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EXACT PARAMETER COMPARISON, TRUE-ERROR MONOTONICITY, AND SECOND-ORDER ASYMPTOTICS FOR ANALYTIC-WEIGHT SZÁSZ–MIRAKJAN–KANTOROVICH OPERATORS

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Abstract. We develop a unified finite-scale and asymptotic theory for Szász–Mirakjan–Kantorovich operators with analytic weights and a power parameter. Exact moments yield a necessary-and-sufficient criterion for improvement of the second central moment. In the exponential specialization, a probabilistic coupling strengthens this comparison to monotone ordering of the true approximation error for nondecreasing convex functions and identifies the limiting discrete Szász operator. Allowing the parameter to grow with the approximation index suppresses the first-order Kantorovich drift. For a genuinely non-Poisson perturbed exponential generator, an exact Poisson–Bernoulli decomposition exposes the generator correction and leads to an explicit second-order expansion. Supporting convergence estimates and numerical tests verify the distinct finite-scale, true-error, drift-suppression, and second-order regimes.

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1. Introduction

Approximation on the half-line poses features that are absent on compact intervals, since moment behavior, weighted convergence, and asymptotic localization must be controlled simultaneously. Among the classical positive approximation schemes on $[0,\infty)$, the Mirakjan–Favard–Szász construction [1–3] serves as the standard discrete model. Convergence in weighted spaces is commonly established through Korovkin-type criteria [4–6], while quantitative estimates are usually expressed by moduli of smoothness and related approximation functionals [7,8]. Mortici [9] broadened the Szász–Mirakjan setting by introducing analytically generated weights, and Bărbosu, Pop and Miclăuş [10] subsequently developed the corresponding Kantorovich form. This extension is especially useful when the generating function is intended to remain visible in the moment structure; the usual exponential model is recovered as a specialization. Voronovskaja-type results for Szász–Mirakjan–Kantorovich operators

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were obtained in [11], and subsequent studies developed higher-order variants and more detailed asymptotic expansions [12–14].

Parameter-dependent modifications provide another established route for controlling approximation quality while preserving the positive structure of the underlying operators. Mohammad and Hassan [18] studied simultaneous approximation for an integral-type sequence on the half-line together with a Voronovskaja-type description, and the two-parameter variant in [19] made the role of the auxiliary parameters explicit in the analysis. Related parameter-dependent constructions have been considered for Bernstein–Kantorovich operators [20], modified Baskakov operators [21], and Baskakov–Beta operators [22]. A different strategy is to combine operators: Muslim, Keidan and Hassan [23] used a Micchelli-type combination to increase the approximation order of half-Bernstein polynomials while retaining an explicit asymptotic description. Despite the different kernels involved, these works point to a common issue that is central here: an adjustable parameter is mathematically informative only when its effect is quantified, for instance through moment identities, direct comparison results, or identifiable asymptotic coefficients.

Recent Szász-type work makes this distinction particularly important. Resen and Hassan considered modified Szász–Mirakjan operators generated through a continuously infinitely differentiable function and established Korovkin convergence, a Voronovskaja-type formula, and numerical error comparisons [24]. Their subsequent modification of a class of positive linear operators continues the same program of altering the operator structure while retaining positivity and approximation control [25]. These results sit naturally beside other parameterized Szász–Kantorovich modifications. In particular, for the exponential generator, Baytunç, Aktuğlu and Mahmudov introduced the power-type Szász–Mirakjan–Kantorovich family used below as the classical specialization [15]; related parameterized forms and improved approximants appear in [16, 17]. Consequently, the presence of a power parameter or a modified sampling node, by itself, is not sufficient to constitute the novelty of the present work.

The purpose of this paper is instead to develop a unified *comparison and asymptotic theory* for the analytic-weight power-Kantorovich family

$$K_{n,\gamma}(f;x) := \sum_{k=0}^{\infty} W_{n,k}(x) \int_0^1 f\left(\frac{k+t^\gamma}{b_n}\right) dt, \tag{1}$$

where

$$W_{n,k}(x) = \frac{1}{\varphi(b_n x)} \frac{\varphi^{(k)}(0)}{k!} (b_n x)^k. \tag{2}$$

For $\varphi(y) = e^y$ and $b_n = n$, (1) reduces exactly to the parameterized family of [15]. Thus, no novelty is claimed for the power node itself. The analytic generator is retained because it permits the finite-scale parameter effect and the asymptotic generator effect to be studied within the same operator model rather than in disconnected special cases.

