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ON CONNECTED CLOSED GEODETIC DOMINATION IN GRAPHS

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Abstract. Let G be a simple, undirected and connected graph of order n and $S \subseteq V(G)$. The connected closed geodetic dominating set S of a graph G is a geodetic closure $I_G[S]$ containing all the vertices lying on u-v geodesics (shortest path) where $u, v \in S$. We select vertices of G sequentially as follows: Select a vertex v_1 and let $S_1 = \{v_1\}$. Select a vertex $v_2 \neq v_1$ and $S_2 = \{v_1, v_2\}$. Then determine $I_G[S_2]$. If $I_G[S_2] \neq V(G)$, then successively select vertex $v_i \notin I_G[S_{i-1}]$ and let $S_i = \{v_1, v_2, \ldots, v_i\}$. Since the set $V(G)$ is finite, there is a k for which $I_G[S_k] = V(G)$. Also, the induced subgraph $\langle S \rangle$ must be connected and a dominating set of G. The minimum cardinality of a connected closed geodetic dominating set of G is called the connected closed geodetic domination number of a graph G, denoted by $\gamma_{ccg}(G)$.

In this paper, we introduce and investigate the concept of connected closed geodetic domination and determine the connected closed geodetic domination number of some special graphs. We also present some characterizations of a connected closed geodetic dominating set of graphs resulting from the join of two connected graphs and determine its connected closed geodetic domination number.

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1. Introduction

Graph theory is one of the ever growing fields of research in the literature of Mathematics. Out of its numerous topics, domination in graph has been one of the most recent studied concepts. The origin of the study of the dominating sets in graphs was the "five Queens" problem. The problem of determining the minimum number of queens that can be placed on a chess board, so that all the squares are either attacked by a queen was shown in 1850's with five as the minimum number of queens that can dominate all the squares of a chess board. The dominating queen problem can be stated in general, as the dominating set of vertices of a graph. The very concept on closed geodesic was based from the graphical games called the achievement and avoidance games.

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Let $G = (V(G), E(G))$ be a connected graph. For $S \subseteq V(G)$, the geodetic closure $I_G[S]$ of S is the set of all vertices on geodesic (shortest path) between two vertices of S. We select vertices of G sequentially as follows: Player A selects a vertex v_1 and let $S_1 = \{v_1\}$. Player B selects a vertex $v_2 \neq v_1$ and let $S_2 = \{v_1, v_2\}$. Then determine $I_G[S_1]$, if $I_G[S_2] \neq V(G)$ then the two players will alternately select a vertex $v_i \notin I_G[S_{i-1}]$ and let $S_i = \{v_1, v_2, \ldots, v_i\}$. Since the set $V(G)$ is finite, there is a k for which $I_G[S_k] = V(G)$.

In this paper, the authors will combine both connected closed geodesic and domination in graphs and define a new parameter called connected closed geodetic domination number.

2. Preliminary Concepts and Some Known Results

Definition 2.1. [6] For a graph G, the neighborhood of a vertex $x \in V(G)$ is the set $N_G(x)$ consisting of all vertices y which are adjacent to x, that is,

$$N_G(x) = \{y \in V(G) : xy \in E(G)\}.$$

In addition, a vertex $y \in N_G(x)$ is called a *neighbor of* x.

Definition 2.2. [6] The vertices u, v of a graph G are *adjacent* or *neighbors*, if uv is an edge in G. The set $N_G(u)$ is also called the *open neighborhood* of u in G. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{u \in X} N_G(u)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

Definition 2.3. [5] The *join* $G + H$ of two graphs G and H is the graph with vertex set $V(G + H) = V(G) \dot{\cup} V(H)$ (disjoint union of $V(G)$ and $V(H)$) and edge set

$$E(G + H) = E(G) \dot{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

Definition 2.4. [6] The *removal* of a vertex v from a graph G results in the subgraph G - v that contains all vertices of G except v and all edges not incident with v. If more than one vertex are removed, then we let S be the set of vertices such that $G \backslash S$ is the subgraph of G consisting of all vertices of G except S and all edges not incident with the vertices in S.

Definition 2.5. [5] Let G be a graph . A set $S \subseteq V(G)$ is a *dominating set* of G if for every $v \in V(G) \backslash S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. Any dominating set of G of cardinality $\gamma(G)$ is referred to as a γ*-set* of G.

