

REGULARITY AND GREEN'S RELATIONS FOR TRANSFORMATION SEMIGROUPS DETERMINED BY AN EQUIVALENCE AND A ZIGZAG ORDER

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ABSTRACT. Consider a finite ordered set $(X; \preceq)$ whose order forms a zigzag. For an equivalence E on X , let $EOP(X)$ be the semigroup of all transformations preserving the relation $E \cap \preceq$, that is, $EOP(X) = \{\alpha \in T(X) \mid \forall x, y \in X, [(x, y) \in E \wedge x \preceq y] \Rightarrow [(x\alpha, y\alpha) \in E \wedge x\alpha \preceq y\alpha]\}$. In this paper, we completely describe regular elements of $EOP(X)$. A characterization of regular semigroups $EOP(X)$ is given. Green's relations $\mathcal{L}$ and $\mathcal{R}$ on $EOP(X)$ are determined.

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1. INTRODUCTION AND PRELIMINARIES

For a non-empty set X , the set of all transformations on X is a semigroup, denoted by $T(X)$ and called the *full transformation semigroup* on X , with respect to the composition. Subsemigroups of $T(X)$ are called *transformation semigroups*. The semigroup $T(X)$ and its subsemigroups have been intensively studied since every semigroup can be represented as a transformation semigroup.

Regularity and Green's relations are one of the most-studied topics in semigroup theory. An element a in a semigroup S is said to be *regular* if there is an element $b \in S$ with $a = aba$. A *regular* semigroup is a semigroup in which every element is regular. The semigroup $T(X)$ is regular, however, its subsemigroups do not need to be. The different types of regular subsemigroups have been studied (see for example: [3, 8, 12, 19]). One of interesting subsemigroups of $T(X)$ is an α -semigroup (see in [6, 7]). An α -semigroup is a transformation semigroup containing all constant maps and the identity map, such as the semigroup $OT(X)$ of all order-preserving transformations on an ordered set $(X; \leq)$ and the semigroup $T_E(X)$ of all transformation preserving an equivalence E . The regularity and Green's relations for transformation semigroups determined by a totally ordered set were widely investigated (see for example: [8, 9, 12, 14, 19]). We now focus on the semigroups $EOP(X)$ of all transformations

preserving the relation $E \cap \leq$ when $(X; \leq)$ is an ordered set. A semigroup $EOP(X)$ is a generalization of the semigroup $OT(X)$. For a finite totally ordered set $(X; \leq)$ and an equivalence E on X , Ma et al [14] described the regularity and Green's relations of the semigroup $EOP(X)$.

Our interest focuses on important ordered sets whose simplicity is "next" to that of totally ordered sets. A *fence* $(X; \preceq)$ is an ordered set that the order forms a zigzag. Indeed, the only comparability relations in X are either

$$x_1 \prec x_2 \succ x_3, x_3 \prec x_4 \succ x_5, \dots, x_{2m-1} \prec x_{2m} \succ x_{2m+1}, \dots \text{ OR}$$

$$x_1 \succ x_2 \prec x_3, x_3 \succ x_4 \prec x_5, \dots, x_{2m-1} \succ x_{2m} \prec x_{2m+1}, \dots$$

where $X = \{x_1, x_2, x_3, \dots\}$. A subordered set of a fence $(X; \preceq)$ is called a *subfence* if it is a fence. For a fence $(X; \preceq)$, the semigroups $EOP(X)$ is adapted from an analogous condition for $EOP(X)$ where $(X; \leq)$ is a totally ordered set. The regularities of $OT(X)$ where $(X; \preceq)$ is a fence and its subsemigroups have been studied (see for example: [1, 2, 4, 10, 11, 13, 15–18]).

Throughout this paper the word "fence" will mean a finite fence. Our main purpose is to investigate the regularity and Green's relations for the semigroup $EOP(X)$ where $(X; \preceq)$ is a fence. Section 2 is devoted to the study of the regularity for $EOP(X)$. Regular elements in $EOP(X)$ are completely determined and a characterization for $EOP(X)$ to be regular is given. Finally, in Section 3, Green's relations $\mathfrak{L}$ and $\mathfrak{R}$ on $EOP(X)$ are described.

2. REGULARITY FOR A SEMIGROUP $EOP(X)$

In this section, we investigate the regularity for a semigroup $EOP(X)$. Since the regularities for $T(X)$ and $OT(X)$ where $(X; \preceq)$ is a fence are fully described, we first focus on the case of $EOP(X) = T(X)$ and $EOP(X) = OT(X)$, as shown in the following propositions.

An ordered set is called an *antichain* if any two distinct elements are incomparable.

Proposition 2.1. *Let $(X; \preceq)$ be a fence. Then $EOP(X) = T(X)$ if and only if every class $A \in X/E$ is an antichain.*

Proof. Assume first that $EOP(X) = T(X)$ and there is a class $A \in X/E$ which is not an antichain. Let $x, y \in A$ with $x \prec y$. Then $(x, y) \in E \cap \preceq$. Consider a map $\alpha : X \rightarrow X$ with $x\alpha = y$ and $y\alpha = x$. We have $\alpha \in EOP(X)$ and so, $(x\alpha, y\alpha) \in E \cap \preceq$. It follows that $y = x\alpha \preceq y\alpha = x$. By the symmetricity of $\preceq$, $x = y$ and hence to a contradiction. Thus every class $A \in X/E$ is an antichain. Conversely, assume that every class $A \in X/E$ is an antichain. Let $\alpha \in T(X)$ and let $(x, y) \in E \cap \preceq$. Then $x, y \in [x]_E$ and $x \preceq y$. From $[x]_E$ is an antichain, $x = y$ and then $(x\alpha, y\alpha) \in E \cap \preceq$. Hence, $\alpha \in EOP(X)$ and therefore, $EOP(X) = T(X)$. $\square$

Proposition 2.1 implies that for a fence $(X; \preceq)$, the semigroup $EOP(X)$ is regular if every class $A \in X/E$ is an antichain.