The contributions are fourfold. First, for the full analytic-weight family, the second central moment is computed in a form that yields an exact finite-n difference identity. This identity leads to a necessary-and-sufficient threshold describing precisely when increasing γ decreases the second central moment. The result is therefore a finite-scale comparison theorem, not merely an asymptotic statement or a sufficient monotonicity condition. Second, in the exponential specialization, a coupling based on a common Poisson index and a common uniform random variable converts the parameter ordering into a monotone ordering of the *actual approximation error* for every nondecreasing convex target. Accordingly, the dispersion comparison and the ordering of the actual approximation error are treated as distinct statements, with the latter obtained only under explicit shape restrictions on the target function.

Third, the first-order asymptotic analysis is formulated without suppressing the contribution of the analytic generator. The resulting Voronovskaja limit isolates two sources in the leading term: the correction inherited from the generator and the displacement introduced by the power-Kantorovich cell. In the exponential specialization, the latter contribution can be attenuated by taking $\gamma = \gamma_n \to \infty$; thus the varying parameter has an asymptotic function that is separate from its finite-n comparison role. Fourth, the perturbed exponential choice $\varphi_\eta(y) = e^y(1 + \eta y)$ yields a genuinely non-Poisson example for which the index distribution factorizes exactly into independent Poisson and Bernoulli components. This representation makes the generator correction explicit and permits the derivation of a closed second-order expansion. In contrast with the higher-order Kantorovich expansions in [13], the additional term obtained here comes from the interaction of the analytic weight with the power-Kantorovich cell, rather than from increasing the order of the Kantorovich construction.

The subsequent analysis therefore keeps four issues separate: finite-scale reduction of the second central moment, ordering of the actual error for a shape-restricted class, removal of the leading Kantorovich drift through a varying parameter, and the second-order contribution generated by a non-Poisson weight. Weighted Korovkin convergence and local error bounds are used only as supporting approximation results. The numerical examples are organized around the preceding theoretical claims and include a nonmonotone test function whose pointwise error curves cross; this demonstrates why second-moment improvement alone cannot justify an unrestricted ordering of the actual approximation error.

2. Preliminaries

Let $C[0,\infty)$ denote the real-valued continuous functions on $[0,\infty)$ and $C_B[0,\infty)$ the bounded continuous functions, equipped with

$$\|f\|_\infty = \sup_{x\geq 0} |f(x)|.$$

For an integer $r \geq 1$, let

$$C_B^r[0,\infty) = \{f \in C_B[0,\infty) : f^{(j)} \in C_B[0,\infty),\ 1 \leq j \leq r\}.$$

For $M > 0$ and $0 < \nu \leq 1$, define the Lipschitz class

$$\mathrm{Lip}_M(\nu) = \{f \in C_B[0,\infty) : |f(u) - f(v)| \leq M|u-v|^\nu \text{ for all } u, v \geq 0\}.$$

For $f \in C_B[0,\infty)$, define the modulus of continuity

$$\omega(f;\delta) = \sup_{\substack{x,y\geq 0 \\ |x-y|\leq\delta}} |f(x) - f(y)|.$$

For weighted approximation we use

$$\rho(x) = 1 + x^2. \tag{3}$$

Set

$$B_\rho = \{f : [0,\infty) \to \mathbb{R} : |f(x)| \leq M_f \rho(x) \text{ for some } M_f > 0\},$$

$$C_\rho = B_\rho \cap C[0,\infty), \qquad \|f\|_\rho = \sup_{x\geq 0} \frac{|f(x)|}{1+x^2},$$

and

$$C_\rho^* = \left\{ f \in C_\rho : \lim_{x\to\infty} \frac{f(x)}{1+x^2} \text{ exists and is finite} \right\}.$$

We shall use Gadjiev's weighted Korovkin theorem [5,6] for the test functions $e_i(x) = x^i$, $i = 0, 1, 2$.

Throughout the paper, $\gamma > 0$ is fixed and

$$\alpha_\gamma = \frac{1}{\gamma+1}, \qquad \beta_\gamma = \frac{1}{2\gamma+1}, \qquad \mu_\gamma = \frac{1}{3\gamma+1}, \qquad \theta_\gamma = \frac{1}{4\gamma+1}. \tag{4}$$

To justify the positivity statement in Section 3 without changing its formula, we impose the following standing hypothesis on the analytic generator:

$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_k = \frac{\varphi^{(k)}(0)}{k!}, \qquad a_0 > 0, \quad a_k \geq 0 \ (k \geq 1),$$

with an infinite radius of convergence. Thus $\varphi(y) > 0$ for $y \geq 0$ and its Maclaurin series represents φ at every nonnegative argument. Define

$$R_j(y) = \frac{\varphi^{(j)}(y)}{\varphi(y)}, \qquad R_{j,n}(x) = R_j(b_n x), \qquad j \geq 1.$$

For the weighted theorem we additionally assume

$$M_j := \sup_{y\geq 0} R_j(y) < \infty \quad (j = 1, 2)$$

and

$$\varepsilon_{1,n} := \sup_{x\geq 0} \frac{x}{1+x^2} |R_{1,n}(x) - 1| \to 0, \qquad \varepsilon_{2,n} := \sup_{x\geq 0} \frac{x^2}{1+x^2} |R_{2,n}(x) - 1| \to 0.$$

These hypotheses restrict the admissible generators; they do not alter the operator defined below.

