

STABILITY OF THE INDEX FOR CLOSED LINEAR RELATIONS VIA THE NUMERICAL RANGE

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ABSTRACT. This paper investigates the stability of the index for closed linear relations (multivalued linear operators) on Hilbert spaces under relatively bounded perturbations. We introduce the numerical range as a novel and powerful tool to analyze the behavior of the nullity, deficiency, and index of a linear relation A when subjected to a perturbation of the form $A - \lambda B$. Our primary contribution is the establishment of a clear criterion for index stability, demonstrating the existence of an open disk $D = \{\lambda \in \mathbb{C} : |\lambda| < r\}$ in the complex plane, centered at the origin, within which the index $\kappa(A - \lambda B)$ remains constant. The radius of this stability disk is explicitly determined by the minimum modulus of A and the constants defining the relative bound of the perturbation B .

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1. INTRODUCTION

The study of how the spectral properties of an operator change under perturbation is a cornerstone of operator theory, with profound implications across various fields of applied mathematics. For Fredholm operators, the stability of the index is a particularly critical property, as it guarantees the robustness of the operator's essential solvability characteristics against minor alterations. While the perturbation theory for single-valued bounded operators is extensively developed, numerous problems arising in areas such as differential equations, control theory, and optimization necessitate the consideration of operators that are unbounded or even multivalued, commonly referred to as linear relations.

This paper focuses on the stability of the index for closed linear relations defined on Banach spaces. Specifically, we examine a closed linear relation A possessing a finite-dimensional null space and a closed range, and we analyze its behavior under a bounded perturbation of the form $A - \lambda B$, where

B is another linear relation and λ is a complex parameter. The central question we address is: under what conditions does the index of the perturbed relation, $\kappa(A - \lambda B)$, maintain its stability as λ varies?

Existing literature has explored various conditions for the stability of the nullity, deficiency, and index for linear relations [4], [6], [7], [9]. However, the application of geometric tools, such as the numerical range, to rigorously analyze these stability properties for linear relations remains an evolving area of research.

In this work, we introduce a novel methodological approach that effectively leverages the numerical range of one linear relation relative to another to establish a precise criterion for index stability. Our main result (Theorem 23) rigorously demonstrates that for all λ within a specific open disk centered at the origin, the index $\kappa(A - \lambda B)$ is constant. The radius of this disk is explicitly quantifiable in terms of the minimum modulus of A and the constants defining the perturbation bound. A significant finding from our analysis is the observation that the index can exhibit stability even in scenarios where the nullity and deficiency are not individually stable. This phenomenon, which is contingent upon the properties of the operator pair (A, B) , provides a more comprehensive and detailed understanding of spectral stability, a perspective uniquely afforded by the theory of linear relations.

The remainder of this paper is structured as follows: Section 1 provides a concise overview of the fundamental definitions and properties pertinent to linear relations. Section 2 is dedicated to the stability of the nullity and deficiency, presenting the foundational theorems underpinning our analysis. Finally, in Section 3, we formally introduce the numerical range for linear relations and employ it to prove our main results concerning the stability of the index.

2. LINEAR RELATIONS

We present a brief overview of the necessary definitions and properties of linear relations. For more details regarding these notions, we refer the reader to the monographs [1], [2], [3], [5] and [11].

2.1. Some basic definitions.

Definition 1. Let X and Y be normed linear spaces. A **linear relation** A from X to Y is a linear subspace of the product space $X \times Y$.

Note that the concept of a linear relation generalizes the notion of the graph of a linear operator. We say that A is closed if it is closed as a subspace of $X \times Y$.

We denote by $LR(X, Y)$ the class of all linear relations from X to Y while $CLR(X, Y)$ will denote the class of all closed linear relations from X to Y . For $A \in LR(X, Y)$ and $(x, y) \in \mathcal{G}$, we define $A(x)$ to be the subset of Y given by $A(x) := \{y : (x, y) \in \mathcal{G}\}$ and A^{-1} to be the linear relation in $LR(Y, X)$ given by $A^{-1} := \{(y, x) : y \in A(x)\}$.

Definition 2. Let A be a linear relation from X to Y . The domain $D(A)$, range $R(A)$, null space $N(A)$, and the multi-valued part $\text{mult}(A)$ of A are defined by:

- (i) $D(A) = \{x \in X \mid (x, y) \in A \text{ for some } y \in Y\},$
- (ii) $R(A) = \{y \in Y \mid (x, y) \text{ for some } x \in X\},$
- (iii) $N(A) = \{x \in X \mid (x, 0) \in A\} = \{x \in X : A(x) = A(0)\},$
- (iv) $\text{mult}(A) = A(0) = \{y \in Y \mid (0, y) \in A\}.$

We note here that a linear relation A is the graph of a single-valued linear operator if and only if its multi-valued part is trivial, that is, $A(0) = \{0\}$. In other words, a linear relation is simply a multivalued linear operator.

2.2. Normed Linear Relations. In order to handle the notion of boundedness for linear relations, we introduce the concept of the quotient norm. Let A be a linear relation from a normed space X to a normed space Y . By A_q we denote the single-valued operator defined on $D(A)$ with values in the quotient space $Y/\overline{A(0)}$ by

$$A_q(x) = [y] \quad \text{where } (x, y) \in A \text{ and } [y] \text{ is the equivalence class of } y \text{ in } Y/A(0).$$

This is well-defined because if (x, y_1) and (x, y_2) are both in A , then $(0, y_1 - y_2) \in A$, which means $y_1 - y_2 \in A(0)$, so $[y_1] = [y_2]$ in the quotient space. The operator A_q will be called the Operator induced by A .

