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A NEW STATISTICAL MODEL GENERATED BY POWER TRANSFORMATION: PROPERTIES, SIMULATION, AND APPLICATIONS

HAZAR A. KHOGEER[1], IBRAHIM HASSAN ALKHAIRY[2], SULE OMEIZA BASHIRU[3], CANER TANIŞ[4], LAXMI PRASAD SAPKOTA[5,*], DIVINE-FAVOUR N. EKEMEZIE[6], HASSAN ALSUHABI[2], ESLAM HUSSAM[7,8], AHMED M. GEMEAY[9]

[1]Department of Mathematics, College of Science, Umm Al-Qura University, Makkah, Saudi Arabia

[2]Department of Mathematics, Al-Qunfudah University College, Umm Al-Qura University, Mecca, Saudi Arabia

[3]Department of Mathematics and Statistics, Confluence University of Science and Technology, Osara, Kogi State, Nigeria

[4]Department of Statistics, Necmettin Erbakan University, Konya, Turkey

[5]Department of Statistics, Tribhuvan University, Tribhuvan Multiple Campus, Palpa, Nepal

[6]Department of Statistics, Faculty of Physical Sciences, Nnamdi Azikiwe University, P.O. Box 5025, Awka 420007, Nigeria

[7]Department of Accounting, College of Business Administration in Hawtat Bani Tamim, Prince Sattam bin Abdulaziz University, Saudi Arabia

[8]Department of Mathematics, Faculty of Science, Capital University, Cairo, Egypt

[9]Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt

[*]Corresponding author: laxmisapkota75@gmail.com

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Abstract. Real-world data may have features that classical probability distributions cannot adequately describe. These include skewness, heavy tails, and different hazard rate shapes, which occur in reliability engineering, medical research, and environmental studies. An unsuitable model may give misleading conclusions, underestimate risks, or lead to poor policy decisions. In this paper, we propose the power exponential Komal distribution (PEKD) by applying the power transformation to the exponential Komal distribution. This adds a shape parameter to the baseline model. We derive its moments and other statistical properties and estimate its parameters using nine methods, including maximum likelihood, Anderson–Darling, and maximum product of spacings. A simulation study examines the performance of the estimators. We also apply the PEKD to air-conditioning failure times and annual flood discharges. For both datasets, the PEKD provides the best fit among the models considered.

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Key words and phrases. exponential Komal distribution; power transformation; moments; estimation; real data.

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1. Introduction

Probability distributions are used in statistics and the applied sciences to describe uncertainty and random behavior in data. Their applications cover many fields. Reliability engineers use them to study the failure times of machines and components, while medical researchers use them to analyze the survival and recovery times of patients. Environmental scientists also use them to model pollutant concentrations in soil, water, and air. In these applications, the choice of distribution matters. An unsuitable model can give incorrect predictions and lead to poor decisions, sometimes with economic or social consequences.

Classical distributions such as the Weibull [8], exponential, and Gompertz [23] remain useful for many practical problems. They are simple and easy to interpret. However, some datasets have features that these models cannot adequately describe, such as pronounced skewness, heavy tails, and different hazard rate shapes. This has led researchers to develop extensions of existing distributions. The power transformation is one of the approaches used. It adds a shape parameter to the baseline distribution, allowing greater flexibility while retaining mathematical tractability. Examples include the power Darna distribution [19], power half-logistic distribution [17], power Zeghdoudi distribution [9], power type II Topp-Leone half logistic distribution [10], power Chris-Jerry distribution [18], power unit inverse Lindley [11], power new XLindley distribution [11], power Erlang distribution [21], power Quasi-XLindley distribution [22], and power quasi-Xgamma distribution [20]. These distributions have been used to improve the fit of datasets with different characteristics.

Many studies (see [15], [16], [12], [13], [14]) have emphasized that no single distribution can describe all types of data. There is therefore a continuing need for models that can accommodate different data patterns.

In this study, we propose the power exponential Komal distribution (PEKD). We obtain it by applying the power transformation to the exponential Komal distribution introduced by [2]. The additional shape parameter extends the flexibility of the baseline model. The PEKD has a right-skewed probability density function and can be used to model right-skewed lifetimes and waiting times. Its hazard function can be decreasing or have an upside-down bathtub shape. These hazard shapes are useful in reliability studies, where the risk of failure may change over time.

We study the theoretical properties of the PEKD, including its moments. We also estimate its parameters using several methods, including maximum likelihood, Anderson–Darling, and maximum product of spacings. A simulation study examines the performance of the estimators at different sample sizes. The results show that the methods provide reliable estimates under the simulation settings considered. Two datasets are used to assess the fit of the PEKD: air-conditioning failure times and annual flood discharges. For both datasets, the PEKD provides the best fit among the models considered. The main aim of this study is to introduce the PEKD, examine its statistical properties, compare its

estimation methods, and assess its performance using real data. Figure 1 shows the organization of the remaining sections.

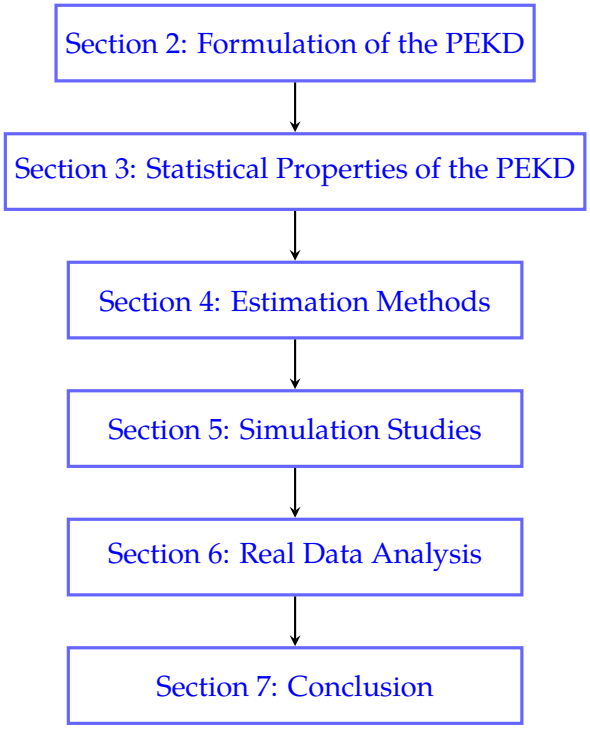


FIGURE 1. Flowchart of the Remaining Sections of the Study.

2. Model Formulation

Shanker and Ray [2] recently introduced a novel probability model known as the exponential Komal distribution (EKD), which is defined on the positive real line. The cumulative distribution function (CDF) of the EKD is given by

$$G(x) = \frac{x\left[(\tau+1)\tau(\tau+x)+(2\tau+x)\right]}{(\tau^2+\tau+1)(\tau+x)^2}, \quad \tau>0,\ x>0. \tag{1}$$

Building upon this foundation, we propose a new flexible distribution by applying the power transformation to the CDF in Equation (1). The resulting model is referred to as the power exponential Komal distribution (PEKD). The CDF and the corresponding probability density function (PDF) of the PEKD are defined as follows:

$$F(x) = \frac{x^{\alpha}\left(2\tau+\tau(\tau+1)(\tau+x^{\alpha})+x^{\alpha}\right)}{(\tau^2+\tau+1)(\tau+x^{\alpha})^2};\ \tau>0,\ \alpha>0,\ x>0, \tag{2}$$

$$f(x) = \frac{\alpha\tau^2 x^{\alpha-1}\left(\tau^2+\tau+(\tau+1)x^{\alpha}+2\right)}{(\tau^2+\tau+1)(\tau+x^{\alpha})^3}. \tag{3}$$

The hazard function (HRF), also known as the failure rate, is widely used in various fields. It helps measure the risk or chance that an event (such as a failure or death) will occur at a particular time,

assuming the subject has survived until that point. For PEKD, the HRF can be defined as

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\alpha\tau^2 x^{\alpha-1}\left(\tau^2+\tau+(\tau+1)x^\alpha+2\right)(\tau+x^\alpha)^{-1}}{(\tau^2+\tau+1)(\tau+x^\alpha)^2 - x^\alpha\left(2\tau+\tau(\tau+1)(\tau+x^\alpha)+x^\alpha\right)}. \tag{4}$$

Figure 2 shows some possible shapes of the PEKD density and hazard functions. The density can be unimodal or decreasing, while the hazard function can have an upside-down bathtub shape or be decreasing for the parameter values shown.

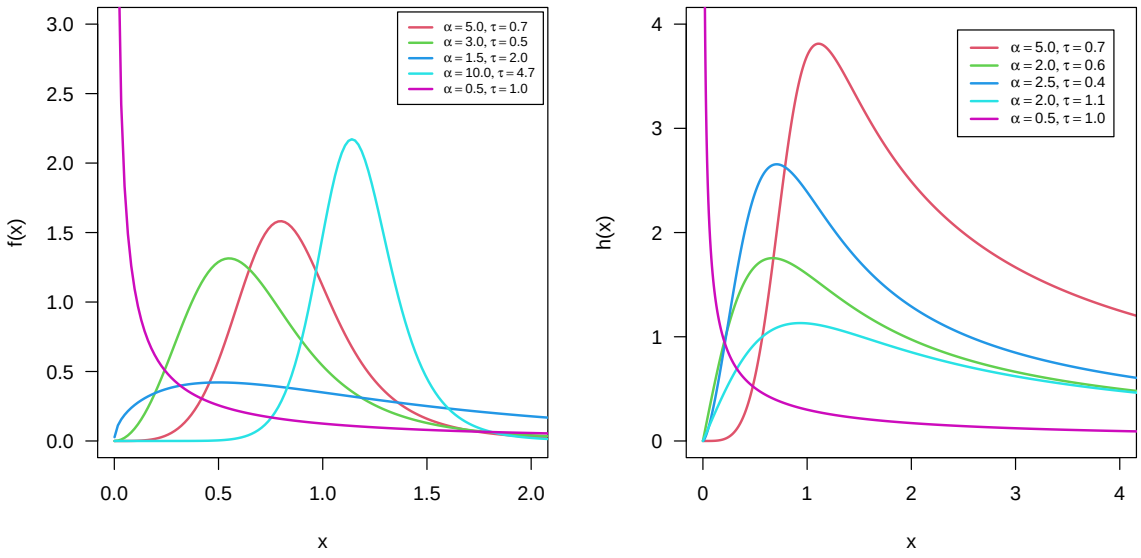


FIGURE 2. Some possible shapes of PDF and HRF of PEKD.

3. Statistical Properties

3.1. **Quantile function (QF).** The QF also plays a crucial role in determining various descriptive measures, including the median, interquartile range, and percentiles, and is essential in performing probabilistic risk assessments and constructing prediction intervals. The QF of PEKD is

$$Q(p) = \left\{\frac{\tau\left[2pA - C + \sqrt{C^2-4Ap}\right]}{2A(1-p)}\right\}^{1/\alpha}, \qquad 0<p<1, \tag{5}$$

where $A = \tau^2+\tau+1$ and $C = \tau^2+\tau+2 = A+1$. Numerical representation of the PEKD quantiles is presented in Table 1. Table 1 presents the PEKD quantiles $Q_{0.1}$ through $Q_{0.95}$ for six distinct values of τ and eleven distinct values of α. With a constant τ, increasing α, each quantile is drawn toward the median, significantly diminishing the proportion between the higher and lower tails. At a constant α, increasing τ elevates the entire distribution, and at minimal α, significantly amplifies the top quantiles

by numerous orders of magnitude, transitioning from $Q_{0.95} = 24.51$ at $\tau = 0.4$ to $Q_{0.95} = 62309.4$ at $\tau = 2.75$. As α surpasses 1, the quantile profile exhibits increased symmetry, and the median approaches the interquartile midpoint.

