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BACKWARDS MARTINGALE AND BACKWARDS ITÔ-HENSTOCK INTEGRAL FOR THE HILBERT SCHMIDT-VALUED STOCHASTIC PROCESS

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Abstract. It is known that $\mathbb{P}(W_t \in U_Q) = 0$, $t > 0$, if $\dim U_Q = \infty$ [4, Theorem 2.4.7]. Hence, the Riemann sum $(D)\sum f_\xi(W_\xi - W_u)$ in [24, Definition 3.3] does not make sense pathwise as an application of an operator $f_\xi \in L_2(U_Q, V)$ to a vector in a Hilbert space. In this paper, we introduce the backwards Itô-Henstock integral for backwards adapted Hilbert-Schmidt-valued stochastic processes. We also provide a correction to [23, Definition 1].

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1. Introduction and Preliminaries

The Henstock integral, also known as the Henstock-Kurzweil integral, was developed independently by Henstock and Kurzweil in the 1950s. It is one of the most significant extensions of the classical theory of integration, addressing certain limitations of the Lebesgue integral, particularly on compact intervals. For instance, the derivative of the function $f(x) = x^2 \sin x^{-2}$ if $x \neq 0$ and $f(0) = 0$ is not Lebesgue integrable on $[0, 1]$. However, it is Henstock integrable on $[0, 1]$; see [8, Theorem 9.6].

A real-valued function f is said to be *Henstock integrable* [13, Definition 2.2] to $A \in \mathbb{R}$ on $[a, b]$ if for every $\varepsilon > 0$, there is a function $\delta(\xi) > 0$ such that whenever a division D given by $a = x_0 < x_1 < \cdots < x_n = b$ and $\{\xi_i, \xi_2, \ldots, \xi_n\}$ satisfies $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, \ldots, n$ we have $|\sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) - A| < \varepsilon$. The Henstock integral of f on $[a, b]$ is given by A and it is uniquely determined.

It is evident that this definition closely resembles that of the Riemann integral. Consequently, the Henstock integral provides a more direct and intuitive approach to integration, without requiring

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the deep machinery of measure theory. Moreover, measurability of the integrand is not assumed in the definition. It is known that the Henstock integral includes the improper Riemann, Lebesgue, and Newton integrals, and it is particularly effective for integrating highly oscillatory functions. Owing to these features, the Henstock integral has attracted considerable attention from researchers; see [8–11,13–17]. This approach to integration is known as the generalized Riemann approach or Henstock-Kurzweil approach, which has also been used to deal with stochastic integration for real-valued cases, see [5,18,19,25–28].

In 2009, Arcede and Cabral [2,3] introduced the concept of backwards Itô integral with respect to a Brownian motion. Unlike the classical Itô integral, this formulation considers backwards filtrations and *backwards δ-fine* partial divisions. In this framework, the tags correspond to the right endpoints of a collection of non-overlapping subintervals of $[0,T]$. Motivated by the work of Arcede and Cabral, the authors in [23] extended the backwards Itô-Henstock integral to operator-valued stochastic processes with respect to a Hilbert space-valued Q-Wiener process. However, the Riemann sum $(D)\sum f_\xi(W_\xi - W_u)$ in [23, Definition 1] does not make sense pathwise as an application of an operator $f_\xi \in L_2(U_Q,V)$ to a vector in a Hilbert space since $\mathbb{P}(W_t \in U_Q) = 0$, $t > 0$, if $\dim U_Q = \infty$ [4, Theorem 2.4.7].

In the present paper, we define the backwards Itô-Henstock integral for backwards adapted Hilbert-Schmidt-valued stochastic processes and provide a correction to [23, Definition 1]. We also establish a version of the Itô-isometry and demonstrate that the stochastic integral, with a fixed upper bound and a varying lower bound, belongs to the space of continuous, square-integrable backward martingales.

Throughout this paper, the closed interval $[0,T]$ is nondegenerate, i.e. $0 < T$ and can be replaced with any closed interval $[a,b]$ and $\{\Omega,\mathcal{G},\mathbb{P}\}$ denotes a probability space. Let $\{\mathcal{G}_t : 0 \le t \le T\}$ be a family of sub σ-field of $\mathcal{G}$. Then $\{\mathcal{G}_t : 0 \le t \le T\}$ is called a *backwards filtration* if $\mathcal{G}_t \subseteq \mathcal{G}_s$ for all $0 \le s \le t \le T$. If in addition, $\{\mathcal{G}_t : 0 \le t \le T\}$ satisfies the following conditions: (i) $\mathcal{G}_T$ contains all sets of $\mathbb{P}$-measure zero in $\mathcal{G}$; and (ii) for each $t \in (0,T]$, $\mathcal{G}_t = \mathcal{G}_{t-} := \bigcap_{s<t}\mathcal{G}_s$. Then $\{\mathcal{G}_t : 0 \le t \le T\}$ is called a *standard backwards filtration*. We often write $\{\mathcal{G}_t\}$ instead of $\{\mathcal{G}_t : 0 \le t \le T\}$, see [1].

