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BIPOLAR FUZZY TRI-IDEALS OF Γ-SEMIGROUPS

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Abstract. In this paper, we introduce the new concept of bipolar fuzzy tri-ideals in the algebraic context of Γ-semigroups. We identify the key features of these structures and propose a criterion based on level sets. We further prove that a bipolar fuzzy tri-ideal is also the set-theoretic intersection of two such ideals and their union under an inclusion relation. The central results show that the defining features of a bipolar fuzzy tri-ideal are preserved in both its homomorphic pre-image and image, thereby confirming the algebraic invariance of these ideals under homomorphisms.

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1. Introduction

Zadeh [15] introduced fuzzy set theory in 1965 as a framework for addressing uncertainty. Numerous fields have used it, including topology, logic, set theory, medical science, robotics, computer science, information science, control engineering, and measure theory. Zhang [16] introduced bipolar fuzzy sets in 1994.

In 2000, Lee [8] expanded fuzzy set theory to include bipolar fuzzy sets whose membership values fall in $[-1, 0] \cup [0, 1]$. Decision-making effectiveness and efficiency are impacted by the implementation of the bipolar set theory. As a result, it is employed to address issues in decision-making, organizational and economic evaluation, risk management, and social and environmental impact assessment. The bipolar fuzzy set in Γ-semigroups and the integration features of bipolar fuzzy ideals in Γ-semigroups

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were examined later in 2012 by Majumder [9]. Many scholars have used their understanding of ideals in Γ-semigroups to study fuzzy semigroups. The ideal theory is a fundamental structure in semigroups. Bipolar fuzzy weakly interior ideals in semigroups were examined by Gaketem and Khamrot [3] in 2021. The same year, Simuen et al. [14] investigated fuzzy ideals of Γ-semigroups and a new class of ideals. In 2022-2023, Gaketem and Khamrot [4,6,7] examined the properties of a new class of ideals on intuitionistic fuzzy ideals, interval-valued fuzzy ideals of Γ-semigroups, and cubic ideals.

In order to study the properties of Γ-semigroups, we extend the new fuzzy tri-ideal to the bipolar fuzzy tri-ideal in this research. We establish a relationship between bipolar fuzzy tri-ideals and tri-ideals of Γ-semigroups.

2. Preliminaries

The purpose of this section is to establish the basic terminology and definitions that underpin the results presented later.

Definition 2.1. [13] *Let $\mathcal{Y}$ and Γ be non-empty sets. The set $\mathcal{Y}$ is called a Γ-semigroup if there exists a mapping*

$$\mathcal{Y} \times \Gamma \times \mathcal{Y} \longrightarrow \mathcal{Y}$$

whose value at $(\ddot{\varsigma}, \alpha, \ddot{\kappa})$ is denoted by $\ddot{\varsigma}\alpha\ddot{\kappa}$ for all $\ddot{\varsigma}, \ddot{\kappa} \in \mathcal{Y}$ and $\alpha \in \Gamma$, and which satisfies the associativity condition

$$\ddot{\varsigma}\alpha(\ddot{\kappa}\beta\ddot{\upsilon}) = (\ddot{\varsigma}\alpha\ddot{\kappa})\,\beta\ddot{\upsilon}$$

for all $\ddot{\varsigma}, \ddot{\kappa}, \ddot{\upsilon} \in \mathcal{Y}$ and all $\alpha, \beta \in \Gamma$.

Definition 2.2. [2] *A non-empty subset $\mathcal{G}$ of Γ-semigroup $\mathcal{Y}$ is called*

(*i*) *a Γ-subsemigroup of $\mathcal{Y}$ if $\mathcal{G}\Gamma\mathcal{G} \subseteq \mathcal{G}$.*

(*ii*) *a left (right) ideal of $\mathcal{Y}$ if $\mathcal{Y}\Gamma\mathcal{G} \subseteq \mathcal{G}(\mathcal{G}\Gamma\mathcal{Y} \subseteq \mathcal{G})$.*

(*iii*) *an ideal if $\mathcal{G}\Gamma\mathcal{Y} \subseteq \mathcal{G}$ and $\mathcal{Y}\Gamma\mathcal{G} \subseteq \mathcal{G}$.*

Definition 2.3. [15] *Given a non-empty universe of discourse $\vee$, a fuzzy set on $\vee$ is formally defined by a membership function $\aleph : \vee \to [0,1]$, which assigns to each element of $\vee$ a degree of membership in the fuzzy set.*

Definition 2.4. [16] *Given a universe of discourse $\vee$, a bipolar fuzzy set $\aleph$ associates with each element $v \in \vee$ an ordered pair of membership degrees $(\aleph^-(v), \aleph^+(v))$, where the negative membership function $\aleph^- : \vee \to [-1,0]$ and the positive membership function $\aleph^+ : \vee \to [0,1]$ quantify, respectively, the degrees of counter-membership and membership of v with respect to $\aleph$.*

For notational convenience, the bipolar fuzzy set

$$\aleph := \{(v, \aleph^-(v), \aleph^+(v)) : v \in \vee\}$$

will, throughout this paper, be equivalently represented in the concise form

$$\aleph = \{\vee; \aleph^-, \aleph^+\}.$$

Definition 2.5. [16] *Let $\aleph = \{\vee; \aleph^-, \aleph^+\}$ be a bipolar fuzzy set and consider any pair $(\ddot{\lambda}, \ddot{\epsilon}) \in [-1, 0] \times [0, 1]$. Define*

$$\aleph_{\ddot{\lambda}}^{\mathcal{H}} = \{v \in \vee : \aleph^-(v) \leq \ddot{\lambda}\} \quad \textit{and} \quad \aleph_{\ddot{\epsilon}}^{\mathcal{B}} = \{v \in \vee : \aleph^+(v) \geq \ddot{\epsilon}\}.$$

The set $\aleph_{\ddot{\lambda}}^{\mathcal{H}}$ is called the negative $\ddot{\lambda}$-cut of $\aleph$, while $\aleph_{\ddot{\epsilon}}^{\mathcal{B}}$ is called the positive $\ddot{\epsilon}$-cut of $\aleph$. The intersection

$$\aleph_{(\ddot{\lambda}, \ddot{\epsilon})} = \aleph_{\ddot{\lambda}}^{\mathcal{H}} \cap \aleph_{\ddot{\epsilon}}^{\mathcal{B}}$$

is termed the $(\ddot{\lambda}, \ddot{\epsilon})$-cut associated with the bipolar fuzzy set $\aleph$.

Definition 2.6. [16] *Let $\aleph = \{\vee; \aleph^-, \aleph^+\}$ and $\eta = \{\vee; \eta^-, \eta^+\}$ be bipolar fuzzy sets defined on the same universe $\vee$. For each $v \in \vee$, the bipolar fuzzy intersection of $\aleph$ and η is given by*

$$(\aleph \cap \eta)^-(v) = \min\{\aleph^-(v), \eta^-(v)\} \quad \textit{and} \quad (\aleph \cap \eta)^+(v) = \min\{\aleph^+(v), \eta^+(v)\}.$$

Similarly, the bipolar fuzzy union of $\aleph$ and η is defined pointwise by

$$(\aleph \cup \eta)^-(v) = \max\{\aleph^-(v), \eta^-(v)\} \quad \textit{and} \quad (\aleph \cup \eta)^+(v) = \max\{\aleph^+(v), \eta^+(v)\}.$$

Moreover, $\aleph$ is said to be a bipolar fuzzy set of η, if for every $v \in \vee$,

$$\aleph^-(v) \geq \eta^-(v) \quad \textit{and} \quad \aleph^+(v) \leq \eta^+(v).$$

This is denoted by $\aleph \subseteq \eta$.