3. Definition of the Operators

Let (b_n) be a positive sequence satisfying

$$\lim_{n\to\infty} b_n = \infty. \tag{5}$$

For $x \geq 0$ and $k \in \mathbb{N}_0$, set

$$W_{n,k}(x) = \frac{1}{\varphi(b_n x)} \frac{\varphi^{(k)}(0)}{k!} (b_n x)^k. \tag{6}$$

Thus $W_{n,k}(x) \geq 0$ and

$$\sum_{k=0}^{\infty} W_{n,k}(x) = 1, \qquad x \geq 0. \tag{7}$$

Definition 3.1. For every $f \in C_\rho$, define

$$K_{n,\gamma}(f;x) = \sum_{k=0}^{\infty} W_{n,k}(x) \int_0^1 f\left(\frac{k+t^\gamma}{b_n}\right) dt, \qquad x \geq 0. \tag{8}$$

For the polynomial moment identities used below, the same formula (8) is also applied to $e_j(t) = t^j$, $0 \leq j \leq 4$. These functions need not belong to C_ρ when $j > 2$, but the defining series is finite because the analytic weights have finite polynomial moments of every fixed order. Thus the moment computations are well defined independently of the weighted-space mapping statement.

The series in (8) is absolutely convergent for $f \in C_\rho$. Indeed, because $0 \leq t^\gamma \leq 1$,

$$\left| f\left(\frac{k+t^\gamma}{b_n}\right) \right| \leq \|f\|_\rho \left(1 + \frac{(k+1)^2}{b_n^2}\right),$$

and the corresponding weighted sums of 1, k, and k^2 are finite. To verify continuity, fix $A > 0$. Since φ is positive and continuous, $\min_{0\leq x\leq A} \varphi(b_n x) > 0$. The kth summand in (8) is continuous in x and, uniformly for $0 \leq x \leq A$, is dominated by a constant multiple of

$$a_k (b_n A)^k \left(1 + \frac{(k+1)^2}{b_n^2}\right).$$

The sum of these majorants is finite because the entire Maclaurin series and its first two differentiated series converge at $b_n A$. The Weierstrass test therefore gives local uniform convergence, so $K_{n,\gamma} f$ is continuous. Positivity and linearity follow from (6)–(7). Finally, under the standing weighted assumptions, Lemma 4.2 yields $K_{n,\gamma}(1+t^2;x) \leq C_n(1+x^2)$ for a finite constant C_n; hence $K_{n,\gamma} : C_\rho \to C_\rho$.

4. Some Basic Results

Lemma 4.1. *For $y = b_n x \geq 0$,*

$$\varphi(y) = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} y^k. \tag{9}$$

Moreover, for $j = 1, 2, 3, 4$,

$$\sum_{k=j}^{\infty} \frac{\varphi^{(k)}(0)}{(k-j)!} y^{k-j} = \varphi^{(j)}(y). \tag{10}$$

If

$$R_{j,n}(x) = \frac{\varphi^{(j)}(b_n x)}{\varphi(b_n x)}, \qquad j = 1, 2, 3, 4, \tag{11}$$

then the pointwise asymptotic hypothesis used below is

$$R_{j,n}(x) \longrightarrow 1 \qquad (n \to \infty), \quad j = 1, 2, 3, 4, \tag{12}$$

whenever it is explicitly invoked.

Proof. The standing hypothesis in Section 2 gives global convergence of the Maclaurin series. Termwise differentiation yields (10). □

Lemma 4.2 (Raw moments)**.** *For* $e_j(t) = t^j$*, the operator* (8) *satisfies*

