-almost ideals, quasi-almost ideals, and bi-quasi-almost ideals. We establish basic constructions and relationships among these classes. In particular, the defining right and left bi-quasi conditions reduce to the corresponding bi-condition. Consequently, the notions of bi-quasi ideal and bi-ideal coincide, and the notions of bi-quasi-almost ideal and bi-almost ideal also coincide in both ordered semiring reducts. We further combine the Type 1 and Type 2 conditions to define simultaneous ideal structures in an ordered bisemiring. Consequently, every ordered ideal belongs successively to the classes represented in the following chain:

$$I^d \implies QI^d \implies BiI^d = BiQI^d \implies Bi\mathfrak{I}I^d = BiQ\mathfrak{I}I^d \implies Q\mathfrak{I}I^d \implies \mathfrak{I}I^d.$$

A finite example shows that a bi-quasi-almost ideal need not be a bi-quasi ideal. We also prove that arbitrary nonempty unions of almost ideals are almost ideals, while nonempty chains are used for the classes whose definitions require subsemiring closure.

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1. INTRODUCTION

A semiring is an algebraic structure equipped with two associative binary operations in which multiplication distributes over addition from both sides. The concept was introduced by Vandiver [11]. A standard example is the set $\mathbb{N}_0$ of nonnegative integers under the usual addition and multiplication. Another example is

$$([0, 1], \max, \min),$$

where 0 is the identity element for max and 1 is the identity element for min. Semirings provide a common framework for studying several algebraic systems that have ring-like and semigroup-like properties without requiring additive inverses.

An ordered semiring is a semiring equipped with a partial order that is compatible with both operations. Gan and Jiang [1] studied ordered ideals in ordered semirings under the additional assumption that the zero element is the least element. They introduced the notions of ordered ideals, minimal ideals, maximal ideals, and simple ordered semirings. They also established several basic properties and characterizations of minimal ideals and examined matrix semirings over ordered semirings. Their work provides a useful ideal-theoretic framework for subsequent studies of generalized ideal structures in ordered semirings. Han et al. [3] investigated order relations associated with semiring structures. Mandal [5] studied fuzzy ideals and fuzzy interior ideals in ordered semirings and obtained related characterizations of regularity. Palakawong na Ayutthaya and Pibajommee [6,7] investigated regular and k -regular ordered semirings by means of ordered quasi-ideals. Moreover, Gao et al. [2] considered fuzzy orders and related fuzzy structures in semirings.

Different classes of ideals are useful for describing the internal structure of semirings. In particular, right ideals, left ideals, bi-ideals, and quasi-ideals provide several forms of absorption under multiplication. Almost ideals weaken the usual inclusion conditions by requiring only a nonempty intersection with the subset under consideration. Palanikumar and Arulmozhi [8] studied almost ideals, bi-almost ideals, quasi-almost ideals, and related generalizations in semirings. These notions motivate the study of corresponding structures when an order relation is also present.

A bisemiring consists of three associative binary operations such that the first two operations form a Type 1 semiring reduct and the last two operations form a Type 2 semiring reduct. The notion of a bisemiring was introduced by Sen and Ghosh [10]. More recently, bisemirings and subbisemirings have also been considered in connection with fuzzy, neutrosophic, and related set-theoretic structures [4,9]. These developments suggest studying ideal conditions simultaneously in the two semiring reducts of a bisemiring.

In this paper, we equip a bisemiring with a partial order that is compatible with all three operations and investigate almost ideal structures in its Type 1 and Type 2 ordered semiring reducts. For each reduct, we study right almost ideals, left almost ideals, bi-almost ideals, quasi-almost ideals, and bi-quasi-almost ideals. We establish basic constructions and relationships among these classes. In particular, the natural subset inclusions

$$(A \bar{\cap} S \bar{\cap} A) \subseteq (A \bar{\cap} S) \cap (S \bar{\cap} A)$$

and

$$(A * S * A) \subseteq (A * S) \cap (S * A)$$

show that the right and left bi-quasi conditions reduce to the corresponding bi-conditions. Consequently, in both semiring reducts, the classes of bi-almost ideals and bi-quasi-almost ideals coincide.

We also examine the relationships between ordinary ordered ideals and almost ideals. Under the closure conditions specified in the definitions, the following implications are obtained in each reduct:

$$\text{Bi}\mathfrak{I}^d = \text{BiQ}\mathfrak{I}^d \implies \text{Q}\mathfrak{I}^d \implies \mathfrak{I}^d.$$

Finite examples are provided to show that a bi-quasi-almost ideal need not be a bi-quasi ideal. We further investigate the behavior of these classes under the passage to larger lower subsets and under unions. For almost ideals, arbitrary nonempty unions are considered, whereas chains are required for the classes whose definitions include subsemiring conditions.

Finally, we combine the Type 1 and Type 2 conditions to define simultaneous ideal structures in an ordered bisemiring. In this setting, a subset is an ideal of a given type when it satisfies the corresponding conditions in both ordered semiring reducts. This leads to the unified relations

$$\text{Bi}I^d = \text{BiQ}I^d$$

and

$$\text{Bi}\mathfrak{I}^d = \text{BiQ}\mathfrak{I}^d \implies \text{Q}\mathfrak{I}^d \implies \mathfrak{I}^d.$$

The paper is organized as follows. Section 2 recalls the basic notions of semirings, ordered semirings, ordered ideals, almost ideals, bisemirings, and ordered bisemirings. Section 3 studies almost ideals associated with the Type 1 ordered semiring reduct. The corresponding structures in the Type 2 ordered semiring reduct are investigated in Section 4. Section 5 combines the two reducts and develops simultaneous ideal structures in ordered bisemirings. Section 6 summarizes the main results and indicates possible directions for further study.

List of abbreviations.

Type 1 semiring reduct	T1S
Type 2 semiring reduct	T2S
Ordered right ideal	RI^d
Ordered left ideal	LI^d
Ordered ideal	I^d
Ordered bi-ideal	$\text{Bi}I^d$
Ordered quasi-ideal	QI^d
Ordered right bi-quasi ideal	RBIQI^d
Ordered left bi-quasi ideal	LBIQI^d

Ordered bi-quasi ideal	$\text{BiQ}I^d$
Right almost ideal	$\text{R}\mathfrak{I}^d$
Left almost ideal	$\text{L}\mathfrak{I}^d$
Almost ideal	$\mathfrak{I}^d$
Bi-almost ideal	$\text{Bi}\mathfrak{I}^d$
Quasi-almost ideal	$\text{Q}\mathfrak{I}^d$
Right bi-quasi-almost ideal	$\text{RBiQ}\mathfrak{I}^d$
Left bi-quasi-almost ideal	$\text{LBiQ}\mathfrak{I}^d$
Bi-quasi-almost ideal	$\text{BiQ}\mathfrak{I}^d$

The prefixes T1S- and T2S- indicate that the corresponding ideal condition is considered in the Type 1 and Type 2 ordered semiring reducts, respectively.

2. PRELIMINARIES

This section recalls the basic notions of semirings, ordered semirings, ordered ideals, almost ideals, and bisemirings. We also fix the notation for products of subsets and lower closures. Finally, we introduce the notion of an ordered bisemiring and describe its Type 1 and Type 2 semiring reducts.

Definition 2.1. Following [7], an algebraic system $(S, +, \cdot)$ is called a **semiring** if $(S, +)$ and $(S, \cdot)$ are semigroups and multiplication distributes over addition from both sides; that is,

$$x(y + z) = xy + xz \quad \text{and} \quad (x + y)z = xz + yz$$

for all $x, y, z \in S$.

A semiring $(S, +, \cdot)$ equipped with a partial order $\preceq$ is called an **ordered semiring** if, for all $x, y, z \in S$,

$$x \preceq y$$

implies

$$x + z \preceq y + z, \quad z + x \preceq z + y,$$

and

$$xz \preceq yz, \quad zx \preceq zy.$$

Throughout this paper, neither identity elements nor the commutativity of addition are assumed unless they are explicitly stated.

Let A and B be nonempty subsets of a semiring S . For a binary operation $\star$ on S , we write

$$A \star B = \{a \star b \mid a \in A, b \in B\}.$$

In particular, when $\star$ is the multiplication of S , we write

$$AB = \{ab \mid a \in A, b \in B\}.$$

Since multiplication is associative, products of three or more subsets are written without parentheses.