Definition 2.6. [7] Let G be a connected graph. A dominating set $S \subseteq V(G)$ is called a *connected dominating set* of G if $\langle S \rangle$, the subgraph induced by S, is connected. The *connected domination number*

of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set of G. A connected dominating set of cardinality $\gamma_c(G)$ is called a *minimum connected dominating set* or a γ_c*-set* of G.

Definition 2.7. [3] For every two vertices u and v of G, the symbol $I_G[u,v]$ is used to denote the set of vertices lying on any of the u-v geodesics. Any vertex in $I_G[u,v]$ which is different from u and v is called a *geodetic interior vertex* of u and v. If $S \subseteq V(G)$, we define the geodetic closure of S to be the set $I_G[S]$ given by $I_G[S] = \cup\{I_G[u,v] : u,v \in S\}$.

Definition 2.8. [3] A set $S \subseteq V(G)$ is called a *geodetic cover* if
$I_G[S] = V(G)$. The number $gn(G)$ given by

$$gn(G) = \min\{|S| : I_G[S] = V(G)\}$$

is called the *geodetic number* of G. A geodetic cover S of G for which $|S| = gn(G)$ is called the *minimum geodetic cover or basis* of G.

Definition 2.9. [3] A vertex of a connected graph G is an *extreme vertex* if its neighbors induce a complete subgraph of G. The set of extreme vertices of G is denoted by $Ext(G)$.

Definition 2.10. [2] Given a connected graph G and $S \subseteq V(G)$, the set S is a *closed geodetic cover* of G if $S = \{v_1, v_2, \ldots, v_k\}$ and is obtained by choosing the vertices $v_1, v_2, \ldots, v_k$ such that the following hold:

(i) $v_1 \neq v_2$

(ii) $v_i \notin I_G[S_{i-1}]$ for $3 \leq i \leq k$; and

(iii) $I_G[S_k] = V(G)$,

where $S_i = \{v_1, v_2, \ldots v_i\}$ for all $i = 1, 2, \ldots, k$. The collection of all closed geodetic covers of G is denoted by $C^*(G)$. The *closed geodetic number of* G is given by

$$cgn(G) = \min\{|S| : S \in C^*(G)\}.$$

A set $S \in C^*(G)$ with $|S| = cgn(G)$ is called the closed geodetic basis of G denoted by $cgb(G)$.

Definition 2.11. [1] Let G be a connected graph of order n. Let $S \subseteq V(G)$ such that $S \in C^*(G)$. Then the *connected closed geodetic number* of a graph G, denoted by $ccgn(G)$ is given by

$$ccgn(G) = \{|S| : \langle S\rangle \text{ is connected}\}.$$

Definition 2.12. [1] A set $S \subseteq V(G)$ with $S \in C^*(G)$ is called *connected closed geodetic dominating set* if S is both connected closed geodetic and a dominating set of G. That is, $S \in C^*(G)$ induces a subgraph $\langle S\rangle$ that is connected, and S is dominating. The minimum cardinality of a connected closed geodetic dominating set of G is called the *connected closed geodetic domination number* of a graph G denoted by $\gamma_{ccg}(G)$ and is given by

$$\gamma_{ccg}(G) = \min\{|S| : S \in C^*(G), N_G[S] = V(G) \text{ and } \langle S\rangle \text{ is connected}\}.$$

If $S \subseteq V(G)$ is a connected closed geodetic dominating set of G with $|S| = \gamma_{ccg}(G)$, then S is called a γ_{ccg}-set of G.

Lemma 2.13. [4] *Every geodetic cover of a connected graph G contains its extreme vertices.*

Theorem 2.14. [2] *Let $G = K_{m,n}$, where , $n \geq 2$ and let U and W be the partite sets of G. Let $S \subset V(G)$. Then $S \in C^*(G)$ if and only if S is any of the following:*

(*i*) $S = U$

(*ii*) $S = W$

(*iii*) $S = U \cup \{w\}$ *for some* $w \in W$

(*iv*) $S = W \cup \{u\}$ *for some* $u \in U$.

Theorem 2.15. [2] *Let $G = H + K$, where H and K are connected noncomplete graphs. If either*

(*i*) $S \subseteq V(H), P_2[S]_H = V(H)$ *and* $S \in C^*(H)$ *or*

(*ii*) $S \subseteq V(K), P_2[S]_K = V(K)$ *and* $S \in C^*(K)$,

then $S \in C^*(G)$.