$$\begin{aligned}
K_{n,\gamma}(e_0; x) &= 1, \\
K_{n,\gamma}(e_1; x) &= x R_{1,n}(x) + \frac{\alpha_\gamma}{b_n}, \\
K_{n,\gamma}(e_2; x) &= x^2 R_{2,n}(x) + \frac{x(1+2\alpha_\gamma) R_{1,n}(x)}{b_n} + \frac{\beta_\gamma}{b_n^2}, \\
K_{n,\gamma}(e_3; x) &= x^3 R_{3,n}(x) + \frac{3(1+\alpha_\gamma) x^2 R_{2,n}(x)}{b_n} + \frac{(1+3\alpha_\gamma+3\beta_\gamma) x R_{1,n}(x)}{b_n^2} + \frac{\mu_\gamma}{b_n^3}, \\
K_{n,\gamma}(e_4; x) &= x^4 R_{4,n}(x) + \frac{(6+4\alpha_\gamma) x^3 R_{3,n}(x)}{b_n} + \frac{(7+12\alpha_\gamma+6\beta_\gamma) x^2 R_{2,n}(x)}{b_n^2} \\
&\quad + \frac{(1+4\alpha_\gamma+6\beta_\gamma+4\mu_\gamma) x R_{1,n}(x)}{b_n^3} + \frac{\theta_\gamma}{b_n^4}.
\end{aligned}$$

Proof. Let $N_{n,x}$ have probabilities $\Pr(N_{n,x} = k) = W_{n,k}(x)$, and let T be independent and uniform on $[0, 1]$. Then

$$K_{n,\gamma}(f; x) = \mathbb{E} f\left(\frac{N_{n,x} + T^\gamma}{b_n} \right),$$

while (10) gives

$$\mathbb{E}(N_{n,x})_j = (b_n x)^j R_{j,n}(x), \qquad j \geq 1.$$

Together with $\mathbb{E}(T^{j\gamma}) = (j\gamma + 1)^{-1}$, expansion of $(N_{n,x} + T^\gamma)^m$ gives the five formulas. □

Lemma 4.3. *Under the standing weighted assumptions,*

$$\|K_{n,\gamma}(e_0) - e_0\|_\rho = 0, \tag{13}$$

$$\|K_{n,\gamma}(e_1) - e_1\|_\rho \longrightarrow 0, \tag{14}$$

$$\|K_{n,\gamma}(e_2) - e_2\|_\rho \longrightarrow 0. \tag{15}$$

If, in addition, (12) *holds at a fixed* $x \geq 0$*, then*

$$K_{n,\gamma}(e_3;x) \longrightarrow x^3, \tag{16}$$

$$K_{n,\gamma}(e_4;x) \longrightarrow x^4. \tag{17}$$

Proof. The first statement is exact. From Lemma 4.2,

$$\|K_{n,\gamma}(e_1) - e_1\|_\rho \leq \varepsilon_{1,n} + \frac{\alpha_\gamma}{b_n},$$

and

$$\|K_{n,\gamma}(e_2) - e_2\|_\rho \leq \varepsilon_{2,n} + \frac{(1+2\alpha_\gamma)M_1}{2b_n} + \frac{\beta_\gamma}{b_n^2},$$

which tend to zero. The last two assertions follow directly from the third and fourth raw moments and (12). □

Corollary 4.4 (Central moments)**.** *Let* $\psi_x(t) = t - x$*. Then*

$$K_{n,\gamma}(1;x) = 1, \tag{18}$$

$$K_{n,\gamma}(\psi_x;x) = x(R_{1,n}(x) - 1) + \frac{\alpha_\gamma}{b_n}, \tag{19}$$

$$K_{n,\gamma}(\psi_x^2;x) = x^2(R_{2,n}(x) - 2R_{1,n}(x) + 1) + \frac{x\{(1+2\alpha_\gamma)R_{1,n}(x) - 2\alpha_\gamma\}}{b_n} + \frac{\beta_\gamma}{b_n^2}, \tag{20}$$

$$\begin{aligned} K_{n,\gamma}(\psi_x^3;x) &= x^3(R_{3,n}(x) - 3R_{2,n}(x) + 3R_{1,n}(x) - 1) \\ &\quad + \frac{3x^2\{(1+\alpha_\gamma)R_{2,n}(x) - (1+2\alpha_\gamma)R_{1,n}(x) + \alpha_\gamma\}}{b_n} \\ &\quad + \frac{x\{(1+3\alpha_\gamma+3\beta_\gamma)R_{1,n}(x) - 3\beta_\gamma\}}{b_n^2} + \frac{\mu_\gamma}{b_n^3}, \end{aligned} \tag{21}$$

$$\begin{aligned} K_{n,\gamma}(\psi_x^4;x) &= x^4(R_{4,n}(x) - 4R_{3,n}(x) + 6R_{2,n}(x) - 4R_{1,n}(x) + 1) \\ &\quad + \frac{x^3\{(6+4\alpha_\gamma)R_{3,n}(x) - 12(1+\alpha_\gamma)R_{2,n}(x) + 6(1+2\alpha_\gamma)R_{1,n}(x) - 4\alpha_\gamma\}}{b_n} \\ &\quad + \frac{x^2\{(7+12\alpha_\gamma+6\beta_\gamma)R_{2,n}(x) - (4+12\alpha_\gamma+12\beta_\gamma)R_{1,n}(x) + 6\beta_\gamma\}}{b_n^2} \\ &\quad + \frac{x\{(1+4\alpha_\gamma+6\beta_\gamma+4\mu_\gamma)R_{1,n}(x) - 4\mu_\gamma\}}{b_n^3} + \frac{\theta_\gamma}{b_n^4}. \end{aligned} \tag{22}$$