For $A \in LR(X, Y)$ and $x \in D(A)$, we set $\|A(x)\| = \|A_q(x)\|$ and $\|A\| = \|A_q\|$. We say that A is bounded if We say that a linear relation A from X to Y is bounded if

$$\|A_s\| < \infty. \quad (1)$$

Note that $\|A(x)\| = \|A_q(x)\|$ together with $\|A\| = \|A_q\|$ do not define a norm since nonzero relations can have zero norm.

Let M and N be nonempty subsets of a normed linear space X . We denote by $\text{dist}(M, N)$ the distance between M and N , that is

$$\text{dist}(M, N) := \inf\{\|m - n\| : m \in M, n \in N\}. \quad (2)$$

Similarly, we denote by $\text{dist}(m, N)$ the distance between $m \in M$ and N , that is,

$$\text{dist}(m, N) := \inf\{\|m - n\| : n \in N\}. \quad (3)$$

Lemma 3. [5, II.1.4] Let $A \in LR(X, Y)$.

- (a) If $x \in D(A)$ and $y_1, y_2 \in A(x)$ then $\text{dist}(y_1, A(0)) = \text{dist}(y_2, A(0))$,
- (b) For all $x \in D(A)$

$$\inf_{y \in A(x)} \text{dist}(y, A(0)) = \sup_{y \in A(x)} \text{dist}(y, A(0))$$

(c) $\|A_q(x)\| = \text{dist}(y, A(0))$ for every $y \in A(x)$ and $x \in D(A)$.

(d) $\|A_q(x)\| = \text{dist}(A(x), A(0)) = \text{dist}(\mathcal{A}(x), 0)$, $x \in D(A)$.

Lemma 4. [5, II.1.5] For $A, B \in LR(X, Y)$ and $\alpha \in \mathbb{F}$, we have

(a) $\|A_q(x) + B_q(x)\| \leq \|A_q(x)\| + \|B_q(x)\|$ for $x \in D(A + B)$,

(b) For all $x \in D(A)$, we have $\|\alpha A_q(x)\| = |\alpha| \|A_q(x)\|$.

Lemma 5. Let $A, B \in LR(X)$ be such that $D(B) \supset D(A)$ and $B(0) \subset A(0)$. Then for all $x \in D(A)$

$$\|A_q(x)\| - \|B_q(x)\| \leq \|A_q(x) - B_q(x)\| \quad (4)$$

Proof. Let $x \in D(A)$ and let y and z be such that $y \in A(x)$ and $z \in B(x)$. Then

$$\begin{aligned} \text{dist}(y, A(0)) &= \text{dist}(y - z + z, A(0)) \\ &\leq \text{dist}(y - z, A(0)) + \text{dist}(z, A(0)) \\ &= \text{dist}(y - z, A(0) - B(0)) + \text{dist}(z, A(0)) \\ &\leq \text{dist}(y - z, A(0) - B(0)) + \text{dist}(z, B(0)), \end{aligned}$$

so that

$$\text{dist}(y, A(0)) - \text{dist}(z, B(0)) \leq \text{dist}(y - z, A(0) - B(0)).$$

The inequality

$$\|A_q(x)\| - \|B_q(x)\| \leq \|A_q(x) - B_q(x)\|$$

then follows from Cross II.1.4(c). □

Let $A \in LR(X, Y)$. The minimum modulus of A is the quantity $\gamma(A)$ defined by

$$\gamma(A) := \sup\{\lambda : \|A_q(x)\| \geq \lambda d(x, N(A)) \text{ for } x \in D(A)\}.$$

A useful equivalent formulation is the following:

$$\gamma(A) = \begin{cases} \infty, & \text{if } D(A) \subset \overline{N(A)} \\ \inf \left\{ \frac{\|A_q(x)\|}{d(x, N(A))} : x \in D(A), x \notin \overline{N(A)} \right\} & \text{otherwise.} \end{cases} \quad (5)$$

Lemma 6. [6, Lemma 15] If A is a closed linear relation then $R(A)$ is closed if and only if $\gamma(A) > 0$.

2.3. The number $\nu(A : B)$. Let X and Y be two linear spaces and let $A, B \in LR(X, Y)$ with $B(0) \subset A(0)$. For $n \in \mathbb{N}$, let M_n and N_n be the linear subsets of X defined by

$$M_0 = X, M_n = B^{-1}(A(M_{n-1})) \text{ for } n = 1, 2, \dots, \text{ and} \quad (6)$$

$$N_1 = A^{-1}(0), N_n = A^{-1}(B(N_{n-1})) \text{ for } n = 2, 3, \dots. \quad (7)$$

It can be shown [6] that

$$M_0 \supset M_1 \supset M_2 \supset \dots \supset N(B)$$

and

$$N_1 \subset N_2 \subset N_3 \subset \cdots \subset D(A).$$

Note that $N_1 = N(A)$ and if $N_1 \subset M_n$ then $N_1 \subset M_{n'}$ for all $n' < n$. We denote by $\nu(A : B)$ the smallest number n for which the condition $N_1 \subset M_n$ is not satisfied. We set $\nu(A : B) = \infty$ if there is no such n .

3. STABILITY OF THE NULLITY AND DEFICIENCY.

For a linear relation $A \in LR(X, Y)$, its nullity $\alpha(A)$, deficiency $\beta(A)$, and index $\kappa(A)$ are the quantities

$$\alpha(A) = \dim N(A),$$

$$\beta(A) = \text{codim}(Y/R(A)), \text{ and}$$

$$\kappa(A) = \alpha(A) - \beta(A).$$

The index $\kappa(A)$ is considered well-defined if at least one of $\alpha(A)$ or $\beta(A)$ is finite, otherwise, A is said to have no index.