Theorem 2.16. [2] *Let $G = H + K$, where H and K are connected noncomplete graphs. If S is a closed geodetic basis of G, then either*

(*i*) $S \subseteq V(H), P_2[S]_H = V(H)$ *or*

(*ii*) $S \subseteq V(K), P_2[S]_K = V(K)$.,

Remark 2.17. [1] *For any connected graph G, $cgn(G) \leq ccgn(G)$. That is, every connected closed geodetic cover is a closed geodetic cover.*

Theorem 2.18. [1] *For any connected nontrivial graph G of order n,*

$$2 \leq cgn(G) \leq ccgn(G) \leq n.$$

Theorem 2.19. [1] *Let G be a connected graph of order p. Then $ccgn(G) = p$ if and only if $G = K_p$.*

Theorem 2.20. [8] *Let G be a connected graph with cut-vertices. If $S \subseteq V(G)$ is a connected closed geodetic basis of G and w is a cut-vertex of G, then every component of $G - w$ contains an element in S.*

Theorem 2.21. [8] *Let G be a graph with $|V(G)| = p$. If every vertex of G is either an extreme or a cut-vertex of G, then $ccgn(G) = n$.*

Theorem 2.22. [8] *Let H and K be connected noncomplete graphs. If either*

(*i*) $S = \{T \cup \{v\} : v \in V(H)$ *and* $T \subset V(K)\}, P_2[T]_K = V(K)$ *and* $T \in C^*(K)$, *or*

(*ii*) $S = \{T \cup \{w\} : w \in V(K)$ *and* $T \subset V(H)\}, P_2[T]_H = V(H)$ *and* $T \in C^*(H)$,

then $S \in C^*(H + K)$.

Theorem 2.23. [8] *Let* H *and* K *be connected noncomplete graphs. If* S *is a connected closed geodetic basis of* $H + K$. *Then either*

(*i*) $S = \{T \cup \{v\} : v \in V(H)$ *and* $T \subset V(K)\}$, T *is a 2-path closure absorbing in* K, *or*

(*ii*) $S = \{T \cup \{w\} : w \in V(K)$ *and* $T \subset V(H)\}$, T *is a 2-path closure absorbing in* H.

3. Connected Closed Geodetic Domination Number of Some Common Graphs

In this section, we establish the connected closed geodetic domination numbers of several common classes of graphs. We begin by presenting a fundamental result that describes the relationship between the connected closed geodetic number and the connected closed geodetic domination number of a connected graph G. Remark 3.1 serves as the basis for determining the connected closed geodetic domination numbers of the graph families considered in the succeeding results.

Remark 3.1. *For any connected graph* G, $ccgn(G) \leq \gamma_{ccg}(G)$. *That is, every connected closed geodetic domination cover is a connected closed geodetic cover.*

Lemma 3.2. *For any connected nontrivial graph* G *of order* n,

$$2 \leq ccgn(G) \leq \gamma_{ccg}(G) \leq n.$$

Proof: Note that any connected closed geodetic set needs at least two vertices and by Theorem 2.18, $2 \leq ccgn(G)$. Let S be any connected closed geodetic dominating set of G with minimum cardinality. By Remark 3.1, it follows that, $2 \leq ccgn(G) \leq \gamma_{ccg}(G)$. Since $V(G)$ induces a connected closed geodetic dominating set of G, it implies that $\gamma_{ccg}(G) \leq n$. Therefore, the inequality holds. □

Lemma 3.3. *Let* G *be a connected graph with cut-vertices. If* $S \subseteq V(G)$ *is a* $\gamma_{ccg} - set$ *of* G *and* x *is a cut-vertex of* G, *then every component of* $G - x$ *contains an element in* S.

Proof: Let G be a connected graph and $x \in V(G)$ be a cut-vertex of G. Let $S \subseteq V(G)$ be a γ_{ccg}-set of G. Then S is both connected closed geodetic set and a dominating set. Thus, by Remark 3.1, S is a connected closed geodetic cover. Hence, by Theorem 2.20, every component of $G - x$ contains an element in S. □

Theorem 3.4. *Let* G *be a connected graph of order* n. *Then*

(*i*) *every connected closed geodetic dominating set of graph* G *contains its extreme vertices*

(*ii*) *every end vertex belongs to every connected closed geodetic dominating set of* G

(*iii*) *every cut-vertex belongs to every connected closed geodetic dominating set of* G.

Proof: Let G be a connected graph of order n.

(i) Let v be an extreme vertex and let S be a connected closed geodetic dominating set of G. Note that by Remark 3.1, S is a closed geodetic cover of G. Thus S is a closed geodetic dominating set of G. Suppose that $v \notin S$, then by Lemma 2.13, S is not a geodetic set of G. Thus, S is not a closed geodetic dominating set of G, which is a contradiction. Hence, each extreme vertex of G belongs to every closed geodetic dominating set of G.