Definition 2.7. [5] *Assume that $\mathcal{W} : \mathcal{V} \to \mathcal{D}$ is a surjective homomorphism. Let $\aleph = \{\mathcal{V}; \aleph^-, \aleph^+\}$ be a bipolar fuzzy set on $\mathcal{V}$ and $\eta = \{\mathcal{D}; \eta^-, \eta^+\}$ be a bipolar fuzzy set on $\mathcal{D}$.*

The bipolar fuzzy set induced by $\aleph$ through $\mathcal{W}$, denoted by $\mathcal{W}(\aleph)$, is defined on $\mathcal{D}$ as

$$\mathcal{W}(\aleph) = \{\mathcal{D}; (\mathcal{W}(\aleph))^-, (\mathcal{W}(\aleph))^+\},$$

where, for each element $\hat{v} \in \mathcal{D}$, the membership values are assigned by

$$(\mathcal{W}(\aleph))^-(\hat{v}) = \begin{cases} \inf\{\aleph^-(u) : u \in \mathcal{W}^{-1}(\hat{v})\}, & \textit{if } \mathcal{W}^{-1}(\hat{v}) \neq \emptyset, \\ 0, & \textit{otherwise}, \end{cases}$$

and

$$(\mathcal{W}(\aleph))^+(\hat{v}) = \begin{cases} \sup\{\aleph^+(u) : u \in \mathcal{W}^{-1}(\hat{v})\}, & \textit{if } \mathcal{W}^{-1}(\hat{v}) \neq \emptyset, \\ 0, & \textit{otherwise}. \end{cases}$$

On the other hand, the bipolar fuzzy pre-image of η under $\mathcal{W}$ is the bipolar fuzzy set $\mathcal{W}^{-1}(\eta)$ defined on $\mathcal{V}$ by

$$(\mathcal{W}^{-1}(\eta))^-(u) = \eta^-(\mathcal{W}(u)) \quad \textit{and} \quad (\mathcal{W}^{-1}(\eta))^+(u) = \eta^+(\mathcal{W}(u)),$$

for every $u \in \mathcal{V}$.

Definition 2.8. [1] *Let* $\vee \subseteq \Re$, *where* $\Re$ *is a semigroup. Consider the mapping* $\delta_\vee : \Re \to [0,1]$ *defined by*

$$\delta_\vee(x) = \begin{cases} 1, & x \in \vee, \\ 0, & \textit{otherwise}. \end{cases}$$

This mapping represents the fuzzy characteristic function corresponding to the subset $\vee$.

Definition 2.9. [12] *Let* $\vee$ *be a non-empty subset of a semigroup* $\Re$. *Associate with* $\vee$ *a bipolar fuzzy mapping* $\delta_\vee = (\delta_\vee^-, \delta_\vee^+) : \Re \to [-1,0] \times [0,1]$ *defined by*

$$\delta_\vee^+(x) = \begin{cases} 1, & \textit{when } x \in \vee, \\ 0, & \textit{otherwise}, \end{cases} \qquad \delta_\vee^-(x) = \begin{cases} -1, & \textit{when } x \in \vee, \\ 0, & \textit{otherwise}. \end{cases}$$

This mapping $\delta_\vee$ *represents the bipolar fuzzy characteristic function corresponding to the subset* $\vee$.

Definition 2.10. [17] *Let* $\aleph = \{\Re; \aleph^-, \aleph^+\}$ *be a bipolar fuzzy set defined on a* Γ*-semigroup* $\Re$. *The set* $\aleph$ *is referred to as a* bipolar fuzzy semigroup *of* $\Re$ *whenever it fulfills the conditions listed below for all* $\ddot{\theta}, \ddot{\sigma} \in \Re$ *and* $\alpha \in \Gamma$.

(*i*) $\aleph^-(\ddot{\theta}\alpha\ddot{\sigma}) \leq \max\{\aleph^-(\ddot{\theta}), \aleph^-(\ddot{\sigma})\}$,

(*ii*) $\aleph^+(\ddot{\theta}\alpha\ddot{\sigma}) \geq \min\{\aleph^+(\ddot{\theta}), \aleph^+(\ddot{\sigma})\}$.

Definition 2.11. [10] *Let* $\vee$ *be a non-empty subset of a* Γ*-semigroup* $\Re$. *We call* $\vee$ *a* tri-ideal *of* $\Re$ *if* $\vee$ *is closed under the* Γ*-semigroup operations* (*that is,* $\vee$ *is a sub-semigroup of* $\Re$) *and, in addition, satisfies the absorption conditions*

$$\vee\Gamma\vee\Gamma\Re\Gamma\vee \subseteq \vee \quad \textit{and} \quad \vee\Gamma\Re\Gamma\vee\Gamma\vee \subseteq \vee.$$

Definition 2.12. [11] *Let* $\aleph$ *be a fuzzy subset of a* Γ*-semigroup* $\Re$. *The fuzzy set* $\aleph$ *is said to be a* fuzzy tri-ideal *of* $\Re$ *if it satisfies the following conditions for all* $\ddot{\theta}, \ddot{\sigma}, \dot{\ddot{\pi}}, \dot{\ddot{\mu}} \in \Re$ *and* $\alpha, \beta, \gamma \in \Gamma$:

(*i*) $\aleph(\ddot{\theta}\alpha\ddot{\sigma}\beta\dot{\ddot{\pi}}\gamma\dot{\ddot{\mu}}) \geq \min\{\aleph(\ddot{\theta}), \aleph(\ddot{\sigma}), \aleph(\dot{\ddot{\mu}})\}$,

(*ii*) $\aleph(\ddot{\theta}\alpha\ddot{\sigma}\beta\dot{\ddot{\pi}}\gamma\dot{\ddot{\mu}}) \geq \min\{\aleph(\ddot{\theta}), \aleph(\dot{\ddot{\pi}}), \aleph(\dot{\ddot{\mu}})\}$.

3. Main Results

This section presents the notion of bipolar fuzzy tri-ideal (BPFTI) and investigates its fundamental properties. Throughout the remainder of this section, $\vee$ is assumed to be a Γ-semigroup.

Definition 3.1. *Let* $\aleph = (\Re; \aleph^-, \aleph^+)$ *be a bipolar fuzzy set*(*BPFS*) *on* $\Re$. *We say that* $\aleph$ *is a* bipolar fuzzy tri-ideal (*BPFTI*) *of* $\Re$ *if the following are satisfied for all* $\ddot{\theta}, \ddot{\sigma}, \ddot{\pi}, \ddot{\mu} \in \Re$ *and* $\alpha, \beta, \gamma \in \Gamma$:

(*i*) $\aleph^-(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\mu}) \leq \max\{\aleph^-(\ddot{\theta}), \aleph^-(\ddot{\sigma}), \aleph^-(\ddot{\mu})\}$,

(*ii*) $\aleph^-(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\mu}) \leq \max\{\aleph^-(\ddot{\theta}), \aleph^-(\ddot{\pi}), \aleph^-(\ddot{\mu})\}$,

(*iii*) $\aleph^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\mu}) \geq \min\{\aleph^+(\ddot{\theta}), \aleph^+(\ddot{\sigma}), \aleph^+(\ddot{\mu})\}$,

(*iv*) $\aleph^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\mu}) \geq \min\{\aleph^+(\ddot{\theta}), \aleph^+(\ddot{\pi}), \aleph^+(\ddot{\mu})\}$.

Example 3.1. *Let $\vee$ denote the set of all negative integers, and let Γ denote the set of all non-positive even integers. When endowed with the usual integer multiplication as the defining binary operation, $\vee$ forms a Γ-semigroup with respect to Γ. Define a bipolar fuzzy set $\aleph = (\vee; \aleph^-, \aleph^+)$ on $\vee$ by the following membership functions:*

$$\aleph^-(\wp) = \begin{cases} -0.4 & \text{if } \wp = -1 \\ -0.5 & \text{if } \wp = -2 \\ -0.7 & \text{otherwise} \end{cases} \quad \text{and} \quad \aleph^+(\wp) = \begin{cases} 0.4 & \text{if } \wp = -1 \\ 0.5 & \text{if } \wp = -2 \\ 0.6 & \text{otherwise} \end{cases}$$

Then $\aleph$ is a BPFTI of $\vee$.