Table 1: Numerical values of PEKD quantiles.

Parameters		Measures									
τ	α	$Q_{0.1}$	$Q_{0.2}$	$Q_{0.25}$	$Q_{0.35}$	$Q_{0.4}$	$Q_{0.5}$	$Q_{0.65}$	$Q_{0.75}$	$Q_{0.85}$	$Q_{0.95}$
0.4	0.35	0.0000317652	0.000305877	0.00067612	0.00249596	0.00443191	0.0130003	0.06493	0.220522	1.10043	24.5123
0.4	0.7	0.00563607	0.0174893	0.0260023	0.0499595	0.0665726	0.114019	0.254814	0.469597	1.04902	4.95099
0.4	0.95	0.0220204	0.0507226	0.0679382	0.109923	0.135817	0.201902	0.365157	0.572946	1.03589	3.24996
0.4	1.3	0.0615159	0.113188	0.140134	0.199185	0.232482	0.310612	0.478932	0.665636	1.0261	2.36627
0.4	1.5	0.0892186	0.151342	0.182112	0.246995	0.282404	0.363013	0.528328	0.702757	1.02258	2.10955
0.4	1.75	0.126006	0.198203	0.232276	0.301611	0.338313	0.419556	0.578747	0.739078	1.01933	1.89617
0.4	2.	0.163246	0.242645	0.278775	0.350362	0.387394	0.467672	0.619695	0.767545	1.01689	1.75042
0.4	2.35	0.213835	0.29962	0.337191	0.409595	0.446162	0.52372	0.665473	0.798392	1.01436	1.61038
0.4	2.5	0.23457	0.322089	0.359918	0.432129	0.468298	0.544444	0.681934	0.809251	1.01349	1.565
0.4	2.8	0.273996	0.363658	0.401562	0.472775	0.507953	0.581091	0.710486	0.827811	1.01204	1.49167
0.4	3.2	0.322126	0.412674	0.450074	0.519186	0.552835	0.62189	0.7415	0.847598	1.01052	1.41894
0.85	0.35	0.000451106	0.00445983	0.0100056	0.0382111	0.069189	0.212511	1.16723	4.3545	25.0934	731.397
0.85	0.7	0.0212393	0.0667819	0.100028	0.195477	0.263038	0.460989	1.08039	2.08674	5.00933	27.0444
0.85	0.95	0.0585283	0.136132	0.183336	0.300365	0.373805	0.565188	1.05863	1.71948	3.27814	11.3557
0.85	1.3	0.125669	0.232876	0.28947	0.415223	0.487194	0.659038	1.04251	1.48601	2.38125	5.90362
0.85	1.5	0.165704	0.282818	0.3415	0.46685	0.536219	0.696715	1.03674	1.40957	2.12111	4.6591
0.85	1.75	0.214218	0.338738	0.398152	0.520521	0.586148	0.733628	1.03141	1.34211	1.90507	3.73965
0.85	2.	0.259716	0.38782	0.446727	0.564785	0.626623	0.762591	1.02743	1.29364	1.75761	3.17122
0.85	2.35	0.317469	0.446579	0.503691	0.614946	0.671799	0.794004	1.0233	1.24498	1.61601	2.6704
0.85	2.5	0.340094	0.46871	0.524849	0.63315	0.688027	0.805069	1.02189	1.22872	1.57014	2.51757
0.85	2.8	0.381755	0.508352	0.562381	0.664927	0.716151	0.823991	1.01952	1.2019	1.49605	2.28044
0.85	3.2	0.430588	0.553215	0.604333	0.699724	0.746671	0.844175	1.01706	1.17458	1.42258	2.05715
1.25	0.35	0.00180258	0.0180362	0.0407323	0.157854	0.288178	0.901654	5.12841	19.7212	118.192	3634.95
1.25	0.7	0.0424567	0.134299	0.201823	0.397309	0.536822	0.949555	2.2646	4.44086	10.8716	60.2906
1.25	0.95	0.097502	0.227786	0.307519	0.506548	0.632306	0.962578	1.8263	2.99972	5.80209	20.5004
1.25	1.3	0.182474	0.339235	0.422428	0.608339	0.715359	0.972513	1.55292	2.2317	3.61413	9.09067
1.25	1.5	0.228933	0.391833	0.473863	0.650019	0.748033	0.976134	1.46441	2.00517	3.04511	6.77303
1.25	1.75	0.282604	0.447949	0.527215	0.691276	0.779708	0.979508	1.38674	1.81546	2.59727	5.15347
1.25	2.	0.330964	0.495251	0.571136	0.723927	0.804342	0.982046	1.33121	1.68505	2.30517	4.19843
1.25	2.35	0.390214	0.549891	0.620826	0.759611	0.830852	0.9847	1.27568	1.55906	2.03556	3.39068
1.25	2.5	0.412881	0.569981	0.638839	0.772246	0.840142	0.985611	1.25718	1.51807	1.95057	3.15116
1.25	2.8	0.453928	0.605366	0.670259	0.79393	0.855968	0.987143	1.22673	1.45167	1.81582	2.78652
1.25	3.2	0.50103	0.644564	0.704632	0.817164	0.872771	0.988741	1.19579	1.38559	1.68535	2.45149
1.6	0.35	0.00430153	0.0432797	0.0980345	0.3824	0.700582	2.20906	12.7386	49.5297	300.739	9396.47
1.6	0.7	0.0655861	0.208038	0.313105	0.618385	0.837008	1.48629	3.56912	7.03773	17.3418	96.9354
1.6	0.95	0.134331	0.314469	0.425014	0.701764	0.87713	1.3391	2.55359	4.21138	8.18497	29.0887
1.6	1.3	0.230621	0.429384	0.535116	0.771971	0.908642	1.23786	1.98396	2.85963	4.64734	11.7394
1.6	1.5	0.280444	0.480618	0.581641	0.799075	0.920323	1.20314	1.81076	2.48581	3.78655	8.45328
1.6	1.75	0.336298	0.53365	0.628457	0.825094	0.931305	1.17177	1.66351	2.18259	3.13067	6.2315
1.6	2.	0.385375	0.577231	0.666026	0.845163	0.939627	1.14878	1.56098	1.97971	2.71446	4.95757
1.6	2.35	0.444182	0.62646	0.707587	0.866606	0.948382	1.12529	1.46081	1.78824	2.33933	3.90587
1.6	2.5	0.466345	0.644288	0.722425	0.874083	0.951402	1.11735	1.42796	1.72696	2.22304	3.59928
1.6	2.8	0.506061	0.67536	0.748036	0.886778	0.956494	1.10414	1.37449	1.62876	2.04067	3.13777
1.6	3.2	0.551033	0.709322	0.775679	0.900198	0.961827	1.09056	1.32091	1.53241	1.8666	2.71984
2.1	0.35	0.0108982	0.110093	0.249908	0.979244	1.79839	5.70014	33.1607	129.799	794.072	25022.8
2.1	0.7	0.104394	0.331803	0.499908	0.989567	1.34104	2.3875	5.75853	11.3929	28.1793	158.186
2.1	0.95	0.1892	0.443572	0.59997	0.992302	1.24138	1.89882	3.6327	6.00584	11.7049	41.7292
2.1	1.3	0.296207	0.552093	0.688437	0.994369	1.17118	1.59775	2.56684	3.70645	6.03575	15.2816
2.1	1.5	0.348377	0.5976	0.723573	0.995118	1.14676	1.50097	2.26366	3.11241	4.74935	10.6238
2.1	1.75	0.405015	0.643209	0.757803	0.995814	1.12454	1.41637	2.0143	2.64639	3.80165	7.57997
2.1	2.	0.453458	0.679685	0.784534	0.996336	1.10817	1.35606	1.84548	2.34327	3.21718	5.88449
2.1	2.35	0.510142	0.719919	0.813407	0.996881	1.09134	1.29592	1.68452	2.06416	2.7033	4.5193
2.1	2.5	0.531165	0.734254	0.823549	0.997068	1.08563	1.27592	1.63263	1.97632	2.54672	4.12826
2.1	2.8	0.56842	0.758962	0.840858	0.997382	1.07612	1.24304	1.54909	1.83721	2.304	3.54644
2.1	3.2	0.610008	0.785584	0.859275	0.997709	1.0663	1.20969	1.46662	1.7027	2.07574	3.02739
2.75	0.35	0.0264783	0.268082	0.609252	2.39315	4.40069	13.9859	81.7213	320.902	1969.96	62309.4
2.75	0.7	0.162722	0.517767	0.780546	1.54698	2.09778	3.73978	9.03998	17.9137	44.3842	249.619

(continued on next page)

Table 1 (continued)

Parameters		Measures									
τ	α	$Q_{0.1}$	$Q_{0.2}$	$Q_{0.25}$	$Q_{0.35}$	$Q_{0.4}$	$Q_{0.5}$	$Q_{0.65}$	$Q_{0.75}$	$Q_{0.85}$	$Q_{0.95}$
2.75	0.95	0.262397	0.61569	0.833134	1.37918	1.72618	2.643	5.06459	8.38302	16.3586	58.4002
2.75	1.3	0.376178	0.701572	0.875106	1.26483	1.49024	2.03449	3.27235	4.72925	7.70849	19.5363
2.75	1.5	0.428556	0.735522	0.890812	1.22582	1.41304	1.85067	2.7939	3.84433	5.87093	13.144
2.75	1.75	0.483702	0.768517	0.905648	1.19068	1.34494	1.69488	2.4125	3.17157	4.55923	9.09726
2.75	2.	0.529671	0.794231	0.916937	1.16498	1.29603	1.5867	2.16102	2.74546	3.77164	6.90314
2.75	2.35	0.582252	0.821956	0.928856	1.13879	1.24693	1.48127	1.92671	2.36205	3.09502	5.17705
2.75	2.5	0.601457	0.831683	0.932978	1.12994	1.23053	1.44676	1.85237	2.24332	2.89217	4.6907
2.75	2.8	0.635128	0.848269	0.939939	1.11525	1.20348	1.39063	1.73397	2.05729	2.58111	3.97484
2.75	3.2	0.672208	0.865898	0.947245	1.10014	1.17594	1.33447	1.61868	1.8799	2.29262	3.34506

3.2. **Moments.** Using the PDF of PEKD defined in 3, we have defined the rth raw moment about the origin, which can be presented as

$$\begin{aligned}\mu'_r = \mathbb{E}[X^r] &= \int_0^\infty x^r \cdot \frac{\alpha\tau^2 x^{\alpha-1}\left(\tau^2+\tau+(\tau+1)x^\alpha+2\right)}{(\tau^2+\tau+1)(\tau+x^\alpha)^3}dx \\ &= \frac{\alpha\tau^2}{\tau^2+\tau+1}\int_0^\infty \frac{x^{r+\alpha-1}\left(\tau^2+\tau+(\tau+1)x^\alpha+2\right)}{(\tau+x^\alpha)^3}dx \end{aligned} \tag{6}$$