Proposition 2.2. *Let $(X; \preceq)$ be a fence. Then $EOP(X) = OT(X)$ if and only if $E = X \times X$.*

Proof. If $E = X \times X$, then $E \cap \preceq = \preceq$ and so, $EOP(X) = OT(X)$. Conversely, assume that $EOP(X) = OT(X)$ and $E \neq X \times X$. Then there are elements $a, b \in X$ with $(a, b) \notin E$. So, $[a]_E \neq [b]_E$. Define the map $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} a, & x \in [b]_E, \\ b, & \text{otherwise.} \end{cases}$$

Let $(x, y) \in E \cap \preceq$. Then $[x]_E = [y]_E$ implies $x\alpha = y\alpha$ and so, $(x\alpha, y\alpha) \in E \cap \preceq$. Hence, $\alpha \in EOP(X)$. By the assumption, α is order-preserving. Since $(X; \preceq)$ is a fence, $\text{ran } \alpha = \{a, b\}$ is a 2-element fence. We may let $a \prec b$. Then $a\alpha \preceq b\alpha$. Since $a \notin [b]_E$, we have $a\alpha = b$ and $b\alpha = a$ implying $b \preceq a$. By the symmetricity of $\preceq$, $a = b$ which is a contradiction. Therefore, $E = X \times X$ $\square$

Jitman, Srithus and Worawannotai described regular elements of $OT(X)$ where $(X; \preceq)$ is a fence. They showed that α is regular if and only if there exists a subfence $(F; \preceq)$ of $(X; \preceq)$ such that $\alpha|_F$ is a bijection onto $\text{ran } \alpha$ where $\text{ran } \alpha$ is the range of α . A characterization of regular semigroups $OT(X)$ where X is a fence is given in [18] as stated below.

Theorem 2.3. *Let $(X; \preceq)$ be a fence. Then $OT(X)$ is regular if and only if $|X| \leq 4$.*

Proposition 2.2 and Theorem 2.3 imply that for a fence $(X; \preceq)$, the semigroup $EOP(X)$ is regular if $|X| \leq 4$ and $E = X \times X$.

In the theorem below, a necessary and sufficient condition for $EOP(X)$ to be regular is given.

Theorem 2.4. *Let $(X; \preceq)$ be a fence. Then $EOP(X)$ is regular if and only if one of the following conditions is satisfied:*

- (i) $|X| \leq 4$ and $E = X \times X$.
- (ii) Every class $A \in X/E$ is an antichain.

Proof. Assume that $EOP(X)$ is regular and Condition (i)-(ii) are not satisfied. Then $|X| > 4$ or $E \neq X \times X$ and there exists a class $A \in X/E$ which is not an antichain. If $E = X \times X$, then by Theorem 2.2, $EOP(X) = OT(X)$ and so, $|X| \leq 4$ by Theorem 2.3 and hence to a contradiction. Thus, $E \neq X \times X$ and then $A \neq X$. Because A is not an antichain, there are elements $a, b \in A$ with $a \prec b$ and hence, $(a, b) \in E \cap \preceq$. Define the map $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} a, & x \in A, \\ b, & \text{otherwise.} \end{cases}$$

Let $(x, y) \in E \cap \preceq$. Then $[x]_E = [y]_E$ and so, $x\alpha = y\alpha$ implies $(x\alpha, y\alpha) \in E \cap \preceq$. Hence, $\alpha \in EOP(X)$ and therefore, α is regular. So, there is a map $\beta \in EOP(X)$ with $\alpha\beta\alpha = \alpha$. Since $A \neq X$, there is an element $x \in X \setminus A$. By the definition of α , we have $(b\beta)\alpha = x\alpha\beta\alpha = x\alpha = b$ and $(a\beta)\alpha = a\alpha\beta\alpha = a\alpha = a$ implying $b\beta \notin A$ and $a\beta \in A$. So, $(a\beta, b\beta) \notin E$. Since $\beta \in EOP(X)$ and $(a, b) \in E \cap \preceq$, we have $(a\beta, b\beta) \in E$, a contradiction. Thus, (i) or (ii) is satisfied. The converse is a consequence of Theorem 2.3 and Theorem 2.1. $\square$

To finish this section, it remains to describe regular elements in $EOP(X)$. Before doing so, we mention some basic knowledge involving order-preserving maps. An ordered set P is called *connected* if for all $a, b \in P$, there is a fence $F \subseteq P$ with endpoints a and b . It is well known that if P is connected and $\alpha : P \rightarrow Q$ is order-preserving from P to an ordered set Q , then $P\alpha$ is connected. Because order-connected subsets of a fence X are precisely the subfences, an order-preserving map $\alpha : X \rightarrow X$ maps subfences to subfences. We also need the following technical lemmas.

An element a in a semigroup S is said to be *regular with respect to b* if $a = aba$.

Lemma 2.5. *Let α be regular in $T(X)$ with respect to β . Assume that $(F; \preceq)$ is a subfence of a fence $(X; \preceq)$ and $(\beta\alpha)|_F$ is order-preserving. If $F\beta\alpha$ is a subfence of $(X; \preceq)$, then $F = F\beta\alpha$ and so, $F \subseteq \text{ran } \alpha$.*

Proof. Let $a, b \in \text{ran } \alpha$ be the endpoints of F . Then there are elements $a', b' \in X$ with $a'\alpha = a$ and $b'\alpha = b$, respectively. Since $\alpha\beta\alpha = \alpha$, we have $a\beta\alpha = a'\alpha\beta\alpha = a'\alpha = a$ and $b\beta\alpha = b'\alpha\beta\alpha = b'\alpha = b$ implying $a, b \in F\beta\alpha$. Since $(\beta\alpha)|_F$ is order-preserving, $F\beta\alpha$ is a subfence of X . But $|F\beta\alpha| \leq |F|$. Since $a, b \in F\beta\alpha$ and a, b are the endpoints of F , $F \subseteq F\beta\alpha$ implies $|F| \leq |F\beta\alpha|$. So, $|F\beta\alpha| = |F|$. $\square$

For a fence X and $x, y \in X$, let $F_{x,y}$ be the subfence of X having x and y as the endpoints. Clearly, $F_{x,y} = F_{y,x}$.