Let U and V be separable Hilbert spaces, $L(U,V)$ be the space of all bounded linear operators from U to V, $L(U) := L(U,U)$, $Qu := Q(u)$ if $Q \in L(U,V)$, and $L^2(\Omega,V)$ be the space of all square-integrable random variables from Ω to V. If $Q \in L(U)$ is a symmetric nonnegative definite trace-class operator, then there exists an orthonormal basis (abbrev. as ONB) $\{e_j\} \subset U$ and a sequence of nonnegative real numbers $\{\lambda_j\}$ such that $Qe_j = \lambda_j e_j$ for all $j \in \mathbb{N}$, $\{\lambda_j\} \in \ell^1$, see [21, p.203]. We shall call the sequence of pairs $\{\lambda_j, e_j\}$ an eigensequence defined by Q. The subspace U_Q of U equipped with the inner product $\langle u,v\rangle_{U_Q} = \langle Q^{-1/2}u, Q^{-1/2}v\rangle_U$, where $Q^{1/2}$ is being restricted to $[\mathrm{Ker}\, Q^{1/2}]^\perp$, is a separable Hilbert space with $\{\sqrt{\lambda_j}e_j\}$ as its ONB, see [6, p.90], [7, p.23]. The space $L_2(U_Q,V)$ of all Hilbert-Schmidt operators from U_Q to V is a separable Hilbert space with norm $\|S\|_{L_2(U_Q,V)} = \sqrt{\sum_{j=1}^\infty \|Sf_j\|_V^2}$, see [20, p.112]. The Hilbert-Schmidt operator $S \in L_2(U_Q,V)$ and the norm $\|S\|_{L_2(U_Q,V)}$ may be defined in terms of an

arbitrary ONB, see [6, p.418], [20, p.111]. We note that $L(U,V)$ is properly contained in $L_2(U_Q,V)$, see [7, p.25]. We also note that $L_2(U_Q,V)$ contains genuinely unbounded linear operators from U to V.

Let $Q:U\to U$ be a symmetric nonnegative definite trace-class operator, $\{\lambda_j,e_j\}$ be an eigensequence defined by Q, and $\{B^{(j)}\}$ be a sequence of independent Brownian motions (abbrev. as BM) defined on a filtered probability space $(\Omega,\mathcal{G},\{\mathcal{G}_t\},\mathbb{P})$. A process $\tilde{W}_t:=\sum_{j=1}^{\infty}\sqrt{\lambda_j}B_t^{(j)}e_j$, is called a Q-Wiener process in U, with the series converging in $L^2(\Omega,U)$. For every $u\in U$, denote $\tilde{W}_t(u):=\sum_{j=1}^{\infty}\sqrt{\lambda_j}B_t^{(j)}\langle e_j,u\rangle_U$, with the series converging in $L^2(\Omega,\mathbb{R})$. Then there exists a U-valued process W, known as a U-valued Q-Wiener process, such that $\tilde{W}_t(u)(\omega)=\langle W_t(\omega),u\rangle_U$ $\mathbb{P}$-almost surely (abbrev. as $\mathbb{P}$-a.s.). It should be noted that the process W is an infinite-dimentional BM, and if we assume that $\lambda_j>0$ for all j, $\frac{W_t(e_j)}{\sqrt{\lambda_j}}$, $j=1,2,\ldots$, is a sequence of real-valued BM defined on $(\Omega,\mathcal{G},\{\mathcal{G}_t\},\mathbb{P})$, see [6, p.87].

Let $\mathcal{N}:=\{A\in\mathcal{G}:\mathbb{P}(A)=0\}$, $\tilde{\mathcal{G}}_t^W:=\sigma\{W_T-W_s:t\le s\le T\}$, $\tilde{\mathcal{G}}_t^{W,0}:=\sigma(\tilde{\mathcal{G}}_t^W\cup\mathcal{N})$, and

$$\mathcal{G}_t:=\mathcal{G}_t^W:=\bigcap_{s<t}\tilde{\mathcal{G}}_s^{W,0},\quad t\in(0,T]. \tag{1}$$

Then $\{\mathcal{G}_t:0\le t\le T\}$ is a standard backwards filtration, see [23, Proposition 1]. In view of [23, Proposition 2], a U-valued Q-Wiener process W is a Q-Wiener process with respect to a standard backwards filtration $\{\mathcal{G}_t:0\le t\le T\}$ given in (1). From now onwards, if no confusion arises, a filtered probability space $(\Omega,\mathcal{G},\{\mathcal{G}_t\},\mathbb{P})$ shall mean a probability space equipped with the standard backwards filtration given in (1).

2. Backwards Martingale and Backwards Itô-Henstock Integral

In the following discussions, if no confusion arises, we may write $(D)\sum$ instead of $\sum_{i=1}^{n}$ for the given finite collection D.

A stochastic process $M:[0,T]\times\Omega\to V$ is said to be a *backwards martingale* (or *reverse martingale*) if (i) M is backwards adapted; (ii) for all $t\in[0,T]$, M_t is Bochner integrable, i.e. $\mathbb{E}[\|M_t\|_V]<\infty$; and (iii) for any $0\le s\le t\le T$, $\mathbb{E}\left[M_s\middle|\mathcal{G}_t\right]=M_t$ $\mathbb{P}$-a.s.. A backwards martingale $M:[0,T]\times\Omega\to V$ is said to be *square-integrable* if $M_t\in L^2(\Omega,V)$ for all $t\in[0,T]$. Denote by $\mathcal{BM}_T^2:=\mathcal{BM}_T^2(V)$, the collection of all V-valued continuous square-integrable backwards martingales. Following the same argument as in [20, p.21], one can verify that $\mathcal{BM}_T^2:=\mathcal{BM}_T^2(V)$ is a Banach space with norm $\|M\|_{\mathcal{BM}_T^2}:=\sup_{t\in[0,T]}\left(\mathbb{E}\left[\|M_t\|_V^2\right]\right)^{\frac{1}{2}}$.

Before defining the backwards Itô-Henstock integral of a backwards adapted process, we shall consider first the following concepts and results.

Definition 2.1. [3] Let δ be a positive function defined on $[0,T]$. A finite collection $D=\{([u_i,\xi_i],\xi_i)\}_{i=1}^n$ of interval-point pairs is a *backwards δ-fine partial division* of $[0,T]$ if $\{[u_i,\xi_i]\}_{i=1}^n$ is a collection of non-overlapping subintervals of $[0,T]$ and each $[u_i,\xi_i]$ is *backwards δ-fine*, that is, $[u_i,\xi_i]\subset(\xi_i-\delta(\xi_i),\xi_i]$.