Let us transform $w = x^\alpha \Rightarrow x = w^{1/\alpha},\ dx = \frac{1}{\alpha}w^{\frac{1}{\alpha}-1}dw$. Then,

$$x^{r+\alpha-1} = w^{\frac{r+\alpha-1}{\alpha}}.$$

So, we get

$$x^{r+\alpha-1}dx = w^{\frac{r+\alpha-1}{\alpha}}\cdot\frac{1}{\alpha}w^{\frac{1}{\alpha}-1}dw = \frac{1}{\alpha}w^{\frac{r}{\alpha}}dw.$$

Substituting into the integral defined in 6, we have

$$\begin{aligned}\mu'_r &= \frac{\tau^2}{\tau^2+\tau+1}\int_0^\infty \frac{w^{\frac{r}{\alpha}}\left(\tau^2+\tau+(\tau+1)w+2\right)}{(\tau+w)^3}dw \\ &= \frac{\tau^2}{\tau^2+\tau+1}\left[(\tau^2+\tau+2)\int_0^\infty \frac{w^{\frac{r}{\alpha}}}{(\tau+w)^3}dw + (\tau+1)\int_0^\infty \frac{w^{\frac{r}{\alpha}+1}}{(\tau+w)^3}dw\right].\end{aligned}$$

Using the standard integral formula defined by [1]

$$\int_0^\infty \frac{w^{c-1}}{(w+a)^d}dw = a^{c-d}B(c,d-c), \quad \text{for } a>0,\ c>0,\ d>c,$$

we have obtained both terms as

$$\begin{aligned}I_1 &= \int_0^\infty \frac{w^{\frac{r}{\alpha}}}{(\tau+w)^3}dw = \tau^{\frac{r}{\alpha}-2}B\left(\frac{r}{\alpha}+1,\ 2-\frac{r}{\alpha}\right), \\ I_2 &= \int_0^\infty \frac{w^{\frac{r}{\alpha}+1}}{(\tau+w)^3}dw = \tau^{\frac{r}{\alpha}-1}B\left(\frac{r}{\alpha}+2,\ 1-\frac{r}{\alpha}\right).\end{aligned}$$

Finally we get the r^{th} raw moment about origin as

$$\mu'_r = \frac{\tau^2}{\tau^2+\tau+1}\left[(\tau^2+\tau+2)\tau^{\frac{r}{\alpha}-2}\cdot B\left(\frac{r}{\alpha}+1,\ 2-\frac{r}{\alpha}\right) + (\tau+1)\tau^{\frac{r}{\alpha}-1}\cdot B\left(\frac{r}{\alpha}+2,\ 1-\frac{r}{\alpha}\right)\right]. \tag{7}$$

where $0 < \frac{r}{\alpha} < 1$. Using the expression (7), we can obtain the mean, variance, skewness, and kurtosis, respectively as follows

$$\mu_1 = \frac{\tau^2}{\tau^2+\tau+1}\left((\tau^2+\tau+2)\tau^{\frac{1}{\alpha}-2}\mathcal{B}\left(1+\frac{1}{\alpha},2-\frac{1}{\alpha}\right)+(\tau+1)\tau^{\frac{1}{\alpha}-1}\mathcal{B}\left(2+\frac{1}{\alpha},1-\frac{1}{\alpha}\right)\right),$$

$$\sigma^2 = \mu_2 - \mu_1^2,$$

$$\gamma_1 = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{\sigma^3},$$

$$\gamma_2 = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{\sigma^4}.$$

We have displayed the nature of skewness and kurtosis of PEKD in Figure 3 and found that both skewness and kurtosis decrease rapidly as α increases and are nearly invariant in τ. Table 2 enumerates the initial four raw moments, variance, skewness, and kurtosis of the suggested distribution for five values of τ paired with eleven values of α, all confined to the domain $\alpha > 1$, ensuring a finite mean. Two patterns arise from the data. As one progresses along a row (increasing α while keeping τ constant), both the variance and the shape metrics consistently decline; for $\tau = 1.95$, σ^2 decreases from 0.369437 at $\alpha = 4.2$ to 0.0692618 at $\alpha = 7.8$, whereas γ_1 and γ_2 diminish from 3.70857 and 165.633 to 1.26551 and 8.68236 within the same interval.

This suggests that increased values of the shape parameter α drive the distribution toward a more compact, less skewed, and lighter-tailed configuration, along with the observed reduction in the quantile range for elevated α. Advancing down a column (increasing τ while keeping α constant) inversely influences location and dispersion: both μ_1 and σ^2 escalate with τ - at $\alpha = 4.2$, μ_1 ascends from 0.60571 at $\tau = 0.2$ to 1.36977 at $\tau = 2.75$, and σ^2 from 0.0625206 to 0.450095 - whereas γ_1 remains approximately 3.7 and γ_2 around 165 throughout this interval, indicating that τ predominantly affects scale rather than form. In all documented combinations, $\gamma_1 > 0$ and γ_2 significantly exceeds 3, resulting in a distribution that is consistently positively skewed and markedly leptokurtic across the analyzed region, with the heightened kurtosis being most evident when α approaches its minimum threshold, as the heavier tail amplifies μ_3 and particularly μ_4 in comparison to the lower-order moments.

Table 2: Numerical values of the first four moments and related measures.

Parameters		Measures						
τ	α	μ_1	μ_2	μ_3	μ_4	σ^2	γ_1	γ_2
0.2	4.2	0.60571	0.429405	0.385454	1.00544	0.0625206	3.17414	156.822
0.2	4.5	0.623411	0.444848	0.382414	0.552919	0.0562064	2.62742	58.0871
0.2	4.75	0.637059	0.45763	0.384696	0.459936	0.051786	2.30615	37.2404
0.2	5.	0.649799	0.47019	0.389467	0.418674	0.0479514	2.0591	27.273
0.2	5.4	0.668493	0.489639	0.400061	0.391028	0.042756	1.76182	19.1631
0.2	5.8	0.685357	0.50816	0.412587	0.382895	0.0384451	1.54138	14.9258
0.2	6.3	0.704232	0.529925	0.429405	0.384335	0.0339833	1.33298	11.8896

(continued on next page)

Table 2 (continued)

Parameters		Measures						
τ	α	μ_1	μ_2	μ_3	μ_4	σ^2	γ_1	γ_2
0.2	6.7	0.717812	0.546238	0.443133	0.390041	0.0309839	1.20204	10.358
0.2	7.1	0.730224	0.561611	0.456783	0.397883	0.028384	1.0934	9.27092
0.2	7.45	0.740237	0.57433	0.468531	0.405772	0.0263793	1.01224	8.55496
0.2	7.8	0.749545	0.586405	0.480024	0.41423	0.024587	0.941277	7.9905
0.7	4.2	0.900356	0.990609	1.49907	8.07724	0.179967	3.70811	170.593
0.7	4.5	0.901528	0.964407	1.32621	3.53285	0.151654	3.10414	63.9876
0.7	4.75	0.902884	0.948488	1.23083	2.53353	0.133288	2.74887	41.312
0.7	5.	0.904461	0.936379	1.16173	2.04019	0.11833	2.47558	30.3979
0.7	5.4	0.907248	0.922591	1.08491	1.62562	0.099492	2.1467	21.4472
0.7	5.8	0.910183	0.913519	1.03359	1.40227	0.0850868	1.90296	16.7252
0.7	6.3	0.913875	0.90648	0.990609	1.23899	0.0713133	1.67281	13.3087
0.7	6.7	0.916764	0.903141	0.966887	1.15589	0.0626841	1.52841	11.5692
0.7	7.1	0.919557	0.901186	0.949418	1.09696	0.0556009	1.40877	10.325
0.7	7.45	0.921907	0.900297	0.937812	1.05843	0.0503846	1.31951	9.49989
0.7	7.8	0.924162	0.899974	0.928764	1.02841	0.0458975	1.24158	8.84531
1.15	4.2	1.05897	1.38355	2.51913	16.643	0.262124	3.71692	167.5
1.15	4.5	1.04866	1.31557	2.14493	6.87835	0.215874	3.11642	62.9481
1.15	4.75	1.04173	1.27176	1.93616	4.73642	0.18656	2.76312	40.6923
1.15	5.	1.03595	1.23632	1.78292	3.68006	0.163135	2.49129	29.973
1.15	5.4	1.02854	1.19217	1.60891	2.79139	0.134282	2.16412	21.1743
1.15	5.8	1.02281	1.15891	1.48878	2.31005	0.112762	1.92159	16.5277
1.15	6.3	1.01734	1.12766	1.38355	1.95413	0.0926832	1.69253	13.1619
1.15	6.7	1.01394	1.10845	1.32219	1.76984	0.0803779	1.5488	11.4463
1.15	7.1	1.01119	1.09295	1.27439	1.63657	0.0704497	1.4297	10.2183
1.15	7.45	1.0092	1.08172	1.24065	1.54747	0.0632447	1.34084	9.40317
1.15	7.8	1.00751	1.0722	1.21261	1.47638	0.0571236	1.26324	8.7561
1.95	4.2	1.24379	1.91645	4.13543	32.5718	0.369437	3.70857	165.633
1.95	4.5	1.2184	1.78218	3.40204	12.8367	0.297685	3.11075	62.2819
1.95	4.75	1.20072	1.69489	2.99448	8.53786	0.253162	2.75901	40.2773
1.95	5.	1.18552	1.62367	2.69589	6.43108	0.2182	2.48838	29.6768
1.95	5.4	1.16524	1.53378	2.35705	4.67128	0.175998	2.16262	20.9738
1.95	5.8	1.14874	1.46483	2.1227	3.72521	0.145232	1.92113	16.3763
1.95	6.3	1.13203	1.39865	1.91645	3.02952	0.117158	1.69304	13.0452
1.95	6.7	1.12102	1.35699	1.79531	2.67042	0.100297	1.5499	11.3467
1.95	7.1	1.11163	1.32263	1.70016	2.41092	0.086905	1.4313	10.1307
1.95	7.45	1.10448	1.2972	1.63239	2.23728	0.077316	1.34279	9.32337
1.95	7.8	1.09816	1.27521	1.57558	2.09846	0.0692618	1.26551	8.68236
2.75	4.2	1.36977	2.32635	5.5385	48.1796	0.450095	3.70536	165.174
2.75	4.5	1.33316	2.13535	4.46722	18.4894	0.358022	3.10824	62.1145
2.75	4.75	1.30759	2.01137	3.87537	12.0594	0.301591	2.7569	40.1714
2.75	5.	1.28553	1.9103	3.44371	8.92555	0.257726	2.48657	29.6004
2.75	5.4	1.25595	1.78279	2.95617	6.32527	0.20539	2.16118	20.9211
2.75	5.8	1.23176	1.68496	2.62057	4.93839	0.167735	1.91996	16.3361
2.75	6.3	1.20712	1.59095	2.32635	3.92614	0.133816	1.69212	13.0138
2.75	6.7	1.19079	1.53167	2.15401	3.40697	0.113679	1.54914	11.3198
2.75	7.1	1.1768	1.48268	2.01885	3.0335	0.0978301	1.43067	10.1069
2.75	7.45	1.1661	1.44636	1.92268	2.78449	0.0865694	1.34226	9.30163
2.75	7.8	1.15659	1.41488	1.84208	2.58594	0.0771732	1.26506	8.66224