If $(S, +, \cdot, \preceq)$ is an ordered semiring and $A \subseteq S$, then the *lower closure* of A is defined by

$$[A] = \{x \in S \mid x \preceq a \text{ for some } a \in A\}.$$

A subset A of S is called a *lower subset* if $A = [A]$.

A nonempty subset A of a semiring S is called a *subsemiring* of S if it is closed under both addition and multiplication.

Definition 2.2. Let A be a nonempty subset of an ordered semiring $(S, +, \cdot, \preceq)$. Following [7], we say that:

(i) A is an ordered right ideal, briefly an RI^d , if $(A, +)$ is a subsemigroup of $(S, +)$,

$$AS \subseteq A,$$

and $A = [A]$;

(ii) A is an ordered left ideal, briefly an LI^d , if $(A, +)$ is a subsemigroup of $(S, +)$,

$$SA \subseteq A,$$

and $A = [A]$;

(iii) A is an ordered ideal, briefly an I^d , if it is both an ordered right ideal and an ordered left ideal;

(iv) A is an ordered quasi-ideal, briefly a QI^d , if $(A, +)$ is a subsemigroup of $(S, +)$,

$$(AS] \cap (SA] \subseteq A,$$

and $A = [A]$;

(v) A is an ordered bi-ideal, briefly a BiI^d , if $(A, +)$ is a subsemigroup of $(S, +)$,

$$ASA \subseteq A,$$

and $A = [A]$.

Proposition 2.1. Every ordered quasi-ideal of an ordered semiring is a subsemiring.

Proof. Let A be an ordered quasi-ideal of an ordered semiring $(S, +, \cdot, \preceq)$. By definition, $(A, +)$ is a subsemigroup of $(S, +)$. It remains to show that A is closed under multiplication.

Let $x, y \in A$. Since $A \subseteq S$, we have

$$xy \in AS \quad \text{and} \quad xy \in SA.$$

Hence

$$xy \in AS \cap SA \subseteq (AS] \cap (SA].$$

Since A is an ordered quasi-ideal,

$$(AS] \cap (SA] \subseteq A.$$

Therefore, $xy \in A$. Thus, A is closed under both addition and multiplication, and hence A is a subsemiring of S . $\square$

Remark 2.1. Let A and B be subsets of an ordered semiring S . Then [6]:

(i)

$$A \subseteq (A] \quad \text{and} \quad ((A]) = (A];$$

(ii) if $A \subseteq B$, then

$$(A] \subseteq (B];$$

(iii)

$$A(B] \subseteq (A](B] \subseteq (AB]$$

and

$$(A]B \subseteq (A](B] \subseteq (AB].$$

Definition 2.3. Let A be a nonempty subset of a semiring $(S, +, \cdot)$. Following [8], we say that:

(i) A is a right almost ideal, briefly an $R\mathfrak{A}^d$, if

$$AS \cap A \neq \emptyset;$$

(ii) A is a left almost ideal, briefly an $L\mathfrak{A}^d$, if

$$SA \cap A \neq \emptyset;$$

(iii) A is an almost ideal, briefly a $\mathfrak{A}^d$, if it is both a right almost ideal and a left almost ideal;

(iv) a subsemiring A of S is called a bi-almost ideal, briefly a $Bi\mathfrak{A}^d$, if

$$ASA \cap A \neq \emptyset;$$

(v) a subsemiring A of S is called a quasi-almost ideal, briefly a $Q\mathfrak{A}^d$, if

$$(AS \cap SA) \cap A \neq \emptyset.$$

Definition 2.4. Following [10], an algebraic system

$$(S, \odot, \bar{\wedge}, *)$$

is called a bisemiring if $(S, \odot)$, $(S, \bar{\wedge})$, and $(S, *)$ are semigroups and the following distributive laws hold for all $x, y, z \in S$:

$$x \bar{\wedge} (y \odot z) = (x \bar{\wedge} y) \odot (x \bar{\wedge} z),$$

$$(x \odot y) \bar{\wedge} z = (x \bar{\wedge} z) \odot (y \bar{\wedge} z),$$

$$x * (y \bar{\wedge} z) = (x * y) \bar{\wedge} (x * z),$$

$$(x \bar{\wedge} y) * z = (x * z) \bar{\wedge} (y * z).$$

The semiring $(S, \odot, \bar{\wedge})$ is called the Type 1 semiring reduct, abbreviated as T1S, while $(S, \bar{\wedge}, *)$ is called the Type 2 semiring reduct, abbreviated as T2S.

Definition 2.5. A system

$$(S, \odot, \bar{\wedge}, *, \preceq)$$

is called an ordered bisemiring if $(S, \odot, \bar{\wedge}, *)$ is a bisemiring, $(S, \preceq)$ is a partially ordered set, and, for all $x, y, z \in S$,

$$x \preceq y$$

implies

$$x * z \preceq y * z \quad \text{and} \quad z * x \preceq z * y$$

for every

$$* \in \{\odot, \bar{\wedge}, *\}.$$

Equivalently, both

$$(S, \odot, \bar{\wedge}, \preceq) \quad \text{and} \quad (S, \bar{\wedge}, *, \preceq)$$

are ordered semirings with respect to the same partial order.

3. TYPE 1 ALMOST IDEALS IN ORDERED BISEMIRINGS

In this section, we introduce and investigate several classes of almost ideals associated with the Type 1 semiring reduct of an ordered bisemiring. We establish basic relationships among these classes and examine the connection between bi-almost ideals and bi-quasi-almost ideals.

Throughout this section, let

$$(S, \odot, \bar{\wedge}, *, \preceq)$$

be an ordered bisemiring. We consider its Type 1 ordered semiring reduct

$$(S, \odot, \bar{\wedge}, \preceq).$$

A nonempty subset of S that is closed under both $\odot$ and $\bar{\wedge}$ is called a T1S-subsemiring. Moreover, the terms $\text{T1S-}I^d$, $\text{T1S-Q}I^d$, and $\text{T1S-Bi}I^d$ refer to the corresponding ordered ideals of the Type 1 semiring reduct, as defined in Definition 2.2.

Definition 3.1. Let A be a nonempty lower subset of S . Then:

(i) A is called a T1S-right almost ideal, briefly a T1S-R $\mathfrak{A}I^d$, if

$$(A \bar{\wedge} S] \cap A \neq \emptyset;$$

(ii) A is called a T1S-left almost ideal, briefly a $T1S-L\mathfrak{I}^d$, if

$$(S \bar{\wedge} A] \cap A \neq \emptyset;$$

(iii) A is called a T1S-almost ideal, briefly a $T1S-\mathfrak{I}^d$, if it is both a $T1S-R\mathfrak{I}^d$ and a $T1S-L\mathfrak{I}^d$.

Lemma 3.1. Let A be a nonempty subset of S . Then:

- (i) $(A \bar{\wedge} S]$ is a $T1S-R\mathfrak{I}^d$;
- (ii) $(S \bar{\wedge} A]$ is a $T1S-L\mathfrak{I}^d$;
- (iii) $(S \bar{\wedge} A \bar{\wedge} S]$ is a $T1S-\mathfrak{I}^d$.

Proof. We first prove (i). Put

$$B = (A \bar{\wedge} S].$$

Clearly, B is a nonempty lower subset of S . Choose $a \in A$ and $x, y \in S$. Then $a \bar{\wedge} x \in B$, and, by the associativity of $\bar{\wedge}$,

$$(a \bar{\wedge} x) \bar{\wedge} y = a \bar{\wedge} (x \bar{\wedge} y) \in A \bar{\wedge} S \subseteq B.$$

Moreover,

$$(a \bar{\wedge} x) \bar{\wedge} y \in B \bar{\wedge} S.$$

Therefore,

$$(B \bar{\wedge} S] \cap B \neq \emptyset.$$

Hence B is a $T1S-R\mathfrak{I}^d$.

The proof of (ii) is similar.