Remark 7. For the rest of this section, we fix Banach spaces X and Y and consider linear relations $A, B \in LR(X, Y)$ with the property that A has closed range, $D(B) \supset D(A)$, $B(0) \subset A(0)$, and

$$\|B_q(x)\| \leq \sigma\|x\| + \tau\|A_q(x)\| \quad \text{for every } x \in D(A), \quad (8)$$

where σ and τ are non negative constants.

Theorem 8. [6, Theorem 34] Let X and Y be Banach spaces and let $A, B \in CLR(X, Y)$ be linear relations satisfy all the conditions given in Remark 7. If σ and τ are such that

$$\sigma + \tau\gamma(A) < \gamma(A), \quad (9)$$

then the linear relation $A + B$ is closed and has closed range. If in addition $\alpha(A) < \infty$ then

$$\alpha(A + B) \leq \alpha(A), \text{ and } \beta(A + B) \leq \beta(A). \quad (10)$$

Theorem 9. [6, Theorem 38] Let X and Y be Banach spaces and let $A, B \in CLR(X, Y)$ be linear relations satisfy all the conditions given in Remark 7. If $\nu(A : B) = \infty$, there exists a constant $\rho > 0$ such that $\alpha(A - \lambda B) = \alpha(A)$ and $\beta(A - \lambda B) = \beta(A)$ for all λ for which $|\lambda| < \rho$.

Unlike the stability of both the nullity and deficiency of A shown in Theorem 9, the following theorem proved in [7] establishes some sort of non-stability of both the nullity and deficiency inside the region $0 < |z| \leq \rho$.

Theorem 10. [7] Let X and Y be Banach spaces and let $A, B \in LR(X, Y)$ be linear relations satisfy all the conditions given in Remark 7. Assume further that A is closed and is such that $\alpha(A) < \infty$. If $\nu(A : B) < \infty$ then exists a constant $\rho > 0$ and a positive integer r such that, for $0 < |\lambda| < \rho$, $A - \lambda B \in CLR(X, Y)$ and has closed range, and

$$\alpha(A - \lambda B) = \alpha(A) - r, \quad \beta(A - \lambda B) = \beta(A) - r \quad \text{and} \quad (11)$$

$$\nu(A - \lambda B : B) = \infty. \quad (12)$$

Lemma 11. Let X and Y be Banach spaces and let $A, B \in CLR(X, Y)$ be linear relations satisfy all the conditions given in Remark 7. Then for

$$|\lambda| < \frac{\gamma(A)}{\sigma + \tau\gamma(A)}, \quad (13)$$

$A - \lambda B$ is a closed linear relation with a closed range.

Proof. That $A - \lambda B$ is a closed linear relation with a closed range for any λ satisfying the condition in (13) is a direct consequence of Theorem 8. To see this, note that for λ satisfying (13), we can write $A - \lambda B = A + (-\lambda B)$. We see from (8) that the perturbation $-\lambda B$ satisfies the bound

$$\|(-\lambda B)_q(x)\| = |\lambda| \|B_q(x)\| \leq |\lambda| (\sigma \|x\| + \tau \|A_q(x)\|) \quad (14)$$

for all $x \in D(A)$. Setting $\sigma' = |\lambda| \sigma$ and $\tau' = |\lambda| \tau$, we need to verify that $\sigma' + \tau'\gamma(A) < \gamma(A)$. This follows from

$$\sigma' + \tau'\gamma(A) = |\lambda| (\sigma + \tau\gamma(A)) \quad (15)$$

$$< \frac{\gamma(A)}{\sigma + \tau\gamma(A)} \cdot (\sigma + \tau\gamma(A)) \quad (16)$$

$$= \gamma(A). \quad (17)$$

By Theorem 8, $A - \lambda B = \lambda A + (-\lambda B)$ is closed with closed range. $\square$

Lemma 12. Let X and Y be Banach spaces and let $A, B \in CLR(X, Y)$ be linear relations satisfy all the conditions given in Remark 7. The inequalities

$$(1 - \tau |\lambda_0|) \|B_q(x)\| \leq \sigma \|x\| + \tau \|(A - \lambda_0 B)_q x\|. \quad (18)$$

and

$$\|B_q(x)\| \leq \frac{\sigma}{(1 - \tau |\lambda_0|)} \|x\| + \frac{\tau}{(1 - \tau |\lambda_0|)} \|(A - \lambda_0 B)_q x\|. \quad (19)$$

hold for any λ_0 satisfying

$$|\lambda_0| < \frac{\gamma(A)}{\sigma + \tau\gamma(A)}. \quad (20)$$

Proof. For λ_0 satisfying (20), the perturbation bound (8) implies that

$$\begin{aligned} (1 - \tau|\lambda_0|)\|B_q(x)\| &= \|B_q(x)\| - \tau|\lambda_0|\|B_q(x)\| \\ &\leq \sigma\|x\| + \tau\|A_q(x)\| - \tau|\lambda_0|\|B_q(x)\| \\ &\leq \sigma\|x\| + \tau\|(A - \lambda_0 B)_q(x)\| \quad \text{by Lemma 5,} \end{aligned} \quad (21)$$

which proves (18). Since $(1 - \tau|\lambda_0|) > 0$, inequality (19) is obtained by dividing both sides of (18) by $(1 - \tau|\lambda_0|)$. That $(1 - \tau|\lambda_0|) > 0$ follows from (20) since that inequality implies that $|\lambda_0|(\sigma + \tau\gamma(A)) < \gamma(A)$, which is the same as $|\lambda_0|\sigma < \gamma(A) - \tau|\lambda_0|\gamma(A)$. Consequently, $|\lambda_0|\sigma < \gamma(A)(1 - \tau|\lambda_0|)$. Since $|\lambda_0|\sigma > 0$, we have that $\gamma(A)(1 - \tau|\lambda_0|) > 0$ and by Lemma 6 ($\gamma(A) > 0$), we conclude that $1 - \tau|\lambda_0| > 0$. $\square$