3.3. **Order Statistics.** Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ represent the order statistics from a random sample of size n drawn from the PEKD distribution, with CDF and PDF $F(x)$ and $f(x)$ given by Equations (2)

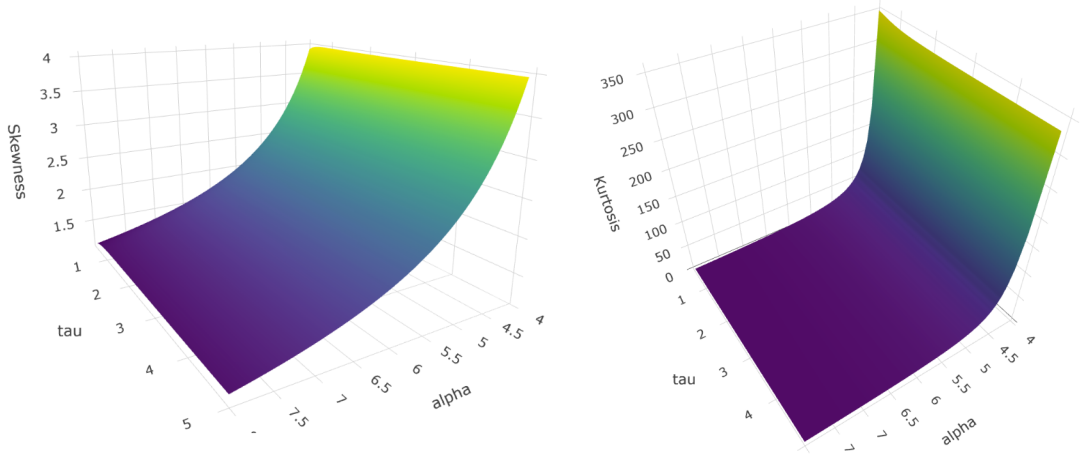


FIGURE 3. Skewness and Kurtosis plots for various values of the parameters of PEKD.

and (3), respectively. The PDF of the s-th order statistic $X_{s:n}$ is given by

$$f_{X_{s:n}}(x) = \frac{\Gamma(n+1)}{\Gamma(s)\Gamma(n-s+1)}[F(x)]^{s-1}[1-F(x)]^{n-s}f(x), \quad x > 0. \tag{8}$$

Substituting the expressions of $F(x)$ and $f(x)$ into the above equation yields

$$\begin{aligned} f_{X_{s:n}}(x) = \frac{\Gamma(n+1)}{\Gamma(s)\Gamma(n-s+1)} &\left[\frac{x^{\alpha}\left(2\tau+\tau(\tau+1)\left(\tau+x^{\alpha}\right)+x^{\alpha}\right)}{\left(\tau^2+\tau+1\right)\left(\tau+x^{\alpha}\right)^2}\right]^{s-1} \\ &\times\left[1-\frac{x^{\alpha}\left(2\tau+\tau(\tau+1)\left(\tau+x^{\alpha}\right)+x^{\alpha}\right)}{\left(\tau^2+\tau+1\right)\left(\tau+x^{\alpha}\right)^2}\right]^{n-s} \\ &\times\frac{\alpha\tau^2x^{\alpha-1}\left(\tau^2+\tau+(\tau+1)x^{\alpha}+2\right)}{\left(\tau^2+\tau+1\right)\left(\tau+x^{\alpha}\right)^3}, \quad x>0. \end{aligned}$$

3.4. **Mean Residual Life (MRL) Function.** The MRL function, denoted by $m(t)$, is a key reliability metric that captures the expected remaining lifetime of a system given survival up to time t. Formally, it is defined as

$$m(t) = \mathbb{E}[X - t \mid X > t] = \frac{1}{S(t)}\int_t^{\infty} S(x)\,dx, \tag{9}$$

where $S(x) = 1 - F(x)$ represents the survival function of the proposed distribution. Using the CDF from Equation (2), the survival function takes the form

$$S(x) = 1 - \frac{x^{\alpha}\left(2\tau+\tau(\tau+1)(\tau+x^{\alpha})+x^{\alpha}\right)}{\left(\tau^2+\tau+1\right)\left(\tau+x^{\alpha}\right)^2}. \tag{10}$$

Thus, the MRL function becomes

$$m(t) = \frac{1}{S(t)}\int_t^{\infty}\left[1-\frac{x^{\alpha}\left(2\tau+\tau(\tau+1)\left(\tau+x^{\alpha}\right)+x^{\alpha}\right)}{\left(\tau^2+\tau+1\right)\left(\tau+x^{\alpha}\right)^2}\right]dx. \tag{11}$$

where $\alpha > 1$, so that $\mathbb{E}[X] < \infty$ and the integral in Equation (11) converges.

The mean residual life (MRL) function serves as a critical tool in reliability analysis, quantifying the expected remaining lifetime of a system beyond a given time t. Figure 4 displays the MRL curves for the proposed distribution under five distinct combinations of the shape parameter α and scale parameter τ.

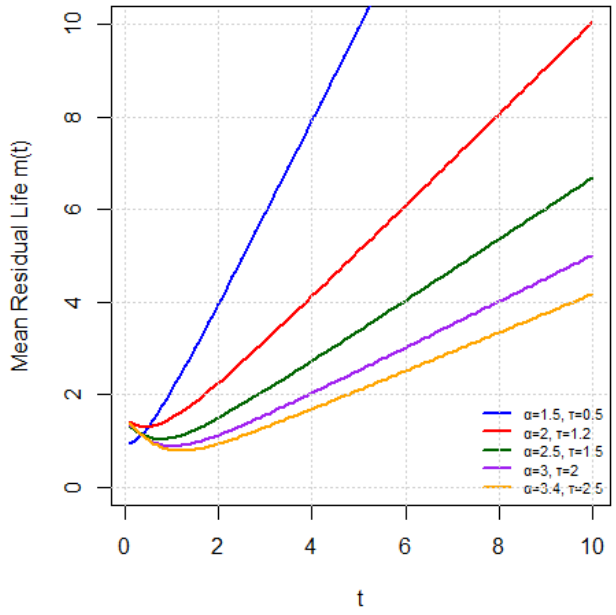


FIGURE 4. MRL curves of the PEKD for different combinations of parameters (α, τ).

4. Estimation Methods

This section is concerned with the parameter estimation of the PEKD by employing a broad spectrum of statistical techniques. In particular, nine distinct estimation techniques are considered including maximum likelihood estimation, Cramér–von Mises estimation, maximum product of spacings estimation, least squares estimation, weighted least squares estimation, Anderson–Darling and right-tailed Anderson–Darling estimations, as well as the minimum spacing absolute distance and minimum spacing absolute-log distance estimations.

4.1. Maximum Likelihood Estimation.

Let $X_1, X_2, \ldots, X_n$ represent a random sample from the PEKD with observed values $x_1, x_2, \ldots, x_n$, where the PDF is specified by $f(x; \alpha, \tau)$ in Eq. (3). The associated likelihood function $\mathcal{L}(\alpha, \tau)$ is then given by:

$$\mathcal{L}(x; \alpha, \tau) = \prod_{i=1}^{n} \frac{\alpha \tau^2 x_i^{\alpha-1} \left(\tau^2 + \tau + (\tau+1)x_i^{\alpha} + 2\right)}{(\tau^2 + \tau + 1)\left(\tau + x_i^{\alpha}\right)^3}. \tag{12}$$

The corresponding log-likelihood function, $\ell(x; \alpha, \tau)$, can be expressed as

$$\ell(\alpha,\tau) = n\log\alpha + 2n\log\tau + (\alpha-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\left(\tau^2+\tau+(\tau+1)x_i^{\alpha}+2\right)$$
$$-3\sum_{i=1}^{n}\log\left(\tau+x_i^{\alpha}\right) - n\log\left(\tau^2+\tau+1\right).$$

The derivatives of $\ell(x;\alpha,\tau)$ according to α and τ are given as:

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\frac{(\tau+1)x_i^{\alpha}\log x_i}{\tau^2+\tau+(\tau+1)x_i^{\alpha}+2} - 3\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{\tau+x_i^{\alpha}},$$

and

$$\frac{\partial\ell}{\partial\tau} = \frac{2n}{\tau} + \sum_{i=1}^{n}\frac{2\tau+1+x_i^{\alpha}}{\tau^2+\tau+(\tau+1)x_i^{\alpha}+2} - 3\sum_{i=1}^{n}\frac{1}{\tau+x_i^{\alpha}} - n\cdot\frac{2\tau+1}{\tau^2+\tau+1}.$$

The maximum likelihood estimates (MLEs) are obtained by simultaneously solving the system of equations $\frac{\partial}{\partial\alpha}\ell(x;\alpha,\tau)=0$ and $\frac{\partial}{\partial\tau}\ell(x;\alpha,\tau)=0$. As these equations typically do not allow closed-form solutions, numerical optimization techniques such as the Newton–Raphson method or the Nelder–Mead algorithm are implemented.

4.2. **Cramér-von Mises Estimation.** Macdonald [6] introduced the Cramér–von Mises estimator (CVME). In the present study, we use the CVME to estimate the parameters of the PEKD. The CVME of the unknown parameters is obtained by minimizing the expression in Eq. (13).

$$C(\alpha,\tau) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{2i-1}{2n}\right]^2, \tag{13}$$

where $F(x_{i:n})$ is the PEKD CDF evaluated at the ith ordered observation.

4.3. **Maximum Product of Spacing Estimation.** Cheng and Amin [4] proposed the maximum product of spacings estimator (MPSE) as an alternative to the MLE. The corresponding MPSEs are derived by maximizing the expression in Eq. (14).

$$M(\alpha,\tau) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log(F(x_{i:n}) - F(x_{i-1:n})), \tag{14}$$

where $F(x_{0:n}) = 0$ and $F(x_{n+1:n}) = 1$.