The term *partial division* is used in Definition 2.1, to emphasize that the finite collection of non-overlapping intervals of $[0,T]$ may not cover the entire interval $[0,T]$. Using the Vitali covering theorem, the following concept can be defined.

Definition 2.2. [3] Given $\eta>0$, a given backwards δ-fine partial division $D=\{([u,\xi],\xi)\}$ of $[0,T]$ is said to be a *backwards (δ,η)-fine partial division* of $[0,T]$ if it fails to cover $[0,T]$ by at most length η, that is,

$$\left|T-(D)\sum(\xi-u)\right|\leq\eta.$$

The following lemma is important before defining the backwards Itô-Henstock integral. The proof of the following lemma is analogous with the proof of [12, Lemma 2.3]. A stochastic process $f:[0,T]\times\Omega\to L_2(U_Q,V)$ means a process measurable as mapping from $([0,T]\times\Omega,\mathcal{B}([0,T])\times\mathcal{G})$ to $(L_2(U_Q,V),\mathcal{B}(L_2(U_Q,V)))$.

Lemma 2.3. *Let $f:[0,T]\times\Omega\to L_2(U_Q,V)$ be a backwards adapted process. For $0\leq u\leq\xi\leq T$, $\sum_{j=1}^{\infty}(B_\xi^{(j)}-B_u^{(j)})f_\xi(\sqrt{\lambda_j}e_j)\in L^2(\Omega,V)$.*

Proof. Let $m,n\in\mathbb{N}$ with $n>m$. Then

$$\begin{aligned}
&\mathbb{E}\left[\left\|\sum_{j=m}^{n}(B_\xi^{(j)}-B_u^{(j)})f_\xi(\sqrt{\lambda_j}e_j)\right\|_V^2\right]\\
=&\sum_{j=m}^{n}\mathbb{E}\left[\mathbb{E}\left[(B_\xi^{(j)}-B_u^{(j)})^2\left\langle f_\xi(\sqrt{\lambda_j}e_j),f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V\middle|\mathcal{G}_\xi\right]\right]\\
&+\sum_{j\neq j'}\mathbb{E}\left[\mathbb{E}\left[(B_\xi^{(j)}-B_u^{(j)})(B_\xi^{(j')}-B_u^{(j')})\left\langle f_\xi(\sqrt{\lambda_j}e_j),f_\xi(\sqrt{\lambda_{j'}}e_{j'})\right\rangle_V\middle|\mathcal{G}_\xi\right]\right]\\
=&\sum_{j=m}^{n}\mathbb{E}\left[\left\langle f_\xi(\sqrt{\lambda_j}e_j),f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V\mathbb{E}\left[(B_\xi^{(j)}-B_u^{(j)})^2\middle|\mathcal{G}_\xi\right]\right]\\
&+\sum_{j\neq j'}\mathbb{E}\left[\left\langle f_\xi(\sqrt{\lambda_j}e_j),f_\xi(\sqrt{\lambda_{j'}}e_{j'})\right\rangle_V\mathbb{E}\left[(B_\xi^{(j)}-B_u^{(j)})(B_\xi^{(j')}-B_u^{(j')})\middle|\mathcal{G}_\xi\right]\right]\\
=&(\xi-u)\sum_{j=m}^{n}\mathbb{E}\left[\left\langle f_\xi(\sqrt{\lambda_j}e_j),f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V\right]=(\xi-u)\sum_{j=m}^{n}\mathbb{E}\left[\left\|f_\xi(\sqrt{\lambda_j}e_j)\right\|_V^2\right].
\end{aligned}$$

Let $S_k = \sum_{j=1}^{k} \left\| f_\xi(\sqrt{\lambda_j} e_j) \right\|_V^2$. Using the Beppo Levi's lemma [22, p.370], we have

$$\lim_{k\to\infty} \mathbb{E}\left[\sum_{j=1}^{k} \left\| f_\xi(\sqrt{\lambda_j} e_j) \right\|_V^2 \right] = \mathbb{E}\left[\lim_{k\to\infty} \sum_{j=1}^{k} \left\| f_\xi(\sqrt{\lambda_j} e_j) \right\|_V^2 \right] < \infty.$$

Hence,

$$\sum_{j=1}^{\infty} \mathbb{E}\left[\left\| f_\xi(\sqrt{\lambda_j} e_j) \right\|_V^2 \right] = \mathbb{E}\left[\sum_{j=1}^{\infty} \left\| f_\xi(\sqrt{\lambda_j} e_j) \right\|_V^2 \right] = \mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)} \right] < \infty.$$

It follows that $M_n := \sum_{j=1}^{n} (B_\xi^{(j)} - B_u^{(j)}) f_\xi(\sqrt{\lambda_j} e_j)$ is Cauchy in $L^2(\Omega, V)$, hence convergent in $L^2(\Omega, V)$.

Thus, $\sum_{j=1}^{\infty} (B_\xi^{(j)} - B_u^{(j)}) f_\xi(\sqrt{\lambda_j} e_j) \in L^2(\Omega, V)$. □

We shall now define the backwards Itô-Henstock integral of backwards adapted process. Again, we need to note that $[0, T]$ can be replaced with any closed interval $[a, b]$.