(ii) Follows directly from (i).

(iii) Let G be a connected graph and $S \subseteq V(G)$ be a γ_{ccg}-set of G and $C_1, C_2, \cdots, C_r, r \geq 2$, be the components of $G - x$. By Lemma 3.3, S contains at least one vertex from each component C_i, $i = 1, 2, \cdots, r$. Since the induced subgraph $\langle S \rangle$ of G is connected, it follows that $x \in S$. □

Lemma 3.5. *Let G be a connected graph of order n. If every vertex of G is either an extreme vertex or a cut-vertex then $\gamma_{ccg}(G) = n$.*

Proof: Let G be a connected graph of order n. Note that by Remark 3.1, S is a connected closed geodetic set of G. Let any $v \in V(G)$ be either an extreme vertex or a cut-vertex. Then by Theorem 2.21, $v \in S$ and $ccgn(G) = n$. This means that $S = V(G)$. Hence, S is a dominating set of G. Therefore, $\gamma_{ccg}(G) = n$. □

Lemma 3.6. *For any path graph P_n, $\gamma_{ccg}(P_n) = n$ for all $n \geq 2$.*

Corollary 3.7. *If $n = 2$, then $\gamma_{ccg}(P_n) = ccgn(P_n)$.*

Corollary 3.8. *The connected closed geodetic domination number of some special graphs are given as follows:*

(*i*) *The cycle graph C_n of n vertices with $n \geq 6$ has $\gamma_{ccg}(C_n) = n - 2$*

(*ii*) *The complement of C_n with $n \geq 5$ has $\gamma_{ccg}(\overline{C_n}) = 4$.*

(*iii*) *The Petersen graph G has $\gamma_{ccg}(G) = 6$.*

(*iv*) *The star graph $K_{1,n-1}$ of $m = n + 1$ vertices has $\gamma_{ccg}(K_{1,n-1}) = m$.*

(*v*) *The wheel graph W_n of $m = n + 1$ with $n \geq 5$ vertices has $\gamma_{ccg}(W_n) = \lceil \frac{n}{2} \rceil + 1$.*

(*vi*) *The fan graph F_n of $m = n + 1$ vertices with $n \geq 4$ has $\gamma_{ccg}(F_n) = \lfloor \frac{n}{2} \rfloor + 1$.*

(*vii*) *The helm graph H_n of $m = 2n + 1$ vertices has $\gamma_{ccg}(H_n) = m - 1$.*

Theorem 3.9. *Let G be a nontrivial connected graph of order p. Then $\gamma_{ccg}(G) = p$ if and only if $G = K_p$.*

Proof: Let G be a nontrivial connected graph or order p. Let $\gamma_{ccg}(G) = p$ and suppose that $G \neq K_p$. Then there exists $u, v \in V(G)$ such that $d_G(u, v) = 2$. Now we construct a set of vertices in G, say $S = \{v_1, v_2, \ldots, v_k\} \in C^*(G)$, where $k < p$ such that $v_1 = u$ and $v_2 = v$. Since $I_G[v_1, v_2] \neq \{v_1, v_2\}$ we have $I_G[S] \neq S$. In fact $I_G[S] = V(G)$. Thus $k < p$ which means that $\gamma_{ccg}(G) < p$. A contradiction to

our assumption that $\gamma_{ccg}(G) = p$. Therefore, $G = K_p$.

For the converse, let $G = K_p$. Since G is a complete graph for any $u, v \in V(G)$, $d_G(u, v) = 1$ and $I_G[u, v] = \{u, v\}$. Thus every vertex of $V(G)$ is an extreme vertex. By Theorem 3.4, $V(G)$ is a connected closed geodetic dominating set of G. Therefore, $\gamma_{ccg}(G) = p$. □

Theorem 3.10. *Let G be a nontrivial connected graph with order p. Then $ccgn(G) = \gamma_{ccg}(G)$ if and only if $G = K_p$ a complete graph of order $p \geq 2$.*