4.4. **Least Square Estimation and Weighted Least Square Estimation.** In this subsection, we discuss the least-squares and weighted least-squares estimation methods introduced by [5]. The least squares estimators (LSEs) of parameters can be obtained by minimizing the function $S(\alpha,\tau)$ in Eq. (15) with respect to the parameters α and τ. Similarly, the weighted least squares estimators (WLSEs) of parameters can be derived by minimizing the function $W(\alpha,\tau)$ in Eq. (16) with respect to the parameters α and τ. These functions $(S(\alpha,\tau), W(\alpha,\tau))$ are respectively given by

$$S(\alpha,\tau)=\sum_{i=1}^{n}\left[F(x_{i:n})-\frac{i}{n+1}\right]^2, \tag{15}$$

and

$$W(\alpha,\tau)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n})-\frac{i}{n+1}\right]^2. \tag{16}$$

4.5. **Anderson-Darling Estimation.** In this subsection, we provide another estimation method, the Anderson-Darling, based on the test statistics proposed by [7]. The Anderson-Darling estimators (ADEs) of the parameters for the PEKD can be obtained by minimizing the function in Eq. (17)

$$A(\alpha,\tau)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_{i:n})+\log(1-F(x_{n+1-i:n}))\right]. \tag{17}$$

4.6. **Right-Tailed Anderson-Darling Estimation.** In this subsection, we employ Right-Tailed Anderson-Darling estimation method to estimate the parameters α and τ of the PEKD. The right-tailed Anderson-Darling estimators (RTADEs) can be derived by minimizing the function in Eq. (18)

$$R(\alpha,\tau)=\frac{n}{2}-2\sum_{i=1}^{n}F(x_{i:n})-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log(1-F(x_{n+1-i:n})). \tag{18}$$

4.7. **Minimum spacing Absolute Distance Estimation.** In this subsection, the parameters α and τ are estimated using the minimum spacing absolute distance estimator (MSADE). Minimizing the function presented in Eq. (19) produces the associated MSADEs.

$$MS(\alpha,\tau)=\sum_{i=1}^{n+1}\left|\left[F(x_{i:n})-F(x_{i-1:n})\right]-\frac{1}{n+1}\right|. \tag{19}$$

4.8. **Minimum Spacing Absolute-Log Distance Estimation.** Torabi [3] introduced the minimum spacing absolute-log distance estimator (MSALDE). This technique estimates the parameter by minimizing the sum of the absolute differences between the logarithms of consecutive gaps in the ordered data. Hence, the MSALDEs of the parameters are obtained by minimizing the expression in Eq. (20).

$$MSL(\alpha,\tau)=\sum_{i=1}^{n+1}\left|\left[F(x_{i:n})-F(x_{i-1:n})\right]-\log\left(\frac{1}{n+1}\right)\right|. \tag{20}$$

5. Numerical Simulation

In this section, we carried out a comprehensive Monte Carlo (MC) simulation study to assess and contrast the performance of the estimators introduced in Section 4 for the point estimation of the PEKD. For sample sizes $n = 50, 100, 250, 500, 750, 1000$, 5000 independent replications are performed. We specify the initial values allocated to the parameters α and τ as follows:

$Case_I = (\alpha = 0.7, \tau = 0.5)$,

$Case_{II} = (\alpha = 0.9, \tau = 1.2)$,

$Case_{III} = (\alpha = 1.5, \tau = 0.8)$,

$Case_{IV} = (\alpha = 2, \tau = 2.5)$.

The performances of the estimators are measured through bias, mean squared error (MSE), and mean relative error (MRE). The corresponding formulas for these measures are presented as follows:

$$Bias = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{\Gamma} - \Gamma\right),$$

$$MSE = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{\Gamma} - \Gamma\right)^2,$$

$$MRE = \frac{1}{5000}\sum_{i=1}^{5000}\frac{|\hat{\Gamma} - \Gamma|}{\Gamma},$$

where $\Gamma = (\alpha, \tau)$.

5.1. Random sample generation.

This subsection presents a procedure for generating random samples from the $PEKD(\alpha, \tau)$. Random samples are produced from $PEKD(\alpha, \tau)$ using an acceptance-rejection (AR) sampling algorithm. In this context, the Weibull distribution is adopted as the proposal distribution. The steps of the AR algorithm are described below.

Algorithm 1.

A1. Simulate observations of the random variable $Y \sim$ Weibull(θ, β), whose PDF g is defined as follows:

$$g(\theta, \beta) = \theta\beta x^{\beta-1}e^{-\theta x^{\beta}}.$$

A2. Generate U from standard uniform distribution(independent of Y).

A3. If

$$U < \frac{f(Y; \alpha, \tau)}{k \times g(Y; \theta, \beta)}$$

then set $X = Y$ ("accept"); otherwise go back to A1 ("reject"), where the PDF $f(.)$ is given as in Eq. (3) and

$$k = \max_{z \in \mathbb{R}_+}\frac{f(z; \alpha, \tau)}{g(z; \theta, \beta)}.$$

The output of the Algorithm 1 provides a random data on X from the $PEKD(\alpha, \tau)$. The Algorithm 1 is used in all MC simulations.

We present the results of MC simulation study via Tables 3-6.

TABLE 3. The simulation results under Case I

		Bias		MSE		MRE	
Estimator	n	$\hat{\alpha}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\tau}$
MLE	50	0.0492	-0.0371	0.0094	0.0090	0.1073	0.1535
	100	0.0399	-0.0377	0.0049	0.0050	0.0791	0.1156
	250	0.0353	-0.0386	0.0025	0.0030	0.0583	0.0907
	500	0.0341	-0.0387	0.0018	0.0023	0.0510	0.0818
	750	0.0332	-0.0389	0.0015	0.0020	0.0486	0.0793
	1000	0.0326	-0.0388	0.0014	0.0019	0.0472	0.0784
LSE	50	0.0134	-0.0114	0.0089	0.0090	0.1047	0.1505
	100	0.0144	-0.0187	0.0045	0.0046	0.0750	0.1088
	250	0.0156	-0.0235	0.0019	0.0023	0.0493	0.0779
	500	0.0166	-0.0251	0.0011	0.0015	0.0375	0.0636
	750	0.0163	-0.0257	0.0008	0.0012	0.0327	0.0586
	1000	0.0163	-0.0261	0.0007	0.0011	0.0302	0.0565
WLSE	50	0.0251	-0.0222	0.0080	0.0083	0.0989	0.1455
	100	0.0259	-0.0295	0.0042	0.0045	0.0718	0.1083
	250	0.0276	-0.0348	0.0021	0.0027	0.0518	0.0850
	500	0.0291	-0.0368	0.0015	0.0021	0.0454	0.0781
	750	0.0293	-0.0377	0.0013	0.0019	0.0437	0.0770
	1000	0.0294	-0.0383	0.0012	0.0018	0.0431	0.0772
ADE	50	0.0293	-0.0253	0.0082	0.0085	0.1002	0.1483
	100	0.0250	-0.0287	0.0042	0.0046	0.0726	0.1095
	250	0.0231	-0.0313	0.0019	0.0026	0.0497	0.0825
	500	0.0231	-0.0320	0.0012	0.0018	0.0403	0.0716
	750	0.0225	-0.0324	0.0010	0.0016	0.0366	0.0681
	1000	0.0222	-0.0325	0.0008	0.0014	0.0347	0.0668
CvME	50	0.0350	-0.0252	0.0105	0.0095	0.1126	0.1562
	100	0.0251	-0.0257	0.0051	0.0049	0.0793	0.1129
	250	0.0198	-0.0263	0.0021	0.0025	0.0514	0.0804
	500	0.0187	-0.0265	0.0012	0.0016	0.0390	0.0654
	750	0.0177	-0.0266	0.0009	0.0013	0.0338	0.0600
	1000	0.0174	-0.0268	0.0007	0.0011	0.0311	0.0576
MPSE	50	0.0048	-0.0113	0.0061	0.0076	0.0882	0.1391
	100	0.0152	-0.0233	0.0033	0.0041	0.0648	0.1034
	250	0.0242	-0.0322	0.0018	0.0025	0.0486	0.0825
	500	0.0281	-0.0353	0.0014	0.0020	0.0442	0.0762
	750	0.0291	-0.0365	0.0013	0.0018	0.0434	0.0750
	1000	0.0294	-0.0370	0.0012	0.0017	0.0430	0.0750
RTADE	50	0.0499	-0.0313	0.0115	0.0090	0.1173	0.1527
	100	0.0422	-0.0342	0.0062	0.0050	0.0869	0.1142
	250	0.0386	-0.0364	0.0031	0.0029	0.0645	0.0886
	500	0.0380	-0.0371	0.0023	0.0022	0.0570	0.0794
	750	0.0370	-0.0374	0.0019	0.0019	0.0542	0.0767
	1000	0.0363	-0.0374	0.0017	0.0018	0.0526	0.0758
MSADE	50	0.0167	-0.0164	0.0101	0.0111	0.1099	0.1653
	100	0.0254	-0.0282	0.0059	0.0066	0.0862	0.1322
	250	0.0249	-0.0301	0.0027	0.0035	0.0589	0.0967
	500	0.0260	-0.0313	0.0017	0.0023	0.0474	0.0784
	750	0.0257	-0.0318	0.0014	0.0019	0.0431	0.0726
	1000	0.0255	-0.0314	0.0012	0.0016	0.0405	0.0684
MSALDE	50	0.0140	-0.0184	0.0083	0.0109	0.1019	0.1662
	100	0.0189	-0.0242	0.0051	0.0065	0.0811	0.1318
	250	0.0149	-0.0196	0.0022	0.0030	0.0534	0.0884
	500	0.0145	-0.0183	0.0012	0.0017	0.0392	0.0663
	750	0.0138	-0.0181	0.0008	0.0012	0.0332	0.0565
	1000	0.0133	-0.0175	0.0007	0.0009	0.0293	0.0499

TABLE 4. The simulation results under Case II

		Bias		MSE		MRE	
Estimator	n	$\hat{\alpha}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\tau}$
MLE	50	0.1523	-0.1570	0.0389	0.0614	0.1794	0.1742
	100	0.1418	-0.1728	0.0276	0.0475	0.1600	0.1577
	250	0.1291	-0.1867	0.0195	0.0419	0.1435	0.1570
	500	0.1280	-0.1868	0.0177	0.0380	0.1422	0.1557
	750	0.1262	-0.1861	0.0169	0.0368	0.1403	0.1551
	1000	0.1254	-0.1869	0.0164	0.0366	0.1394	0.1558
LSE	50	0.0938	-0.1290	0.0274	0.0533	0.1406	0.1600
	100	0.0968	-0.1478	0.0186	0.0398	0.1213	0.1420
	250	0.0933	-0.1632	0.0121	0.0339	0.1060	0.1387
	500	0.0955	-0.1641	0.0108	0.0301	0.1062	0.1369
	750	0.0937	-0.1634	0.0099	0.0289	0.1041	0.1361
	1000	0.0936	-0.1643	0.0096	0.0287	0.1040	0.1369
WLSE	50	0.1271	-0.1552	0.0321	0.0592	0.1576	0.1712
	100	0.1338	-0.1756	0.0255	0.0479	0.1517	0.1591
	250	0.1340	-0.1937	0.0208	0.0444	0.1490	0.1625
	500	0.1382	-0.1956	0.0204	0.0413	0.1536	0.1630
	750	0.1385	-0.1958	0.0201	0.0405	0.1539	0.1632
	1000	0.1391	-0.1971	0.0200	0.0405	0.1546	0.1642
ADE	50	0.1182	-0.1532	0.0302	0.0581	0.1519	0.1693
	100	0.1145	-0.1693	0.0210	0.0455	0.1340	0.1543
	250	0.1065	-0.1834	0.0143	0.0404	0.1191	0.1542
	500	0.1070	-0.1837	0.0129	0.0367	0.1189	0.1531
	750	0.1051	-0.1830	0.0120	0.0356	0.1168	0.1525
	1000	0.1048	-0.1839	0.0117	0.0354	0.1164	0.1532
CvME	50	0.1253	-0.1334	0.0358	0.0567	0.1634	0.1652
	100	0.1124	-0.1504	0.0222	0.0410	0.1348	0.1444
	250	0.0995	-0.1643	0.0133	0.0343	0.1123	0.1397
	500	0.0985	-0.1647	0.0114	0.0303	0.1096	0.1374
	750	0.0958	-0.1637	0.0103	0.0291	0.1064	0.1365
	1000	0.0952	-0.1646	0.0099	0.0288	0.1057	0.1372
MPSE	50	0.0866	-0.1505	0.0212	0.0553	0.1252	0.1652
	100	0.1050	-0.1685	0.0179	0.0449	0.1236	0.1534
	250	0.1124	-0.1846	0.0154	0.0409	0.1252	0.1552
	500	0.1189	-0.1856	0.0154	0.0375	0.1321	0.1547
	750	0.1199	-0.1853	0.0153	0.0365	0.1332	0.1544
	1000	0.1206	-0.1863	0.0152	0.0363	0.1340	0.1553
RTADE	50	0.1958	-0.1259	0.0599	0.0624	0.2246	0.1728
	100	0.1834	-0.1492	0.0440	0.0437	0.2050	0.1488
	250	0.1724	-0.1670	0.0336	0.0363	0.1915	0.1431
	500	0.1712	-0.1685	0.0311	0.0320	0.1902	0.1407
	750	0.1694	-0.1677	0.0299	0.0307	0.1882	0.1398
	1000	0.1688	-0.1688	0.0294	0.0304	0.1875	0.1407
MSADE	50	0.1140	-0.1426	0.0381	0.0734	0.1665	0.1849
	100	0.1194	-0.1579	0.0272	0.0534	0.1473	0.1620
	250	0.1131	-0.1736	0.0178	0.0418	0.1282	0.1509
	500	0.1140	-0.1735	0.0153	0.0355	0.1268	0.1453
	750	0.1126	-0.1732	0.0142	0.0337	0.1252	0.1444
	1000	0.1130	-0.1742	0.0140	0.0331	0.1256	0.1452
MSALDE	50	0.0887	-0.1358	0.0289	0.0684	0.1468	0.1806
	100	0.0886	-0.1419	0.0189	0.0468	0.1217	0.1522
	250	0.0789	-0.1443	0.0105	0.0317	0.0947	0.1294
	500	0.0794	-0.1402	0.0083	0.0248	0.0895	0.1184
	750	0.0783	-0.1383	0.0075	0.0226	0.0874	0.1157
	1000	0.0782	-0.1383	0.0071	0.0217	0.0870	0.1154