For (iii), put

$$C = (S \bar{\wedge} A \bar{\wedge} S].$$

Clearly, C is a nonempty lower subset of S . Choose $a \in A$ and $x, y, z \in S$. Then

$$x \bar{\wedge} a \bar{\wedge} y \in C.$$

By associativity,

$$(x \bar{\wedge} a \bar{\wedge} y) \bar{\wedge} z = x \bar{\wedge} a \bar{\wedge} (y \bar{\wedge} z) \in S \bar{\wedge} A \bar{\wedge} S \subseteq C,$$

and

$$z \bar{\wedge} (x \bar{\wedge} a \bar{\wedge} y) = (z \bar{\wedge} x) \bar{\wedge} a \bar{\wedge} y \in S \bar{\wedge} A \bar{\wedge} S \subseteq C.$$

It follows that

$$(C \bar{\wedge} S] \cap C \neq \emptyset$$

and

$$(S \bar{\wedge} C] \cap C \neq \emptyset.$$

Thus, C is a $T1S-\mathfrak{I}^d$. □

Definition 3.2. Let A be a nonempty lower subset of S . Then:

(i) A is called a T1S-bi-almost ideal, briefly a $T1S\text{-Bi}\mathfrak{I}^d$, if A is a T1S-subsemiring and

$$(A \bar{\cap} S \bar{\cap} A) \cap A \neq \emptyset;$$

(ii) A is called a T1S-quasi-almost ideal, briefly a $T1S\text{-Q}\mathfrak{I}^d$, if A is a T1S-subsemiring and

$$(A \bar{\cap} S] \cap (S \bar{\cap} A) \cap A \neq \emptyset;$$

(iii) A is called a T1S-right bi-quasi ideal, briefly a $T1S\text{-R}\mathfrak{B}\mathfrak{I}^d$, if $(A, \odot)$ is a subsemigroup of $(S, \odot)$,

$$(A \bar{\cap} S] \cap (A \bar{\cap} S \bar{\cap} A) \subseteq A,$$

and $A = (A]$;

(iv) A is called a T1S-left bi-quasi ideal, briefly a $T1S\text{-L}\mathfrak{B}\mathfrak{I}^d$, if $(A, \odot)$ is a subsemigroup of $(S, \odot)$,

$$(S \bar{\cap} A] \cap (A \bar{\cap} S \bar{\cap} A) \subseteq A,$$

and $A = (A]$;

(v) A is called a T1S-bi-quasi ideal, briefly a $T1S\text{-Bi}\mathfrak{Q}^d$, if it is both a $T1S\text{-R}\mathfrak{B}\mathfrak{I}^d$ and a $T1S\text{-L}\mathfrak{B}\mathfrak{I}^d$;

(vi) A is called a T1S-right bi-quasi-almost ideal, briefly a $T1S\text{-R}\mathfrak{B}\mathfrak{I}^d$, if A is a T1S-subsemiring and

$$[(A \bar{\cap} S] \cap (A \bar{\cap} S \bar{\cap} A)] \cap A \neq \emptyset;$$

(vii) A is called a T1S-left bi-quasi-almost ideal, briefly a $T1S\text{-L}\mathfrak{B}\mathfrak{I}^d$, if A is a T1S-subsemiring and

$$[(S \bar{\cap} A] \cap (A \bar{\cap} S \bar{\cap} A)] \cap A \neq \emptyset;$$

(viii) A is called a T1S-bi-quasi-almost ideal, briefly a $T1S\text{-Bi}\mathfrak{Q}\mathfrak{I}^d$, if it is both a $T1S\text{-R}\mathfrak{B}\mathfrak{I}^d$ and a $T1S\text{-L}\mathfrak{B}\mathfrak{I}^d$.

Lemma 3.2. For every nonempty subset A of S ,

$$(A \bar{\cap} S \bar{\cap} A) \subseteq (A \bar{\cap} S] \cap (S \bar{\cap} A).$$

Consequently,

$$(A \bar{\cap} S] \cap (A \bar{\cap} S \bar{\cap} A) = (A \bar{\cap} S \bar{\cap} A),$$

$$(S \bar{\cap} A] \cap (A \bar{\cap} S \bar{\cap} A) = (A \bar{\cap} S \bar{\cap} A).$$

Proof. By the associativity of $\bar{\cap}$, we have

$$A \bar{\cap} S \bar{\cap} A \subseteq A \bar{\cap} S$$

and

$$A \bar{\cap} S \bar{\cap} A \subseteq S \bar{\cap} A.$$

Taking lower closures gives

$$(A \bar{\cap} S \bar{\cap} A] \subseteq (A \bar{\cap} S]$$

and

$$(A \bar{\cap} S \bar{\cap} A] \subseteq (S \bar{\cap} A].$$

Therefore,

$$(A \bar{\cap} S \bar{\cap} A] \subseteq (A \bar{\cap} S] \cap (S \bar{\cap} A].$$

The two equalities follow immediately. $\square$

Proposition 3.1. *Let A be a nonempty lower subset of S . Then the following conditions are equivalent:*

- (i) A is a $T1S\text{-}Bi\mathfrak{I}^d$;
- (ii) A is a $T1S\text{-}RBiQ\mathfrak{I}^d$;
- (iii) A is a $T1S\text{-}LBiQ\mathfrak{I}^d$;
- (iv) A is a $T1S\text{-}BiQ\mathfrak{I}^d$.

Moreover, the notions of $T1S\text{-}RBiQI^d$, $T1S\text{-}LBiQI^d$, $T1S\text{-}BiQI^d$, and $T1S\text{-}BiI^d$ coincide.

Proof. By Lemma 3.2,

$$(A \bar{\cap} S] \cap (A \bar{\cap} S \bar{\cap} A] = (A \bar{\cap} S \bar{\cap} A]$$

and

$$(S \bar{\cap} A] \cap (A \bar{\cap} S \bar{\cap} A] = (A \bar{\cap} S \bar{\cap} A].$$

Thus, the right and left bi-quasi-almost conditions are both equivalent to

$$(A \bar{\cap} S \bar{\cap} A] \cap A \neq \emptyset.$$

This proves the first assertion.

Similarly, the right and left bi-quasi ideal conditions are both equivalent to

$$(A \bar{\cap} S \bar{\cap} A] \subseteq A,$$

which is precisely the defining absorption condition of a $T1S\text{-}BiI^d$. $\square$

Theorem 3.1. *Every $T1S\text{-}I^d$ is a $T1S\text{-}\mathfrak{I}^d$. Moreover:*

- (i) *every $T1S\text{-}BiI^d$ that is a $T1S\text{-}subsemiring$ is a $T1S\text{-}Bi\mathfrak{I}^d$;*
- (ii) *every $T1S\text{-}QI^d$ is a $T1S\text{-}Q\mathfrak{I}^d$.*

Proof. Let A be a $T1S\text{-}I^d$. Then

$$A \bar{\cap} S \subseteq A \quad \text{and} \quad S \bar{\cap} A \subseteq A.$$

Since A is a lower subset,

$$(A \bar{\cap} S] \subseteq A \quad \text{and} \quad (S \bar{\cap} A] \subseteq A.$$

The sets $(A \bar{\cap} S]$ and $(S \bar{\cap} A]$ are nonempty. Hence

$$(A \bar{\cap} S] \cap A \neq \emptyset$$

and

$$(S \bar{\cap} A] \cap A \neq \emptyset.$$

Therefore, A is a $T1S\text{-}\mathfrak{I}^d$.

Let A be a $T1S\text{-}BiI^d$ that is a $T1S$ -subsemiring. Since

$$A \bar{\cap} S \bar{\cap} A \subseteq A$$

and $A = (A]$, we have

$$(A \bar{\cap} S \bar{\cap} A] \subseteq A.$$

The set on the left is nonempty, so

$$(A \bar{\cap} S \bar{\cap} A] \cap A \neq \emptyset.$$

Thus, A is a $T1S\text{-}Bi\mathfrak{I}^d$.

Finally, let A be a $T1S\text{-}QI^d$. By Proposition 2.1, A is a $T1S$ -subsemiring. Moreover,

$$(A \bar{\cap} S] \cap (S \bar{\cap} A] \subseteq A.$$

By Lemma 3.2, the intersection on the left is nonempty. Consequently,

$$(A \bar{\cap} S] \cap (S \bar{\cap} A] \cap A \neq \emptyset.$$

Hence A is a $T1S\text{-}Q\mathfrak{I}^d$. □

Theorem 3.2. Every $T1S\text{-}QI^d$ and every $T1S\text{-}BiI^d$ is a $T1S\text{-}BiQI^d$.

Proof. Let A be a $T1S\text{-}QI^d$. Then

$$(A \bar{\cap} S] \cap (S \bar{\cap} A] \subseteq A.$$

By Lemma 3.2,

$$(A \bar{\cap} S \bar{\cap} A] \subseteq (A \bar{\cap} S] \cap (S \bar{\cap} A] \subseteq A.$$

Hence A is a $T1S\text{-}BiI^d$ and therefore, by Proposition 3.1, it is a $T1S\text{-}BiQI^d$.