Definition 2.4. Let $f : [0, T] \times \Omega \to L_2(U_Q, V)$ be a backwards adapted process. Then f is said to be *backwards Itô-Henstock integrable*, or *$\mathcal{IH}_B$-integrable*, on $[0, T]$ with respect to W if there exists $A \in L^2(\Omega, V)$ such that for every $\varepsilon > 0$, there is a positive function δ on $[0, T]$ and a number $\eta > 0$ such that for any backwards (δ, η)-fine partial division $D = \{([u_i, \xi_i], \xi_i)\}_{i=1}^n$ of $[0, T]$, we have

$$\mathbb{E}\left[\|S(f, D, \delta, \eta) - A\|_V^2 \right] < \varepsilon,$$

where

$$\begin{aligned} S(f, D, \delta, \eta) &:= (D) \sum \left\{ \sum_{j=1}^{\infty} (B_\xi^{(j)} - B_u^{(j)}) f_\xi(\sqrt{\lambda_j} e_j) \right\} \\ &:= \sum_{i=1}^{n} \left\{ \sum_{j=1}^{\infty} (B_{\xi_i}^{(j)} - B_{u_i}^{(j)}) f_{\xi_i}(\sqrt{\lambda_j} e_j) \right\}. \end{aligned}$$

In this case, f is $\mathcal{IH}_B$-integrable to A on $[0, T]$ and A is called the backwards Itô-Henstock integral, or *$\mathcal{IH}_B$-integral* of f, which will be denoted by $(\mathcal{IH}_B) \int_0^T f_t \, dW_t$ or $(\mathcal{IH}_B) \int_0^T f \, dW$.

Denote by $\Lambda^2_{\mathcal{IH}_B}(U_Q, V)$, the collection of all backwards Itô-Henstock integrable processes on $[0, T]$.

We have to note here that if the process f is $L(U, V)$-valued, then for f to be backwards Itô-Henstock integrable on $[0, T]$, we need to replace the Riemann sum $S(f, D, \delta, \eta)$ with

$$(D) \sum f_\xi(W_\xi - W_u) := \sum_{i=1}^{n} f_{\xi_i}(W_{\xi_i} - W_{u_i}).$$

In this case, Definition 2.4 is the same with [24, Definition 3.3].

We also remark that since $\mathbb{P}(W_t \in U_Q) = 0$, $t > 0$, if $\dim U_Q = \infty$ [4, Theorem 2.4.7], the Riemann sum $(D)\sum f_\xi(W_\xi - W_u)$ in [24, Definition 3.3] does not make sense pathwise as an application of an operator $f_\xi \in L_2(U_Q, V)$ to a vector in a Hilbert space. Hence, Definition 2.4 is a rectification of [23, Definition 1].

Example 2.5. [23, Example 1] Let $W : [0,T] \times \Omega \to U$ be a Q-Wiener process. Then $\langle W_t, \cdot\rangle_U$ is $\mathcal{IH}_B$-integrable on $[0,T]$ and

$$(\mathcal{IH}_B)\int_0^T \langle W_t, \cdot\rangle_U \, dW_t = \frac{1}{2}\left(\|W_T\|_U^2 + T \cdot \operatorname{tr} Q\right),$$

where $\langle W_t, \cdot\rangle_U (u) = \langle W_t, u\rangle_U$, for all $u \in U$.

The proofs of the following basic (nonstochastic) properties of the backwards Itô-Henstock integral are standard in Henstock-Kurzweil integration, hence omitted.

(1) The backwards Itô-Henstock integral is uniquely determined, in the sense that if A_1 and A_2 are two backwards Itô-Henstock integrals of f in Definition 2.4, then

$$\|A_1 - A_2\|_{L^2(\Omega,V)} = 0.$$

(2) Let $\alpha \in \mathbb{R}$. If $f, g \in \Lambda^2_{\mathcal{IH}_B}(U_Q, V)$, then

(i) $f + g \in \Lambda^2_{\mathcal{IH}_B}(U_Q, V)$ and

$$(\mathcal{IH}_B)\int_0^T (f+g)\, dW = (\mathcal{IH}_B)\int_0^T f\, dW + (\mathcal{IH}_B)\int_0^T g\, dW;$$

(ii) $\alpha f \in \Lambda^2_{\mathcal{IH}_B}(U_Q, V)$ and

$$(\mathcal{IH}_B)\int_0^T (\alpha f)\, dW = \alpha \cdot (\mathcal{IH}_B)\int_0^T f\, dW.$$

(3) If $f : [0,T] \times \Omega \to L_2(U_Q, V)$ is $\mathcal{IH}_B$-integrable on $[0,c]$ and $[c,T]$ where $c \in (0,T)$, then f is $\mathcal{IH}_B$-integrable on $[0,T]$ and

$$(\mathcal{IH}_B)\int_0^T f\, dW = (\mathcal{IH}_B)\int_0^c f\, dW + (\mathcal{IH}_B)\int_c^T f\, dW.$$

(4) If $f : [0,T] \times \Omega \to L_2(U_Q, V)$ is $\mathcal{IH}_B$-integrable on $[0,T]$, then f is also $\mathcal{IH}_B$-integrable on every subinterval $[c,d]$ of $[0,T]$.

(5) (*Sequential definition*). A process $f \in \Lambda^2_{\mathcal{IH}_B}(U_Q, V)$ if and only if there exist $A \in L^2(\Omega, V)$, a decreasing sequence $\{\delta_n\}$ of positive functions defined on $[0,T]$, and a decreasing sequence of positive numbers $\{\eta_n\}$ such that for any backwards (δ_n, η_n)-fine partial division D_n of $[0,T]$, we have

$$\lim_{n\to\infty} S(f, D_n, \delta_n, \eta_n) = A \text{ in } L^2(\Omega, V).$$

In this case,

$$A = (\mathcal{IH}_B)\int_0^T f_t\, dW_t.$$

(6) (*Cauchy criterion*). A process $f \in \Lambda^2_{\mathcal{IH}_B}(U_Q, V)$ if and only if for every $\varepsilon > 0$, there exist a positive function δ on $(0, T]$ and a positive number η such that for any two backwards (δ, η)-fine partial divisions D and D' of $[0, T]$, we have

$$\mathbb{E}\left[\left\|S(f, D, \delta, \eta) - S(f, D', \delta, \eta)\right\|_V^2\right] < \varepsilon.$$

(7) (*Saks-Henstock lemma*). Let $f \in \Lambda^2_{\mathcal{IH}_B}(U_Q, V)$ and let $F[u, v] := (\mathcal{IH}_B)\int_u^v f_t \, dW_t$ for any $[u, v] \subseteq [0, T]$. Then for every $\varepsilon > 0$, there exists a positive function δ on $[0, T]$ such that for any backwards δ-fine partial division $D = \{([u, \xi], \xi)\}$ of $[0, T]$, we have

$$\mathbb{E}\left[\left\|(D)\sum\left\{\sum_{j=1}^{\infty}(B_\xi^{(j)} - B_u^{(j)})f_\xi(\sqrt{\lambda_j}e_j) - F[u, \xi]\right\}\right\|_V^2\right] < \varepsilon.$$

Before proving the Itô-isometry for the backwards Itô-Henstock integral, we shall consider first the backwards Henstock integral [3, Definition 3.1], which is equivalent [3, Corollary 3.1] to the Lebesgue integral for real-valued functions.