Proof. Use $K_{n,\gamma}((t-x)^m;x) = \sum_{j=0}^m \binom{m}{j}(-x)^{m-j}K_{n,\gamma}(e_j;x)$ and Lemma 4.2. □

5. Exact Finite-Scale Parameter Comparison

Put

$$a_{n,\gamma}(x) = K_{n,\gamma}(t-x;x), \qquad \delta_{n,\gamma}(x) = K_{n,\gamma}((t-x)^2;x),$$

with exact expressions (19)–(20). Since $K_{n,\gamma}$ is positive, $\delta_{n,\gamma}(x) \geq 0$.

The dependence on γ is determined exactly by the second central moment.

Theorem 5.1 (Exact comparison with respect to γ)**.** *Let $0 < \gamma_1 < \gamma_2$. Then*

$$\delta_{n,\gamma_2}(x) - \delta_{n,\gamma_1}(x) = \frac{2x\{R_{1,n}(x) - 1\}}{b_n}(\alpha_{\gamma_2} - \alpha_{\gamma_1}) + \frac{\beta_{\gamma_2} - \beta_{\gamma_1}}{b_n^2}. \tag{23}$$

At $x = 0$ one has the strict inequality $\delta_{n,\gamma_2}(0) < \delta_{n,\gamma_1}(0)$. For $x > 0$,

$$\delta_{n,\gamma_2}(x) \le \delta_{n,\gamma_1}(x) \quad \Longleftrightarrow \quad R_{1,n}(x) \ge 1 - \frac{\beta_{\gamma_1} - \beta_{\gamma_2}}{2b_n x\,(\alpha_{\gamma_1} - \alpha_{\gamma_2})}. \tag{24}$$

Consequently, the simpler condition $R_{1,n}(x) \ge 1$ is sufficient for every $x > 0$.

Proof. Subtract (20) for γ_1 and γ_2 to obtain (23). Since both α_γ and β_γ decrease strictly with γ, the differences $\alpha_{\gamma_1} - \alpha_{\gamma_2}$ and $\beta_{\gamma_1} - \beta_{\gamma_2}$ are positive. At $x = 0$, (23) reduces to $(\beta_{\gamma_2} - \beta_{\gamma_1})/b_n^2 < 0$. For $x > 0$, rewriting (23) with positive differences and solving the inequality $\delta_{n,\gamma_2}(x) - \delta_{n,\gamma_1}(x) \le 0$ gives (24). □

Remark 5.2. Theorem 5.1 compares the second central moments and the error bounds derived from them. It does not imply a universal pointwise ordering of the actual errors $|K_{n,\gamma}(f;x) - f(x)|$.

6. True-Error Monotonicity in the Exponential Case

The exact criterion in Theorem 5.1 applies to the second central moment for the full analytic-weight family. A stronger conclusion about the actual error is available under additional shape assumptions in the exponential specialization

$$\varphi(y) = e^y, \qquad b_n = n. \tag{25}$$

Let $N_x \sim \mathrm{Poisson}(nx)$ and let T be uniform on $[0,1]$, independent of N_x. Then

$$K_{n,\gamma}(f;x) = \mathbb{E} f\left(\frac{N_x + T^\gamma}{n}\right). \tag{26}$$

Theorem 6.1 (True-error monotonicity)**.** *Assume (25), let $0 < \gamma_1 < \gamma_2$, and let $f : [0,\infty) \to \mathbb{R}$ be nondecreasing and convex with at most polynomial growth. Then, for every $x \ge 0$,*

$$0 \le K_{n,\gamma_2}(f;x) - f(x) \le K_{n,\gamma_1}(f;x) - f(x). \tag{27}$$

Consequently,

$$|K_{n,\gamma_2}(f;x) - f(x)| \le |K_{n,\gamma_1}(f;x) - f(x)|.$$

Proof. For $0 \le T \le 1$ and $\gamma_2 > \gamma_1$, one has $T^{\gamma_2} \le T^{\gamma_1}$. Since f is nondecreasing, (26) gives $K_{n,\gamma_2}(f;x) \le K_{n,\gamma_1}(f;x)$. Moreover,

$$\mathbb{E}\frac{N_x + T^\gamma}{n} = x + \frac{1}{n(\gamma+1)} \ge x.$$

Jensen's inequality and monotonicity of f therefore imply

$$K_{n,\gamma}(f;x) \ge f\left(x + \frac{1}{n(\gamma+1)}\right) \ge f(x).$$

Combining the two inequalities proves (27). □

Corollary 6.2 (Large-parameter limit). *Under* (25)*, if f has at most polynomial growth, then for each fixed n and x,*

$$\lim_{\gamma\to\infty} K_{n,\gamma}(f;x) = S_n(f;x) := \sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} f\left(\frac{k}{n}\right).$$