If A is a $T1S\text{-}BiI^d$, the conclusion follows directly from Proposition 3.1. □

Corollary 3.1. Every $T1S\text{-}Bi\mathfrak{I}^d$, equivalently every $T1S\text{-}BiQ\mathfrak{I}^d$, is a $T1S\text{-}Q\mathfrak{I}^d$.

Proof. Let A be a T1S-Bi $\mathfrak{J}I^d$. By Proposition 3.1, this is equivalent to A being a T1S-BiQ $\mathfrak{J}I^d$. By the definition of a T1S-Bi $\mathfrak{J}I^d$,

$$(A \bar{\cap} S \bar{\cap} A] \cap A \neq \emptyset.$$

Moreover, Lemma 3.2 gives

$$(A \bar{\cap} S \bar{\cap} A] \subseteq (A \bar{\cap} S] \cap (S \bar{\cap} A].$$

Therefore,

$$\emptyset \neq (A \bar{\cap} S \bar{\cap} A] \cap A \subseteq (A \bar{\cap} S] \cap (S \bar{\cap} A] \cap A.$$

Hence,

$$(A \bar{\cap} S] \cap (S \bar{\cap} A] \cap A \neq \emptyset.$$

Since A is a lower T1S-subsemiring, it follows that A is a T1S-Q $\mathfrak{J}I^d$. □

Theorem 3.3. *Every T1S-Q $\mathfrak{J}I^d$ is a T1S- $\mathfrak{J}I^d$.*

Proof. Let A be a T1S-Q $\mathfrak{J}I^d$. Then

$$(A \bar{\cap} S] \cap (S \bar{\cap} A] \cap A \neq \emptyset.$$

Consequently,

$$(A \bar{\cap} S] \cap A \neq \emptyset$$

and

$$(S \bar{\cap} A] \cap A \neq \emptyset.$$

Thus, A is both a T1S-R $\mathfrak{J}I^d$ and a T1S-L $\mathfrak{J}I^d$. Therefore, A is a T1S- $\mathfrak{J}I^d$. □

Theorem 3.4. *Every T1S-BiQ I^d that is a T1S-subsemiring is a T1S-BiQ $\mathfrak{J}I^d$.*

Proof. Let A be a T1S-BiQ I^d that is a T1S-subsemiring. By Proposition 3.1, A is a T1S-Bi I^d . Hence

$$(A \bar{\cap} S \bar{\cap} A] \subseteq A.$$

Since $(A \bar{\cap} S \bar{\cap} A]$ is nonempty,

$$(A \bar{\cap} S \bar{\cap} A] \cap A \neq \emptyset.$$

Thus, A is a T1S-Bi $\mathfrak{J}I^d$. Another application of Proposition 3.1 shows that A is a T1S-BiQ $\mathfrak{J}I^d$. □

The converse of Theorem 3.4 does not hold, as shown in the following example.

Example 3.1. *Let*

$$S = \{\iota, \chi, \delta, \pi, \lambda, \alpha\}.$$

Define the operations $\odot$ and $\bar{\cap}$ on S by the following tables.

$\odot$	ι	χ	δ	π	λ	$\propto$	$\bar{\wedge}$	ι	χ	δ	π	λ	$\propto$
ι	ι	χ	δ	π	λ	$\propto$	ι	ι	ι	ι	ι	ι	ι
χ	χ	χ	δ	π	λ	$\propto$	χ	ι	χ	δ	χ	δ	δ
δ	δ	δ	δ	$\propto$	λ	$\propto$	δ	ι	χ	δ	χ	δ	δ
π	π	π	$\propto$	π	λ	$\propto$	π	ι	π	λ	π	λ	λ
λ	λ	λ	λ	λ	λ	λ	λ	ι	π	λ	π	λ	λ
$\propto$	$\propto$	$\propto$	$\propto$	$\propto$	λ	$\propto$	$\propto$	ι	π	λ	π	λ	λ

Define

$$x * y = \iota$$

for all $x, y \in S$, and let $\preceq$ be the equality relation on S , that is,

$$\preceq = \{(x, x) \mid x \in S\}.$$

A direct verification shows that

$$(S, \odot, \bar{\wedge}, *, \preceq)$$

is an ordered bisemiring.

Let

$$A = \{\lambda\}.$$

Clearly, A is a T1S-subsemiring and $A = (A]$. Moreover,

$$A \bar{\wedge} S = \{\iota, \pi, \lambda\},$$

$$S \bar{\wedge} A = \{\iota, \delta, \lambda\},$$

$$A \bar{\wedge} S \bar{\wedge} A = \{\iota, \lambda\}.$$

Thus,

$$(A \bar{\wedge} S \bar{\wedge} A) \cap A = \{\lambda\} \neq \emptyset.$$

Hence A is a T1S-Bi $\mathfrak{I}^d$ and therefore, by Proposition 3.1, a T1S-BiQ $\mathfrak{I}^d$.

However,

$$(A \bar{\wedge} S \bar{\wedge} A) = \{\iota, \lambda\} \not\subseteq A.$$

Therefore, A is not a T1S-Bi I^d and hence is not a T1S-BiQ I^d .

Theorem 3.5. Let A and B be nonempty subsets of S such that

$$A \subseteq B \subseteq S.$$

Then:

- (i) if A is a T1S- $\mathfrak{I}^d$ and B is a lower subset of S , then B is a T1S- $\mathfrak{I}^d$;

(ii) if A is a $T1S\text{-}Bi\mathfrak{I}^d$, $T1S\text{-}Q\mathfrak{I}^d$, or $T1S\text{-}BiQ\mathfrak{I}^d$ and B is a lower $T1S$ -subsemiring of S , then B is an almost ideal of the same type as A .

Proof. Suppose first that A is a $T1S\text{-}\mathfrak{I}^d$ and B is a lower subset of S . Since $A \subseteq B$, we have

$$(A \bar{\cap} S] \subseteq (B \bar{\cap} S]$$

and

$$(S \bar{\cap} A] \subseteq (S \bar{\cap} B].$$

Therefore,

$$\emptyset \neq (A \bar{\cap} S] \cap A \subseteq (B \bar{\cap} S] \cap B$$

and

$$\emptyset \neq (S \bar{\cap} A] \cap A \subseteq (S \bar{\cap} B] \cap B.$$

Hence B is a $T1S\text{-}\mathfrak{I}^d$.

Suppose that A is a $T1S\text{-}Bi\mathfrak{I}^d$ and that B is a lower $T1S$ -subsemiring. Then

$$A \bar{\cap} S \bar{\cap} A \subseteq B \bar{\cap} S \bar{\cap} B,$$

and hence

$$\emptyset \neq (A \bar{\cap} S \bar{\cap} A] \cap A \subseteq (B \bar{\cap} S \bar{\cap} B] \cap B.$$

Thus, B is a $T1S\text{-}Bi\mathfrak{I}^d$.

The proofs for $T1S\text{-}Q\mathfrak{I}^d$ and $T1S\text{-}BiQ\mathfrak{I}^d$ are similar. □

Theorem 3.6. *The following statements hold:*

- (i) *the union of a nonempty family of $T1S\text{-}\mathfrak{I}^d$ s is a $T1S\text{-}\mathfrak{I}^d$;*
- (ii) *the union of a nonempty chain of $T1S\text{-}Bi\mathfrak{I}^d$ s, $T1S\text{-}Q\mathfrak{I}^d$ s, or $T1S\text{-}BiQ\mathfrak{I}^d$ s is an almost ideal of the corresponding type.*

Proof. Let $\{A_i\}_{i \in I}$ be a nonempty family of $T1S\text{-}\mathfrak{I}^d$ s and put

$$A = \bigcup_{i \in I} A_i.$$

Since every A_i is a lower subset of S , the set A is also a lower subset of S . Choose $i_0 \in I$. Since

$$A_{i_0} \subseteq A,$$

Theorem 3.5 implies that A is a $T1S\text{-}\mathfrak{I}^d$. This proves (i).