Lemma 2.6. *Let α be regular in $T(X)$ with respect to β and let $x \in X$ with $x \notin \text{ran } \alpha$. Assume that $(X; \preceq)$ is a fence and $(\beta\alpha)|_A$ is order-preserving where A is a subset of X . If $F_{x,a}$ and $F_{x,b}$ are contained in A where $a, b \in \text{ran } \alpha$ and $|F_{x,a}| = |F_{x,b}|$, then $F_{x,a} = F_{x,b}$.*

Proof. Let $a, b \in \text{ran } \alpha$ such that $F_{x,a}$ and $F_{x,b}$ are contained in $A \subseteq X$ and $|F_{x,a}| = |F_{x,b}|$. Assume to the contrary that $F_{x,a} \neq F_{x,b}$. From $|F_{x,a}| = |F_{x,b}|$, the element x is in between a and b . It follows that $F_{x,a} \cup F_{x,b}$ is a fence having a and b as the endpoints. Because $F_{x,a}$ and $F_{x,b}$ are in A , the fence $F_{x,a} \cup F_{x,b}$ is contained in A . Since $(\beta\alpha)|_A$ is order-preserving, $(\beta\alpha)|_{F_{x,a} \cup F_{x,b}}$ is order-preserving. Lemma 2.5 implies that $F_{x,a} \cup F_{x,b} \subseteq \text{ran } \alpha$ and so, $x \in \text{ran } \alpha$ which is a contradiction. Thus, $F_{x,a} = F_{x,b}$. $\square$

Let $(X; \preceq)$ be a fence and let $A \subseteq X$. Consider a map α which is assumed as in Lemma 2.5. For an element $x \notin \text{ran } \alpha$ with $F_{x,y} \subseteq A$ for some $y \in \text{ran } \alpha$, the set $S = \{|F_{x,y}| \in \mathbb{N} \mid y \in \text{ran } \alpha \wedge F_{x,y} \subseteq A\}$ is nonempty where $|F_{x,y}|$ is the number of elements in $F_{x,y}$ and then S has the minimum element. By

Lemma 2.6, there exists a unique element $a \in \text{ran } \alpha$ such that $F_{x,a} \subseteq A$ and $|F_{x,a}|$ is the minimum of S . In the other words, $F_{x,a}$ is the only shortest fence in A having x and an element in $\text{ran } \alpha$ as the endpoints. We denote $F_{x,a}$ and a by F_x and e_x , respectively.

Lemma 2.7. *Let α be regular in $T(X)$ with respect to β . Assume that $(X; \preccurlyeq)$ is a fence and $(\beta\alpha)|_A$ is order-preserving where A is a subset of X and let $x, y \in A$ with $x, y \notin \text{ran } \alpha$ and $x \prec y$. If F_x exists, then F_y exists and $e_y = e_x$, and vice versa.*

Proof. Let $x, y \in A$ with $x, y \notin \text{ran } \alpha$ and $x \prec y$. Assume that F_x exists. Then F_{y,e_x} is a fence in A having y and an element in $\text{ran } \alpha$ as the endpoints. It follows that the shortest fence F_y exists. We obtain that both x and y are not in between e_x and e_y , since otherwise by Lemma 2.6, x and y are in $\text{ran } \alpha$. To show that $e_y = e_x$, we consider these 2 cases.

Case 1 $x \in F_y$. The fence $F_y \setminus \{y\} = F_{x,e_y}$ is contained in F_y . By the shortest of F_x , we have $|F_x| \leq |F_{x,e_y}|$. Since x and y are not in between e_x and e_y , the fence $F_x = F_{x,e_x}$ is contained in F_y and then $F_x \cup \{y\} \subseteq F_y$ having $y, e_x \in \text{ran } \alpha$ as the endpoints. So, $F_y = F_x \cup \{y\}$ and hence, $e_y = e_x$.

Case 2 $x \notin F_y$. Consider the fence $F_y \cup \{x\} = F_{x,e_y}$ which is contained in A . We have $|F_x| \leq |F_{x,e_y}|$. Since x and y are not in between e_x and e_y , we have $F_x \subseteq F$ is contained in F_{x,e_y} and then $F_x \setminus \{x\} \subseteq F_y$. Because $F_x \setminus \{x\}$ is the fence in A having y and $e_x \in \text{ran } \alpha$ as the endpoints, $F_x \setminus \{x\} = F_y$ and hence, $e_y = e_x$. $\square$

To finish this section, it remains to describe regular elements in $EOP(X)$.

Theorem 2.8. *Let $(X; \preccurlyeq)$ be a fence. Then $\alpha \in EOP(X)$ is regular if and only if each class $A \in X/E$ with $A \cap \text{ran } \alpha \neq \emptyset$ satisfies the following conditions.*

- (i) *If $(F; \preccurlyeq)$ is a subfence of $(X; \preccurlyeq)$ such that $F \subseteq A$ and the endpoints of $(F; \preccurlyeq)$ are in $\text{ran } \alpha$, then $F \subseteq \text{ran } \alpha$.*
- (ii) *There exists a subset $C \subseteq X$ satisfying the following conditions.*
 - (a) *The restriction $\alpha|_C : C \rightarrow A \cap \text{ran } \alpha$ is bijective.*
 - (b) *For all $x, y \in C$, $(x, y) \in E \cap \preccurlyeq$ if and only if $(x\alpha, y\alpha) \in E \cap \preccurlyeq$.*

Proof. Assume that $\alpha \in EOP(X)$ is regular. Then there is a map $\beta \in EOP(X)$ with $\alpha\beta\alpha = \alpha$.

Let $A \in X/E$ with $A \cap \text{ran } \alpha \neq \emptyset$.