Proof: Let $G = K_p$ be a complete graph of order p. Then by Theorem 2.19 and Theorem 3.9, $ccgn(G) = p = \gamma_{ccg}(G)$. Conversely, let $ccgn(G) = \gamma_{ccg}(G)$. Suppose that $G \neq K_p$. Then there exists $u, v \in V(G)$ such that $d_G(u, v) = 2$. Let $[u, w, v]$ be a u-v geodesic. Then we can construct a set $S = \{v_1, v_2, \ldots, v_k\} = V(G)\backslash\{w\}$, since $w \in I_G[u, v]$ for which $v_1 = u$ and $v_{k-1} = u$ and $I_G[S] = V(G)$. Since $\langle S\rangle$ is connected and also the set S dominates all the vertices in G including w, $k < p$ which means that $\gamma_{ccg}(G) < p$. A contradiction to our assumption that $\gamma_{ccg}(G) = p$. Therefore, $G = K_p$. □

Corollary 3.11. *If $G = K_2 = P_2$, then $\gamma_{ccg}(G) = 2$.*

Lemma 3.12. *For any complete graph $K_n (n \geq 2)$, $\gamma_{ccg}(K_n) = n$.*

Theorem 3.13. *For any two integers $m, n \geq 2$, the connected closed geodetic domination number of the complete bipartite graph $K_{m,n}$ is $\gamma_{ccg}(K_{m,n}) = \min\{m, n\} + 1$.*

Proof: Let $G = K_{m,n}$. Let U and W be the partite sets of $V(G)$ with orders m and n, respectively. Note that by Theorem 2.14 the only possible closed geodetic covers S of G are when $S = U$; $S = W$; $S = U \cup \{w\}$ for some $w \in W$; and $S = W \cup \{u\}$ for some $u \in U$. Observe that the only connected sets are the sets when $S = U \cup \{w\}$ for some $w \in W$ and $S = W \cup \{u\}$ for some $u \in U$. Thus, it remains to show that the set S is a dominating set of G. Now we consider two cases:

Case 1: If $S = U \cup \{w\}$ for some $w \in W$. Note that w is adjacent to every vertex in U. Thus, $N_G[w] = N_G(w) \cup \{w\}$.

Case 2: If $S = W \cup \{u\}$ for some $u \in U$. Similarly, since u is adjacent to every vertex in W. Thus $N_G[u] = W \cup \{u\}$.

Therefore, $\gamma_{ccg}(K_{m,n}) = \min\{m, n\} + 1$. □

Corollary 3.14. *If $m = n$, then $\gamma_{ccg}(K_{m,n}) = n + 1$ for $n \geq 2$.*

Corollary 3.15. *The connected closed geodetic domination number of the complete multipartite graph $K_{p_1,p_2,\ldots p_k}$ with order $\sum_{i=1}^{k} p_i$ is $\gamma_{ccg}(K_{p_1,p_2,\ldots p_k}) = \min\{p_1, p_2, \ldots p_k\} + 1$.*

Lemma 3.16. *Let T be a tree with n vertices, then the connected closed geodetic domination number of T is $\gamma_{ccg}(T) = n$.*

Remark 3.17. For any tree, $\gamma_{ccg}(T) = \gamma_c(T) + ccgn(T)$.

4. Connected Closed Geodetic Domination Number of the Join of Two Graphs

In this section, we discuss the connected closed geodetic domination number of a graph obtained from the join of two graphs.

Lemma 4.1. *The connected closed geodetic domination number of a graph G, $\gamma_{ccg}(G) = 3$ if and only if either $G = K_3$ or $G = \bar{K_2} + H$ for some connected graph H.*

Proof: Suppose that $\gamma_{ccg}(G) = 3$ and let S be the minimum connected closed geodetic dominating set of G. By Theorem 3.10, $\gamma_{ccg}(G) = 3$ implies $G = K_3$. Otherwise, if S is a path of order 3, say $I_G[S] = \{v_1, v_2, v_3\}$ with $N_G[S] = V(G)$ and $\langle S\rangle$ is connected. Thus each vertex in $V(G)\backslash\{v_1, v_3\}$ must be adjacent to both v_1 and v_3. Hence there exists a connected graph say H such that v_1 and v_3 are adjacent to every vertex in H.Therefore $G = \bar{K_2} + H$. The necessity of this proof immediately follows. □

Corollary 4.2. *For any complete graphs K_m and K_n. Then*

$$\gamma_{ccg}(\bar{K_m} + \bar{K_n}) = \min\{m, n\} + 1.$$

Since the join of two complete graphs is a complete graph, we have the following consequence of Theorem 3.10.

Corollary 4.3. *Let K_m and K_n be complete graphs of orders m and n respectively. Then $\gamma_{ccg}(K_m+K_n) = m+n$.*

The next results will give the necessary condition for the graphs $G = H + K_p$ that has a connected closed geodetic dominating set where H is a connected noncomplete graph and K_p a complete graph. First let us consider the collection of minimal connected closed geodetic dominating set of a connected graph G denoted by $C^*(G)$.