For (ii), suppose that $\{A_i\}_{i \in I}$ is a nonempty chain under set inclusion, and put

$$A = \bigcup_{i \in I} A_i.$$

The union A is a lower subset of S . We show that it is a T1S-subsemiring. Let $x, y \in A$. Then $x \in A_i$ and $y \in A_j$ for some $i, j \in I$. Since the family is a chain, either

$$A_i \subseteq A_j \quad \text{or} \quad A_j \subseteq A_i.$$

Thus, x and y belong to one common member of the chain. Since that member is a T1S-subsemiring,

$$x \odot y \in A \quad \text{and} \quad x \bar{\wedge} y \in A.$$

Hence A is a T1S-subsemiring.

Choose $i_0 \in I$. Since $A_{i_0} \subseteq A$, the conclusion follows from Theorem 3.5. $\square$

4. TYPE 2 ALMOST IDEALS IN ORDERED BISEMIRINGS

In this section, we introduce and investigate several classes of almost ideals associated with the Type 2 semiring reduct of an ordered bisemiring. We establish basic relationships among these classes and examine the connection between bi-almost ideals and bi-quasi-almost ideals. The results obtained in this section are parallel to those established for the Type 1 semiring reduct in Section 3.

Throughout this section, let

$$(S, \odot, \bar{\wedge}, *, \preceq)$$

be an ordered bisemiring. We consider its Type 2 ordered semiring reduct

$$(S, \bar{\wedge}, *, \preceq).$$

A nonempty subset of S that is closed under both $\bar{\wedge}$ and $*$ is called a *T2S-subsemiring*. Moreover, the terms $\text{T2S-}I^d$, $\text{T2S-Q}I^d$, and $\text{T2S-Bi}I^d$ refer to the corresponding ordered ideals of the Type 2 semiring reduct, as defined in Definition 2.2.

Definition 4.1. Let A be a nonempty lower subset of S . Then:

(i) A is called a T2S-right almost ideal, briefly a $\text{T2S-R}\mathfrak{I}^d$, if

$$(A * S] \cap A \neq \emptyset;$$

(ii) A is called a T2S-left almost ideal, briefly a $\text{T2S-L}\mathfrak{I}^d$, if

$$(S * A] \cap A \neq \emptyset;$$

(iii) A is called a T2S-almost ideal, briefly a $\text{T2S-}\mathfrak{I}^d$, if it is both a $\text{T2S-R}\mathfrak{I}^d$ and a $\text{T2S-L}\mathfrak{I}^d$.

Lemma 4.1. Let A be a nonempty subset of S . Then:

(i) $(A * S]$ is a $\text{T2S-R}\mathfrak{I}^d$;

(ii) $(S * A]$ is a $\text{T2S-L}\mathfrak{I}^d$;

(iii) $(S * A * S]$ is a $\text{T2S-}\mathfrak{I}^d$.

Proof. We first prove (i). Put

$$B = (A * S].$$

Clearly, B is a nonempty lower subset of S . Choose $a \in A$ and $x, y \in S$. Then $a * x \in B$, and, by the associativity of $*$,

$$(a * x) * y = a * (x * y) \in A * S \subseteq B.$$

Moreover,

$$(a * x) * y \in B * S.$$

Therefore,

$$(B * S] \cap B \neq \emptyset.$$

Hence B is a T2S-R $\mathfrak{J}I^d$.

The proof of (ii) is similar.

For (iii), put

$$C = (S * A * S].$$

Clearly, C is a nonempty lower subset of S . Choose $a \in A$ and $x, y, z \in S$. Then

$$x * a * y \in C.$$

By associativity,

$$(x * a * y) * z = x * a * (y * z) \in S * A * S \subseteq C,$$

and

$$z * (x * a * y) = (z * x) * a * y \in S * A * S \subseteq C.$$

It follows that

$$(C * S] \cap C \neq \emptyset$$

and

$$(S * C] \cap C \neq \emptyset.$$

Thus, C is a T2S- $\mathfrak{J}I^d$. □

Definition 4.2. Let A be a nonempty lower subset of S . Then:

(i) A is called a T2S-bi-almost ideal, briefly a T2S-Bi $\mathfrak{J}I^d$, if A is a T2S-subsemiring and

$$(A * S * A] \cap A \neq \emptyset;$$

(ii) A is called a T2S-quasi-almost ideal, briefly a T2S-Q $\mathfrak{J}I^d$, if A is a T2S-subsemiring and

$$(A * S] \cap (S * A] \cap A \neq \emptyset;$$

(iii) A is called a T2S-right bi-quasi ideal, briefly a $T2S-RBiQI^d$, if $(A, \bar{\cap})$ is a subsemigroup of $(S, \bar{\cap})$,

$$(A * S] \cap (A * S * A] \subseteq A,$$

and $A = (A]$;

(iv) A is called a T2S-left bi-quasi ideal, briefly a $T2S-LBiQI^d$, if $(A, \bar{\cap})$ is a subsemigroup of $(S, \bar{\cap})$,

$$(S * A] \cap (A * S * A] \subseteq A,$$

and $A = (A]$;

(v) A is called a T2S-bi-quasi ideal, briefly a $T2S-BiQI^d$, if it is both a $T2S-RBiQI^d$ and a $T2S-LBiQI^d$;

(vi) A is called a T2S-right bi-quasi-almost ideal, briefly a $T2S-RBiQ3I^d$, if A is a T2S-subsemiring and

$$[(A * S] \cap (A * S * A)] \cap A \neq \emptyset;$$

(vii) A is called a T2S-left bi-quasi-almost ideal, briefly a $T2S-LBiQ3I^d$, if A is a T2S-subsemiring and

$$[(S * A] \cap (A * S * A)] \cap A \neq \emptyset;$$

(viii) A is called a T2S-bi-quasi-almost ideal, briefly a $T2S-BiQ3I^d$, if it is both a $T2S-RBiQ3I^d$ and a $T2S-LBiQ3I^d$.

Lemma 4.2. For every nonempty subset A of S ,

$$(A * S * A] \subseteq (A * S] \cap (S * A].$$

Consequently,

$$(A * S] \cap (A * S * A] = (A * S * A],$$

$$(S * A] \cap (A * S * A] = (A * S * A].$$

Proof. By the associativity of $*$, we have

$$A * S * A \subseteq A * S$$

and

$$A * S * A \subseteq S * A.$$

Taking lower closures gives

$$(A * S * A] \subseteq (A * S]$$

and

$$(A * S * A] \subseteq (S * A].$$

Therefore,

$$(A * S * A] \subseteq (A * S] \cap (S * A].$$

The two equalities follow immediately. □

Proposition 4.1. *Let A be a nonempty lower subset of S . Then the following conditions are equivalent:*

- (i) A is a $T2S\text{-}Bi\mathfrak{I}^d$;
- (ii) A is a $T2S\text{-}RBiQ\mathfrak{I}^d$;
- (iii) A is a $T2S\text{-}LBiQ\mathfrak{I}^d$;
- (iv) A is a $T2S\text{-}BiQ\mathfrak{I}^d$.

Moreover, the notions of $T2S\text{-}RBiQ\mathfrak{I}^d$, $T2S\text{-}LBiQ\mathfrak{I}^d$, $T2S\text{-}BiQ\mathfrak{I}^d$, and $T2S\text{-}Bi\mathfrak{I}^d$ coincide.

Proof. By Lemma 4.2,

$$(A * S] \cap (A * S * A] = (A * S * A]$$

and

$$(S * A] \cap (A * S * A] = (A * S * A].$$

Thus, the right and left bi-quasi-almost conditions are both equivalent to

$$(A * S * A] \cap A \neq \emptyset.$$

This proves the first assertion.

Similarly, the right and left bi-quasi ideal conditions are both equivalent to

$$(A * S * A] \subseteq A,$$

which is precisely the defining absorption condition of a $T2S\text{-}Bi\mathfrak{I}^d$. □

Theorem 4.1. *Every $T2S\text{-}\mathfrak{I}^d$ is a $T2S\text{-}\mathfrak{I}^d$. Moreover:*

- (i) *every $T2S\text{-}Bi\mathfrak{I}^d$ that is a $T2S\text{-}subsemiring$ is a $T2S\text{-}Bi\mathfrak{I}^d$;*
- (ii) *every $T2S\text{-}Q\mathfrak{I}^d$ is a $T2S\text{-}Q\mathfrak{I}^d$.*

Proof. Let A be a $T2S\text{-}\mathfrak{I}^d$. Then

$$A * S \subseteq A \quad \text{and} \quad S * A \subseteq A.$$

Since A is a lower subset,

$$(A * S] \subseteq A \quad \text{and} \quad (S * A] \subseteq A.$$

The sets $(A * S]$ and $(S * A]$ are nonempty. Hence

$$(A * S] \cap A \neq \emptyset$$

and

$$(S * A] \cap A \neq \emptyset.$$

Therefore, A is a $T2S\text{-}\mathfrak{I}^d$.