Lemma 13. Let X and Y be Banach spaces and let $A, B \in CLR(X, Y)$ be linear relations satisfy all the conditions given in Remark 7. For any λ_0 satisfying $|\lambda_0| < \frac{\gamma(A)}{\sigma + \tau\gamma(A)}$, there exists a constant $\rho > 0$ such that $\alpha(A - \lambda B)$ and $\beta(A - \lambda B)$ are constant for all λ satisfying $|\lambda - \lambda_0| < \rho$.

Proof. Setting $A_0 := A - \lambda_0 B$, we see from the refined bound in (19) that

$$\|B_q(x)\| \leq \sigma' \|x\| + \tau' \|(A_0)_q(x)\| \quad \text{where } \sigma' = \frac{\sigma}{1 - \tau|\lambda_0|}, \quad \tau' = \frac{\tau}{1 - \tau|\lambda_0|}.$$

Moreover, $D(B) \supseteq D(A) = D(A_0)$ and $B(0) \subseteq A(0) \subseteq A_0(0)$. Lemma 11 therefore implies that A_0 is closed and has a closed range. We conclude, using Theorem 10 that $\alpha(A_0) < \infty$ and that $\nu(A_0 : B) = \infty$. It then follows from Theorem 9 that there exists a constant $\tilde{\rho} > 0$ such that both $\alpha(A_0 - \tilde{\lambda}B)$ and $\beta(A_0 - \tilde{\lambda}B)$ are constant for all $\tilde{\lambda}$ with $|\tilde{\lambda}| < \tilde{\rho}$.

Since $A_0 - \tilde{\lambda}B = A - (\lambda_0 + \tilde{\lambda})B = A - \lambda B$ where $\lambda = \lambda_0 + \tilde{\lambda}$, we are able to conclude that there exists a neighborhood $B_\rho(\lambda_0) = \{\lambda \in \mathbb{C} : |\lambda_0 - \lambda| < \rho\}$ of λ_0 on which $\alpha(A - \lambda B)$ and $\beta(A - \lambda B)$ are constant. $\square$

We conclude this section with the following theorem.

Theorem 14. Let X and Y be Banach spaces and let $A, B \in CLR(X, Y)$ be such that A has a closed range, $\alpha(A) < \infty$, $D(B) \supset D(A)$, $B(0) \subset A(0)$, and

$$\|B_q(x)\| \leq \sigma\|x\| + \tau\|A_q(x)\| \quad x \in D(A) \quad (22)$$

holds for some nonnegative constants σ and τ . Then for

$$|\lambda| < \frac{\gamma(A)}{\sigma + \tau\gamma(A)}, \quad (23)$$

$A - \lambda B$ is a closed linear relation with closed range, and both $\alpha(A - \lambda B)$ and $\beta(A - \lambda B)$ are constant for all values of λ for which (23) holds.

Proof. That $A - \lambda B$ is a closed linear relation with closed range for λ satisfying (23) is proved in Theorem 8. By Lemma 13, for every point λ_0 in the disk $D = \{\lambda \in \mathbb{C} : |\lambda| < \frac{\gamma(A)}{\sigma + \tau\gamma(A)}\}$, there exists a neighborhood $B_{r_{\lambda_0}}(\lambda_0)$ of λ_0 in which $\alpha(A - \lambda B)$ and $\beta(A - \lambda B)$ are constant.

Consider any two points $\lambda_1, \lambda_2 \in D$. Since D is a connected open set in $\mathbb{C}$ and therefore path-connected, there exists a continuous path $\gamma(t)$ for $t \in [0, 1]$ with $\gamma(0) = \lambda_1$ and $\gamma(1) = \lambda_2$, entirely contained in D . The image $\gamma([0, 1])$ is a compact subset of D and therefore, we can cover $\gamma([0, 1])$ with a finite number of neighborhoods from the collection $\{B_{r_{\lambda_0}}(\lambda_0) : \lambda_0 \in D\}$, which can be chosen in such a way that they overlap.

Within each neighborhood, $\alpha(A - \lambda B)$ and $\beta(A - \lambda B)$ are constant and since the neighborhoods overlap, these values must match. Therefore, $\alpha(A - \lambda B)$ and $\beta(A - \lambda B)$ have the same value at λ_1 and λ_2 . Since λ_1 and λ_2 were arbitrary, the functions are constant throughout D , except at a countable set of isolated exceptional points. $\square$

4. THE NUMERICAL RANGE AND STABILITY OF INDEX

The theory of the numerical range for linear operators on a Hilbert space H is a very well known concept and is well established in the literature. However this is not the case for the numerical range for linear relations. The papers [8], [10] and [11] are among the few that deal with this concept.

For linear relations A and B on a Hilbert space H , we define the numerical range $\Theta_B(A)$ of A relative to B to be the subset of $\mathbb{C}$ given by

$$\Theta_B(A) = \{\langle v, w \rangle : v \in A(u), w \in B(u) \text{ with } \|w\| = 1 \text{ \& } u \in D(A) \cap D(B)\}. \quad (24)$$

If B is the identity relation I on H , then $\Theta_I(A)$ is simply the classical numerical range $\Theta(A)$ of A given by

$$\Theta(A) = \{\langle v, u \rangle : v \in A(u), u \in D(A) \text{ with } \|u\| = 1\}. \quad (25)$$

A linear relation B on a Hilbert space H is said to be injective if B^{-1} is single valued, that is, $N(B) = \{0\}$.