Theorem 3.1. *A BPFS $\aleph = (\Re; \aleph^-, \aleph^+)$ of $\Re$ is a BPFTI of $\Re$ if and only if for all $s \times \ddot{\epsilon} \in [-1, 0] \times [0, 1]$, $\emptyset \neq \aleph^{\mathcal{H}}_{\ddot{\lambda}}$ and $\emptyset \neq \aleph^{\mathcal{B}}_{\ddot{\epsilon}}$ are tri-ideals of $\Re$.*

Proof. Let $\aleph = (\Re; \aleph^-, \aleph^+)$ be a BPFTI of $\Re$. Let $\ddot{\lambda} \times \ddot{\epsilon} \in [-1, 0] \times [0, 1]$ be such that $\aleph^{\mathcal{H}}_{\ddot{\lambda}} \neq \emptyset$ and $\aleph^{\mathcal{B}}_{\ddot{\epsilon}} \neq \emptyset$. Let $\ddot{\hbar}, \ddot{\varepsilon}, \ddot{\rho}, \ddot{\imath} \in \aleph^{\mathcal{H}}_{\ddot{\lambda}}$, $\ddot{\theta}, \ddot{\sigma}, \ddot{\pi}, \ddot{\varpi} \in \aleph^{\mathcal{B}}_{\ddot{\epsilon}}$ and $\alpha, \beta, \gamma \in \Gamma$. Then $\aleph^-(\ddot{\hbar}) \leq \ddot{\lambda}$, $\aleph^-(\ddot{\varepsilon}) \leq \ddot{\lambda}$, $\aleph^-(\ddot{\rho}) \leq \ddot{\lambda}$, $\aleph^-(\ddot{\imath}) \leq \ddot{\lambda}$ and $\aleph^+(\ddot{\theta}) \geq \ddot{\epsilon}$, $\aleph^+(\ddot{\sigma}) \geq t$, $\aleph^+(\ddot{\pi}) \geq \ddot{\epsilon}$ and $\aleph^+(\ddot{\varpi}) \geq \ddot{\epsilon}$. Since $\aleph = (\Re; \aleph^-, \aleph^+)$ is a BPFTI of $\Re$, we have

(i) $\aleph^-(\ddot{\hbar}\alpha\ddot{\varepsilon}\beta\ddot{\rho}\gamma\ddot{\imath}) \leq \max\{\aleph^-(\ddot{\hbar}), \aleph^-(\ddot{\varepsilon}), \aleph^-(\ddot{\imath})\} \leq \ddot{\lambda}$,

(ii) $\aleph^-(\ddot{\hbar}\alpha\ddot{\varepsilon}\beta\ddot{\rho}\gamma\ddot{\imath}) \leq \max\{\aleph^-(\ddot{\hbar}), \aleph^-(\ddot{\rho}), \aleph^-(\ddot{\imath})\} \leq \ddot{\lambda}$,

(iii) $\aleph^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq \min\{\aleph^+(\ddot{\theta}), \aleph^+(\ddot{\pi}), \aleph^+(\ddot{\varpi})\} \geq \ddot{\epsilon}$,

(iv) $\aleph^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq \min\{\aleph^+(\ddot{\theta}), \aleph^+(\ddot{\sigma}), \aleph^+(\ddot{\varpi})\} \geq \ddot{\epsilon}$.

Then $\ddot{\hbar}\alpha\ddot{\varepsilon}\beta\ddot{\rho}\gamma\ddot{\imath} \in \aleph^{\mathcal{H}}_{\ddot{\lambda}}$ and $\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi} \in \aleph^{\mathcal{B}}_{\ddot{\epsilon}}$. Thus, $\aleph^{\mathcal{H}}_{\ddot{\lambda}}$ and $\aleph^{\mathcal{B}}_{\ddot{\epsilon}}$ are tri-ideals of $\Re$.

Conversely, consider the case where, for all $\ddot{\lambda} \times \ddot{\epsilon} \in [-1, 0] \times [0, 1]$, $\emptyset \neq \aleph^{\mathcal{H}}_{\ddot{\lambda}}$, $\emptyset \neq \aleph^{\mathcal{B}}_{\ddot{\epsilon}}$ are tri-ideals of $\Re$. Let $\ddot{\hbar}, \ddot{\varepsilon}, \ddot{\rho}, \ddot{\imath} \in \aleph^{\mathcal{H}}_{\ddot{\lambda}}$, $\ddot{\theta}, \ddot{\sigma}, \ddot{\pi}, \ddot{\varpi} \in \aleph^{\mathcal{B}}_{\ddot{\epsilon}}$, and $\alpha, \beta, \gamma \in \Gamma$. Then $\ddot{\hbar}\alpha\ddot{\varepsilon}\beta\ddot{\rho}\gamma\ddot{\imath} \in \aleph^{\mathcal{H}}_{\ddot{\lambda}}$ and $\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi} \in \aleph^{\mathcal{B}}_{\ddot{\epsilon}}$. Choose $\ddot{\lambda} = \max\{\aleph^-(\ddot{\hbar}), \aleph^-(\ddot{\varepsilon}), \aleph^-(\ddot{\rho}), \aleph^-(\ddot{\imath})\}$. Let $\aleph^-(\ddot{\hbar}) \leq \ddot{\lambda}_1$, $\aleph^-(\ddot{\varepsilon}) \leq \ddot{\lambda}_2$, $\aleph^-(\ddot{\rho}) \leq \ddot{\lambda}_3$, $\aleph^-(\ddot{\imath}) \leq \ddot{\lambda}_4$ and $\ddot{\lambda}_1 \leq \ddot{\lambda}_2 \leq \ddot{\lambda}_3 \leq \ddot{\lambda}_4 \leq \ddot{\lambda}$.

(i) $\aleph^-(\ddot{\hbar}\alpha\ddot{\varepsilon}\beta\ddot{\rho}\gamma\ddot{\imath}) \leq \max\{\aleph^-(\ddot{\hbar}), \aleph^-(\ddot{\rho}), \aleph^-(\ddot{\imath})\} \leq \max\{\ddot{\lambda}_1, \ddot{\lambda}_3, \ddot{\lambda}_4\} = \ddot{\lambda}_4 \leq \ddot{\lambda}$,

(ii) $\aleph^-(\ddot{\hbar}\alpha\ddot{\varepsilon}\beta\ddot{\rho}\gamma\ddot{\imath}) \leq \max\{\aleph^-(\ddot{\hbar}), \aleph^-(\ddot{\varepsilon}), \aleph^-(\ddot{\imath})\} \leq \max\{\ddot{\lambda}_1, \ddot{\lambda}_2, \ddot{\lambda}_4\} = \ddot{\lambda}_4 \leq \ddot{\lambda}$.

Choose $\ddot{\epsilon} = \min\{\aleph^+(\ddot{\theta}), \aleph^+(\ddot{\sigma}), \aleph^+(\ddot{\pi}), \aleph^+(\ddot{\varpi})\}$. Let $\aleph^+(\ddot{\theta}) \geq \ddot{\epsilon}_1$, $\aleph^+(\ddot{\sigma}) \geq \ddot{\epsilon}_2$, $\aleph^+(\ddot{\pi}) \geq \ddot{\epsilon}_3$, $\aleph^+(\ddot{\varpi}) \geq \ddot{\epsilon}_4$ and $\ddot{\epsilon}_1 \geq \ddot{\epsilon}_2 \geq \ddot{\epsilon}_3 \geq \ddot{\epsilon}_4 \geq \ddot{\epsilon}$.

(iii) $\aleph^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq \min\{\aleph^+(\ddot{\theta}), \aleph^+(\ddot{\pi}), \aleph^+(\ddot{\varpi})\} \geq \min\{\ddot{\epsilon}_1, \ddot{\epsilon}_3, \ddot{\epsilon}_4\} = \ddot{\epsilon}_4 \geq \ddot{\epsilon}$,

(iv) $\aleph^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq \min\{\aleph^+(\ddot{\theta}), \aleph^+(\ddot{\sigma}), \aleph^+(\ddot{\varpi})\} \geq \min\{\ddot{\epsilon}_1, \ddot{\epsilon}_2, \ddot{\epsilon}_4\} = \ddot{\epsilon}_4 \geq \ddot{\epsilon}$.