Let A be a $T2S\text{-}Bi\mathfrak{I}^d$ that is a $T2S\text{-}subsemiring$. Since

$$A * S * A \subseteq A$$

and $A = (A]$, we have

$$(A * S * A] \subseteq A.$$

The set on the left is nonempty, so

$$(A * S * A] \cap A \neq \emptyset.$$

Thus, A is a $T2S\text{-}Bi\mathfrak{I}^d$.

Finally, let A be a $T2S\text{-}QI^d$. By Proposition 2.1, A is a $T2S$ -subsemiring. Moreover,

$$(A * S] \cap (S * A] \subseteq A.$$

By Lemma 4.2, the intersection on the left is nonempty. Consequently,

$$(A * S] \cap (S * A] \cap A \neq \emptyset.$$

Hence A is a $T2S\text{-}Q\mathfrak{I}^d$. □

Theorem 4.2. *Every $T2S\text{-}QI^d$ and every $T2S\text{-}BiI^d$ is a $T2S\text{-}BiQI^d$.*

Proof. Let A be a $T2S\text{-}QI^d$. Then

$$(A * S] \cap (S * A] \subseteq A.$$

By Lemma 4.2,

$$(A * S * A] \subseteq (A * S] \cap (S * A] \subseteq A.$$

Hence A is a $T2S\text{-}BiI^d$ and therefore, by Proposition 4.1, it is a $T2S\text{-}BiQI^d$.

If A is a $T2S\text{-}BiI^d$, the conclusion follows directly from Proposition 4.1. □

Corollary 4.1. *Every $T2S\text{-}Bi\mathfrak{I}^d$, equivalently every $T2S\text{-}BiQ\mathfrak{I}^d$, is a $T2S\text{-}Q\mathfrak{I}^d$.*

Proof. Let A be a $T2S\text{-}Bi\mathfrak{I}^d$. By Proposition 4.1, this is equivalent to A being a $T2S\text{-}BiQ\mathfrak{I}^d$. By the definition of a $T2S\text{-}Bi\mathfrak{I}^d$,

$$(A * S * A] \cap A \neq \emptyset.$$

Moreover, Lemma 4.2 gives

$$(A * S * A] \subseteq (A * S] \cap (S * A].$$

Therefore,

$$\emptyset \neq (A * S * A] \cap A \subseteq (A * S] \cap (S * A] \cap A.$$

Hence,

$$(A * S] \cap (S * A] \cap A \neq \emptyset.$$

Since A is a lower $T2S$ -subsemiring, it follows that A is a $T2S\text{-}Q\mathfrak{I}^d$. □

Theorem 4.3. *Every $T2S\text{-}Q\mathfrak{I}^d$ is a $T2S\text{-}\mathfrak{I}^d$.*

Proof. Let A be a $T2S-Q\mathfrak{I}^d$. Then

$$(A * S] \cap (S * A] \cap A \neq \emptyset.$$

Consequently,

$$(A * S] \cap A \neq \emptyset$$

and

$$(S * A] \cap A \neq \emptyset.$$

Thus, A is both a $T2S-R\mathfrak{I}^d$ and a $T2S-L\mathfrak{I}^d$. Therefore, A is a $T2S-\mathfrak{I}^d$. $\square$

Theorem 4.4. Every $T2S-BiQ\mathfrak{I}^d$ that is a $T2S$ -subsemiring is a $T2S-BiQ\mathfrak{I}^d$.

Proof. Let A be a $T2S-BiQ\mathfrak{I}^d$ that is a $T2S$ -subsemiring. By Proposition 4.1, A is a $T2S-Bi\mathfrak{I}^d$. Hence

$$(A * S * A] \subseteq A.$$

Since $(A * S * A]$ is nonempty,

$$(A * S * A] \cap A \neq \emptyset.$$

Thus, A is a $T2S-Bi\mathfrak{I}^d$. Another application of Proposition 4.1 shows that A is a $T2S-BiQ\mathfrak{I}^d$. $\square$

The converse of Theorem 4.4 does not hold, as shown in the following example.

Example 4.1. Let

$$S = \{\hbar, \wp, \angle, \times, \mathfrak{O}, \ell\}.$$

Define the operations $\odot$, $\bar{\wedge}$, and $*$ on S by the following tables.

$\odot$	$\hbar$	$\wp$	$\angle$	$\times$	$\mathfrak{O}$	ℓ	$\bar{\wedge}$	$\hbar$	$\wp$	$\angle$	$\times$	$\mathfrak{O}$	ℓ
$\hbar$	$\hbar$	$\wp$	$\angle$	$\times$	$\mathfrak{O}$	ℓ	$\hbar$	$\hbar$	$\wp$	$\angle$	$\times$	$\mathfrak{O}$	ℓ
$\wp$	$\wp$	$\wp$	$\angle$	$\times$	$\mathfrak{O}$	ℓ	$\wp$	$\hbar$	$\wp$	$\angle$	$\times$	$\mathfrak{O}$	ℓ
$\angle$	$\angle$	$\angle$	$\angle$	ℓ	$\mathfrak{O}$	ℓ	$\angle$	$\hbar$	$\wp$	$\angle$	$\times$	$\mathfrak{O}$	ℓ
$\times$	$\times$	$\times$	ℓ	$\times$	$\mathfrak{O}$	ℓ	$\times$	$\hbar$	$\wp$	$\angle$	$\times$	$\mathfrak{O}$	ℓ
$\mathfrak{O}$	$\mathfrak{O}$	$\mathfrak{O}$	$\mathfrak{O}$	$\mathfrak{O}$	$\mathfrak{O}$	$\mathfrak{O}$	$\mathfrak{O}$	$\hbar$	$\wp$	$\angle$	$\times$	$\mathfrak{O}$	ℓ
ℓ	ℓ	ℓ	ℓ	$\mathfrak{O}$	ℓ		ℓ	$\hbar$	$\wp$	$\angle$	$\times$	$\mathfrak{O}$	ℓ

$*$	$\hbar$	$\wp$	$\angle$	$\times$	$\mathfrak{O}$	ℓ
$\hbar$	$\hbar$	$\hbar$	$\hbar$	$\hbar$	$\hbar$	$\hbar$
$\wp$	$\hbar$	$\wp$	$\angle$	$\wp$	$\angle$	$\angle$
$\angle$	$\hbar$	$\wp$	$\angle$	$\wp$	$\angle$	$\angle$
$\times$	$\hbar$	$\times$	$\mathfrak{O}$	$\times$	$\mathfrak{O}$	$\mathfrak{O}$
$\mathfrak{O}$	$\hbar$	$\times$	$\mathfrak{O}$	$\times$	$\mathfrak{O}$	$\mathfrak{O}$
ℓ	$\hbar$	$\times$	$\mathfrak{O}$	$\times$	$\mathfrak{O}$	$\mathfrak{O}$

Let $\preceq$ be the equality relation on S , that is,

$$\preceq = \{(x, x) \mid x \in S\}.$$

It can be verified directly that

$$(S, \odot, \bar{\wedge}, *, \preceq)$$

is an ordered bisemiring.

Let

$$A = \{\bar{\partial}\}.$$

Clearly, A is a T2S-subsemiring and $A = (A]$. Moreover,

$$A * S = \{\bar{h}, \ltimes, \bar{\partial}\},$$

$$S * A = \{\bar{h}, \angle, \bar{\partial}\},$$

$$A * S * A = \{\bar{h}, \bar{\partial}\}.$$

Thus,

$$(A * S * A] \cap A = \{\bar{\partial}\} \neq \emptyset.$$

Hence A is a T2S-Bi $\mathfrak{I}^d$ and therefore, by Proposition 4.1, a T2S-BiQ $\mathfrak{I}^d$.

However,

$$(A * S * A] = \{\bar{h}, \bar{\partial}\} \not\subseteq A.$$

Therefore, A is not a T2S-Bi I^d and hence is not a T2S-BiQ I^d .