The following result establishes the convexity of $\Theta_B(A)$.

Theorem 15. [8] *Let A and B be linear relations on a Hilbert space H . If B is injective then the numerical range of A relative to B , $\Theta_B(A)$ is convex.*

Let A and B be linear relations on a Hilbert space H with B injective. We use the convexity of the numerical range $\Theta_B(A)$ of A relative to B , to discuss the effect of $-\lambda B$ on the nullity, the deficiency, and the index of A when perturbed by $-\lambda B$ for some $\lambda \in \mathbb{C}$. We recall that the null space $N(A)$ of A is defined by

$$N(A) := \{x \in D(A) : A(x) = A(0)\}. \quad (26)$$

This is a consequence of the fact that

$$N(A) := \{x \in D(A) : 0 \in A(x)\}. \quad (27)$$

We denote by $N(A)^\perp$ the orthogonal complement of the null space of A .

Let $A \in CLR(H)$. Recall from (5) that the lower bound/minimum modulus $\gamma(A)$ of A is the quantity defined by

$$\gamma(A) =: \sup\{\lambda : \|A(x)\| \geq \lambda \operatorname{dist}(x, N(A)) \text{ for all } x \in D(A)\}$$

that is, $\gamma(A)$ is the greatest number λ for which the inequality

$$\gamma \operatorname{dist}(x, N(A)) \leq \|A(x)\| \text{ for every } x \in D(A)$$

is satisfied. Note that $\gamma(A) = \infty$ if $x \in N(A)$. Therefore

$$\gamma(A) = \inf \left\{ \frac{\|A(x)\|}{\operatorname{dist}(x, N(A))}, x \in D(A) \right\}$$

and

$$\gamma(A|_{N(A)^\perp}) = \inf \left\{ \frac{\|A(x)\|}{\|P_A(x)\|}, x \in D(A) \right\}, \quad (28)$$

where P_A denotes the projection $P_A : H \rightarrow N(A)^\perp$ of H onto the orthogonal complement of $N(A)$ and we set $\frac{0}{0} = \infty$.

Remark 16. Let H be a Hilbert space and let $A \in CLR(H)$. Then $N(A)$ is a closed subspace of H in the sense that for any convergent sequence $(u_n)_{n=0}^\infty \in N(A)$ converging to $u \in H$, $u \in D(A)$ and $0 \in A(u)$. This follows from the fact that for each $n \in \mathbb{N}$, $(u_n, 0)$ belongs to the graph of A and since the sequence $(u_n, 0)$ converges to $(u, 0)$ and the graph of A is a closed subspace of $H \times H$, it follows that $(u, 0)$ belongs to the graph of A and that $0 \in A(u)$.

If H is a Hilbert space and $x \in H$, then by Remark 16 we have that $x = x_1 + x_2$ for $x_1 \in N(A)^\perp$ and $x_2 \in N(A)$. Therefore we have that

$$\begin{aligned} \|x\|^2 &= \langle x_1 + x_2, x_1 + x_2 \rangle \\ &= \langle x_1, x_1 \rangle + \langle x_1, x_2 \rangle + \langle x_2, x_1 \rangle + \langle x_2, x_2 \rangle \\ &= \|x_1\|^2 + 0 + 0 + \|x_2\|^2 = \|x_1\|^2 + \|x_2\|^2 \end{aligned} \quad (29)$$

and for $x \in D(A)$,

$$\begin{aligned} \operatorname{dist}(x, N(A)) &= \inf\{\|x - n\|, n \in N(A)\} \\ &= \inf\left\{\left(\|x_1\|^2 + \|x_2 - n\|^2\right)^{\frac{1}{2}}, n \in N(A)\right\} \\ &= \left(\|x_1\|^2\right)^{\frac{1}{2}} = \|x_1\|. \end{aligned} \quad (30)$$

We see from (30) that $\|P_A(x)\| = \|P_A(x_1 + n)\| = \|P_A(x_1)\| = \|x_1\| = \text{dist}(x, N(A))$. In other words, if H is a Hilbert space and $A \in CLR(H)$, then

$$\gamma(A) = \gamma(A|_{N(A)^\perp}). \quad (31)$$

Lemma 17. *Let A be a linear relation on a Hilbert space H with null space $N(A)$ and let P_A be the projection of H onto $N(A)^\perp$ defined by $P_A(x) \rightarrow x_1$ for $x = x_1 + x_2$ with $x_1 \in N(A)^\perp$ and $x_2 \in N(A)$. Then $A(x) = AP_A(x)$ for $x \in D(A)$. Moreover $\gamma(A)\|P_A(x)\| \leq \|z\|$ for $z \in A(x)$.*

Proof. We show that $A(x) \subset AP_A(x)$ and $A(x) \supset AP_A(x)$. Let $x \in D(A)$ be decomposed as $x = x_1 + x_2$ for $x_1 \in N(A)^\perp$ and $x_2 \in N(A)$. Suppose that $y \in A(x)$. Then

$$y \in A(x_1 + x_2) = A(x_1) + A(x_2) = A(x_1) + A(0) = A(x_1) = AP_A(x).$$

This implies that $y \in AP_A(x)$ hence $A(x) \subset AP_A(x)$.