Hence, $\aleph = (\Re; \aleph^-, \aleph^+)$ is a BPFTI of $\Re$. □

Theorem 3.2. *If $\aleph = (\Re; \aleph^-, \aleph^+)$, $\Im = (\Re; \Im^-, \Im^+)$ are BPFTIs of $\Re$, then $\aleph \cap \Im$ is a BPFTI of $\Re$.*

Proof. Assume that $\aleph = (\Re; \aleph^-, \aleph^+)$ and $\Im = (\Re; \Im^-, \Im^+)$ are BPFTIs of $\Re$. Let $\ddot{\theta}, \ddot{\sigma}, \ddot{\pi}, \ddot{\varpi} \in \Re$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$$
\begin{aligned}
&(\aleph^- \cap \Im^-)(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&= \min\{\aleph^-(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}), \Im^-(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})\} \\
&\leq \min\{\max\{\aleph^-(\ddot{\theta}), \aleph^-(\ddot{\pi}), \aleph^-(\ddot{\varpi})\}, \max\{\Im^-(\ddot{\theta}), \Im^-(\ddot{\pi}), \Im^-(\ddot{\varpi})\}\} \\
&\leq \min\{\max\{\aleph^-(\ddot{\theta}), \Im^-(\ddot{\theta})\}, \max\{\aleph^-(\ddot{\pi}), \Im^-(\ddot{\pi})\}, \max\{\aleph^-(\ddot{\varpi}), \Im^-(\ddot{\varpi})\}\} \\
&\leq \max\{\min\{\aleph^-(\ddot{\theta}), \Im^-(\ddot{\theta})\}, \min\{\aleph^-(\ddot{\pi}), \Im^-(\ddot{\pi}), \{\aleph^-(\ddot{\varpi}), \Im^-(\ddot{\varpi})\}\}\} \\
&= \max\{(\aleph^- \cap \Im^-)(\ddot{\theta}), (\aleph^- \cap \Im^-)(\ddot{\pi}), (\aleph^- \cap \Im^-)(\ddot{\varpi})\},
\end{aligned}
$$

$$
\begin{aligned}
&(\aleph^- \cap \Im^-)(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&= \min\{\aleph^-(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}), \Im^-(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})\} \\
&\leq \min\{\max\{\aleph^-(\ddot{\theta}), \aleph^-(\ddot{\sigma}), \aleph^-(\ddot{\varpi})\}, \max\{\Im^-(\ddot{\theta}), \Im^-(\ddot{\sigma}), \Im^-(\ddot{\varpi})\}\} \\
&\leq \min\{\max\{\aleph^-(\ddot{\theta}), \Im^-(\ddot{\theta})\}, \max\{\aleph^-(\ddot{\sigma}), \Im^-(\ddot{\sigma})\}, \max\{\aleph^-(\ddot{\varpi}), \Im^-(\ddot{\varpi})\}\} \\
&\leq \max\{\min\{\aleph^-(\ddot{\theta}), \Im^-(\ddot{\theta})\}, \min\{\aleph^-(\ddot{\sigma}), \Im^-(\ddot{\sigma}), \{\aleph^-(\ddot{\varpi}), \Im^-(\ddot{\varpi})\}\}\} \\
&= \max\{(\aleph^- \cap \Im^-)(\ddot{\theta}), (\aleph^- \cap \Im^-)(\ddot{\pi}), (\aleph^- \cap \Im^-)(\ddot{\varpi})\},
\end{aligned}
$$

$$
\begin{aligned}
&(\aleph^+ \cap \Im^+)(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&= \min\{\aleph^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}), \Im^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})\} \\
&\geq \min\{\min\{\aleph^+(\ddot{\theta}), \aleph^+(\ddot{\pi}), \aleph^+(\ddot{\varpi})\}, \min\{\Im^+(\ddot{\theta}), \Im^+(\ddot{\pi}), \Im^+(\ddot{\varpi})\}\} \\
&= \min\{\min\{\aleph^+(\ddot{\theta}), \Im^+(\ddot{\theta})\}, \min\{\aleph^+(\ddot{\pi}), \Im^+(\ddot{\pi})\}, \min\{\aleph^+(\ddot{\varpi}), \Im^+(\ddot{\varpi})\}\} \\
&= \min\{(\aleph^+ \cap \Im^+)(\ddot{\theta}), (\aleph^+ \cap \Im^+)(\ddot{\pi}), (\aleph^+ \cap \Im^+)(\ddot{\varpi})\},
\end{aligned}
$$

$$
\begin{aligned}
&(\aleph^+ \cap \Im^+)(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&= \min\{\aleph^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}), \Im^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})\} \\
&\geq \min\{\min\{\aleph^+(\ddot{\theta}), \aleph^+(\ddot{\sigma}), \aleph^+(\ddot{\varpi})\}, \min\{\Im^+(\ddot{\theta}), \Im^+(\ddot{\sigma}), \Im^+(\ddot{\varpi})\}\} \\
&= \min\{\min\{\aleph^+(\ddot{\theta}), \Im^+(\ddot{\theta})\}, \min\{\aleph^+(\ddot{\sigma}), \Im^+(\ddot{\sigma})\}, \min\{\aleph^+(\ddot{\varpi}), \Im^+(\ddot{\varpi})\}\} \\
&= \min\{(\aleph^+ \cap \Im^+)(\ddot{\theta}), (\aleph^+ \cap \Im^+)(\ddot{\sigma}), (\aleph^+ \cap \Im^+)(\ddot{\varpi})\}.
\end{aligned}
$$

Hence, $\aleph \cap \Im$ is a BPFTI of $\Re$. □

Example 3.2. *Consider two BPFTIs* $\aleph$ *and* $\varkappa$ *on* $\Re$*, given by*

$$\aleph^-(\wp) = \begin{cases} -0.64 \text{ if } \wp = 0 \\ -0.51 \text{ if } \wp > 0, \\ -0.42 \text{ if } \wp < 0 \end{cases} \quad \aleph^+(\wp) = \begin{cases} 0.72 \text{ if } \wp = 0 \\ 0.63 \text{ if } \wp > 0 \\ 0.2 \text{ if } \wp < 0 \end{cases}$$

and

$$\varkappa^-(\wp) = \begin{cases} -0.73 \text{ if } \wp = 0 \\ -0.47 \text{ if } \wp > 0, \\ -0.35 \text{ if } \wp < 0 \end{cases} \quad \varkappa^+(\wp) = \begin{cases} 0.83 \text{ if } \wp = 0 \\ 0.54 \text{ if } \wp > 0 \\ 0.19 \text{ if } \wp < 0 \end{cases}$$

Then $\aleph \cup \varkappa$ *is not a BPFTI of* $\Re$*.*

Theorem 3.3. *If* $\aleph = (\Re; \aleph^-, \aleph^+)$ *and* $\Im = (\Re; \Im^-, \Im^+)$ *are two BPFTIs of* $\Re$*, then* $\aleph \cup \Im$ *is a BPFTI of* $\Re$ *when* $\aleph \subseteq \Im$ *or* $\Im \subseteq \aleph$*.*

Proof. Suppose $\aleph \subseteq \Im$. Let $\ddot{\theta}, \ddot{\sigma}, \ddot{\pi}, \ddot{\varpi} \in \Re$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$$\begin{aligned}
&(\aleph^- \cup \Im^-)(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&= \max\{\aleph^-(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}), \Im^-(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}\} \\
&= \aleph^-(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&\leq \max\{\aleph^-(\ddot{\theta}), \aleph^-(\ddot{\pi}), \aleph^-(\ddot{\varpi})\} \\
&\leq \max\{\max\{\aleph^-(\ddot{\theta}), \Im^-(\ddot{\theta})\}, \max\{\aleph^-(\ddot{\pi}), \Im^-(\ddot{\pi}), \max\{\aleph^-(\ddot{\varpi}), \Im^-(\ddot{\varpi})\}\} \\
&= \max\{(\aleph^- \cup \Im^-)(\ddot{\theta}), (\aleph^- \cup \Im^-)(\ddot{\pi}), (\aleph^- \cup \Im^-)(\ddot{\varpi})\},
\end{aligned}$$

$$\begin{aligned}
&(\aleph^- \cup \Im^-)(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&= \max\{\aleph^-(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}), \Im^-(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})\} \\
&= \aleph^-(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&\leq \max\{\aleph^-(\ddot{\theta}), \aleph^-(\ddot{\sigma}), \aleph^-(\ddot{\varpi})\} \\
&\leq \max\{\max\{\aleph^-(\ddot{\theta}), \Im^-(\ddot{\theta})\}, \max\{\aleph^-(\ddot{\sigma}), \Im^-(\ddot{\sigma}), \max\{\aleph^-(\ddot{\varpi}), \Im^-(\ddot{\varpi})\}\} \\
&= \max\{(\aleph^- \cup \Im^-)(\ddot{\theta}), (\aleph^- \cup \Im^-)(\ddot{\sigma}), (\aleph^- \cup \Im^-)(\ddot{\varpi})\},
\end{aligned}$$

$$\begin{aligned}
&(\aleph^+ \cup \Im^+)(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&= \max\{\aleph^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}), \Im^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})\} \\
&= \Im^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&\geq \min\{\Im^+(\ddot{\theta}), \Im^+(\ddot{\pi}), \Im^+(\ddot{\varpi})\}
\end{aligned}$$