TABLE 5. The simulation results under Case III

		Bias		MSE		MRE	
Estimator	n	$\hat{\alpha}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\tau}$
MLE	50	0.0491	0.0078	0.0381	0.0210	0.1014	0.1416
	100	0.0273	0.0035	0.0170	0.0106	0.0693	0.1016
	250	0.0142	-0.0002	0.0067	0.0044	0.0436	0.0658
	500	0.0111	-0.0023	0.0032	0.0020	0.0304	0.0443
	750	0.0092	-0.0014	0.0022	0.0014	0.0250	0.0374
	1000	0.0084	-0.0028	0.0016	0.0010	0.0213	0.0321
LSE	50	0.0032	0.0189	0.0452	0.0215	0.1113	0.1426
	100	-0.0005	0.0102	0.0209	0.0108	0.0766	0.1020
	250	0.0002	0.0033	0.0084	0.0046	0.0490	0.0670
	500	0.0015	0.0000	0.0041	0.0021	0.0341	0.0448
	750	0.0017	0.0005	0.0028	0.0015	0.0280	0.0381
	1000	0.0019	-0.0012	0.0020	0.0011	0.0239	0.0326
WLSE	50	0.0150	0.0154	0.0400	0.0209	0.1043	0.1411
	100	0.0096	0.0075	0.0181	0.0106	0.0714	0.1014
	250	0.0055	0.0016	0.0072	0.0044	0.0450	0.0658
	500	0.0052	-0.0012	0.0034	0.0020	0.0313	0.0442
	750	0.0044	-0.0004	0.0023	0.0014	0.0257	0.0374
	1000	0.0040	-0.0020	0.0017	0.0010	0.0218	0.0321
ADE	50	0.0230	0.0132	0.0381	0.0208	0.1020	0.1408
	100	0.0121	0.0068	0.0177	0.0106	0.0707	0.1012
	250	0.0055	0.0016	0.0071	0.0044	0.0448	0.0658
	500	0.0049	-0.0011	0.0034	0.0020	0.0312	0.0442
	750	0.0041	-0.0004	0.0023	0.0014	0.0257	0.0374
	1000	0.0038	-0.0019	0.0017	0.0010	0.0218	0.0321
CvME	50	0.0512	0.0080	0.0513	0.0220	0.1162	0.1449
	100	0.0230	0.0046	0.0222	0.0109	0.0783	0.1028
	250	0.0096	0.0011	0.0087	0.0046	0.0493	0.0672
	500	0.0062	-0.0012	0.0041	0.0021	0.0343	0.0450
	750	0.0048	-0.0002	0.0028	0.0015	0.0281	0.0381
	1000	0.0042	-0.0018	0.0021	0.0011	0.0240	0.0327
MPSE	50	-0.0519	0.0285	0.0340	0.0203	0.0991	0.1383
	100	-0.0300	0.0154	0.0159	0.0104	0.0677	0.1001
	250	-0.0122	0.0052	0.0064	0.0043	0.0430	0.0651
	500	-0.0034	0.0006	0.0031	0.0020	0.0296	0.0439
	750	-0.0009	0.0007	0.0021	0.0014	0.0244	0.0372
	1000	0.0006	-0.0012	0.0015	0.0010	0.0207	0.0319
RTADE	50	0.0393	0.0142	0.0493	0.0216	0.1148	0.1428
	100	0.0197	0.0072	0.0222	0.0107	0.0785	0.1020
	250	0.0106	0.0017	0.0088	0.0044	0.0497	0.0660
	500	0.0079	-0.0010	0.0043	0.0020	0.0349	0.0443
	750	0.0071	-0.0003	0.0028	0.0014	0.0283	0.0375
	1000	0.0060	-0.0019	0.0021	0.0010	0.0244	0.0322
MSADE	50	-0.0329	0.0321	0.0516	0.0321	0.1191	0.1713
	100	-0.0232	0.0193	0.0260	0.0177	0.0857	0.1310
	250	-0.0107	0.0059	0.0102	0.0069	0.0538	0.0819
	500	-0.0024	0.0016	0.0048	0.0032	0.0370	0.0556
	750	0.0011	0.0008	0.0032	0.0022	0.0300	0.0462
	1000	0.0033	-0.0015	0.0024	0.0016	0.0260	0.0403
MSALDE	50	-0.0425	0.0311	0.0427	0.0292	0.1104	0.1647
	100	-0.0273	0.0170	0.0211	0.0147	0.0780	0.1194
	250	-0.0101	0.0045	0.0086	0.0060	0.0498	0.0766
	500	-0.0006	0.0004	0.0041	0.0028	0.0341	0.0519
	750	0.0018	0.0002	0.0027	0.0019	0.0279	0.0435
	1000	0.0032	-0.0022	0.0021	0.0015	0.0245	0.0381

TABLE 6. The simulation results under Case IV

		Bias		MSE		MRE	
Estimator	n	$\hat{\alpha}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\tau}$
MLE	50	-0.0212	0.0349	0.0537	0.3717	0.0925	0.1852
	100	-0.0451	-0.0325	0.0271	0.1711	0.0667	0.1302
	250	-0.0575	-0.0668	0.0132	0.0656	0.0464	0.0826
	500	-0.0646	-0.0828	0.0092	0.0373	0.0395	0.0631
	750	-0.0650	-0.0826	0.0075	0.0270	0.0364	0.0537
	1000	-0.0644	-0.0844	0.0066	0.0216	0.0346	0.0481
LSE	50	-0.1221	-0.0603	0.0783	0.3439	0.1148	0.1801
	100	-0.1183	-0.1010	0.0444	0.1662	0.0870	0.1307
	250	-0.1137	-0.1194	0.0248	0.0720	0.0660	0.0878
	500	-0.1140	-0.1293	0.0191	0.0455	0.0601	0.0710
	750	-0.1126	-0.1273	0.0167	0.0354	0.0574	0.0628
	1000	-0.1113	-0.1287	0.0153	0.0304	0.0562	0.0587
WLSE	50	-0.0837	-0.0232	0.0636	0.3424	0.1024	0.1792
	100	-0.0841	-0.0674	0.0336	0.1632	0.0749	0.1285
	250	-0.0831	-0.0889	0.0173	0.0665	0.0539	0.0836
	500	-0.0861	-0.1012	0.0127	0.0395	0.0478	0.0654
	750	-0.0856	-0.1003	0.0108	0.0295	0.0449	0.0565
	1000	-0.0845	-0.1015	0.0097	0.0243	0.0434	0.0515
ADE	50	-0.0722	-0.0134	0.0603	0.3435	0.0992	0.1797
	100	-0.0812	-0.0650	0.0329	0.1636	0.0741	0.1286
	250	-0.0845	-0.0901	0.0175	0.0667	0.0544	0.0838
	500	-0.0879	-0.1028	0.0130	0.0397	0.0486	0.0656
	750	-0.0875	-0.1019	0.0111	0.0298	0.0457	0.0568
	1000	-0.0864	-0.1031	0.0100	0.0246	0.0442	0.0518
CvME	50	-0.0621	-0.0045	0.0723	0.3870	0.1079	0.1866
	100	-0.0888	-0.0756	0.0395	0.1713	0.0812	0.1317
	250	-0.1019	-0.1097	0.0225	0.0711	0.0623	0.0869
	500	-0.1081	-0.1246	0.0178	0.0446	0.0577	0.0702
	750	-0.1086	-0.1241	0.0158	0.0347	0.0556	0.0621
	1000	-0.1084	-0.1263	0.0147	0.0299	0.0548	0.0580
MPSE	50	-0.1509	-0.0872	0.0694	0.2934	0.1095	0.1716
	100	-0.1189	-0.0976	0.0374	0.1572	0.0802	0.1273
	250	-0.0917	-0.0959	0.0179	0.0666	0.0554	0.0839
	500	-0.0834	-0.0987	0.0118	0.0391	0.0461	0.0650
	750	-0.0781	-0.0937	0.0093	0.0285	0.0415	0.0554
	1000	-0.0746	-0.0929	0.0080	0.0229	0.0388	0.0497
RTADE	50	-0.0628	0.0277	0.0725	0.4606	0.1080	0.1974
	100	-0.0806	-0.0505	0.0387	0.1941	0.0802	0.1380
	250	-0.0894	-0.0895	0.0204	0.0744	0.0588	0.0885
	500	-0.0942	-0.1055	0.0152	0.0440	0.0522	0.0691
	750	-0.0935	-0.1048	0.0129	0.0327	0.0490	0.0596
	1000	-0.0934	-0.1076	0.0117	0.0271	0.0479	0.0545
MSADE	50	-0.1240	-0.0253	0.0961	0.4596	0.1260	0.1964
	100	-0.1050	-0.0662	0.0523	0.2205	0.0941	0.1445
	250	-0.0878	-0.0806	0.0233	0.0954	0.0625	0.0975
	500	-0.0823	-0.0885	0.0143	0.0536	0.0498	0.0740
	750	-0.0767	-0.0900	0.0110	0.0383	0.0440	0.0634
	1000	-0.0743	-0.0906	0.0093	0.0313	0.0409	0.0576
MSALDE	50	-0.1346	-0.0456	0.0824	0.4235	0.1179	0.2032
	100	-0.1075	-0.0753	0.0441	0.2157	0.0864	0.1486
	250	-0.0864	-0.0866	0.0207	0.0882	0.0592	0.0961
	500	-0.0801	-0.0916	0.0132	0.0495	0.0477	0.0726
	750	-0.0738	-0.0895	0.0099	0.0348	0.0416	0.0611
	1000	-0.0722	-0.0915	0.0086	0.0284	0.0395	0.0554

The outcomes of the MC simulation study can be summarized as follows:

- Considering all nine estimators, the bias, MSE and MRE tend to decrease and approach zero as the sample size increases.
- In scenarios considering all parameters simultaneously, the MPSE exhibits superior performance for small sample sizes $(n = 50, 100)$. As the sample size increases, the differences in performance among the estimators decrease, although the MSADE generally demonstrates a slight advantage.
- Although MSADE remains the most consistent estimator in general, the LSE can serve as a viable alternative, particularly for large sample sizes $(n = 1000)$, since it produces low bias, MSE, and MRE values. Additionally, the MSALDE may be recommended for scenarios with large sample sizes.
- For small samples, the MLE shows comparatively higher bias, MSE, and MRE values than the other estimators.
- The lowest values of bias, MSE, and MRE are observed under the Case III parameter setting.
- Across all parameter settings, the parameter α is estimated more accurately than the parameter τ, as evidenced by consistently smaller bias, MSE, and MRE values. In contrast, the τ parameter exhibits relatively larger error measures across all estimators.