Conversely, suppose $y \in AP_A(x)$, then

$$y \in A(x_1) = A(x_1) + A(0) = A(x_1) + A(x_2) = A(x_1 + x_2) = A(x).$$

This shows that $y \in A(x)$ meaning that $A(x) \supset AP_A(x)$. We conclude that $A(x) = AP_A(x)$. We also see from (28) and (31) that $\gamma(A)\|P_A(x)\| \leq \|z\|$ for $z \in A(x)$. $\square$

corollary 18. *Let $A \in CLR(H)$ with $\alpha(A) < \infty$ and let P_A be the projection $P_A : H \rightarrow N(A)^\perp$. Then $\alpha(AP_A) < \infty$.*

Remark 19. *Let H be a Hilbert space and let $B \in CLR(H)$. Assume that B has a closed range. If A is a bounded linear operator defined on the whole of H then $BA \in CLR(H)$. To show that this is the case, we show that if $\{x_n\}$ is a sequence in $D(BA)$ such that $x_n \rightarrow x$ and if $\{y_n\}$ is a sequence in $R(BA)$ with $y_n \in BA(x_n)$ for each $n \in \mathbb{N}$ is such that $y_n \rightarrow y$, then $x \in D(BA)$ and $y \in BA(x)$. Since $\{x_n\}$ is a sequence in $D(BA)$ such that $x_n \rightarrow x$ and A is a bounded linear operator, it follows that $\{Ax_n\}$ is a sequence in $D(B)$ such that $Ax_n \rightarrow Ax$. Since B is a closed linear relation and $y_n \in B(Ax_n)$ for each $n \in \mathbb{N}$, it follows that $Ax \in D(B)$ and $y \in B(Ax)$, that is $x \in D(BA)$ and $y \in BA(x)$.*

Lemma 20. *Let $A, B \in LR(H)$, $\xi \in \mathbb{C}$, and let P_A denote the projection of H onto $N(A)^\perp$. If $N((B - \xi A)P_A) \subset N(A)$ then the linear relation $(B - \xi A)P_A$ is one to one and equals $B - \xi A$ on $N(A)^\perp$.*

Proof. That $(B - \xi A)P_A = B - \xi A$ on $N(A)^\perp$ is clear. To show injectivity on $N(A)^\perp$, we let $x_1, x_2 \in N(A)^\perp$ and show that $0 \in (B - \xi A)P_A(x_1 - x_2)$ implies that $x_1 = x_2$. But $0 \in (B - \xi A)P_A(x_1 - x_2)$ implies that $(x_1 - x_2) \in N((B - \xi A)P_A) \subset N(A)$. Since $(x_1 - x_2) \in N(A)$ and $(x_1 - x_2) \in N(A)^\perp$, it follows that $\langle (x_1 - x_2), (x_1 - x_2) \rangle = 0$ and that $x_1 - x_2 = 0$. Hence $x_1 = x_2$ and $(B - \xi A)P_A$ is injective on $N(A)^\perp$. $\square$

Lemma 21. Let H be a Hilbert space, let $A \in CLR(H)$, and let P_A be the projection $P_A : H \rightarrow N(A)^\perp$. Then $\kappa(P_A) = 0$.

Proof. Since P_A is the projection onto $N(A)^\perp$, we have $N(P_A) = N(A)$ and $R(P_A) = N(A)^\perp$. From the direct sum decomposition $H = N(A) \oplus N(A)^\perp$, the quotient space $H/N(A)^\perp$ is isomorphic to $N(A)$. Therefore, $\text{codim } N(A)^\perp = \dim(H/N(A)^\perp) = \dim N(A)$. Substituting this into the index formula gives

$$\kappa(P_A) = \dim N(P_A) - \text{codim } R(P_A) = \dim N(A) - \dim N(A) = 0.$$

□

Theorem 22. [4, Corollary 3.2] Let X, Y, Z be vector spaces, let $A \in LR(X, Y)$, $B \in LR(Y, Z)$, and suppose that A and B have finite indices. Then BA has a finite index and $\kappa(BA) = \kappa(B) + \kappa(A) - \dim A(0) \cap B^{-1}(0)$.

The following theorem, which is our main result is proved using the concept of the numerical range. In particular, we use the convexity of the numerical range to prove stability of the index for $B - \xi A$ for all ξ outside of the numerical range. This is done with the help of Theorem 22 above.

Theorem 23. Let $A, B \in CLR(H)$ be such that $\alpha(A) < \infty$, A has closed range, is injective, $D(B) \supset D(A)$, $B(0) \subset A(0)$, and

$$\|B(x)\| \leq \sigma\|x\| + \tau\|A(x)\| \text{ for every } x \in D(A), \quad (32)$$

where σ and τ are nonnegative constants. If $\overline{\Theta_A(BP_A)} \supset \left\{ \lambda : |\lambda| \leq \frac{\tau\gamma(A)+\sigma}{\gamma(A)} \right\}$, then $B - \xi A$ is closed, has closed range and $\alpha(B - \xi A) < \infty$ for all $\xi \notin \overline{\Theta_A(BP_A)}$. Moreover, for such ξ

$$\alpha(B - \xi A) \leq \alpha(A), \quad \beta(B - \xi A) \leq \beta(A) \text{ and}$$

$$\kappa(B - \xi A) = \kappa(A).$$

Proof. Assume that $\overline{\Theta_A(BP_A)} \supset \left\{ \lambda : |\lambda| \leq \frac{\tau\gamma(A)+\sigma}{\gamma(A)} \right\}$. If $\xi \notin \overline{\Theta_A(BP_A)}$, then $|\xi| > \frac{\tau\gamma(A)+\sigma}{\gamma(A)}$ and therefore $\frac{1}{|\xi|} \leq \frac{\gamma(A)}{\tau\gamma(A)+\sigma}$. That $A - \frac{1}{\xi}B$ (and hence $B - \xi A$) is closed and has closed range follows from Theorem 14. Since $AP_A = A$, it follows from (32) that $\|BP_A(x)\| \leq \sigma\|P_A(x)\| + \tau\|A(x)\|$ for every $x \in D(A)$. Since BP_A is closed by Remark 19 and $AP_A = A$, Theorem 14 implies that $\kappa(B - \xi A)P_A = \kappa(A)$ for any complex number $\xi \notin \overline{\Theta_A(BP_A)}$.