$$\geq \min\{\max\{\aleph^+(\ddot{\theta}), \Im^+(\ddot{\theta})\}, \max\{\aleph^+(\ddot{\pi}), \Im^+(\ddot{\pi})\}, \max\{\aleph^+(\ddot{\varpi}), \Im^+(\ddot{\varpi})\}\}$$
$$= \min\{(\aleph^+ \cup \Im^+)(\ddot{\theta}), (\aleph^+ \cup \Im^+)(\ddot{\pi}), (\aleph^+ \cup \Im^+)(\ddot{\varpi})\},$$

$$(\aleph^+ \cup \Im^+)(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})$$
$$= \max\{\aleph^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}), \Im^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})\}$$
$$= \Im^+(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})$$
$$\geq \min\{\Im^+(\ddot{\theta}), \Im^+(\ddot{\sigma}), \Im^+(\ddot{\varpi})\}$$
$$\geq \min\{\max\{\aleph^+(\ddot{\theta}), \Im^+(\ddot{\theta})\}, \max\{\aleph^+(\ddot{\sigma}), \Im^+(\ddot{\sigma})\}, \max\{\aleph^+(\ddot{\varpi}), \Im^+(\ddot{\varpi})\}\}$$
$$= \min\{(\aleph^+ \cup \Im^+)(\ddot{\theta}), (\aleph^+ \cup \Im^+)(\ddot{\sigma}), (\aleph^+ \cup \Im^+)(\ddot{\varpi})\}.$$

Hence, $\aleph \cup \Im$ is a BPFTI of $\Re$. Similarly, if $\Im \subseteq \aleph$, then $\aleph \cup \Im$ is a BPFTI of $\Re$. □

Theorem 3.4. *Let $\ddot{\varepsilon}$ be a surjective mapping preserving Γ-semigroup operations from $\mathcal{K}$ onto $\Re$. The inverse image of any bipolar fuzzy tri-ideal of $\Re$ under $\ddot{\varepsilon}$ is again a bipolar fuzzy tri-ideal in $\mathcal{K}$.*

Proof. Let $\Im$ be a BPFTI of $\Re$. Consider any elements $\ddot{\theta}, \ddot{\sigma}, \ddot{\pi}, \ddot{\varpi}$ in $\mathcal{K}$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$$(\ddot{\varepsilon}^{-1}(\Im)^-)(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})$$
$$= \Im^-(\ddot{\varepsilon}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}))$$
$$= \Im^-(\ddot{\varepsilon}(\ddot{\theta})\alpha\ddot{\varepsilon}(\ddot{\sigma})\beta\ddot{\varepsilon}(\ddot{\pi})\gamma\ddot{\varepsilon}(\ddot{\varpi}))$$
$$\leq \max\{\Im^-(\ddot{\varepsilon}(\ddot{\theta})), \Im^-(\ddot{\varepsilon}(\ddot{\pi})), \Im^-(\ddot{\varepsilon}(\ddot{\varpi}))\}$$
$$= \max\{(\ddot{\varepsilon}^{-1}(\Im)^-)(\ddot{\theta}), (\ddot{\varepsilon}^{-1}(\Im)^-)(\ddot{\pi}), (\ddot{\varepsilon}^{-1}(\Im)^-)(\ddot{\varpi})\},$$

$$(\ddot{\varepsilon}^{-1}(\Im)^-)(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})$$
$$= \Im^-(\ddot{\varepsilon}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}))$$
$$= \Im^-(\ddot{\varepsilon}(\ddot{\theta})\alpha\ddot{\varepsilon}(\ddot{\sigma})\beta\ddot{\varepsilon}(\ddot{\pi})\gamma\ddot{\varepsilon}(\ddot{\varpi}))$$
$$\leq \max\{\Im^-(\ddot{\varepsilon}(\ddot{\theta})), \Im^-(\ddot{\varepsilon}(\ddot{\sigma})), \Im^-(\ddot{\varepsilon}(\ddot{\varpi}))\}$$
$$= \max\{(\ddot{\varepsilon}^{-1}(\Im)^-)(\ddot{\theta}), (\ddot{\varepsilon}^{-1}(\Im)^-)(\ddot{\sigma}), (\ddot{\varepsilon}^{-1}(\Im)^-)(\ddot{\varpi})\},$$

$$(\ddot{\varepsilon}^{-1}(\Im)^+)(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})$$
$$= \Im^+(\ddot{\varepsilon}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}))$$
$$= \Im^+(\ddot{\varepsilon}(\ddot{\theta})\alpha\ddot{\varepsilon}(\ddot{\sigma})\beta\ddot{\varepsilon}(\ddot{\pi})\gamma\ddot{\varepsilon}(\ddot{\varpi}))$$
$$\geq \min\{\Im^+(\ddot{\varepsilon}(\ddot{\theta})), \Im^+(\ddot{\varepsilon}(\ddot{\pi})), \Im^+(\ddot{\varepsilon}(\ddot{\varpi}))\}$$
$$= \min\{(\ddot{\varepsilon}^{-1}(\Im)^+)(\ddot{\theta}), (\ddot{\varepsilon}^{-1}(\Im)^+)(\ddot{\pi}), (\ddot{\varepsilon}^{-1}(\Im)^+)(\ddot{\varpi})\},$$

$$
\begin{aligned}
&(\ddot{\varepsilon}^{-1}(\Im)^{+})(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&= \Im^{+}(\ddot{\varepsilon}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})) \\
&= \Im^{+}(\ddot{\varepsilon}(\ddot{\theta})\alpha\ddot{\varepsilon}(\ddot{\sigma})\beta\ddot{\varepsilon}(\ddot{\pi})\gamma\ddot{\varepsilon}(\ddot{\varpi})) \\
&\geq \min\{\Im^{+}(\ddot{\varepsilon}(\ddot{\theta})), \Im^{+}(\ddot{\varepsilon}(\ddot{\sigma})), \Im^{+}(\ddot{\varepsilon}(\ddot{\varpi}))\} \\
&= \min\{(\ddot{\varepsilon}^{-1}(\Im)^{+})(\ddot{\theta}), (\ddot{\varepsilon}^{-1}(\Im)^{+})(\ddot{\sigma}), (\ddot{\varepsilon}^{-1}(\Im)^{+})(\ddot{\varpi})\}.
\end{aligned}
$$

Hence, $\ddot{\varepsilon}^{-1}(\Im)$ is a BPFTI of $\mathcal{K}$. □

Theorem 3.5. *Assume there exists an onto homomorphism $\ddot{\varepsilon}$ from the Γ-semigroup $\mathcal{K}$ to the Γ-semigroup $\Re$. Whenever $\aleph$ is a BPFTI in $\mathcal{K}$, its homomorphic image $\ddot{\varepsilon}(\aleph)$ is again a BPFTI in $\Re$.*

Proof. Let $\aleph$ be a BPFTI of $\mathcal{K}$ and choose arbitrary elements $\ddot{\theta}, \ddot{\sigma}, \ddot{\pi}, \ddot{\varpi} \in \Re$. Now, $\ddot{\varepsilon}^{-1}(\ddot{\theta})$, $\ddot{\varepsilon}^{-1}(\ddot{\sigma})$, $\ddot{\varepsilon}^{-1}(\ddot{\pi})$ and $\ddot{\varepsilon}^{-1}(\ddot{\varpi})$ are non-empty. Since $\ddot{\varepsilon}$ is a homomorphism and there exist $\ddot{\hbar}, \ddot{\kappa}, \ddot{\rho}, \ddot{\imath} \in \mathcal{K}$ such that $\ddot{\varepsilon}(\ddot{\hbar}) = \ddot{\theta}$, $\ddot{\varepsilon}(\ddot{\kappa}) = \ddot{\sigma}$, $\ddot{\varepsilon}(\ddot{\rho}) = \ddot{\pi}$ and $\ddot{\varepsilon}(\ddot{\imath}) = \ddot{\varpi}$, we have that $\ddot{\hbar}\alpha\ddot{\kappa}\beta\ddot{\rho}\gamma\ddot{\imath} \in \ddot{\varepsilon}^{-1}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})$, for all $\alpha, \beta, \gamma \in \Gamma$. Then