6. Real Data Analysis

This section assesses the fit of the proposed power exponential Komal distribution (PEKD) using two right-skewed datasets. The competing models are the power Ailamujia distribution (PAD) [24], power type II Topp-Leone half logistic distribution (PT2TLHLD) [10], exponential Komal distribution (EKD) [2], Komal distribution (KD) [25], and exponential distribution (ED).

The models are compared using the log-likelihood (LL), Akaike information criterion (AIC), corrected Akaike information criterion (AICc), Bayesian information criterion (BIC), and Hannan–Quinn information criterion (HQIC). The comparison also uses the root mean square error (RMSE), coefficient of determination (R^2), Kolmogorov–Smirnov (KS), Anderson–Darling (AD), and Cramér–von Mises (CVM) statistics. A model with a higher LL, lower information criteria, lower RMSE, and higher R^2 has a better fit according to those measures. Smaller KS, AD, and CVM statistics, together with larger corresponding p-values, also indicate a better fit. The fitted probability density function (PDF), cumulative distribution function (CDF), and probability–probability (P–P) plots provide visual comparisons.

The first dataset, reported by [27], gives the times between successive failures of air-conditioning equipment in a Boeing 720 airplane. The 15 observations are:

74, 57, 48, 29, 502, 12, 70, 21, 29, 386, 59, 27, 153, 26, 326.

Table 7 presents the descriptive statistics for this dataset. The mean is 121.27, while the median is 57.00. The skewness of 1.52 confirms that the data are skewed to the right. The kurtosis of 3.82 is greater than 3, indicating greater tail weight than a normal distribution.

TABLE 7. Descriptive statistics for the first dataset.

n	Mean	Median	Minimum	Maximum	Skewness	Kurtosis
15	121.27	57.00	12.00	502.00	1.52	3.82

Table 8 presents the parameter estimates and their standard errors for the fitted models. Table 9 reports the model selection and goodness-of-fit results. The PEKD has the highest LL (-85.3097) and the lowest AIC (174.6195), AICc (175.6195), BIC (176.0356), and HQIC (174.6044). It also has the lowest RMSE (0.0652) and the highest reported R^2 (0.9510). Its KS, AD, and CVM statistics are the smallest, with p-values of 0.9037, 0.7469, and 0.7925, respectively. Overall, the reported measures favor the PEKD for this dataset.

Figure 5 shows the fitted PDFs for the first dataset. The PEKD curve follows the shape of the observed data more closely than the competing models. Figure 6 shows that the fitted PEKD CDF is generally closer to the empirical CDF than the competing models. In Figure 7, the PEKD points generally lie closer to the diagonal line. These plots agree with Table 9, where the PEKD has the best overall fit among the models considered.

TABLE 8. Parameter estimates and SEs for all models applied to the first dataset.

Models	Estimates	SE	Estimates	SE
PEKD $(\widehat{\alpha},\widehat{\tau})$	1.5759	0.1091	594.2057	8.3914
PAD $(\widehat{\lambda},\widehat{\sigma})$	0.1192	0.0707	0.6189	0.1137
PT2TLHLD $(\widehat{\lambda},\widehat{\sigma})$	0.0835	0.0216	0.5728	0.0001
EKD $(\widehat{\tau})$	60.6050	5.9316	–	–
KD $(\widehat{\sigma})$	0.0164	0.0030	–	–
ED $(\widehat{\sigma})$	0.0082	0.0021	–	–

TABLE 9. Goodness-of-fit statistics for the first dataset.

Model	LL	AIC	AICc	BIC	HQIC	RMSE	R^2	KS (p-value)	AD (p-value)	CVM (p-value)
PEKD	-85.3097	174.6195	175.6195	176.0356	174.6044	0.0652	0.9510	0.1466 (0.9037)	0.4972 (0.7469)	0.0646 (0.7925)
PAD	-86.1723	176.3446	177.3446	177.7607	176.3295	0.0899	0.8999	0.2232 (0.4439)	0.7671 (0.5030)	0.1312 (0.4565)
PT2TLHLD	-87.1823	178.3647	179.3647	179.7808	178.3496	0.0884	0.8468	0.2373 (0.3670)	0.9462 (0.3851)	0.1558 (0.3755)
EKD	-87.1973	176.3946	176.7023	177.1027	176.3871	0.0744	0.8918	0.1907 (0.6463)	0.7812 (0.4924)	0.1208 (0.4973)
KD	-90.6687	183.3374	183.6451	184.0455	183.3299	0.2083	0.6529	0.3862 (0.0228)	4.0209 (0.0088)	0.6333 (0.0172)
ED	-86.9699	175.9398	176.2475	176.6478	175.9322	0.1190	0.8395	0.2764 (0.2019)	1.1626 (0.2814)	0.2186 (0.2352)

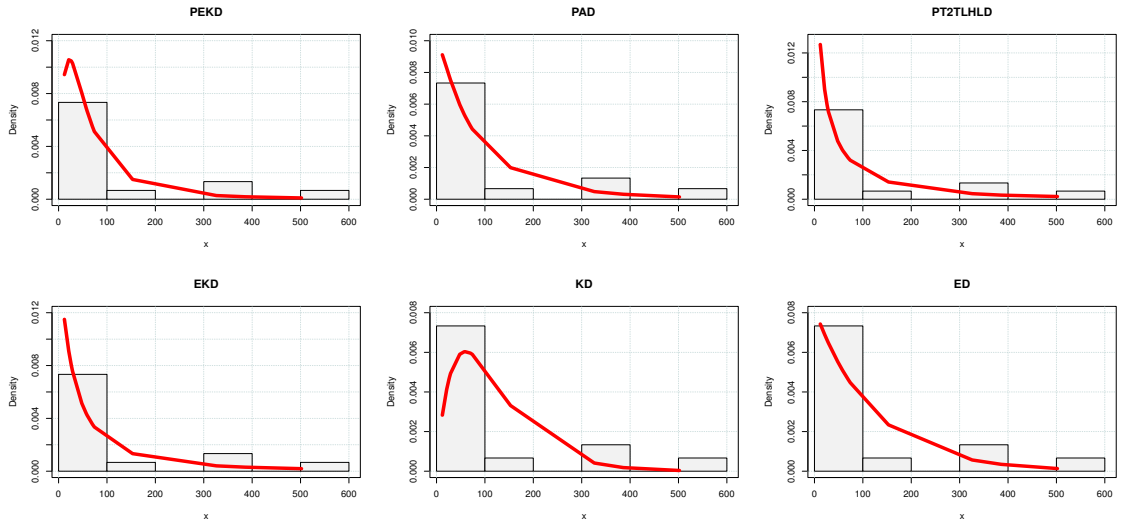


FIGURE 5. Estimated PDFs of the competing models for the first dataset.

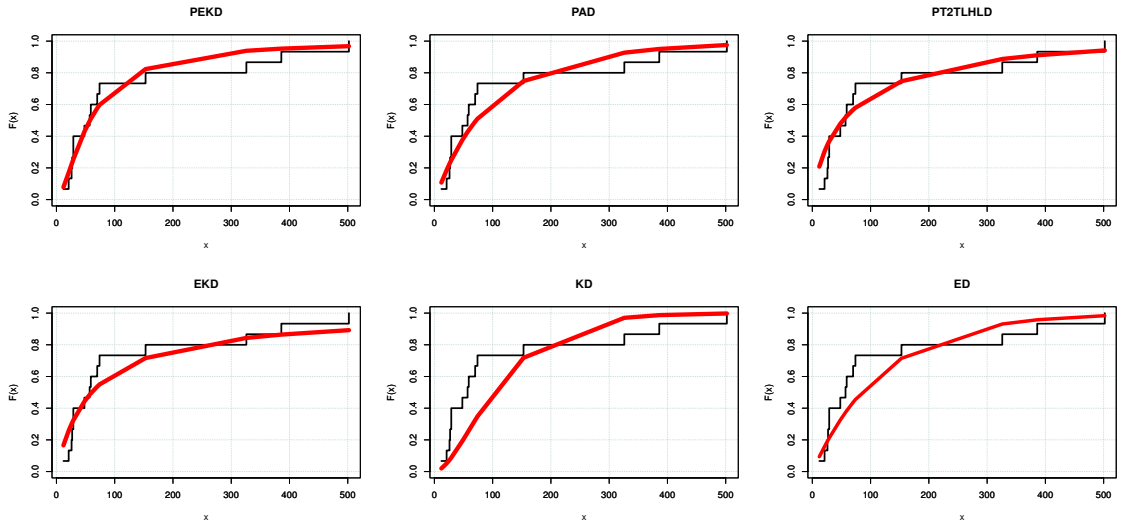


FIGURE 6. Plots comparing the empirical CDF with the fitted CDFs of all models for the first dataset.

The second dataset contains annual flood discharge rates of the Floyd River, as discussed by [26]. The 39 observations are: 1460, 4050, 3570, 2060, 1300, 1390, 1720, 6280, 1360, 7440, 5320, 1400, 3240, 2710, 4520, 4840, 8320, 13900, 71500, 6250, 2260, 318, 1330, 970, 1920, 15100, 2870, 20600, 3810, 726, 7500, 7170, 2000, 829, 17300, 4740, 13400, 2940, 5660.

Table 10 presents the descriptive statistics for the second dataset. The mean is 6771.10 while the median is 3570.00. The skewness of 4.56 shows strong right skewness. The kurtosis of 25.44 is also well above 3, indicating substantial tail weight.