If f is nondecreasing and convex, then

$$0 \le S_n(f;x) - f(x) \le K_{n,\gamma}(f;x) - f(x).$$

Proof. Since $T^\gamma \to 0$ almost surely and the Poisson distribution has moments of every order, dominated convergence applies to (26). The final inequality follows from Theorem 6.1 by passage to the limit. □

Remark 6.3. The shape assumptions in Theorem 6.1 are essential to the conclusion. Theorem 5.1 remains the general analytic-weight comparison, whereas Theorem 6.1 gives a stronger true-error ordering on a narrower class. Both situations are illustrated numerically in Section 9.

7. Supporting Approximation Bounds

Having established the finite-scale comparison, we collect only the convergence and local estimates required in the subsequent analysis. These estimates provide the approximation control used later in the asymptotic arguments and are included for that purpose rather than as separate principal contributions.

Theorem 7.1 (Weighted convergence on C_ρ^*). *Under the standing weighted assumptions, for every $f \in C_\rho^*$,*

$$\lim_{n\to\infty} \|K_{n,\gamma}(f;\cdot) - f\|_\rho = 0. \tag{28}$$

In particular, the convergence is uniform on each fixed compact interval $[0, A]$.

Proof. The estimates in Lemma 4.3 establish the weighted Korovkin conditions for the test functions $1, t, t^2$. Applying Gadjiev's theorem yields (28), and the assertion on $[0, A]$ follows from $\sup_{0\le x\le A} |g(x)| \le (1 + A^2)\|g\|_\rho$. □

Theorem 7.2 (Estimate via the modulus of continuity). *If $f \in C_B[0,\infty)$, then for every $x \ge 0$,*

$$|K_{n,\gamma}(f;x) - f(x)| \le 2\,\omega\left(f;\sqrt{\delta_{n,\gamma}(x)}\right). \tag{29}$$

Proof. If $\delta_{n,\gamma}(x) = 0$, then positivity and $K_{n,\gamma}((t-x)^2;x) = 0$ imply that the representing probability measure is concentrated at x; hence $K_{n,\gamma}(f;x) = f(x)$ and the estimate is immediate. Assume now that $\delta_{n,\gamma}(x) > 0$. For $h > 0$, $|f(u) - f(x)| \le (1 + |u-x|/h)\omega(f;h)$. Positivity and Cauchy–Schwarz give $K_{n,\gamma}(|t-x|;x) \le \sqrt{\delta_{n,\gamma}(x)}$. Taking $h = \sqrt{\delta_{n,\gamma}(x)}$ yields the result. □

Theorem 7.3 (Lipschitz estimate). *If $f \in \mathrm{Lip}_M(\nu)$, $0 < \nu \le 1$, then*

$$|K_{n,\gamma}(f;x) - f(x)| \le M\,[\delta_{n,\gamma}(x)]^{\nu/2}. \tag{30}$$

Proof. By positivity and the Lipschitz condition, $|K_{n,\gamma}(f;x) - f(x)| \le M K_{n,\gamma}(|t-x|^\nu;x)$. Hölder's inequality and $K_{n,\gamma}(1;x) = 1$ give (30). □

Corollary 7.4 (A C^2 estimate)**.** *If $f \in C_B^2[0,\infty)$, then*

$$|K_{n,\gamma}(f;x) - f(x)| \le |a_{n,\gamma}(x)|\,|f'(x)| + \frac{1}{2}\delta_{n,\gamma}(x)\|f''\|_\infty.$$

Proof. Taylor's formula gives $f(t) - f(x) = f'(x)(t-x) + \frac{1}{2}f''(\xi_{x,t})(t-x)^2$ for some $\xi_{x,t}$ between x and t. Apply the positive linear operator, use $K_{n,\gamma}(t-x;x) = a_{n,\gamma}(x)$, and bound $|f''(\xi_{x,t})|$ by $\|f''\|_\infty$. □

8. Generator-Sensitive and Drift-Suppressed Asymptotics

The first-order limit for the analytic-weight operator depends on the rates at which the derivative ratios approach one. Fix $x > 0$ and abbreviate $R_{j,n} = R_{j,n}(x)$.