If $\xi \in \overline{\Theta_A(BP_A)}$, then there exists $\delta > 0$ such that

$$\delta = \text{dist}(\xi, \overline{\Theta_A(BP_A)}) = \inf \left\{ |\mu - \xi| : \mu \in \overline{\Theta_A(BP_A)} \right\} > 0. \quad (33)$$

The fact that $\mu \in \overline{\Theta_A(BP_A)}$ implies that there exists $x \in D(A) = D(B - \xi A) = D(B - \xi A)P_A$ such that μ is of the form $\langle y_1, y_2 \rangle$ for some $y_1 \in BP_A(x)$ and some $y_2 \in A(x)$ with $\|y_2\| = 1$. From (33), we see that

$$|\mu - \xi| = |\langle y_1, y_2 \rangle - \xi| \geq \delta > 0 \quad (34)$$

It follows that

$$\begin{aligned} |\langle y_1, y_2 \rangle - \xi| &= |\langle y_1, y_2 \rangle - \xi \langle y_2, y_2 \rangle| = |\langle (y_1 - \xi y_2), y_2 \rangle| \\ &\leq \|y_1 - \xi y_2\| \|y_2\| = \|y_1 - \xi y_2\|. \end{aligned} \quad (35)$$

From inequalities (34) and (35) we see that

$$\|y_1 - \xi y_2\| \geq \delta \|y_2\|, \quad (36)$$

for $y_1 \in BP_A(x)$ and $y_2 \in AP_A(x)$ for which $\|y_2\| = 1$ and $x \in D(A)$.

Let z be any other element in $D((B - \xi A)P_A)$ and let $\hat{y}_1 \in BP_A(z) = A(z)$ and $\hat{y}_2 \in AP_A(z)$. Then

$$\frac{\hat{y}_1}{\|\hat{y}_2\|} \in BP_A\left(\frac{z}{\|\hat{y}_2\|}\right) \text{ and } \frac{\hat{y}_2}{\|\hat{y}_2\|} \in AP_A\left(\frac{z}{\|\hat{y}_2\|}\right).$$

If we set $y_1 = \frac{\hat{y}_1}{\|\hat{y}_2\|}$ and $y_2 = \frac{\hat{y}_2}{\|\hat{y}_2\|}$, then (36) implies that $\|\frac{\hat{y}_1}{\|\hat{y}_2\|} - \xi \frac{\hat{y}_2}{\|\hat{y}_2\|}\| \geq \delta$ which is the same as

$$\|\hat{y}_1 - \xi \hat{y}_2\| \geq \delta \|\hat{y}_2\|. \quad (37)$$

In order to apply Theorem 14 to arrive at our conclusion, we show that $\alpha(B - \xi A)P_A$ is finite and that $(B - \xi A)P_A$ is closed and has closed range for $\xi \notin \overline{\Theta_A(BP_A)}$.

To show that $\alpha(B - \xi A)P_A$ is finite, suppose that $x' \in N((B - \xi A)P_A)$. Then $0 \in (B - \xi A)P_A(x')$. But elements of $(B - \xi A)P_A(x')$ are of the form $v_1 - \xi v_2$ where $v_1 \in BP_A(x')$ and $v_2 \in AP_A(x')$. Having $0 \in (B - \xi A)P_A(x')$ means that there is at least one element in $(B - \xi A)P_A(x')$ such that $v_1 - \xi v_2 = 0$. By inequality (37), we have that $\|v_1 - \xi v_2\| \geq \delta \|v_2\|$ which implies that $0 \geq \delta \|v_2\|$. But we know that $\delta > 0$ and $\|v_2\| \geq 0$ hence $0 = v_2 \in AP_A(x')$. Therefore we have that $N((B - \xi A)P_A) \subset N(AP_A) = N(A)$ and hence

$$\alpha((B - \xi A)P_A) \leq \alpha(AP_A) = \alpha(A). \quad (38)$$

It therefore follows from (38) that $\alpha((B - \xi A)P_A) < \infty$ for $\xi \notin \overline{\Theta_A(BP_A)}$.

Next we show that $(B - \xi A)P_A$ is closed and has a closed range. We know that P_A is bounded and $B - \xi A$ is closed. Therefore by Remark 19, we have that $(B - \xi A)P_A$ is closed.

To show that $(B - \xi A)P_A$ has closed range, let y_n be a convergent sequence in $R[(B - \xi A)P_A]$ such that $y_n \rightarrow y'$ where

$$y_n = y'_n - \xi y''_n \in (B - \xi A)P_A(x_n) \quad (39)$$

with $x_n \in D[(B - \xi A)P_A]$ for $n \in \mathbb{N}$. It is enough to show that $y' \in R[(B - \xi A)P_A]$. Since $\xi \notin \overline{\Theta_A(BP_A)}$ then for some $\delta > 0$, the inequality (37) implies that

$$\delta \|y''_n\| \leq \|y'_n - \xi y''_n\| \quad (40)$$

for $y'_n \in BP_A(x_n)$, $y''_n \in AP_A(x_n)$. For $n, m \in \mathbb{N}$ there exists $x_n, x_m \in D(B - \xi A)P_A$ such that $y'_n \in BP_A(x_n)$, $y'_m \in BP_A(x_m)$ and $y''_n \in AP_A(x_n)$, $y''_m \in AP_A(x_m)$. This implies that $(y'_n - y'_m) \in$