Theorem 4.5. Let A and B be nonempty subsets of S such that

$$A \subseteq B \subseteq S.$$

Then:

- (i) if A is a T2S- $\mathfrak{I}^d$ and B is a lower subset of S , then B is a T2S- $\mathfrak{I}^d$;
- (ii) if A is a T2S-Bi $\mathfrak{I}^d$, T2S-Q $\mathfrak{I}^d$, or T2S-BiQ $\mathfrak{I}^d$ and B is a lower T2S-subsemiring of S , then B is an almost ideal of the same type as A .

Proof. Suppose first that A is a T2S- $\mathfrak{I}^d$ and B is a lower subset of S . Since $A \subseteq B$, we have

$$(A * S] \subseteq (B * S]$$

and

$$(S * A] \subseteq (S * B].$$

Therefore,

$$\emptyset \neq (A * S] \cap A \subseteq (B * S] \cap B$$

and

$$\emptyset \neq (S * A] \cap A \subseteq (S * B] \cap B.$$

Hence B is a $\text{T2S-}\mathfrak{J}I^d$.

Suppose that A is a $\text{T2S-Bi}\mathfrak{J}I^d$ and that B is a lower T2S-subsemiring . Then

$$A * S * A \subseteq B * S * B,$$

and hence

$$\emptyset \neq (A * S * A] \cap A \subseteq (B * S * B] \cap B.$$

Thus, B is a $\text{T2S-Bi}\mathfrak{J}I^d$.

The proofs for $\text{T2S-Q}\mathfrak{J}I^d$ and $\text{T2S-BiQ}\mathfrak{J}I^d$ are similar. □

Theorem 4.6. *The following statements hold:*

- (i) *the union of a nonempty family of $\text{T2S-}\mathfrak{J}I^d$ s is a $\text{T2S-}\mathfrak{J}I^d$;*
- (ii) *the union of a nonempty chain of $\text{T2S-Bi}\mathfrak{J}I^d$ s, $\text{T2S-Q}\mathfrak{J}I^d$ s, or $\text{T2S-BiQ}\mathfrak{J}I^d$ s is an almost ideal of the corresponding type.*

Proof. Let $\{A_i\}_{i \in I}$ be a nonempty family of $\text{T2S-}\mathfrak{J}I^d$ s and put

$$A = \bigcup_{i \in I} A_i.$$

Since every A_i is a lower subset of S , the set A is also a lower subset of S . Choose $i_0 \in I$. Since

$$A_{i_0} \subseteq A,$$

Theorem 4.5 implies that A is a $\text{T2S-}\mathfrak{J}I^d$. This proves (i).

For (ii), suppose that $\{A_i\}_{i \in I}$ is a nonempty chain under set inclusion, and put

$$A = \bigcup_{i \in I} A_i.$$

The union A is a lower subset of S . We show that it is a T2S-subsemiring . Let $x, y \in A$. Then $x \in A_i$ and $y \in A_j$ for some $i, j \in I$. Since the family is a chain, either

$$A_i \subseteq A_j \quad \text{or} \quad A_j \subseteq A_i.$$

Thus, x and y belong to one common member of the chain. Since that member is a T2S-subsemiring ,

$$x \bar{\wedge} y \in A \quad \text{and} \quad x * y \in A.$$

Hence A is a T2S-subsemiring .

Choose $i_0 \in I$. Since $A_{i_0} \subseteq A$, the conclusion follows from Theorem 4.5. □

5. SIMULTANEOUS IDEAL STRUCTURES IN ORDERED BISEMIRINGS

In Sections 3 and 4, almost ideals were studied separately in the Type 1 and Type 2 ordered semiring reducts of an ordered bisemiring. In this section, we combine these notions by requiring the corresponding conditions to hold simultaneously in both reducts. This approach provides a unified description of ideals and almost ideals in ordered bisemirings.

Throughout this section, let

$$(S, \odot, \bar{\wedge}, *, \preceq)$$

be an ordered bisemiring.

Definition 5.1. A nonempty subset A of S is called a subbisemiring of S if

$$A \odot A \subseteq A, \quad A \bar{\wedge} A \subseteq A, \quad A * A \subseteq A.$$

A subbisemiring A is called a lower subbisemiring if $A = (A]$.

Definition 5.2. Let A be a nonempty subset of S . Then:

- (i) A is called an ordered ideal of S , briefly an I^d , if A is both a $T1S-I^d$ and a $T2S-I^d$;
- (ii) A is called an ordered bi-ideal of S , briefly a BiI^d , if A is both a $T1S-BiI^d$ and a $T2S-BiI^d$;
- (iii) A is called an ordered quasi-ideal of S , briefly a QI^d , if A is both a $T1S-QI^d$ and a $T2S-QI^d$;
- (iv) A is called an ordered bi-quasi ideal of S , briefly a $BiQI^d$, if A is both a $T1S-BiQI^d$ and a $T2S-BiQI^d$;
- (v) A is called an almost ideal of S , briefly a $\mathfrak{I}^d$, if A is both a $T1S-\mathfrak{I}^d$ and a $T2S-\mathfrak{I}^d$;
- (vi) A is called a bi-almost ideal of S , briefly a $Bi\mathfrak{I}^d$, if A is both a $T1S-Bi\mathfrak{I}^d$ and a $T2S-Bi\mathfrak{I}^d$;
- (vii) A is called a quasi-almost ideal of S , briefly a $Q\mathfrak{I}^d$, if A is both a $T1S-Q\mathfrak{I}^d$ and a $T2S-Q\mathfrak{I}^d$;
- (viii) A is called a bi-quasi-almost ideal of S , briefly a $BiQ\mathfrak{I}^d$, if A is both a $T1S-BiQ\mathfrak{I}^d$ and a $T2S-BiQ\mathfrak{I}^d$.

Remark 5.1. Every $Bi\mathfrak{I}^d$, $Q\mathfrak{I}^d$, and $BiQ\mathfrak{I}^d$ of S is a lower subbisemiring. Indeed, the Type 1 condition gives closure under $\odot$ and $\bar{\wedge}$, while the Type 2 condition gives closure under $\bar{\wedge}$ and $*$.

Proposition 5.1. Let A be a nonempty subset of S . Then:

- (i) A is a $BiQI^d$ if and only if it is a BiI^d ;
- (ii) A is a $BiQ\mathfrak{I}^d$ if and only if it is a $Bi\mathfrak{I}^d$.

Proof. By Proposition 3.1, the notions of $T1S-BiQI^d$ and $T1S-BiI^d$ coincide. Similarly, Proposition 4.1 shows that the notions of $T2S-BiQI^d$ and $T2S-BiI^d$ coincide. Hence (i) follows.

The same two propositions show that $T1S-BiQ\mathfrak{I}^d$ coincides with $T1S-Bi\mathfrak{I}^d$ and that $T2S-BiQ\mathfrak{I}^d$ coincides with $T2S-Bi\mathfrak{I}^d$. Therefore, (ii) follows. $\square$

Remark 5.2. In Example 4.1, let $A = \{\bar{0}\}$. Since

$$A \odot A = A, \quad A \bar{\wedge} A = A, \quad A \bar{\wedge} S \bar{\wedge} A = A,$$

the subset A is a T1S-subsemiring and a T1S-Bi I^d , and hence a T1S-Bi QI^d . Moreover, A is a T1S-Bi $\mathfrak{I}^d$ and therefore a T1S-Bi $\mathfrak{Q}\mathfrak{I}^d$. On the other hand, the calculations in Example 4.1 show that A is a T2S-Bi $\mathfrak{Q}\mathfrak{I}^d$ but not a T2S-Bi QI^d . Consequently, A is a Bi $\mathfrak{Q}\mathfrak{I}^d$ of S , but it is not a Bi QI^d of S .

Theorem 5.1. Let A be a nonempty subset of S . Then:

- (i) if A is an I^d , then A is a QI^d ;
- (ii) if A is a QI^d , then A is a Bi I^d and hence a Bi QI^d ;
- (iii) if A is a subbisemiring and a Bi I^d , then A is a Bi $\mathfrak{I}^d$ and hence a Bi $\mathfrak{Q}\mathfrak{I}^d$;
- (iv) if A is a QI^d , then A is a $\mathfrak{Q}\mathfrak{I}^d$;
- (v) if A is a Bi $\mathfrak{I}^d$, then A is a $\mathfrak{Q}\mathfrak{I}^d$;
- (vi) if A is a $\mathfrak{Q}\mathfrak{I}^d$, then A is a $\mathfrak{I}^d$.