(i) Assume that F is a subfence of X contained in A such that the endpoints are in $\text{ran } \alpha$. Then $\preccurlyeq \cap (F \times F) \subseteq E$ and so, $E \cap (\preccurlyeq \cap (F \times F)) = \preccurlyeq \cap (F \times F)$. It follows that $\gamma|_F$ is order-preserving for any $\gamma \in EOP(X)$. Since $\alpha, \beta \in EOP(X)$, the restriction $(\beta\alpha)|_F$ is order-preserving. By Lemma 2.5, $F \subseteq \text{ran } \alpha$.

(ii) Let $C := (A \cap \text{ran } \alpha)\beta$. Then $C \neq \emptyset$ since $A \cap \text{ran } \alpha \neq \emptyset$.

(a) Let $x, y \in C$ with $x\alpha|_C = y\alpha|_C$. Then there are elements $a, b \in A \cap \text{ran } \alpha$ with $x = a\beta$ and $y = b\beta$. Since $a, b \in \text{ran } \alpha$, there are elements $a', b' \in X$ with $a = a'\alpha$ and $b = b'\alpha$. It follows that $x = a\beta = a'\alpha\beta$ and $y = b\beta = b'\alpha\beta$ implies $x\alpha = a'\alpha\beta\alpha = a'\alpha = a$ and $y\alpha = b'\alpha\beta\alpha = b'\alpha = b$. From $x\alpha = y\alpha$, we have $a = b$ implying $x = a\beta = b\beta = y$. Hence, $\alpha|_C$ is injective. To show that $\alpha|_C$ is surjective, let $a \in A \cap \text{ran } \alpha$. Then $a\beta \in (A \cap \text{ran } \alpha)\beta = C$. From $a \in \text{ran } \alpha$, there is an element $x \in X$ with $x\alpha = a$ implying $(a\beta)\alpha|_C = a\beta\alpha = x\alpha\beta\alpha = x\alpha = a$. Therefore, $\alpha|_C$ is bijective.

(b) Let $x, y \in C$. Since $\alpha \in EOP(X)$, we have $(x\alpha, y\alpha) \in E \cap \preceq$ if $(x, y) \in E \cap \preceq$. Conversely, assume that $(x\alpha, y\alpha) \in E \cap \preceq$. By a similar argument as in (a), $x = a\beta, y = b\beta, x\alpha = a$ and $y\alpha = b$ where $a, b \in A \cap \text{ran } \alpha$. Since $(x\alpha, y\alpha) \in E \cap \preceq$, we have $(a, b) \in E \cap \preceq$. Hence, $(x, y) = (a\beta, b\beta) \in E \cap \preceq$.

For the converse, assume that (i) and (ii) are satisfied. Let $A \in X/E$ with $A \cap \text{ran } \alpha \neq \emptyset$. Then $A \times A \subseteq E$ implies $\preceq \cap (A \times A) \subseteq E$ and so, $E \cap (\preceq \cap (A \times A)) = \preceq \cap (A \times A)$. It follows that $\gamma|_A$ is order-preserving for any $\gamma \in EOP(X)$. Consider an element $x \notin \text{ran } \alpha$ with $F_{x,y} \subseteq A$ for some $y \in \text{ran } \alpha$. Since $(\alpha\beta)|_A$ is order-preserving, Lemma 2.6 implies that the fence F_x exists. Define the map $\beta_A : A \rightarrow X$ by

$$x\beta_A = \begin{cases} x\gamma_A^{-1}, & x \in \text{ran } \alpha \\ e_x\gamma_A^{-1}, & x \notin \text{ran } \alpha \text{ and } F_x \text{ exists} \\ a, & x \notin \text{ran } \alpha \text{ and } F_x \text{ does not exist,} \end{cases}$$

where $\gamma_A := \alpha|_C : C \rightarrow A \cap \text{ran } \alpha$ and a is a fixed element in A . For a class $A \in X/E$ with $A \cap \text{ran } \alpha = \emptyset$, the map $\beta_A : A \rightarrow X$ is defined by $x\beta_A = a$ where a is a fixed element in A . The map $\beta : X \rightarrow X$ is defined by $x\beta = x\beta_A$ when $A \in X/E$ and $x \in A$.

To show $\beta \in EOP(X)$, we need to prove that β preserves the relation $E \cap \preceq$. If $x = y$, then $x\beta = y\beta$ and so, $(x\beta, y\beta) \in E \cap \preceq$. Let $(x, y) \in E \cap \preceq$ with $x \neq y$ and let $A := [x]_E$. Then $x, y \in A$. If $A \cap \text{ran } \alpha = \emptyset$, then $x\beta = x\beta_A = a = y\beta_A = y\beta$ and so, $(x\beta, y\beta) \in E \cap \preceq$. Assume that $A \cap \text{ran } \alpha \neq \emptyset$. We consider these 4 cases.

Case 1 $x, y \in \text{ran } \alpha$. By the definition of β , we get $x\beta = x\beta_A = x\gamma_A^{-1}$ and $y\beta = y\beta_A = y\gamma_A^{-1}$ implying $(x\beta)\alpha = x\gamma_A^{-1}\alpha = x\gamma_A^{-1}\alpha|_C = x\gamma_A^{-1}\gamma_A = x$ and similarly, $(y\beta)\alpha = y$. Since $(x, y) \in E \cap \preceq$, we have $((x\beta)\alpha, (y\beta)\alpha) \in E \cap \preceq$. Because $x\beta, y\beta \in C$ and by (ii)(b), $(x\beta, y\beta) \in E \cap \preceq$.

Case 2 $x \in \text{ran } \alpha$ and $y \notin \text{ran } \alpha$. Since $x, y \in A$ and $x \preceq y$, F_y is the 2-element fence $\{x, y\}$ and $e_y = x$. So, $x\beta = x\beta_A = x\gamma_A^{-1} = e_y\gamma_A^{-1} = y\beta_A = y\beta$ implies $(x\beta, y\beta) \in E \cap \preceq$.