$BP_A(x_n) - BP_A(x_m) = BP_A(x_n - x_m)$ and $(y_n'' - y_m'') \in AP_A(x_n) - AP_A(x_m) = AP_A(x_n - x_m)$. Applying (40) and (39) we have that

$$\delta \|y_n'' - y_m''\| \leq \|(y_n' - y_m') - \xi(y_n'' - y_m'')\| = \|y_n - y_m\|. \quad (41)$$

From Lemma 17, we have that

$$\gamma(A) \|P_A x_n\| \leq \|y_n''\| \text{ for } y_n'' \in AP_A(x_n) = A(x_n).$$

This means that

$$\gamma(A) \|P_A(x_n) - P_A(x_m)\| \leq \|y_n'' - y_m''\|. \quad (42)$$

Inequality 41 together with (42) and the convergence of the sequence $(y_n)_{n=0}^\infty$ imply that $\{P_A(x_n)\}_{n=0}^\infty$ is a Cauchy sequence in H and therefore converges to a point of H , say $P_A(x_n) \rightarrow y$. Since $\{P_A(x_n)\}_{n=0}^\infty$ is a sequence in $N(A)^\perp$ and $N(A)^\perp$ is closed, it follows that $y \in N(A)^\perp$ and therefore $y = P_A(x)$ for some $x \in H$. The closedness of $B - \xi A$ implies that $P_A(x) = y \in D(B - \xi A)$ and $y' \in (B - \xi A)P_A(x)$ so that the range of $(B - \xi A)P_A$ is closed when $\xi \notin \overline{\Theta_A(BP_A)}$.

Since $\overline{\Theta_A(BP_A)}$ is a convex subset of the complex plane, its complement is connected. Theorem 14 implies that

$$\kappa((B - \xi A)P_A) = \kappa(A) \quad (43)$$

for every $\xi \notin \overline{\Theta_A(BP_A)}$ because the index $\kappa(\xi) = \kappa(B - \xi A)P_A$ of $(B - \xi A)P_A$ is a constant function on connected subsets of the plane for which $(B - \xi A)P_A$ is closed, has closed range and $(B - \xi A)P_A < \infty$.

Since $N((B - \xi A)P_A) \subset N(A)$ by (37), it follows from Lemma 20 that $(B - \xi A)P_A$ is injective and equal to $B - \xi A$ on $N(A)^\perp$. This means that $\alpha(B - \xi A)$ must be smaller than $\alpha(A) = \alpha(P_A)$.

In fact, suppose that there exists an orthonormal basis

$$\{x_i : i = 1, 2, \dots, n, n+1\}$$

of $N(B - \xi A)$, where $n = \alpha(A)$. Then $0 \in (B - \xi A)x_i$ for $i = 1, 2, \dots, n, n+1$. Let

$$x_i = p_i + q_i \in N(A)^\perp \oplus N(A).$$

Since $B - \xi A$ is one to one on $N(A)^\perp$, then $q_i \neq 0$ for every i . At most n of the q_i are linearly independent. So

$$q_{n+1} = \sum_{i=1}^n \omega_i q_i, \text{ where } \omega_i \text{ are constants.} \quad (44)$$

Since $0 \in (B - \xi A)(x_i) = (B - \xi A)(p_i + q_i) = (B - \xi A)(p_i) + (B - \xi A)(q_i)$, it follows that $(B - \xi A)(p_i)$ and $-(B - \xi A)(q_i)$ have an element in common and therefore

$$(B - \xi A)(p_i) = -(B - \xi A)(q_i). \quad (45)$$

Using (45), (44) and the linearity of $B - \xi A$ we get

$$\begin{aligned}
 (B - \xi A)(p_{n+1}) &= -(B - \xi A)(q_{n+1}) \\
 &= -(B - \xi A) \sum_{i=1}^n \omega_i q_i \\
 &= \sum_{i=1}^n -(B - \xi A) \omega_i q_i \\
 &= \sum_{i=1}^n (B - \xi A) \omega_i p_i \\
 &= (B - \xi A) \sum_{i=1}^n \omega_i p_i.
 \end{aligned} \tag{46}$$

Equality (46) and injectivity of $B - \xi A$ on $N(A)^\perp$ imply that

$$p_{n+1} = \sum_{i=1}^n \omega_i p_i. \tag{47}$$

The equalities (44) and (47) imply that

$$x_{n+1} = \sum_{i=1}^n \omega_i x_i. \tag{48}$$

Since (48) contradicts the assumption that $\{x_i : i = 1, 2, \dots, n+1\}$ is an orthonormal basis of $N(B - \xi A)$, then

$$\alpha(B - \xi A) \leq \alpha(A). \tag{49}$$

Since $B - \xi A$ is closed, has closed range and $\alpha(B - \xi A) < \infty$ for $\xi \in \overline{\Theta_A(BP_A)}$, equality (43) and Theorem 22 imply that

$$\begin{aligned}
 \kappa(A) &= \kappa((B - \xi A)P_A) \\
 &= \kappa(B - \xi A) + \kappa(P_A) - \dim(P_A(0) \cap (B - \xi A)^{-1}(0)) \\
 &= \kappa(B - \xi A) + \kappa(P_A); \text{ since } B - \xi A \text{ is injective and } P_A(0) = \{0\} \\
 &= \kappa(B - \xi A); \text{ since } \kappa(P_A) = 0 \text{ by Lemma 21.}
 \end{aligned} \tag{50}$$

That $\beta(B - \xi A) \leq \beta(A)$ follows from (49) and (50). $\square$

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