$$
\begin{aligned}
&(\ddot{\varepsilon}(\aleph))^{-}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&= \inf\{\aleph^{-}(\ddot{\varsigma}) : \ddot{\varsigma} \in \ddot{\varepsilon}^{-1}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})\} \\
&= \inf\{\aleph^{-}(\ddot{\hbar}\ddot{\kappa}\ddot{\rho}\ddot{\imath}) : \ddot{\hbar} \in \ddot{\varepsilon}^{-1}(\ddot{\theta}), \ddot{\kappa} \in \ddot{\varepsilon}^{-1}(\ddot{\sigma}), \ddot{\rho} \in \ddot{\varepsilon}^{-1}(\ddot{\pi}), \ddot{\imath} \in \ddot{\varepsilon}^{-1}(\ddot{\varpi})\} \\
&\leq \inf\{\max\{\aleph^{-}(\ddot{\hbar}), \aleph^{-}(\ddot{\rho}), \aleph^{-}(\ddot{\imath})\} : \ddot{\hbar} \in \ddot{\varepsilon}^{-1}(\ddot{\theta}), \ddot{\rho} \in \ddot{\varepsilon}^{-1}(\ddot{\pi}), \ddot{\imath} \in \ddot{\varepsilon}^{-1}(\ddot{\varpi})\} \\
&\leq \max\{\inf\{\aleph^{-}(\ddot{\hbar}) : \ddot{\hbar} \in \ddot{\varepsilon}^{-1}(\ddot{\theta})\}, \inf\{\aleph^{-}(\ddot{\rho}) : \ddot{\rho} \in \ddot{\varepsilon}^{-1}(\ddot{\pi})\}, \inf\{\aleph^{-}(\ddot{\imath}) : \ddot{\imath} \in \ddot{\varepsilon}^{-1}(\ddot{\varpi})\}\} \\
&= \max\{(\ddot{\varepsilon}(\aleph))^{-}(\ddot{\theta}), (\ddot{\varepsilon}(\aleph))^{-}(\ddot{\pi}), (\ddot{\varepsilon}(\aleph))^{-}(\ddot{\varpi})\},
\end{aligned}
$$

$$
\begin{aligned}
&(\ddot{\varepsilon}(\aleph))^{-}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&= \inf\{\aleph^{-}(\ddot{\varsigma}) : \ddot{\varsigma} \in \ddot{\varepsilon}^{-1}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})\} \\
&= \inf\{\aleph^{-}(\ddot{\hbar}\ddot{\kappa}\ddot{\rho}\ddot{\imath}) : \ddot{\hbar} \in \ddot{\varepsilon}^{-1}(\ddot{\theta}), \ddot{\kappa} \in \ddot{\varepsilon}^{-1}(\ddot{\sigma}), \ddot{\rho} \in \ddot{\varepsilon}^{-1}(\ddot{\pi}), \ddot{\imath} \in \ddot{\varepsilon}^{-1}(\ddot{\varpi})\} \\
&\leq \inf\{\max\{\aleph^{-}(\ddot{\hbar}), \aleph^{-}(\ddot{\rho}), \aleph^{-}(\ddot{\imath})\} : \ddot{\hbar} \in \ddot{\varepsilon}^{-1}(\ddot{\theta}), \ddot{\kappa} \in \ddot{\varepsilon}^{-1}(\ddot{\sigma}), \ddot{\imath} \in \ddot{\varepsilon}^{-1}(\ddot{\varpi})\} \\
&\leq \max\{\inf\{\aleph^{-}(\ddot{\hbar}) : \ddot{\hbar} \in \ddot{\varepsilon}^{-1}(\ddot{\theta})\}, \inf\{\aleph^{-}(\ddot{\rho}) : \ddot{\kappa} \in \ddot{\varepsilon}^{-1}(\ddot{\sigma})\}, \inf\{\aleph^{-}(\ddot{\imath}) : \ddot{\imath} \in \ddot{\varepsilon}^{-1}(\ddot{\varpi})\}\} \\
&= \max\{(\ddot{\varepsilon}(\aleph))^{-}(\ddot{\theta}), (\ddot{\varepsilon}(\aleph))^{-}(\ddot{\sigma}), (\ddot{\varepsilon}(\aleph))^{-}(\ddot{\varpi})\},
\end{aligned}
$$

$$
\begin{aligned}
&(\ddot{\varepsilon}(\aleph))^{+}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \\
&= \sup\{\aleph^{+}(\ddot{\varsigma}) : \ddot{\varsigma} \in \ddot{\varepsilon}^{-1}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi})\} \\
&= \sup\{\aleph^{+}(\ddot{\hbar}\ddot{\kappa}\ddot{\rho}\ddot{\imath}) : \ddot{\hbar} \in \ddot{\varepsilon}^{-1}(\ddot{\theta}), \ddot{\kappa} \in \ddot{\varepsilon}^{-1}(\ddot{\sigma}), \ddot{\rho} \in \ddot{\varepsilon}^{-1}(\ddot{\pi}), \ddot{\imath} \in \ddot{\varepsilon}^{-1}(\ddot{\varpi})\} \\
&\leq \sup\{\min\{\aleph^{+}(\ddot{\hbar}), \aleph^{+}(\ddot{\rho}), \aleph^{+}(\ddot{\imath})\} : \ddot{\hbar} \in \ddot{\varepsilon}^{-1}(\ddot{\theta}), \ddot{\rho} \in \ddot{\varepsilon}^{-1}(\ddot{\pi}), \ddot{\imath} \in \ddot{\varepsilon}^{-1}(\ddot{\varpi})\}
\end{aligned}
$$

$$\leq \min\{\sup\{\aleph^+(\ddot{\hbar}) : \ddot{\hbar} \in \ddot{\varepsilon}^{-1}(\ddot{\theta})\}, \sup\{\aleph^+(\ddot{\rho}) : \ddot{\rho} \in \ddot{\varepsilon}^{-1}(\ddot{\pi})\}, \sup\{\aleph^+(\ddot{\imath}) : \ddot{\imath} \in \ddot{\varepsilon}^{-1}(\ddot{\varpi})\}\}$$

$$= \min\{(\ddot{\varepsilon}(\aleph))^+(\ddot{\theta}), (\ddot{\varepsilon}(\aleph))^+(\ddot{\pi}), (\ddot{\varepsilon}(\aleph))^+(\ddot{\varpi})\},$$

$$(\ddot{\varepsilon}(\aleph))^+(\ddot{\theta}\ddot{\alpha}\ddot{\sigma}\ddot{\beta}\ddot{\pi}\ddot{\gamma}\ddot{\varpi})$$

$$= \sup\{\aleph^+(\ddot{\varsigma}) : \ddot{\varsigma} \in \ddot{\varepsilon}^{-1}(\ddot{\theta}\ddot{\alpha}\ddot{\sigma}\ddot{\beta}\ddot{\pi}\ddot{\gamma}\ddot{\varpi})\}$$

$$= \sup\{\aleph^+(\ddot{\hbar}\ddot{\kappa}\ddot{\rho}\ddot{\imath}) : \ddot{\hbar} \in \ddot{\varepsilon}^{-1}(\ddot{\theta}), \ddot{\kappa} \in \ddot{\varepsilon}^{-1}(\ddot{\sigma}), \ddot{\rho} \in \ddot{\varepsilon}^{-1}(\ddot{\pi}), \ddot{\imath} \in \ddot{\varepsilon}^{-1}(\ddot{\varpi})\}$$

$$\geq \sup\{\min\{\aleph^+(\ddot{\hbar}), \aleph^+(\ddot{\rho}), \aleph^+(\ddot{\imath})\} : \ddot{\hbar} \in \ddot{\varepsilon}^{-1}(\ddot{\theta}), \ddot{\kappa} \in \ddot{\varepsilon}^{-1}(\ddot{\sigma}), \ddot{\imath} \in \ddot{\varepsilon}^{-1}(\ddot{\varpi})\}$$

$$\geq \min\{\sup\{\aleph^+(\ddot{\hbar}) : \ddot{\hbar} \in \ddot{\varepsilon}^{-1}(\ddot{\theta})\}, \sup\{\aleph^+(\ddot{\kappa}) : \ddot{\kappa} \in \ddot{\varepsilon}^{-1}(\ddot{\sigma})\}, \sup\{\aleph^+(\ddot{\imath}) : \ddot{\imath} \in \ddot{\varepsilon}^{-1}(\ddot{\varpi})\}\}$$

$$= \min\{(\ddot{\varepsilon}(\aleph))^+(\ddot{\theta}), (\ddot{\varepsilon}(\aleph))^+(\ddot{\sigma}), (\ddot{\varepsilon}(\aleph))^+(\ddot{\varpi})\}.$$

Hence, $\ddot{\varepsilon}(\aleph)$ is a BPFTI of $\Re$. □

Theorem 3.6. *Let $\mathcal{X}$ be a non-empty subset of a Γ-semigroup $\Re$. Then $\mathcal{X}$ is a tri-ideal of $\Re$ if and only if its associated bipolar fuzzy characteristic function $\delta_{\mathcal{X}}$ constitutes a BPFTI of $\Re$.*

Proof. Consider $\delta_{\mathcal{X}}$ is a BPFTI of $\Re$. Let $\ddot{\theta}, \ddot{\sigma}, \ddot{\pi}, \ddot{\varpi} \in \Re$ and $\alpha, \beta, \gamma \in \Gamma$.