Theorem 8.1 (Voronovskaja formula)**.** *Let $f \in C_B^2[0,\infty)$ and fix $x > 0$. Assume*

$$R_{j,n} \to 1, \qquad j = 1,2,3,4,$$

that the limits

$$\sigma_1(x) = \lim_{n\to\infty} b_n(R_{1,n} - 1), \qquad \sigma_2(x) = \lim_{n\to\infty} b_n(R_{2,n} - 2R_{1,n} + 1)$$

exist, and that

$$b_n(R_{4,n} - 4R_{3,n} + 6R_{2,n} - 4R_{1,n} + 1) \to 0.$$

Then

$$\lim_{n\to\infty} b_n[K_{n,\gamma}(f;x) - f(x)] = \{x\sigma_1(x) + \alpha_\gamma\}f'(x) + \frac{1}{2}\{x + x^2\sigma_2(x)\}f''(x). \tag{31}$$

Proof. Equations (19) and (20) give

$$b_n K_{n,\gamma}(t-x;x) \to x\sigma_1(x) + \alpha_\gamma, \qquad b_n K_{n,\gamma}((t-x)^2;x) \to x + x^2\sigma_2(x).$$

The fourth central moment (22) and the stated hypotheses imply $b_n K_{n,\gamma}((t-x)^4;x) \to 0$. Taylor's formula gives

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + (t-x)^2\varepsilon_x(t),$$

where $\varepsilon_x(t) \to 0$ as $t \to x$. It remains to show that

$$b_n K_{n,\gamma}((t-x)^2\varepsilon_x(t);x) \longrightarrow 0.$$

Given $\varepsilon > 0$, choose $\delta > 0$ so that $|\varepsilon_x(t)| \le \varepsilon$ whenever $|t-x| \le \delta$. On this region,

$$b_n K_{n,\gamma}((t-x)^2|\varepsilon_x(t)|;x) \le \varepsilon b_n K_{n,\gamma}((t-x)^2;x) = O(\varepsilon).$$

Since $f \in C_B^2[0,\infty)$ and x is fixed, $\varepsilon_x(t)$ is bounded on $\{|t-x| > \delta\}$ by a constant $C_{x,\delta}$. Hence the complementary part is at most

$$\frac{C_{x,\delta} b_n}{\delta^2} K_{n,\gamma}((t-x)^4;x),$$

which tends to zero. Letting $\varepsilon \downarrow 0$ proves that the scaled Taylor remainder vanishes and yields (31). □

8.1. **Exponential specialization and variable-parameter drift suppression.** For the exponential generator in (25), all derivative ratios equal one. Hence Theorem 8.1 yields, for fixed $\gamma > 0$,

$$\lim_{n\to\infty} n[K_{n,\gamma}(f;x) - f(x)] = \alpha_\gamma f'(x) + \frac{x}{2} f''(x). \tag{32}$$

The term $\alpha_\gamma f'(x)$ is the Kantorovich drift. It can be removed without changing the operator family by allowing the existing parameter to depend on n.

Theorem 8.2 (Variable-parameter drift suppression)**.** *Let $\gamma_n > 0$ satisfy $\gamma_n \to \infty$, assume (25), and let $f \in C_B^4[0,\infty)$. For each fixed $x \geq 0$,*

$$\lim_{n\to\infty} n[K_{n,\gamma_n}(f;x) - f(x)] = \frac{x}{2} f''(x). \tag{33}$$

More precisely, writing

$$\alpha_n = \frac{1}{\gamma_n + 1}, \quad \beta_n = \frac{1}{2\gamma_n + 1}, \quad \mu_n = \frac{1}{3\gamma_n + 1}, \quad \theta_n = \frac{1}{4\gamma_n + 1},$$

one has

$$\begin{aligned} &\left| n[K_{n,\gamma_n}(f;x) - f(x)] - \frac{x}{2} f''(x) \right| \\ &\leq \alpha_n \|f'\|_\infty + \frac{\beta_n}{2n} \|f''\|_\infty + \left(\frac{(1+3\alpha_n)x}{6n} + \frac{\mu_n}{6n^2} \right) \|f^{(3)}\|_\infty \\ &\quad + \left(\frac{x^2}{8n} + \frac{(1+4\alpha_n+6\beta_n)x}{24n^2} + \frac{\theta_n}{24n^3} \right) \|f^{(4)}\|_\infty. \end{aligned} \tag{34}$$

Thus the normalized residual is $O(\gamma_n^{-1} + n^{-1})$ for fixed x. In particular, if $\gamma_n \geq cn$ for some $c > 0$, the residual is $O(n^{-1})$.