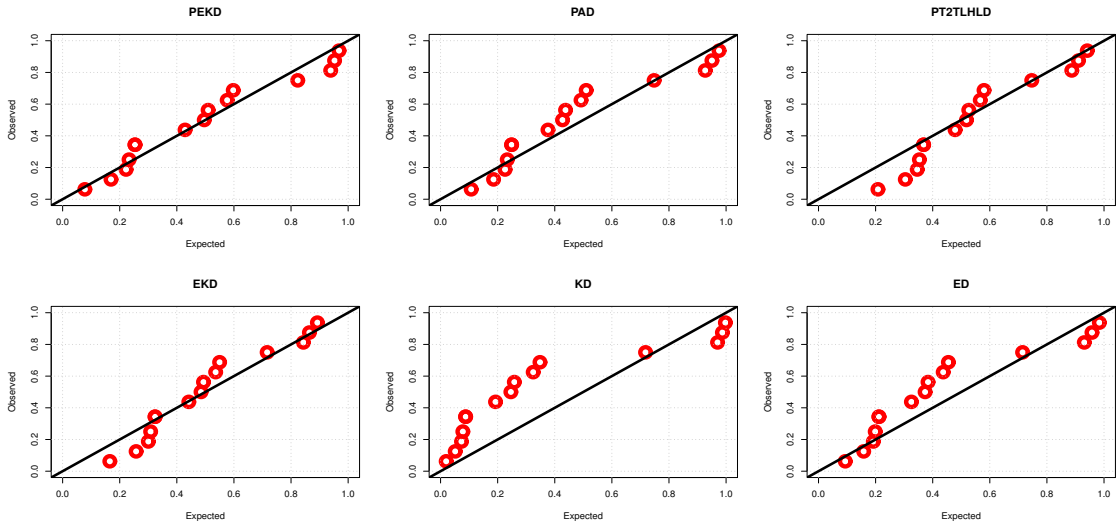


FIGURE 7. P–P plots of the fitted models for the first dataset

TABLE 10. Descriptive statistics for the second dataset.

n	Mean	Median	Minimum	Maximum	Skewness	Kurtosis
39	6771.10	3570.00	318.00	71500.00	4.56	25.44

Table 11 gives the parameter estimates and standard errors for all fitted models. Table 12 presents the model selection and goodness-of-fit results.

The PEKD has the highest LL (-376.7475) and the lowest AIC (757.4949), AICc (757.8283), BIC (760.8221), and HQIC (758.6887). It also has the lowest RMSE (0.0282) and the highest reported R^2 (0.9909). Its KS, AD, and CVM statistics are the smallest, with p-values of 0.9837, 0.9881, and 0.9829, respectively. Thus, the reported measures favor the PEKD for the Floyd River data.

Figure 8 shows the fitted PDFs for the second dataset. The PEKD curve follows the shape of the observed data more closely than the competing models. Figure 9 shows that the fitted PEKD CDF is generally closer to the empirical CDF than the competing models. In Figure 10, the PEKD points generally lie closer to the diagonal line. These plots agree with Table 12, where the PEKD has the best overall fit among the models considered.

TABLE 11. Parameter estimates and SEs for all models applied to the second dataset.

Models	Estimates	SE	Estimates	SE
PEKD $(\widehat{\alpha},\widehat{\tau})$	1.6924	0.0341	989997.4596	4.1943
PAD $(\widehat{\lambda},\widehat{\sigma})$	0.0091	0.0058	0.6295	0.0694
PT2TLHLD $(\widehat{\lambda},\widehat{\sigma})$	0.0741	0.0120	0.3224	0.0033
EKD $(\widehat{\tau})$	3526.6500	882.9110	–	–
KD $(\widehat{\sigma})$	0.0003	3.36×10^{-5}	–	–
ED $(\widehat{\sigma})$	0.0001	2.38×10^{-5}	–	–

TABLE 12. Goodness-of-fit statistics for the second dataset.

Model	LL	AIC	AICc	BIC	HQIC	RMSE	R^2	KS (p-value)	AD (p-value)	CVM (p-value)
PEKD	-376.7475	757.4949	757.8283	760.8221	758.6887	0.0282	0.9909	0.0701 (0.9837)	0.2079 (0.9881)	0.0282 (0.9829)
PAD	-379.9510	763.9020	764.2353	767.2291	765.0957	0.0448	0.9710	0.1075 (0.7180)	0.6808 (0.5741)	0.0940 (0.6175)
PT2TLHLD	-400.3162	804.6325	804.9658	807.9596	805.8262	0.1888	-1.1195	0.3780 (1.63×10^{-5})	7.3253 (0.0003)	1.4818 (0.0002)
EKD	-382.9554	767.9108	768.0189	769.5744	768.5077	0.0741	0.8799	0.1668 (0.2039)	1.7649 (0.1244)	0.2571 (0.1796)
KD	-392.6487	787.2975	787.4056	788.9610	787.8944	0.1577	0.7535	0.2454 (0.0149)	5.9176 (0.0011)	0.9611 (0.0028)
ED	-382.9964	767.9928	768.1009	769.6564	768.5897	0.0766	0.9170	0.1514 (0.3017)	1.3650 (0.2122)	0.2448 (0.1954)

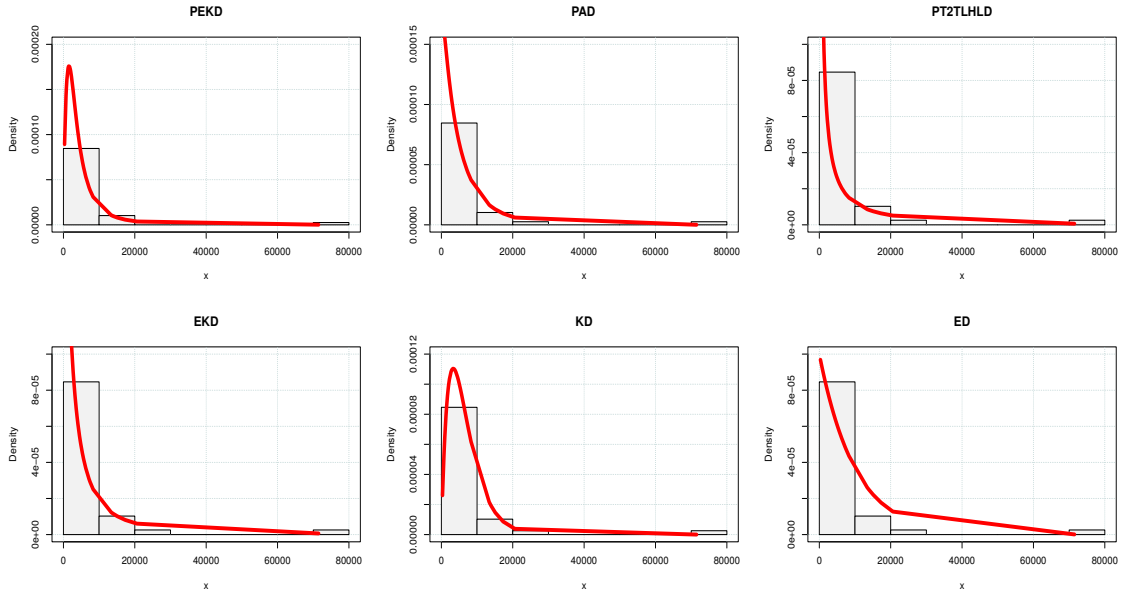


FIGURE 8. Estimated PDFs of the competing models for the second dataset.

7. CONCLUSION

This study introduced the power exponential Komal distribution (PEKD) by applying a power transformation to the exponential Komal distribution. The additional parameter allows the density to take different shapes. The hazard function can be decreasing or have an upside-down bathtub shape for the parameter values examined. Several statistical properties of the model were studied, including its moments, order statistics, quantile function, and mean residual life. Nine methods were considered for estimating the model parameters. The simulation tables report the bias, mean squared error, and mean relative error of the estimators under four parameter settings. The PEKD was also fitted to air-conditioning failure times and annual flood discharges. For both datasets, it had the most favorable

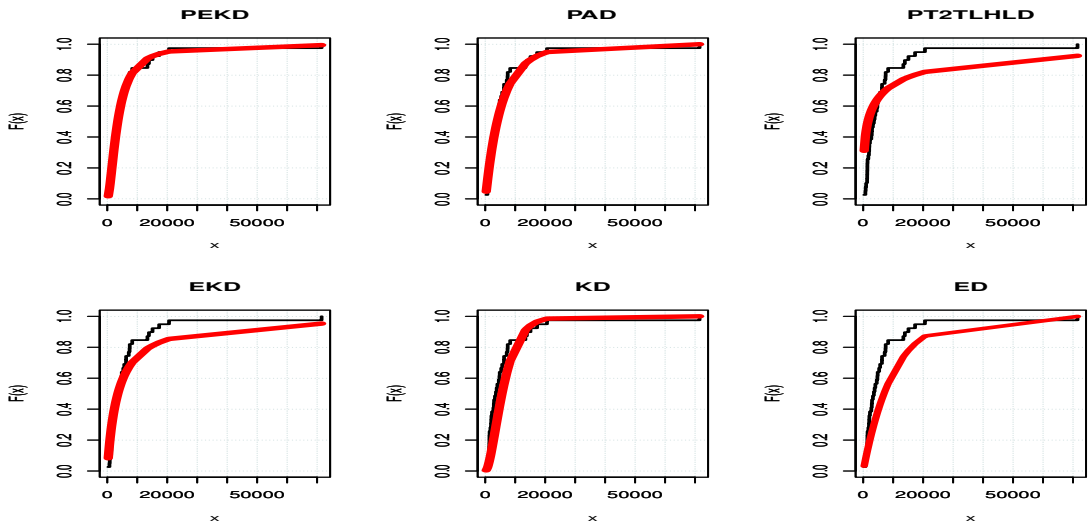


FIGURE 9. Plots comparing the empirical CDF with the fitted CDFs of all models for the second dataset.

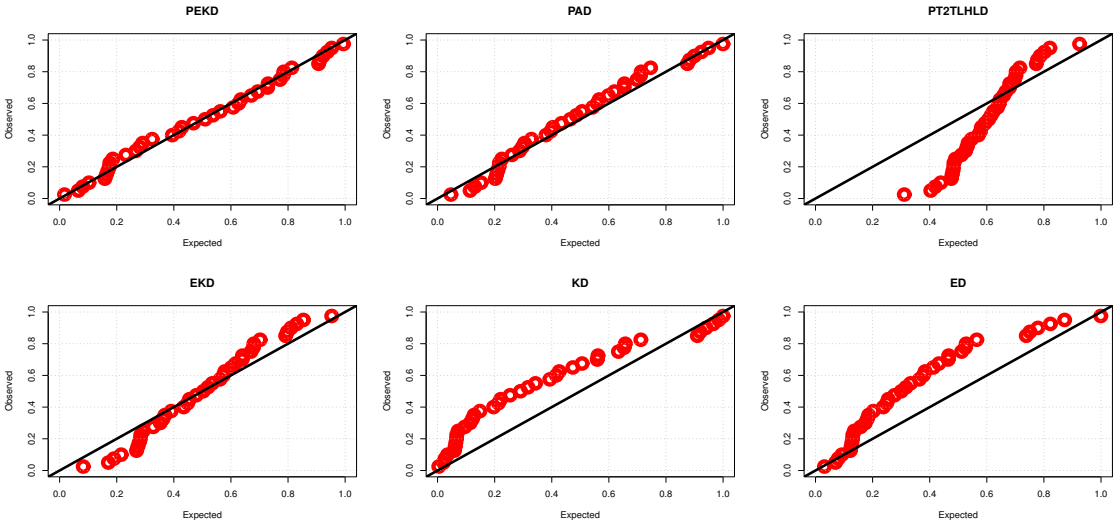


FIGURE 10. P–P plots of the fitted models for the second dataset

model selection and goodness-of-fit results among the models considered. These applications show that the PEKD can be useful for modeling right-skewed data in reliability and environmental studies.

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Data Availability Statement. The data that supports the findings of this study are available within the article.

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