Lemma 4.4. *Let G be a connected noncomplete graph, and $S \subseteq V(G)$. If $S \in C^*(G)$ or a 2-path connected closure absorbing set, then the induced subgraph $\langle S\rangle$ is not complete.*

Proof: Let G be a connected noncomplete graph. Let $S \subseteq V(G)$ be a 2-path closure absorbing set in G. If $S = V(G)$, clearly the result is trivial. Suppose that $S \neq V(G)$. Let $w \in (V(G) \cap S)$. Since S is a connected 2-path closure absorbing set, then there exist $x \in S$ such that $d_{\langle S\rangle}(u, w) = d_{\langle S\rangle}(w, u) = 1$. Let $w \notin S$, then there exist $u, v \in S$ such that $d_{\langle S\rangle}(u, v) = d_G(u, v) = 2$ and $w \in I_G[u, v]$. These show that the subgraph $\langle S\rangle$ of G is not complete. Suppose that $S \in C^*(G)$ and $\langle S\rangle$ is complete. Then $I_G[S] = S$, a contradiction. Therefore, the result follows. □

Theorem 4.5. *Let H be a connected noncomplete graph and let $G = H + K_p$. If S is a 2-path connected closure absorbing in H and $S \subseteq V(H)$ if and only if $S \subseteq V(G)$.*

Proof: Let H be a connected noncomplete graph and let $G = H + K_p$. Suppose that S is a 2-path closure absorbing in H and $S \subseteq V(H)$. Note that by Remark 2.17, S is a closed geodetic cover of G. Then by Theorem 2.15, $S \in C^*(G)$. It remains to show that S dominates the graph G. Since every vertices in H is adjacent to every vertices in K_p, there exist $u, v \in S$ with $d_{\langle H \rangle}(u, v) = 2$ such that every vertices in K_p lies a u-v geodesic in G and $d_G(u, v) = 2$. This implies that $V(K_p) \subseteq N_G[S]$. Thus, $V(G) = V(H) \cup V(K_p) \subseteq N_G[S]$. Hence, S dominates the graph G. Therefore, S is a γ_{ccg}-set, that is $S \subseteq V(G)$.

For the converse, suppose $S \subseteq V(G)$. Let $w \in V(H) \backslash S$, by Lemma 4.4, $\langle S \rangle$ is not complete. Thus there exist $u, v \in S$ such that $d_{\langle H \rangle}(u, v) = 2$ and $w \in I_{\langle H \rangle}[u, v]$. It follows that $u, v \in V(H) \cap S$. Suppose further that $w \in V(K_p) \cap S$. Since S is connected, $d_G(w, u) = 1$ for all $u \in V(H) \cap S$ and $u \neq w$. However, if $w \in V(K_p) \backslash S$, then again by Lemma 4.4, there exist $u, v \in S$ such that $d_{\langle H \rangle}(u, v) = 2$ and that $w \in I_G[u, v]$. Consequently, $I_G[S] = V(G)$. Therefore, S is a 2-path closure absorbing in G. Since all vertices in $V(K_p)$ is an extreme vertices and also by the fact that S is 2-path connected closure absorbing set, these follow that $S \subseteq V(H)$. □

Theorem 4.6. *Let H be a connected noncomplete graph and $G = H + K_p$. If S is a γ_{ccg}-set of G, then $S \subseteq V(H)$ and S is a 2-path connected closure absorbing set in G.*

Proof: Let H be a connected noncomplete graph and $G = H + K_p$. Suppose that $S \subseteq V(G)$ is a γ_{ccg}-set of G. Suppose $w \in V(H) \backslash S$, by Lemma 4.4, $\langle S \rangle$ is not complete. Thus there exist $u, v \in S$ such that $d_H(u, v) = 2$ and $w \in I_H[u, v]$. It follows that, $u, v \in (V(H) \cap S)$. Suppose further that $w \in (K_p \cap S)$, since S is connected $d_G(u, w) = 1$ for all $u \in (V(H) \cap S)$ and $u \neq w$. However, if $w \in (V(K_p) \backslash S)$, then by Lemma 4.4, there exist $u, v \in S$ such that $d_H(u, v) = 2$ and that $w \in I_G[u, v]$. Consequently, $I_G[S] = V(G)$. Therefore, S is a 2-path connected closure absorbing set in G. □