Case 3 $x \notin \text{ran } \alpha$ and $y \in \text{ran } \alpha$. The proof is similar to Case 2.

Case 4 $x \notin \text{ran } \alpha$ and $y \notin \text{ran } \alpha$. Next, we consider x and y as follows. If F_x exists, then by Lemma 2.7, F_y exists and $e_y = e_x$ implying $x\beta = x\beta_A = e_x\gamma_A^{-1} = e_y\gamma_A^{-1} = y\beta_A = y\beta$. If F_x does not exist, then again by Lemma 2.7, F_y does not exist implying $x\beta = x\beta_A = a = y\beta_A = y\beta$. Thus, $(x\beta, y\beta) \in E \cap \preceq$.

Finally, we show that $\alpha\beta\alpha = \alpha$. Let $x \in X$. Then $x\alpha \in \text{ran } \alpha$ and there exists a class $A \in X/E$ with $x\alpha \in A$. So, $A \cap \text{ran } \alpha \neq \emptyset$. By the definition of β , we have $x\alpha\beta\alpha = (x\alpha)\beta_A\alpha = (x\alpha)\gamma_A^{-1}\alpha = (x\alpha)\gamma_A^{-1}\alpha|_C = (x\alpha)\gamma_A^{-1}\gamma_A = x\alpha$. Therefore, α is regular in $EOP(X)$. $\square$

3. GREEN'S RELATIONS ON $EOP(X)$

Recall that Green's relations on a semigroup S are five equivalence relations defined as follows.

(i) a and b are $\mathfrak{L}$ -equivalent if a and b generate the same left principal ideal, that is, $S^1a = S^1b$ or equivalently, $(a, b) \in \mathfrak{L}$ if and only if $a = xb, b = ya$ for some $x, y \in S^1$.

(ii) a and b are $\mathfrak{R}$ -equivalent if a and b generate the same right principal ideal, that is, $aS^1 = bS^1$ or equivalently, $(a, b) \in \mathfrak{R}$ if and only if $a = bx, b = ay$ for some $x, y \in S^1$.

(iii) a and b are $\mathfrak{J}$ -equivalent if a and b generate the same principal ideal, that is, $S^1aS^1 = S^1bS^1$ or equivalently, $(a, b) \in \mathfrak{J}$ if and only if $a = xby, b = uav$ for some $x, y, u, v \in S^1$.

(iv) a and b are $\mathfrak{H}$ -equivalent if $(a, b) \in \mathfrak{L} \cap \mathfrak{R}$, that is, $\mathfrak{H} = \mathfrak{L} \cap \mathfrak{R}$.

(v) a and b are $\mathfrak{D}$ -equivalent if $(a, b) \in \mathfrak{L} \circ \mathfrak{R}$, that is, $\mathfrak{D} = \mathfrak{L} \circ \mathfrak{R}$.

It is well known that the relation $\mathfrak{J}$ is $\mathfrak{D}$ for every finite semigroup and the relations $\mathfrak{H}$ and $\mathfrak{D}$ are determined by the relations $\mathfrak{L}$ and $\mathfrak{R}$. So, in this paper, we consider only the relations $\mathfrak{L}$ and $\mathfrak{R}$. Before doing so, let us mention the following maps for an arbitrary ordered set.

Definition 1. Let $(X; \leq)$ be an ordered set and let E be an equivalence on X . A map $\alpha : A \rightarrow B$, where A and B are subsets of X is said to be E -order preserving if $(x\alpha, y\alpha) \in (E \cap \leq) \cap (B \times B)$ whenever $(x, y) \in (E \cap \leq) \cap (A \times A)$. An E -order preserving α is said to be E -order embedding if α satisfies the condition:

$$\text{If } (x\alpha, y\alpha) \in (E \cap \leq) \cap (B \times B), \text{ then } (x, y) \in (E \cap \leq) \cap (A \times A).$$

An E -order isomorphism is E -order embedding and surjective.

The following propositions gives useful properties of E -order preserving maps that are used later

Proposition 3.1. Let $(X; \leq)$ be an ordered set. Then the following hold:

- (i) A composition of E -order preserving maps is E -order preserving.
- (ii) An E -order embedding map is injective.
- (iii) If α is an E -order isomorphism, then α^{-1} is.
- (iv) Let α be E -order preserving and bijective. If α^{-1} is E -order preserving, then α is an E -order isomorphism.

Proof. Let $\alpha : A \rightarrow B$ be an E -order embedding map.

(i) The proof is straightforward.

(ii) Let $x, y \in A$ with $x\alpha = y\alpha$. By the reflexivity of $\leq$ and E , $x\alpha \leq y\alpha, y\alpha \leq x\alpha$ and $(x\alpha, y\alpha) \in E$ implies $(x\alpha, y\alpha) \in (E \cap \leq) \cap (B \times B)$ and $(y\alpha, x\alpha) \in (E \cap \leq) \cap (B \times B)$. Since α is E -order embedding, $(x, y), (y, x) \in (E \cap \leq) \cap (A \times A)$ implies $x \leq y$ and $y \leq x$. Hence, $x = y$ and therefore, α is injective.

(iii) Let $\alpha : A \rightarrow B$ be an E -order isomorphism. Then α is E -order embedding and surjective. By (ii), α is injective and hence, α is bijective. Therefore, α^{-1} is a bijective map from B onto A . Next we show that α^{-1} is E -order embedding.

Let $(x, y) \in (E \cap \leq) \cap (B \times B)$. By the surjectivity of α , there are elements $a, b \in A$ with $x = a\alpha$ and $y = b\alpha$, that is, $a = x\alpha^{-1}$ and $b = y\alpha^{-1}$. So, $(a\alpha, b\alpha) \in (E \cap \leq) \cap (B \times B)$ implies $(x\alpha^{-1}, y\alpha^{-1}) = (a, b) \in (E \cap \leq) \cap (A \times A)$. Conversely, let $(x\alpha^{-1}, y\alpha^{-1}) = (a, b) \in (E \cap \leq) \cap (A \times A)$. Since α is E -order preserving, $(x\alpha^{-1}\alpha, y\alpha^{-1}\alpha) = (x, y) \in (E \cap \leq) \cap (B \times B)$. Therefore, α^{-1} is E -order embedding and from α^{-1} is surjective, α^{-1} is an E -order isomorphism.