(i) If $\ddot{\theta}, \ddot{\pi}, \ddot{\varpi} \in \mathcal{X}$, then

$$\delta_{\mathcal{X}}^-(\ddot{\theta}\ddot{\alpha}\ddot{\sigma}\ddot{\beta}\ddot{\pi}\ddot{\gamma}\ddot{\varpi}) \leq \max\{\delta_{\mathcal{X}}^-(\ddot{\theta}), \delta_{\mathcal{X}}^-(\ddot{\pi}), \delta_{\mathcal{X}}^-(\ddot{\varpi})\} = \max\{-1, -1, -1\} = -1.$$

(ii) If $\ddot{\theta}, \ddot{\pi}, \ddot{\varpi} \in \mathcal{X}$, then

$$\delta_{\mathcal{X}}^-(\ddot{\theta}\ddot{\alpha}\ddot{\sigma}\ddot{\beta}\ddot{\pi}\ddot{\gamma}\ddot{\varpi}) \leq \max\{\delta_{\mathcal{X}}^-(\ddot{\theta}), \delta_{\mathcal{X}}^-(\ddot{\sigma}), \delta_{\mathcal{X}}^-(\ddot{\varpi})\} = \max\{-1, -1, -1\} = -1.$$

This implies $\ddot{\theta}\ddot{\alpha}\ddot{\sigma}\ddot{\beta}\ddot{\pi}\ddot{\gamma}\ddot{\varpi} \in \mathcal{X}$.

Hence, $\mathcal{X}$ is a tri-ideal of $\Re$.

To establish the converse, take $\mathcal{X}$ to be a tri-ideal within the Γ-semigroup $\Re$. **Case 1.** If $\ddot{\theta}, \ddot{\sigma}, \ddot{\pi}, \ddot{\varpi} \in \mathcal{X}$, then $\ddot{\theta}\ddot{\alpha}\ddot{\sigma}\ddot{\beta}\ddot{\pi}\ddot{\gamma}\ddot{\varpi} \in \mathcal{X}$. Thus,

$$\delta_{\mathcal{X}}^-(\ddot{\theta}\ddot{\alpha}\ddot{\sigma}\ddot{\beta}\ddot{\pi}\ddot{\gamma}\ddot{\varpi}) = -1 = \max\{\delta_{\mathcal{X}}^-(\ddot{\theta}), \delta_{\mathcal{X}}^-(\ddot{\pi})\delta_{\mathcal{X}}^-(\ddot{\varpi})\},$$

$$\delta_{\mathcal{X}}^-(\ddot{\theta}\ddot{\alpha}\ddot{\sigma}\ddot{\beta}\ddot{\pi}\ddot{\gamma}\ddot{\varpi}) = -1 = \max\{\delta_{\mathcal{X}}^-(\ddot{\theta}), \delta_{\mathcal{X}}^-(\ddot{\sigma})\delta_{\mathcal{X}}^-(\ddot{\varpi})\},$$

$$\delta_{\mathcal{X}}^+(\ddot{\theta}\ddot{\alpha}\ddot{\sigma}\ddot{\beta}\ddot{\pi}\ddot{\gamma}\ddot{\varpi}) = 1 = \min\{\delta_{\mathcal{X}}^+(\ddot{\theta}), \delta_{\mathcal{X}}^+(\ddot{\pi}), \delta_{\mathcal{X}}^+(\ddot{\varpi})\},$$

$$\delta_{\mathcal{X}}^+(\ddot{\theta}\ddot{\alpha}\ddot{\sigma}\ddot{\beta}\ddot{\pi}\ddot{\gamma}\ddot{\varpi}) = 1 = \min\{\delta_{\mathcal{X}}^+(\ddot{\theta}), \delta_{\mathcal{X}}^+(\ddot{\sigma}), \delta_{\mathcal{X}}^+(\ddot{\varpi})\}.$$

Case 2. If $\ddot{\theta}, \ddot{\sigma}, \ddot{\pi}, \ddot{\varpi} \notin \mathcal{X}$, then

$$\delta_{\mathcal{X}}^+(\ddot{\theta}) = \delta_{\mathcal{X}}^+(\ddot{\sigma}) = \delta_{\mathcal{X}}^+(\ddot{\pi}) = \delta_{\mathcal{X}}^+(\ddot{\varpi}) = 0,$$

$$\delta_{\mathcal{X}}^-(\ddot{\theta}) = \delta_{\mathcal{X}}^-(\ddot{\sigma}) = \delta_{\mathcal{X}}^-(\ddot{\pi}) = \delta_{\mathcal{X}}^-(\ddot{\varpi}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 3. If $\ddot{\theta}, \ddot{\sigma}, \ddot{\pi} \in \mathcal{X}$, $\ddot{\varpi} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(c) = \delta^+_{\mathcal{X}}(\ddot{\sigma}) = \delta^+_{\mathcal{X}}(\ddot{\pi}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\varpi}) = 0,$$
$$\delta^-_{\mathcal{X}}(\ddot{\theta}) = \delta^-_{\mathcal{X}}(\ddot{\sigma}) = \delta^-_{\mathcal{X}}(\ddot{\pi}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\varpi}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 4. If $\ddot{\theta}, \ddot{\sigma}, \ddot{\varpi} \in \mathcal{X}$, $\ddot{\pi} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(\ddot{\theta}) = \delta^+_{\mathcal{X}}(\ddot{\sigma}) = \delta^+_{\mathcal{X}}(\ddot{\varpi}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\pi}) = 0,$$
$$\delta^-_{\mathcal{X}}(\ddot{\theta}) = \delta^-_{\mathcal{X}}(\ddot{\sigma}) = \delta^-_{\mathcal{X}}(\ddot{\varpi}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\pi}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq -1 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 1 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 5. If $\ddot{\theta}, \ddot{\pi}, \ddot{\varpi} \in \mathcal{X}$, $\ddot{\sigma} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(\ddot{\theta}) = \delta^+_{\mathcal{X}}(\ddot{\pi}) = \delta^+_{\mathcal{X}}(\ddot{\varpi}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\sigma}) = 0,$$
$$\delta^-_{\mathcal{X}}(\ddot{\theta}) = \delta^-_{\mathcal{X}}(\ddot{\pi}) = \delta^-_{\mathcal{X}}(\ddot{\varpi}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\sigma}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq -1 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 1 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 6. If $\ddot{\sigma}, \ddot{\pi}, \ddot{\varpi} \in \mathcal{X}$, $\ddot{\theta} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(\ddot{\sigma}) = \delta^+_{\mathcal{X}}(\ddot{\pi}) = \delta^+_{\mathcal{X}}(\ddot{\varpi}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\theta}) = 0,$$

$$\delta^-_{\mathcal{X}}(\ddot{\sigma}) = \delta^-_{\mathcal{X}}(\ddot{\pi}) = \delta^-_{\mathcal{X}}(\ddot{\varpi}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\theta}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^-_{B}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 7. If $\ddot{\theta}, \ddot{\sigma} \in \mathcal{X}$, $\ddot{\pi}, \ddot{\varpi} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(\ddot{\theta}) = \delta^+_{\mathcal{X}}(\ddot{\sigma}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\pi}) = \delta^+_{\mathcal{X}}(\ddot{\varpi}) = 0,$$