A real-valued function f is Lebesgue integrable on $[0, T]$ to $A := (\mathcal{L})\int_0^T f \in \mathbb{R}$ if for every $\varepsilon > 0$, there exist a positive function δ on $[0, T]$ and $\eta > 0$ such that for every backwards (δ, η)-fine partial division $D = \{([u, \xi], \xi)\}$ of $[0, T]$, we have $\left|(D)\sum f(\xi)(\xi - u) - A\right| < \varepsilon$. Denote by $\Lambda_{\mathcal{L}}$, the collection of all Lebesgue integrable functions on $[0, T]$.

Theorem 2.6. [23, *Theorem 2*] *A function $f : [0, T] \to \mathbb{R}$ is Lebesgue integrable to $A \in \mathbb{R}$ if and only if there exist a decreasing sequence of positive functions $\{\delta_n\}$ on $[0, T]$ and a decreasing sequence of positive constants $\{\eta_n\}$ such that*

$$\lim_{n\to\infty}\left|(D_n)\sum f(\xi^{(n)})(\xi^{(n)} - u^{(n)}) - A\right| = 0,$$

where $D_n = \{([u^{(n)}, \xi^{(n)}], \xi^{(n)})\}$ is any backwards (δ_n, η_n)-fine partial division of $[0, T]$.

Lemma 2.7. *Let $f : [0, T] \times \Omega \to L_2(U_Q, V)$ be an adapted process and $[u, \xi] \subseteq [0, T]$. Then*

$$\mathbb{E}\left[\sum_{j\neq j'=1}^{\infty}(B_\xi^{(j)} - B_u^{(j)})(B_\xi^{(j')} - B_u^{(j')})\left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_{j'}}e_{j'})\right\rangle_V\right] = 0.$$

Proof. We note that $\{B^{(j)}\}$ is a sequence of independent Brownian motions. Hence,

$$\mathbb{E}\left[\sum_{j\neq j'=1}^{k}(B_\xi^{(j)} - B_u^{(j)})(B_\xi^{(j')} - B_u^{(j')})\left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_{j'}}e_{j'})\right\rangle_V\right] = 0$$

for every $k \in \mathbb{N}$. Thus, the conclusion follows. □

Lemma 2.8. *Let $f : [0, T] \times \Omega \to L_2(U_Q, V)$ be an adapted process and $[u, \xi] \subseteq [0, T]$. Then*

$$\mathbb{E}\left[\left\|\sum_{j=1}^{\infty}(B_\xi^{(j)} - B_u^{(j)})f_\xi(\sqrt{\lambda_j}e_j)\right\|_V^2\right] = (\xi - u)\mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)}\right].$$

Proof. Applying the Monotone convergence theorem (MCT) to

$$\sum_{j=1}^{k}(B_\xi^{(j)}-B_u^{(j)})^2\left\langle f_\xi(\sqrt{\lambda_j}e_j),f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V,$$

we have

$$\begin{aligned}&\sum_{j=1}^{\infty}\mathbb{E}\left[(B_\xi^{(j)}-B_u^{(j)})^2\left\langle f_\xi(\sqrt{\lambda_j}e_j),f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V\right]\\ =\;&\mathbb{E}\left[\sum_{j=1}^{\infty}(B_\xi^{(j)}-B_u^{(j)})^2\left\langle f_\xi(\sqrt{\lambda_j}e_j),f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V\right].\end{aligned}$$

Using Lemma 2.7, we have

$$\begin{aligned}&(\xi-u)\mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)}\right]\\ =\;&\mathbb{E}\left[\sum_{j=1}^{\infty}(B_\xi^{(j)}-B_u^{(j)})^2\left\langle f_\xi(\sqrt{\lambda_j}e_j),f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V\right]\\ &+\mathbb{E}\left[\sum_{j\neq j'=1}^{\infty}(B_\xi^{(j)}-B_u^{(j)})(B_\xi^{(j')}-B_u^{(j')})\left\langle f_\xi(\sqrt{\lambda_j}e_j),f_\xi(\sqrt{\lambda_{j'}}e_{j'})\right\rangle_V\right]\\ =\;&\mathbb{E}\left[\left\|\sum_{j=1}^{\infty}(B_\xi^{(j)}-B_u^{(j)})f_\xi(\sqrt{\lambda_j}e_j)\right\|_V^2\right].\end{aligned}$$

□

In view of Lemma 2.8, we have the following lemma.