Corollary 4.7. *Let H be a connected noncomplete graph and $G = H + K_p$, then*

$$\gamma_{ccg}(G) = \min\{|S| : S = T \cup \{v\} \text{ where } v \in V(K_p), S \in C^*(G) \text{ and } P_2[T]_H = V(H)\}.$$

Proof: Let $G = H + K_p$ where H connected noncomplete graph. Thus, S is a connected closed geodetic and a dominating set. Let T be the set of minimum cardinality such that $T \in C^*(G)$. Then by Theorem 2.16, $P_2[T]_H = V(H)$ for which $T \subseteq V(H)$. Thus $\langle T \rangle$ is not connected in G. Hence, construct $S = T \cup \{v\}$ where $v \in K_p$. Hence $\langle S \rangle$ is connected in G. Note that for every vertex $v \in K_p$, $N_G[S] = V(G)$. Thus S is a dominating set of G. Therefore, the result follows. □

Corollary 4.8. *If H is a connected noncomplete graph and $G = H + K_p$, then $\gamma_{ccg}(G) = ccgn(G)$.*

Corollary 4.9. *Let H be a connected noncomplete graph and let $G = H + K_p$. If $diam(H) = 2$, then $\gamma_{ccg}(G) = \gamma_{ccg}(H)$.*

Proof: Let $G = H + K_p$ where H is a connected noncomplete graph with $diam(H) = 2$. Let $S \subseteq V(H)$ and $\langle S \rangle \in G$. Then by Corollary 4.7, $\gamma_{ccg}(G) = |S| \geq \gamma_{ccg}(H)$. On the other hand suppose S be a γ_{ccg}-set of G. Then by Theorem 4.6, then $S \subseteq V(H)$ and S is a 2-path connected closure absorbing set in G. Since S is a 2-path connected closure set in G and $diam(H) = 2$. Thus, $\gamma_{ccg}(H) = |V(H)|$. Therefore, $\gamma_{ccg}(G) = |S| \leq |V(H)| = \gamma_{ccg}(H)$. Thus, the result follows. □

Corollary 4.10. *The connected closed geodetic domination number of the join of some special graph and a complete graph are given as follow:*

(i) $\gamma_{ccg}(K_p + P_n) = \lceil \frac{n+1}{2} \rceil + 1$, *for* $n \geq 3$.
(ii) $\gamma_{ccg}(K_p + C_n) = \lceil \frac{n+1}{2} \rceil + 1$.
(iii) $\gamma_{ccg}(C_n + K_1) = \gamma_{ccg}(W_n) = \lceil \frac{n+1}{2} \rceil + 1$, *for* $n \geq 3$.
(iv) $\gamma_{ccg}(K_{m,n} + K_p) = \min\{m, n\} + 1$.
(v) $\gamma_{ccg}(F_n + K_p) = \lceil \frac{n+1}{2} \rceil + 1$.
(vi) $\gamma_{ccg}(W_m + K_p) = \lceil \frac{n}{2} \rceil + 1$.

To this end, we will give necessary conditions for those graphs whose join has a connected closed geodetic dominating set. First, let us consider the following results.

Theorem 4.11. *Let $G = H + K$, where H and K are connected noncomplete graphs. If either*

(i) $S = \{T \cup \{v\} : v \in V(H)$ *and* $T \subseteq V(K)\}$, T *is a 2-path closure absorbing set in K and* $\langle T \rangle \in K$, *or*
(ii) $S = \{T \cup \{v\} : v \in V(K)$ *and* $T \subseteq V(H)\}$, T *is a 2-path closure absorbing set in H and* $\langle T \rangle \in K$

then $S \in C^*(G)$.

Proof: Let $S = \{T \cup \{v\} : v \in V(H)$ and $T \subseteq V(K)\}$ and T be a 2-path closure absorbing set in K. Note that $\langle T \rangle \in K$. Then $T \in C^*(H)$, by Theorem 2.22, $S \in C^*(G)$. Since T is a 2-path closure absorbing set in K and by joining a vertex $v \in V(G) \backslash V(K)$ it implies that $T \cup \{v\}$ is connected in G. Now it remains to show that S dominates G. Since T is a 2-path closure in K and every vertices in K are adjacent to every vertices in H, there exist $u, v \in T$ such that every vertices in H lies on a u-v geodesic with $d_G(u, v) = 2$. Thus T dominates in H. Consequently, S dominates G. Hence $S \in C^*(G)$. A similar argument shows the other case. □

Theorem 4.12. *Let $G = H + K$, where H and K are connected noncomplete graphs. If S is a γ_{ccg}-set of G, then either*

(*i*) $S = \{T \cup \{v\} : v \in V(H) \text{ and } T \subseteq V(K)\}$, T *is a 2-path closure absorbing set in* K, *or*

(*ii*) $S = \{T \cup \{v\} : v \in V(K) \text{ and } T \subseteq V(H)\}$, T *is a 2-path closure absorbing set in* H.