(iv) Let $(x\alpha, y\alpha) \in (E \cap \leq) \cap (B \times B)$. Then $(x, y) = (x\alpha\alpha^{-1}, y\alpha\alpha^{-1}) \in (E \cap \leq) \cap (A \times A)$. Since α is E -order preserving, α is E -order embedding. By the surjectivity of α , the map α is an E -order isomorphism. $\square$

For $\alpha \in EOP(X)$, let $Sub_\alpha = \{X\gamma \mid \gamma \in EOP(X) \text{ and } \gamma\alpha = \alpha\}$. Since $id_X\alpha = \alpha$, we have $X = Xid_X \in Sub_\alpha$ implying $Sub_\alpha \neq \emptyset$.

Consider the set $Car_\alpha = \{|A| \mid A \in Sub_\alpha\} \subseteq \mathbb{N}$. Since $Sub_\alpha \neq \emptyset$, we have $Car_\alpha \neq \emptyset$ and hence, Car_α has a least element. A subset $A \subseteq X$ is said to be α -minimal if $|A|$ is the least element of Car_α .

Remark 1. (i) If A_1 and A_2 are α -minimal, then $|A_1| = |A_2|$.

(ii) If A is α -minimal, then $|A| \leq |B|$ for all $B \in Sub_\alpha$.

Proposition 3.2. Let $(X; \leq)$ be an ordered set and let E be an equivalence on X . Assume that $\alpha \in EOP(X)$. If A and B are α -minimal, then there is an E -order isomorphism γ from A onto B .

Proof. Since A and B are α -minimal, $|A| = |B|$ and there are E -order preserving maps $\gamma_1, \gamma_2 \in EOP(X)$ with $X\gamma_1 = A$, $X\gamma_2 = B$, $\gamma_1\alpha = \alpha = \gamma_2\alpha$. So, $\gamma_1\gamma_2$ is E -order preserving and $(\gamma_1\gamma_2)\alpha = \gamma_1(\gamma_2\alpha) = \gamma_1\alpha = \alpha$ implies $X\gamma_1\gamma_2 \in Sub_\alpha$. But $X\gamma_1 = A$, we have $A\gamma_2 \in Sub_\alpha$. Since $A \subseteq X$, we have $A\gamma_2 \subseteq X\gamma_2 = B$ implying $|A\gamma_2| \leq |B|$. By the minimality of B , $|A\gamma_2| = |B|$ implies $B = A\gamma_2$. Similarly, $A = B\gamma_1$.

Let $\delta := \gamma_1|B$ and $\gamma := \gamma_2|A$. Then $\delta : B \rightarrow A$ and $\gamma : A \rightarrow B$ are surjective E -order preserving maps. Since $|A| = |B|$, we have that δ and γ are bijective and hence, $\gamma\delta : A \rightarrow A$ is bijective.

Consider the permutation group S_A on A . We have $\gamma\delta \in S_A$. So, $(\gamma\delta)^m = id_A$ when $m = |S_A|$. Since $(\gamma\delta)^m = \gamma\delta(\gamma\delta)^{m-1}$, we get that $\delta(\gamma\delta)^{m-1}$ is an inverse of γ . Since δ and γ are E -order preserving, $\delta(\gamma\delta)^{m-1}$ is E -order preserving which implies that γ is an E -order isomorphism. $\square$

For an ordered set $(X; \leq)$, a characterization of the Green's relation $\mathfrak{L}$ on $EOP(X)$ is stated below:

Theorem 3.3. Let $(X; \leq)$ be an ordered set and let $\alpha, \beta \in EOP(X)$. Assume that A and B are α -minimal and β -minimal. Then $\alpha\mathfrak{L}\beta$ if and only if there is an E -order isomorphism $\varphi : B \rightarrow A$ with $\beta|_B = \varphi\alpha$.

Proof. Assume that $\alpha \mathfrak{L} \beta$. Then there are $\gamma, \delta \in EOP(X)$ with $\beta = \delta\gamma$ and $\alpha = \gamma\beta$. Since A and B are α -minimal and β -minimal, respectively, $A \in Sub_\alpha$, $B \in Sub_\beta$ and then there are $\eta, \lambda \in EOP(X)$ with $A = X\eta$, $B = X\lambda$ and $\eta\alpha = \alpha$, $\lambda\beta = \beta$, respectively.

Let $\varphi := (\delta\eta)|_B$. Since $X\delta \subseteq X$ and $X\eta = A$, we have $X\delta\eta \subseteq X\eta = A$ implying $\varphi : B \rightarrow A$. Since δ, η are E -order preserving, $\delta\eta$ is E -order preserving which implies that φ is E -order preserving. So, $\lambda\varphi\gamma$ is a is E -order preserving transformation on X . Moreover, $\lambda\varphi\gamma\beta = \lambda\varphi(\gamma\beta) = \lambda\varphi\alpha = \lambda\delta(\eta\alpha) = \lambda(\delta\eta)\alpha = \lambda\delta\alpha = \lambda\beta = \beta$, that is, $(\lambda\varphi\gamma)\beta = \beta$ implies $B\varphi\gamma = X\lambda\varphi\gamma \in Sub_\beta$. Because B is β -minimal, $|B| \leq |B\varphi\gamma| \leq |B\varphi|$ implies $|B| = |B\varphi|$. Hence, φ is injective.