Lemma 2.9. *Let $f:[0,T]\times\Omega\to L_2(U_Q,V)$ be an adapted process and $D=\{[u,\xi]\}$ be a finite collection of non-overlapping subintervals of $[0,T]$. Then*

$$\mathbb{E}\left[\left\|(D)\sum\left\{\sum_{j=1}^{\infty}(B_\xi^{(j)}-B_u^{(j)})f_\xi(\sqrt{\lambda_j}e_j)\right\}\right\|_V^2\right]=(D)\sum(\xi-u)\mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)}\right].$$

Theorem 2.10 (Itô-isometry)**.** *Let $f\in\Lambda^2_{\mathcal{IH}_B}(U_Q,V)$. Then $\mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right]\in\Lambda_{\mathcal{L}}$ and*

$$\mathbb{E}\left[\left\|(\mathcal{IH}_B)\int_0^T f_t\,dW_t\right\|_V^2\right]=(\mathcal{L})\int_0^T\mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right]\,dt<\infty.$$

Proof. By the sequential definition of $\mathcal{IH}_B$-integral, there exist a decreasing sequence $\{\delta_n\}$ of positive functions defined on $[0,T]$, and a decreasing sequence of positive numbers $\{\eta_n\}$ such that for any backwards (δ_n,η_n)-fine partial division $D_n=\{([u_i^{(n)},\xi_i^{(n)}],\xi_i^{(n)})\}_{i=1}^{p(n)}$ of $[0,T]$, we have

$$\lim_{n\to\infty}S(f,D_n,\delta_n,\eta_n)=(\mathcal{IH}_B)\int_0^T f_t\,dW_t\qquad\text{in } L^2(\Omega,V).$$

Let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left\| S(f, D_n, \delta_n, \eta_n) - (\mathcal{IH}_B) \int_0^T f_t \, dW_t \right\|_{L^2(\Omega,V)} < \varepsilon.$$

Note that

$$\begin{aligned} & \left| \|S(f, D_n, \delta_n, \eta_n)\|_{L^2(\Omega,V)} - \left\| (\mathcal{IH}_B) \int_0^T f_t \, dW_t \right\|_{L^2(\Omega,V)} \right| \\ \leq & \left\| S(f, D_n, \delta_n, \eta_n) - (\mathcal{IH}_B) \int_0^T f_t \, dW_t \right\|_{L^2(\Omega,V)}. \end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|S(f, D_n, \delta_n, \eta_n)\|_{L^2(\Omega,V)} = \left\| (\mathcal{IH}_B) \int_0^T f_t \, dW_t \right\|_{L^2(\Omega,V)},$$

or equivalently,

$$\lim_{n\to\infty} \sqrt{\mathbb{E}\left[\|S(f, D_n, \delta_n, \eta_n)\|_V^2 \right]} = \sqrt{\mathbb{E}\left[\left\| (\mathcal{IH}_B) \int_0^T f_t \, dW_t \right\|_V^2 \right]}.$$

Using Lemma 2.9, we have

$$\begin{aligned} \sqrt{\mathbb{E}\left[\left\| (\mathcal{IH}_B) \int_0^T f_t \, dW_t \right\|_V^2 \right]} &= \lim_{n\to\infty} \sqrt{\mathbb{E}\left[\|S(f, D_n, \delta_n, \eta_n)\|_V^2 \right]} \\ &= \lim_{n\to\infty} \sqrt{\mathbb{E}\left[\left\| \sum_{i=1}^{p(n)} \left\{ \sum_{j=1}^{\infty} (B^{(j)}_{\xi_i^{(n)}} - B^{(j)}_{u_i^{(n)}}) f_{\xi_i^{(n)}}(\sqrt{\lambda_j} e_j) \right\} \right\|_V^2 \right]} \\ &= \lim_{n\to\infty} \sqrt{\sum_{i=1}^{p(n)} \left(\xi_i^{(n)} - u_i^{(n)} \right) \mathbb{E}\left[\left\| f_{\xi_i^{(n)}} \right\|_{L_2(U_Q,V)}^2 \right]}. \end{aligned}$$

This implies that

$$\lim_{n\to\infty} \sum_{i=1}^{p(n)} \left(\xi_i^{(n)} - u_i^{(n)} \right) \mathbb{E}\left[\left\| f_{\xi_i^{(n)}} \right\|_{L_2(U_Q,V)}^2 \right] = \mathbb{E}\left[\left\| (\mathcal{IH}_B) \int_0^T f_t \, dW_t \right\|_V^2 \right].$$

Since the above equality holds for any backwards (δ_n, η_n)-fine partial division of $[0, T]$, by Theorem 2.6, $\mathbb{E}\left[\|f_t\|_{L_2(U_Q,V)} \right]$ is Lebesgue integrable on $[0, T]$ and

$$\mathbb{E}\left[\left\| (\mathcal{IH}_B) \int_0^T f_t \, dW_t \right\|_V^2 \right] = (\mathcal{L}) \int_0^T \mathbb{E}\left[\|f_t\|_{L_2(U_Q,V)}^2 \right] \, dt < \infty.$$

This completes the proof. □

Finally, we will show that the backwards Itô-Henstock integral, with a fixed upper bound and a varying lower bound belongs to $\mathcal{BM}_T^2(V)$, the space of continuous square-integrable backward martingales. Firstly, by applying the same reasoning as in [23, Theorem 4], we immediately see that the $\mathcal{IH}_B$-integral satisfies the $AC^2[0, T]$-*property*.

Definition 2.11. A process $F : [0,T] \times \Omega \to V$ is said to be $AC^2[0,T]$ if for every $\varepsilon > 0$, there exists $\eta > 0$ such that for any finite collection $D = \{[u,\xi]\}$ of non-overlapping subintervals of $[0,T]$ with $(D)\sum(\xi - u) < \eta$, we have

$$\int_\Omega \left\| (D)\sum F([u,\xi],\omega) \right\|_V^2 d\mathbb{P}(\omega) := \mathbb{E}\left[\left\| (D)\sum F[u,\xi] \right\|_V^2 \right] < \varepsilon,$$

where $F[u,\xi] := F([u,\xi],\cdot) = F(\xi,\cdot) - F(u,\cdot)$.