Proof: Let $G = H + K$, where H and K are connected noncomplete graphs. Suppose that S is a γ_{ccg}-set of G. Then S is also a ccg-set of G and at the same time dominating set of G. Thus, by Theorem 2.23, the conclusion follows. □

Theorem 4.13. *Let $G = H + K$, where H and K are connected noncomplete graphs. Then*

$$\gamma_{ccg}(G) = \min\{\Gamma, \Omega\}$$

where $\Gamma = \{|S| : S = \{T \cup \{v\} : v \in V(K) \text{ and } T \subseteq V(H)\}, T \in C^*(G) \text{ and } T \text{ is a 2-path closure absorbing set in } H\}$ *and* $\Omega = \{|S| : S = \{T \cup \{v\} : v \in V(H) \text{ and } T \subseteq V(K)\}, T \in C^*(G) \text{ and } T \text{ is a 2-path closure absorbing set in } K\}$.

Proof: Let H and K be connected noncomplete graphs and $G = H + K$. Suppose that $S \subseteq V(G)$ is a γ_{ccg}-set of G. Then by Theorem 4.12, we have $S = \{T \cup \{v\} : v \in V(H)$ and $T \subseteq V(K)\}$ and T is a 2-path closure absorbing set in K or $S = \{T \cup \{v\} : v \in V(K)$ and $T \subseteq V(H)\}$ and T is a 2-path closure absorbing set in H. Thus, $\gamma_{ccg}(G) \geq \min\{\Gamma, \Omega\}$, where $\Gamma = \{|S| : S = \{T \cup \{v\} : v \in V(K)$ and $T \subseteq V(H)\}, T \in C^*(G)$ and T is a 2-path closure absorbing set in $H\}$ and $\Omega = \{|S| : S = \{T \cup \{v\} : v \in V(H)$ and $T \subseteq V(K)\}, T \in C^*(G)$ and T is a 2-path closure absorbing set in $K\}$. By Theorem 4.11, $\gamma_{ccg} \leq \min\{\Gamma, \Omega\}$. Combining the two inequalities, the results follows. □

Corollary 4.14. *The connected closed geodetic domination number of the join of some special graphs are given as follows:*

(*i*) $\gamma_{ccg}(P_m + P_n) = \min\{\lceil \frac{m+1}{2} \rceil, \lceil \frac{n+1}{2} \rceil + 1$ *for* $m, n > 2$.

(*ii*) $\gamma_{ccg}(C_m + C_n) = \min\{m, n\} + 1$.

(*iii*) $\gamma_{ccg}(C_m + C_n) = \min\{m, n\} + 1$.

(*iv*) $\gamma_{ccg}(P_m + C_n) = \min\{\lceil \frac{m+1}{2} \rceil, \lceil \frac{n}{2} \rceil + 1$.

Corollary 4.15. *Let $G = H + K$, where H and K are connected noncomplete graphs. If $diam(H) = diam(K) = 2$, then $\gamma_{ccg}(G) = 3$.*

Proof: Let $G = H + K$, where H and K are connected noncomplete graphs with $diam(H) = diam(K) = 2$. Note that $H, K \in \{P_3, K_{1,n-1}\}$. The only possible joins of graph are when $P_3 + P_3$, $P_3 + K_{1,n-1}$ and $K_{1,n-1} + K_{1,n-1}$, either case the diam $H + K$ is 3. Then by applying Theorem 4.13, we can have the result $\gamma_{ccg}(G) = 3$. □

5. Conclusion

This paper introduced the concept of the connected closed geodetic domination number of a connected graph, which combines the properties of geodetic closure, connectivity, and domination into a single graph parameter. Fundamental properties of connected closed geodetic dominating sets were established, providing a framework for analyzing this new parameter. The connected closed geodetic domination numbers of several classes of special graphs were determined, illustrating how the parameter behaves in different graph structures. Moreover, characterizations of connected closed geodetic dominating sets in graphs obtained from the join of two connected graphs were presented, leading to explicit results for their connected closed geodetic domination numbers. These findings broaden the study of geodetic domination in graphs and provide a foundation for further investigations on this parameter in other graph families and graph operations.

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