Let $\psi := (\gamma\lambda)|_A$. By the similar argument as φ , the map $\psi : A \rightarrow B$ is injective E -order preserving and hence, $\varphi\psi : B \rightarrow B$ is injective E -order preserving. So, $\varphi\psi$ is bijective and then $(\varphi\psi)^m = id_B$ where $m = |S_B|$ and S_B is the permutation group on B . Since $id_B = (\varphi\psi)^m = \varphi\psi(\varphi\psi)^{m-1}$, we have $\psi(\varphi\psi)^{m-1}$ is an inverse of φ . Because φ, ψ are E -order preserving, $\varphi^{-1} = \psi(\varphi\psi)^{m-1}$ is E -order preserving. By Proposition 3.1(iv), φ is an E -order isomorphism. Since $x\varphi\alpha = x\delta\eta\alpha = x\delta\alpha = x\beta$, we have $\varphi\alpha = \beta|_B$.

Conversely, assume that there is an E -order isomorphism $\varphi : B \rightarrow A$ with $\beta|_B = \varphi\alpha$. Since $B \in Sub_\beta$, there is $\lambda \in EOP(X)$ with $B = X\lambda$ and $\lambda\beta = \beta$. So, $\lambda : X \rightarrow B$ is E -order preserving which implies that $\lambda\varphi : X \rightarrow X$ is E -order preserving, that is, $\lambda\varphi \in EOP(X)$. Let $\delta := \lambda\varphi$. Then $x\lambda \in B$ and $x\delta\alpha = x\lambda\varphi\alpha = x\lambda\beta|_B = x\lambda\beta = x\beta$ for all $x \in X$. So, $\delta\alpha = \beta$. Since φ is an E -order isomorphism, $\varphi^{-1} : A \rightarrow B$ is an E -order isomorphism. From $A \in Sub_\alpha$, there is a map $\eta \in EOP(X)$ with $A = X\eta$ and $\eta\alpha = \alpha$. Let $\gamma := \eta\varphi^{-1}$. Then $\gamma \in EOP(X)$ and $\alpha = \gamma\beta$. Therefore, $\alpha \mathfrak{L} \beta$. $\square$

For the rest of this paper, we focus on an ordered set which is a fence. The following theorem shows a characterization of the Green's relation $\mathfrak{R}$ on $EOP(X)$ where $(X; \preceq)$ is a fence.

Theorem 3.4. Let $(X; \preceq)$ be a fence where $X = \{a_1, a_2, \dots, a_m\}$ and $a_1 \prec a_2 \succ a_3 \prec \dots \succ (\prec) a_m$. Assume that $\alpha, \beta \in EOP(X)$. Then $\alpha \mathfrak{R} \beta$ if and only if $\ker \alpha = \ker \beta$ and assume that

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ a_1 & a_2 & \dots & a_m \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ b_1 & b_2 & \dots & b_m \end{pmatrix}$$

where $A_i = a_i\alpha^{-1}$ and $\leq$ is the usual order of numbers. Then the following hold:

- (i) $(a_i, a_j) \in E \cap \preceq$ if and only if $(b_i, b_j) \in E \cap \preceq$ for all $i, j \in \{1, 2, 3, \dots, m\}$.
- (ii) For $i \in \{1, 2, 3, \dots, m-1\}$, if $|F_{a_i, a_{i+1}}| \geq 3$ and $\preceq_{F_{a_i, a_{i+1}}} \subseteq E$, then there is an order preserving map $\theta_i : F_{a_i, a_{i+1}} \rightarrow X$ with $a_i\theta_i = b_i$ and $a_{i+1}\theta_i = b_{i+1}$ and $(x\theta_i, y\theta_i) \in E$ for all $x, y \in F_{a_i, a_{i+1}}$ with $x \preceq y$.
- (iii) For $i \in \{1, 2, 3, \dots, m-1\}$, if $|F_{b_i, b_{i+1}}| \geq 3$ and $\preceq_{F_{b_i, b_{i+1}}} \subseteq E$, then there is an E -order preserving map $\theta'_i : F_{b_i, b_{i+1}} \rightarrow X$ with $b_i\theta'_i = a_i$ and $b_{i+1}\theta'_i = a_{i+1}$ and $(x\theta'_i, y\theta'_i) \in E$ for all $x, y \in F_{b_i, b_{i+1}}$ with $x \preceq y$.

Proof. Assume that $\alpha \Re \beta$. Then there are $\delta, \gamma \in EOP(X)$ with $\beta = \alpha\delta$ and $\alpha = \beta\gamma$. It can be proved that $\ker \alpha = \ker \beta$. Consider α and β as follows:

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ a_1 & a_2 & \dots & a_m \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ b_1 & b_2 & \dots & b_m \end{pmatrix}$$

(i) Let $(a_i, a_j) \in E \cap \Re$. Then there are $x \in A_i$ and $y \in A_j$ with $a_i = x\alpha$, $a_j = y\alpha$ and $b_i = x\beta$, $b_j = y\beta$. So, $(x\alpha, y\alpha) \in E \cap \Re$ implies $(x\alpha\delta, y\alpha\delta) \in E \cap \Re$. From $\beta = \alpha\delta$, we have $(b_i, b_j) = (x\beta, y\beta) \in E \cap \Re$. Similarly, $(a_i, a_j) \in E \cap \Re$ if $(b_i, b_j) \in E \cap \Re$.

(ii) Let $i \in \{1, 2, 3, \dots, m-1\}$ and assume that $|F_{a_i, a_{i+1}}| \geq 3$ and $\Re_{F_{a_i, a_{i+1}}} \subseteq E$. Then $\Re_{F_{a_i, a_{i+1}}} \subseteq E \cap \Re$ implies $\Re_{F_{a_i, a_{i+1}}} \cap E = \Re_{F_{a_i, a_{i+1}}}$. It follows that for every $\varphi \in EOP(X)$, the map $\varphi|_{F_{a_i, a_{i+1}}} : F_{a_i, a_{i+1}} \rightarrow X$ is order-preserving. Since $\delta \in EOP(X)$ and $b_i = x\beta = x\alpha\delta = a_i\delta$ for all $x \in A_i$, we consider $\delta|_{F_{a_i, a_{i+1}}}$ as the map θ . Condition (ii) holds.

(iii) The proof is similar as (ii).