Theorem 2.12. *Let $f \in \Lambda^2_{\mathcal{IH}_B}(U_Q,V)$ and define $G_t := (\mathcal{IH}_B)\int_t^T f_s\, dW_s$. Then $G \in \mathcal{BM}^2_T(V)$.*

Proof. By Definition 2.4, G is square integrable. First, we show that G is $\mathbb{P}$-a.s. continuous on $[0,T]$. Let $\varepsilon > 0$. Since G is $AC^2[0,T]$, there exists an $\eta > 0$ such that for any finite collection $D = \{[u,\xi]\}$ of non-overlapping subintervals of $[0,T]$ with $(D)\sum(\xi - u) < \eta$, we have $\mathbb{E}\left[\left\| (D)\sum(\mathcal{IH}_B)\int_u^\xi f\, dW \right\|_V^2 \right] < \varepsilon^3$. Let $t \in [0,T]$ and let $t' \le t$ such that $|t - t'| < \eta$. Then $\mathbb{E}\left[\left\| (\mathcal{IH}_B)\int_{t'}^t f_s\, dW_s \right\|_V^2 \right] < \varepsilon^3$. By Chebychev's inequality,

$$\mathbb{P}\left\{ \omega \in \Omega : \left\| (\mathcal{IH}_B)\int_{t'}^t f_s\, dW_s \right\|_V \ge \varepsilon \right\} \le \frac{1}{\varepsilon^2}\mathbb{E}\left[\left\| (\mathcal{IH}_B)\int_{t'}^t f_s\, dW_s \right\|_V^2 \right] < \varepsilon.$$

Hence, G is $\mathbb{P}$-a.s. continuous on $[0,T]$.

Next, we show that G is backwards adapted. We note that f is also $\mathcal{IH}_B$-integrable on $[t,T]$. From the sequential definition of $\mathcal{IH}_B$-integral, there exist a decreasing sequence $\{\delta_n\}$ of positive functions on $[t,T]$ and a decreasing sequence of positive numbers η_n such that for any backwards (δ_n,η_n)-fine partial division D_n of $[t,T]$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\|S(f,D_n,\delta_n,\eta_n) - G_t\|_V^2 \right] = 0.$$

Note that for $t \le u \le \xi \le T$, $M_n := \sum_{j=1}^n (B_\xi^{(j)} - B_u^{(j)}) f_\xi(\sqrt{\lambda_j} e_j)$ is $\mathcal{G}_t$-measurable. Let $M := \sum_{j=1}^\infty (B_\xi^{(j)} - B_u^{(j)}) f_\xi(\sqrt{\lambda_j} e_j)$. By Chebychev's inequality and using the fact that the filtration is a standard backwards filtration, we have

$$\{\omega \in \Omega : M(\omega) \in O_r(v)\} = \bigcup_{n,N\in\mathbb{N}} \left[\bigcap_{k\ge N} \left\{ \omega \in \Omega : M_k(\omega) \in O_{r-\frac{1}{n}}(v) \right\} \right],$$

where $O_r(v)$ is an open ball in $\mathcal{B}(V)$ centered at $v \in V$ and with radius $r > 0$. This means that M is $\mathcal{G}_t$-measurable. It follows that $S(f,D_n,\delta_n,\eta_n)$ is also $\mathcal{G}_t$-measurable. Since $S(f,D_n,\delta_n,\eta_n) \to G_t$ in $L^2(\Omega,V)$, there exists a subsequence converging almost surely to G_t. Since $\{\mathcal{G}_t : 0 \le t \le T\}$ is a standard backwards filtration, G_t is $\mathcal{G}_t$-measurable. Accordingly, G is backwards adapted.

Lastly, we show that G is a backwards martingale. Let $0 \leq s \leq t \leq T$ and $A \in \mathcal{G}_t$. Note that

$$\begin{aligned}
&\int_A \lim_{n\to\infty}(D_n) \sum_{s<\xi\leq t} \left(\sum_{j=1}^{\infty} (B_\xi^{(j)} - B_u^{(j)}) f_\xi(\sqrt{\lambda_j} e_j) \right) \\
&= \lim_{n\to\infty}(D_n) \sum_{s<\xi\leq t} \left(\sum_{j=1}^{\infty} \mathbb{E}\left[f_\xi(\sqrt{\lambda_j} e_j) \mathbb{E}\left[(B_\xi^{(j)} - B_u^{(j)}) \middle| \mathcal{G}_\xi \right] \middle| \mathcal{G}_t \right] \right) \\
&= \lim_{n\to\infty}(D_n) \sum_{s<\xi\leq t} \left(\sum_{j=1}^{\infty} \mathbb{E}\left[f_\xi(\sqrt{\lambda_j} e_j) \mathbb{E}\left[B_\xi^{(j)} - B_u^{(j)} \right] \middle| \mathcal{G}_t \right] \right) \\
&= 0.
\end{aligned}$$

Again, using the sequential definition of $\mathcal{IH}_B$-integral, we have

$$\begin{aligned}
\int_A G_s &= \int_A \lim_{n\to\infty} S(f, D_n, \delta_n, \eta_n) \\
&= \int_A \lim_{n\to\infty}(D_n) \sum_{s<\xi\leq t} \left(\sum_{j=1}^{\infty} (B_\xi^{(j)} - B_u^{(j)}) f_\xi(\sqrt{\lambda_j} e_j) \right) \\
&\quad + \int_A \lim_{n\to\infty}(D_n) \sum_{t<\xi\leq T} \left(\sum_{j=1}^{\infty} (B_\xi^{(j)} - B_u^{(j)}) f_\xi(\sqrt{\lambda_j} e_j) \right) \\
&= \int_A \lim_{n\to\infty}(D_n) \sum_{t<\xi\leq T} \left(\sum_{j=1}^{\infty} (B_\xi^{(j)} - B_u^{(j)}) f_\xi(\sqrt{\lambda_j} e_j) \right) \\
&= \int_A G_t.
\end{aligned}$$

Thus, G is a backwards martingale. Accordingly, $G \in \mathcal{BM}_T^2(V)$. □

Conclusion and Recommendation

In this paper, we redefine the backwards Itô-Henstock integral for backwards adapted Hilbert-Schmidt-valued stochastic processes, provide a correction to [23, Definition 1], and formulate a version of Itô-isometry. We also show that the stochastic integral establish in this paper, with a fixed upper bound and a varying lower bound, belongs to the space of continuous, square-integrable backward martingales. A worthwhile direction for further investigation is to consider backwards Itô-Henstock integral to deal with backwards stochastic PDEs.