$$\delta^-_{\mathcal{X}}(\ddot{\theta}) = \delta^-_{\mathcal{X}}(\ddot{\sigma}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\pi}) = \delta^-_{\mathcal{X}}(\ddot{\varpi}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 8. If $\ddot{\theta}, \ddot{\pi} \in \mathcal{X}$, $\ddot{\sigma}, \ddot{\varpi} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(\ddot{\theta}) = \delta^+_{\mathcal{X}}(\ddot{\pi}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\sigma}) = \delta^+_{\mathcal{X}}(\ddot{\varpi}) = 0,$$

$$\delta^-_{\mathcal{X}}(\ddot{\theta}) = \delta^-_{\mathcal{X}}(\ddot{\pi}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\sigma}) = \delta^-_{\mathcal{X}}(\ddot{\varpi}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 9. If $\ddot{\sigma}, \ddot{\pi} \in \mathcal{X}$, $\ddot{\theta}, \ddot{\varpi} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(\ddot{\sigma}) = \delta^+_{\mathcal{X}}(\ddot{\pi}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\theta}) = \delta^+_{\mathcal{X}}(\ddot{\varpi}) = 0,$$

$$\delta^-_{\mathcal{X}}(\ddot{\sigma}) = \delta^-_{\mathcal{X}}(\ddot{\pi}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\theta}) = \delta^-_{\mathcal{X}}(\ddot{\varpi}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 10. If $\ddot{\theta}, \ddot{\varpi} \in \mathcal{X}$, $\ddot{\sigma}, \ddot{\pi} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(\ddot{\theta}) = \delta^+_{\mathcal{X}}(\ddot{\varpi}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\sigma}) = \delta^+_{\mathcal{X}}(\ddot{\pi}) = 0,$$
$$\delta^-_{\mathcal{X}}(\ddot{\theta}) = \delta^-_{\mathcal{X}}(\ddot{\varpi}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\sigma}) = \delta^-_{\mathcal{X}}(\ddot{\pi}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 11. If $\ddot{\sigma}, \ddot{\varpi} \in \mathcal{X}$, $\ddot{\theta}, \ddot{\pi} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(\ddot{\sigma}) = \delta^+_{\mathcal{X}}(\ddot{\varpi}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\theta}) = \delta^+_{\mathcal{X}}(\ddot{\pi}) = 0,$$
$$\delta^-_{\mathcal{X}}(p) = \delta^-_{\mathcal{X}}(\ddot{\varpi}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\theta}) = \delta^-_{\mathcal{X}}(\ddot{\pi}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 12. If $\ddot{\pi}, \ddot{\varpi} \in \mathcal{X}$, $\ddot{\theta}, \ddot{\sigma} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(\ddot{\pi}) = \delta^+_{\mathcal{X}}(\ddot{\varpi}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\theta}) = \delta^+_{\mathcal{X}}(\ddot{\sigma}) = 0,$$
$$\delta^-_{\mathcal{X}}(\ddot{\pi}) = \delta^-_{\mathcal{X}}(\ddot{\varpi}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\theta}) = \delta^-_{\mathcal{X}}(\ddot{\sigma}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$
$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 13. If $\ddot{\varpi} \in \mathcal{X}$, $\ddot{\theta}, \ddot{\sigma}, \ddot{\pi} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(\ddot{\varpi}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\theta}) = \delta^+_{\mathcal{X}}(\ddot{\sigma}) = \delta^+_{\mathcal{X}}(\ddot{\pi}) = 0,$$

$$\delta^-_{\mathcal{X}}(\ddot{\varpi}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\sigma}) = \delta^-_{\mathcal{X}}(\ddot{\pi}) = \delta^-_{\mathcal{X}}(\ddot{\theta}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 14. If $\ddot{\pi} \in \mathcal{X}$, $\ddot{\theta}, \ddot{\sigma}, \ddot{\varpi} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(\ddot{\pi}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\theta}) = \delta^+_{\mathcal{X}}(\ddot{\sigma}) = \delta^+_{\mathcal{X}}(\ddot{\varpi}) = 0,$$

$$\delta^-_{\mathcal{X}}(\ddot{\pi}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\theta}) = \delta^-_{\mathcal{X}}(\ddot{\sigma}) = \delta^-_{\mathcal{X}}(\ddot{\varpi}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 15. If $\ddot{\sigma} \in \mathcal{X}$, $\ddot{\theta}, \ddot{\pi}, \ddot{\varpi} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(\ddot{\sigma}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\theta}) = \delta^+_{\mathcal{X}}(\ddot{\pi}) = \delta^+_{\mathcal{X}}(\ddot{\varpi}) = 0,$$

$$\delta^-_{\mathcal{X}}(\ddot{\sigma}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\theta}) = \delta^-_{\mathcal{X}}(\ddot{\pi}) = \delta^-_{\mathcal{X}}(\ddot{\varpi}) = 0.$$

Thus,

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\sigma}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^-_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta^-_{\mathcal{X}}(\ddot{\theta}), \delta^-_{\mathcal{X}}(\ddot{\pi}), \delta^-_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\pi}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\},$$

$$\delta^+_{\mathcal{X}}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta^+_{\mathcal{X}}(\ddot{\theta}), \delta^+_{\mathcal{X}}(\ddot{\sigma}), \delta^+_{\mathcal{X}}(\ddot{\varpi})\}.$$

Case 16. If $\ddot{\theta} \in \mathcal{X}$, $\ddot{\sigma}, \ddot{\pi}, \ddot{\varpi} \notin \mathcal{X}$, then

$$\delta^+_{\mathcal{X}}(\ddot{\theta}) = 1, \quad \delta^+_{\mathcal{X}}(\ddot{\sigma}) = \delta^+_{\mathcal{X}}(\ddot{\pi}) = \delta^+_{\mathcal{X}}(\ddot{\varpi}) = 0,$$

$$\delta^-_{\mathcal{X}}(\ddot{\theta}) = -1, \quad \delta^-_{\mathcal{X}}(\ddot{\sigma}) = \delta^-_{\mathcal{X}}(\ddot{\pi}) = \delta^-_{\mathcal{X}}(\ddot{\varpi}) = 0.$$

Thus,

$$\delta_{\mathcal{X}}^{-}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta_{\mathcal{X}}^{-}(\ddot{\theta}), \delta_{\mathcal{X}}^{-}(\ddot{\sigma}), \delta_{\mathcal{X}}^{-}(\ddot{\varpi})\},$$

$$\delta_{\mathcal{X}}^{-}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \leq 0 = \max\{\delta_{\mathcal{X}}^{-}(\ddot{\theta}), \delta_{\mathcal{X}}^{-}(\ddot{\pi}), \delta_{\mathcal{X}}^{-}(\ddot{\varpi})\},$$

$$\delta_{\mathcal{X}}^{+}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta_{\mathcal{X}}^{+}(\ddot{\theta}), \delta_{\mathcal{X}}^{+}(\ddot{\pi}), \delta_{\mathcal{X}}^{+}(\ddot{\varpi})\},$$

$$\delta_{\mathcal{X}}^{+}(\ddot{\theta}\alpha\ddot{\sigma}\beta\ddot{\pi}\gamma\ddot{\varpi}) \geq 0 = \min\{\delta_{\mathcal{X}}^{+}(\ddot{\theta}), \delta_{\mathcal{X}}^{+}(\ddot{\sigma}), \delta_{\mathcal{X}}^{+}(\ddot{\varpi})\}.$$

Therefore, δ_B is a bipolar fuzzy tri-ideal of $\Re$. □

4. Conclusion

In this paper, we introduced and investigated bipolar fuzzy tri-ideals in Γ-semigroups. Fundamental characterizations of bipolar fuzzy tri-ideals were established through cut sets, which provide a clear connection between crisp tri-ideals and their bipolar fuzzy counterparts. Several structural properties of bipolar fuzzy tri-ideals were examined, including their behavior under set-theoretic operations such as intersection and union.

We also showed that the class of bipolar fuzzy tri-ideals is preserved under Γ-semigroup homomorphisms. In particular, under suitable conditions, we showed that the homomorphic image and pre-image of a bipolar fuzzy tri-ideal are bipolar fuzzy tri-ideals. The relationship between tri-ideals and bipolar fuzzy characteristic functions was also clarified, yielding a one-to-one correspondence between classical tri-ideals and bipolar fuzzy tri-ideals.

The results presented in this work extend the existing theory of fuzzy algebraic structures and deepen the understanding of bipolar fuzzification in Γ-semigroups. These findings may serve as a foundation for further investigations into related fuzzy ideal structures and their applications in algebraic systems involving uncertainty.

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