Conversely, assume that $\ker \alpha = \ker \beta$ and Condition (i), (ii) hold. For $i \in \{1, 2, 3, \dots, m-1\}$ with $\Re_{F_{a_i, a_{i+1}}} \not\subseteq E$, we define the map $\delta_i : F_{a_i, a_{i+1}} \rightarrow X$ by

$$x\delta_i = \begin{cases} b_k, & (a_i, a_{i+1}) \notin E \text{ and } x \in [a_k] \text{ for some } k \in \{i, i+1\}, \\ b_i, & (a_i, a_{i+1}) \notin E \text{ and } x \notin [a_k] \text{ for all } k \in \{i, i+1\}, \\ b_k, & (a_i, a_{i+1}) \in E \text{ and } \Re_{F_{a_k, x}} \subseteq E \text{ for some } k \in \{i, i+1\}, \\ b_i, & (a_i, a_{i+1}) \in E \text{ and } \Re_{F_{a_k, x}} \not\subseteq E \text{ for all } k \in \{i, i+1\}. \end{cases}$$

where $[a]$ is an equivalence class of E . For $|F_{a_i, a_{i+1}}| = 2$ with $\Re_{F_{a_i, a_{i+1}}} \subseteq E$, we define the map $\theta_i : F_{a_i, a_{i+1}} \rightarrow X$ by $a_i\theta_i = b_i$ and $a_{i+1}\theta_i = b_{i+1}$. By (ii), θ_i is E -order preserving. Define the map $\delta : X \rightarrow X$ by

$$x\delta = \begin{cases} b_1, & x \in F_{a_0, a_1}, \\ x\theta_i, & x \in F_{a_i, a_{i+1}} \text{ and } \Re_{F_{a_i, a_{i+1}}} \subseteq E, \\ x\delta_i, & x \in F_{a_i, a_{i+1}} \text{ and } \Re_{F_{a_i, a_{i+1}}} \not\subseteq E, \\ b_m, & x \in F_{a_m, a_{m+1}} \end{cases}$$

Next, we show that $\delta \in EOP(X)$. Let $(x, y) \in E \cap \Re$.

Case 1 $x, y \in F_{a_i, a_{i+1}}$ where $i \in \{0, 1, 2, \dots, m\}$. If $x, y \in F_{a_0, a_1}$, then $x\delta = b_1 = y\delta$ implying $(x\delta, y\delta) \in E \cap \Re$. Similarly, $(x\delta, y\delta) \in E \cap \Re$ if $x, y \in F_{a_m, a_{m+1}}$. Consider $i \in \{1, 2, 3, \dots, m-1\}$. If $\Re_{F_{a_i, a_{i+1}}} \subseteq E$, then from θ_i is E -order preserving, we have $(x\delta, y\delta) = (x\theta_i, y\theta_i) \in E \cap \Re$. We may let $\Re_{F_{a_i, a_{i+1}}} \not\subseteq E$.

Case 1.1 $(a_i, a_{i+1}) \notin E$. If $x \in [a_k]$ for some $k \in \{i, i+1\}$, then from $(x, y) \in E$, we have $y \in [a_k]$ and so, $x\delta = b_k = y\delta$. If $x \notin [a_k]$ for all $k \in \{i, i+1\}$, then from $(x, y) \in E$, we have $y \notin [a_k]$ for all $k \in \{i, i+1\}$ and so, $x\delta = b_i = y\delta$. Hence, $(x\delta, y\delta) \in E \cap \Re$.

Case 1.2 $(a_i, a_{i+1}) \in E$. If $\preceq_{F_{a_k, x}} \subseteq E$ for some $k \in \{i, i+1\}$, then from $(x, y) \in E$, we have $\preceq_{F_{a_k, y}} \subseteq E$ and so, $x\delta = b_k = y\delta$.

Assume that $\preceq_{F_{a_k, x}} \not\subseteq E$ for all $k \in \{i, i+1\}$. Then there is $(u, v) \in \preceq_{F_{a_k, x}}$ with $(u, v) \notin E$. If $y \notin F_{a_k, x}$, then $F_{a_k, y} = F_{a_k, x} \cup \{y\}$ and so, $(u, v) \in \preceq_{F_{a_k, y}}$. If $y \in F_{a_k, x}$, then $F_{a_k, y} = F_{a_k, x} \setminus \{x\}$ and since $(u, v) \notin E$, we have $(u, v) \neq (x, y)$ implying $(u, v) \in \preceq_{F_{a_k, y}}$. Hence, $\preceq_{F_{a_k, y}} \not\subseteq E$. By the definition of δ , we have $x\delta = b_i = y\delta$ implying $(x\delta, y\delta) \in E \cap \preceq$.

Case 2 $x \in F_{a_i, a_{i+1}}$ and $y \in F_{a_j, a_{j+1}}$ where $i, j \in \{0, 1, 2, \dots, m\}$ and $i \neq j$. Assume that $x \in F_{a_i, a_{i+1}}$. Then from $x \preceq y$, we have $y \in F_{a_{i-1}, a_i}$ or $y \in F_{a_{i+1}, a_{i+2}}$. Moreover, $x = a_i$ or $x = a_{i+1}$ implying $x, y \in F_{a_{i-1}, a_i}$ or $x, y \in F_{a_{i+1}, a_{i+2}}$. By the similar argument in Case 1, $(x\delta, y\delta) \in E \cap \preceq$.

It shows that $\delta \in EOP(X)$. Let $X = \bigcup_{i=1}^m A_i$. Then there is an $i \in \{1, 2, 3, \dots, m\}$ with $x \in A_i$. So, $x\beta = b_i$ and $x\alpha = a_i$. By the definition of δ , we have $a_i\delta = b_i$ implying $x\alpha\delta = a_i\delta = b_i = x\beta$. So, $\beta = \alpha\delta$. We can define γ in the similar way as δ and can prove that $\gamma \in EOP(X)$ and $\alpha = \beta\gamma$. Therefore, $\alpha\mathfrak{R}\beta$. $\square$

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