Proof. Suppose that A is an I^d . Since A is a T1S- I^d ,

$$(A \bar{\wedge} S] \subseteq A \quad \text{and} \quad (S \bar{\wedge} A] \subseteq A.$$

Thus,

$$(A \bar{\wedge} S] \cap (S \bar{\wedge} A] \subseteq A,$$

and hence A is a T1S- QI^d .

Similarly, since A is a T2S- I^d ,

$$(A * S] \cap (S * A] \subseteq A.$$

Therefore, A is a T2S- QI^d , and consequently A is a QI^d . This proves (i).

Assertion (ii) follows from Theorems 3.2 and 4.2, together with Proposition 5.1.

Suppose that A is a subbisemiring and a Bi I^d . Then A is both a T1S-subsemiring and a T2S-subsemiring. By Theorems 3.1 and 4.1, A is both a T1S-Bi $\mathfrak{I}^d$ and a T2S-Bi $\mathfrak{I}^d$. Hence A is a Bi $\mathfrak{I}^d$. The remaining assertion in (iii) follows from Proposition 5.1.

Suppose that A is a QI^d . Then A is both a T1S- QI^d and a T2S- QI^d . By Theorems 3.1 and 4.1, A is both a T1S- $\mathfrak{Q}\mathfrak{I}^d$ and a T2S- $\mathfrak{Q}\mathfrak{I}^d$. Hence A is a $\mathfrak{Q}\mathfrak{I}^d$. This proves (iv).

Assertion (v) follows from Corollaries 3.1 and 4.1, while assertion (vi) follows from Theorems 3.3 and 4.3. □

Corollary 5.1. If A is an I^d of S , then A is also a QI^d , a Bi I^d , a Bi QI^d , a Bi $\mathfrak{I}^d$, a Bi $\mathfrak{Q}\mathfrak{I}^d$, a $\mathfrak{Q}\mathfrak{I}^d$, and a $\mathfrak{I}^d$. Moreover,

$$\text{Bi}I^d = \text{Bi}QI^d \quad \text{and} \quad \text{Bi}\mathfrak{I}^d = \text{Bi}\mathfrak{Q}\mathfrak{I}^d.$$

Proof. Let A be an I^d of S . Since A is a T1S- I^d , $(A, \odot)$ is a subsemigroup of $(S, \odot)$, and hence

$$A \odot A \subseteq A.$$

Moreover,

$$A \bar{\wedge} A \subseteq A \bar{\wedge} S \subseteq A.$$

Since A is also a T2S- I^d ,

$$A * A \subseteq A * S \subseteq A.$$

Thus, A is a subbisemiring of S . The conclusion now follows from Theorem 5.1 and Proposition 5.1. $\square$

Corollary 5.2. *Every QI^d of S belongs successively to the classes represented in the following chain:*

$$QI^d \implies BiI^d = BiQI^d \implies Bi\mathfrak{I}I^d = BiQ\mathfrak{I}I^d \implies Q\mathfrak{I}I^d \implies \mathfrak{I}I^d.$$

Proof. Let A be a QI^d of S . Then A is both a T1S- QI^d and a T2S- QI^d . By Proposition 2.1, A is both a T1S-subsemiring and a T2S-subsemiring, and hence it is a subbisemiring of S . The conclusion now follows from Theorem 5.1 and Proposition 5.1. $\square$

Theorem 5.2. *Let A and B be nonempty subsets of S such that*

$$A \subseteq B \subseteq S.$$

Then:

- (i) *if A is a $\mathfrak{I}I^d$ and B is a lower subset of S , then B is a $\mathfrak{I}I^d$;*
- (ii) *if A is a $Bi\mathfrak{I}I^d$, $Q\mathfrak{I}I^d$, or $BiQ\mathfrak{I}I^d$ and B is a lower subbisemiring of S , then B is an almost ideal of the same type as A .*

Proof. Suppose first that A is a $\mathfrak{I}I^d$ and B is a lower subset of S . Since A is both a T1S- $\mathfrak{I}I^d$ and a T2S- $\mathfrak{I}I^d$, Theorems 3.5 and 4.5 imply that B is both a T1S- $\mathfrak{I}I^d$ and a T2S- $\mathfrak{I}I^d$. Hence B is a $\mathfrak{I}I^d$.

For the second assertion, suppose, for example, that A is a $Bi\mathfrak{I}I^d$. Since B is a lower subbisemiring, it is both a lower T1S-subsemiring and a lower T2S-subsemiring. Applying Theorems 3.5 and 4.5, we conclude that B is both a T1S- $Bi\mathfrak{I}I^d$ and a T2S- $Bi\mathfrak{I}I^d$. Therefore, B is a $Bi\mathfrak{I}I^d$.

The proofs for $Q\mathfrak{I}I^d$ and $BiQ\mathfrak{I}I^d$ are similar. $\square$

Theorem 5.3. *The following statements hold:*

- (i) *the union of a nonempty family of $\mathfrak{I}I^d$ s of S is a $\mathfrak{I}I^d$ of S ;*
- (ii) *the union of a nonempty chain of $Bi\mathfrak{I}I^d$ s, $Q\mathfrak{I}I^d$ s, or $BiQ\mathfrak{I}I^d$ s of S is an almost ideal of the corresponding type.*

Proof. Let $\{A_i\}_{i \in I}$ be a nonempty family of $\mathfrak{I}^d$ s. Then every A_i is both a T1S- $\mathfrak{I}^d$ and a T2S- $\mathfrak{I}^d$. By Theorems 3.6 and 4.6,

$$\bigcup_{i \in I} A_i$$

is both a T1S- $\mathfrak{I}^d$ and a T2S- $\mathfrak{I}^d$. Hence it is a $\mathfrak{I}^d$. This proves (i).

For (ii), let $\{A_i\}_{i \in I}$ be a nonempty chain of almost ideals of one of the stated types. By Theorems 3.6 and 4.6, its union is an almost ideal of the corresponding type in both the Type 1 and Type 2 reducts. Therefore, the union is an almost ideal of the corresponding type in S . $\square$

Remark 5.3. By Proposition 5.1, the classes of bi-almost ideals and bi-quasi-almost ideals coincide. Therefore, no strict implication can occur between $\text{Bi}\mathfrak{I}^d$ and $\text{BiQ}\mathfrak{I}^d$ under the present definitions. The same observation applies to $\text{Bi}I^d$ and $\text{BiQ}I^d$.

6. CONCLUSION

In this paper, we investigated several classes of almost ideals in ordered bisemirings through their Type 1 and Type 2 ordered semiring reducts. For each reduct, we considered right almost ideals, left almost ideals, bi-almost ideals, quasi-almost ideals, and bi-quasi-almost ideals. We established basic constructions and relationships among these classes and examined their behavior under passage to larger lower subsets and under unions.

A main observation is that the bi-quasi conditions reduce to the corresponding bi-conditions. More precisely, in both the Type 1 and Type 2 reducts, the notions of bi-quasi ideal and bi-ideal coincide, and the notions of bi-quasi-almost ideal and bi-almost ideal also coincide. Consequently, for simultaneous ideal structures in an ordered bisemiring, we have

$$\text{Bi}I^d = \text{BiQ}I^d$$

and

$$\text{Bi}\mathfrak{I}^d = \text{BiQ}\mathfrak{I}^d.$$

Moreover, every ordered ideal belongs successively to the classes represented in the following chain:

$$I^d \implies \text{Q}I^d \implies \text{Bi}I^d = \text{BiQ}I^d \implies \text{Bi}\mathfrak{I}^d = \text{BiQ}\mathfrak{I}^d \implies \text{Q}\mathfrak{I}^d \implies \mathfrak{I}^d.$$

A finite example shows that a bi-quasi-almost ideal need not be a bi-quasi ideal.

We also proved that arbitrary nonempty unions of almost ideals remain almost ideals. For bi-almost ideals, quasi-almost ideals, and bi-quasi-almost ideals, the corresponding union result holds for nonempty chains, since their definitions include subsemiring closure. These results provide a consistent description of almost ideal structures in the two ordered semiring reducts of an ordered bisemiring. Future work may consider alternative bi-quasi conditions that remain distinct from the corresponding bi-conditions, as well as other generalized ideal structures in ordered bisemirings.

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