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References

[1] D. Applebaum, Lévy Processes and Stochastic Calculus, Cambridge University Press, 2004. https://doi.org/10.1017/CBO9780511755323.

[2] J.P. Arcede, E.A. Cabral, An Equivalent Definition for the Backwards Itô Integral, Thai J. Math. 9 (2011), 619–630.

[3] J.P. Arcede, E.A. Cabral, Backwards Henstock Integral, Matimyás Mat. 32 (2009), 1–13.

[4] V. Bogachev, Gaussian Measures, American Mathematical Society, 1998. https://doi.org/10.1090/surv/062.

[5] T.-S. Chew, J.-Y. Tay, T.-L. Toh, The Non-Uniform Riemann Approach to Itô's Integral, Real Anal. Exch. 27 (2002), 495–514. https://doi.org/10.14321/realanalexch.27.2.0495.

[6] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, 1992. https://doi.org/10.1017/CBO9780511666223.

[7] L. Gawarecki, V. Mandrekar, Stochastic Differential Equations in Infinite Dimensions, Springer Berlin Heidelberg, 2011. https://doi.org/10.1007/978-3-642-16194-0.

[8] R. Gordon, The Integrals of Lebesgue, Denjoy, Perron, and Henstock, American Mathematical Society, 1994. https://doi.org/10.1090/gsm/004.

[9] R. Henstock, Lectures on the Theory of Integration, World Scientific, 1988. https://doi.org/10.1142/0510.

[10] J. Kurzweil, Henstock-Kurzweil Integration: Its Relation to Topological Vector Spaces, World Scientific, 2000. https://doi.org/10.1142/4333.

[11] J. Kurzweil, Š. Schwabik, McShane Equi-Integrability and Vitali's Convergence Theorem, Math. Bohem. 129 (2004), 141–157. https://doi.org/10.21136/MB.2004.133903.

[12] M.A. Labendia, An Alternative Definition of the Itô Integral for the Hilbert-Schmidt-Valued Stochastic Process, Methods Funct. Anal. Topol. 27 (2021), 370–383. https://doi.org/10.31392/MFAT-npu26_4.2021.10.

[13] P.-Y. Lee, Lanzhou Lectures on Henstock Integration, World Scientific, 1989. https://doi.org/10.1142/0845.

[14] P.-Y. Lee, R. Výborný, The Integral: An Easy Approach after Kurzweil and Henstock, Cambridge University Press, 2000.

[15] T.Y. Lee, Henstock–Kurzweil Integration on Euclidean Spaces, World Scientific, 2011. https://doi.org/10.1142/7933.

[16] J. Lu, P.-Y. Lee, The Primitives of Henstock Integrable Functions in Euclidean Space, Bull. Lond. Math. Soc. 31 (1999), 173–180. https://doi.org/10.1112/S0024609398005347.

[17] Z.-M. Ma, Z. Grande, P.-Y. Lee, Absolute Integration Using Vitali Covers, Real Anal. Exch. 18 (1992), 409–419. https://doi.org/10.2307/44152284.

[18] E.J. McShane, Stochastic Integrals and Stochastic Functional Equations, SIAM J. Appl. Math. 17 (1969), 287–306. https://doi.org/10.1137/0117029.

[19] Z.R. Pop-Stojanovic, On McShane's Belated Stochastic Integral, SIAM J. Appl. Math. 22 (1972), 87–92. https://doi.org/10.1137/0122010.

[20] C. Prévôt, M. Röckner, A Concise Course on Stochastic Partial Differential Equations, Springer Berlin Heidelberg, 2007. https://doi.org/10.1007/978-3-540-70781-3.

[21] M. Reed, B. Simon, Methods of Modern Mathematical Physics. I: Functional Analysis, Academic Press, 1980.

[22] H. Royden, P.M. Fitzpatrick, Real Analysis, Prentice Hall, 2010.

[23] R. Rulete, M.A. Labendia, Backwards Ito-Henstock Integral for the Hilbert-Schmidt-Valued Stochastic Process, Eur. J. Pure Appl. Math. 12 (2019), 58–78. https://doi.org/10.29020/nybg.ejpam.v12i1.3342.

[24] R.F. Rulete, M.A. Labendia, Backwards Itô–Henstock's Version of Itô's Formula, Ann. Funct. Anal. 11 (2020), 208–225. https://doi.org/10.1007/s43034-019-00014-3.

[25] T.-L. Toh, T.-S. Chew, On Belated Differentiation and a Characterization of Henstock-Kurzweil-Ito Integrable Processes, Math. Bohem. 130 (2005), 63–72. https://doi.org/10.21136/MB.2005.134223.

[26] T.-L. Toh, T.-S. Chew, On Ito-Kurzweil-Henstock Integral and Integration-by-Part Formula, Czech. Math. J. 55 (2005), 653–663. https://doi.org/10.1007/s10587-005-0052-7.

[27] T.-L. Toh, T.-S. Chew, On the Henstock-Fubini Theorem for Multiple Stochastic Integrals, Real Anal. Exch. 30 (2005), 295–310. https://doi.org/10.14321/realanalexch.30.1.0295.

[28] T.-L. Toh, T.-S. Chew, The Riemann Approach to Stochastic Integration Using Non-Uniform Meshes, J. Math. Anal. Appl. 280 (2003), 133–147. https://doi.org/10.1016/S0022-247X(03)00059-3.