Proof. In the exponential case the centered moments are obtained from (19)–(22) by setting every $R_{j,n} = 1$:

$$\mu_{1,n} = \frac{\alpha_n}{n}, \qquad \mu_{2,n} = \frac{x}{n} + \frac{\beta_n}{n^2},$$

$$\mu_{3,n} = \frac{(1+3\alpha_n)x}{n^2} + \frac{\mu_n}{n^3}, \qquad \mu_{4,n} = \frac{3x^2}{n^2} + \frac{(1+4\alpha_n+6\beta_n)x}{n^3} + \frac{\theta_n}{n^4}.$$

Apply Taylor's formula through third order, use the fourth derivative to bound the remainder, multiply by n, and subtract $xf''(x)/2$. This gives (34); since $\alpha_n, \beta_n, \mu_n, \theta_n \to 0$, the limit (33) follows. □

8.2. **Non-Poisson generator and second-order expansion.** For $\varphi_\eta(y) = e^y(1+\eta y)$, one has

$$b_n(R_{1,n}-1) \to \frac{1}{x}, \qquad R_{2,n} - 2R_{1,n} + 1 = 0,$$

so (31) reduces to

$$\lim_{n\to\infty} b_n[K_{n,\gamma}^{(\eta)}(f;x) - f(x)] = (1+\alpha_\gamma)f'(x) + \frac{x}{2}f''(x). \tag{35}$$

For the exponential generator, all $R_{j,n} = 1$ and the coefficient of $f'(x)$ is simply α_γ.

Proposition 8.3 (Exact decomposition for the perturbed exponential generator)**.** *Let $\eta > 0$ and $\varphi_\eta(y) = e^y(1+\eta y)$. For $m = b_n x$ define*

$$q_n(x) = \frac{\eta m}{1+\eta m}.$$

If $X \sim \mathrm{Poisson}(m)$ and $B \sim \mathrm{Bernoulli}(q_n(x))$ are independent, then the analytic weight index $N_{n,x}$ associated with $W_{n,k}^{(\eta)}(x)$ satisfies

$$N_{n,x} \overset{d}{=} X + B.$$

Consequently, with $U = T^\gamma$ for T uniform on $[0,1]$ and independent of X, B,

$$K_{n,\gamma}^{(\eta)}(f;x) = \mathbb{E}\, f\left(\frac{X+B+U}{b_n}\right).$$

Proof. The probability generating function of $N_{n,x}$ is

$$\sum_{k\geq 0} W_{n,k}^{(\eta)}(x) z^k = e^{m(z-1)}\frac{1+\eta m z}{1+\eta m} = e^{m(z-1)}\{(1-q_n) + q_n z\},$$

which is the product of the probability generating functions of X and B. The final identity follows directly from the integral representation of the operator. □

The same explicit generator permits one further term. Put

$$A_\gamma(f;x) = (1+\alpha_\gamma)f'(x) + \frac{x}{2}f''(x),$$

and

$$D_{\eta,\gamma}(f;x) = -\frac{f'(x)}{\eta x} + \frac{1+2\alpha_\gamma+\beta_\gamma}{2}f''(x) + \frac{x(4+3\alpha_\gamma)}{6}f'''(x) + \frac{x^2}{8}f^{(4)}(x).$$

Theorem 8.4 (Second-order expansion)**.** *Let $\eta > 0$, $f \in C_B^4[0,\infty)$, and fix $x > 0$. For $\varphi_\eta(y) = e^y(1+\eta y)$,*

$$\lim_{n\to\infty} b_n^2\left[K_{n,\gamma}^{(\eta)}(f;x) - f(x) - \frac{A_\gamma(f;x)}{b_n}\right] = D_{\eta,\gamma}(f;x). \tag{36}$$

Proof. Let $m = b_n x$ and $q_n(x) = \eta m/(1+\eta m)$. By Proposition 8.3, the centered increment is the sum of a centered Poisson variable and two bounded independent terms. Hence

$$K_{n,\gamma}^{(\eta)}(t-x;x) = \frac{q_n + \alpha_\gamma}{b_n},$$

$$K_{n,\gamma}^{(\eta)}((t-x)^2;x) = \frac{x}{b_n} + \frac{q_n(1+2\alpha_\gamma)+\beta_\gamma}{b_n^2},$$

$$K_{n,\gamma}^{(\eta)}((t-x)^3;x) = \frac{x\{1+3(q_n+\alpha_\gamma)\}}{b_n^2} + O(b_n^{-3}),$$
$$K_{n,\gamma}^{(\eta)}((t-x)^4;x) = \frac{3x^2}{b_n^2} + O(b_n^{-3}).$$

Since $q_n = 1-(1+\eta b_n x)^{-1}$, the corresponding scaled limits yield exactly the four coefficients in $D_{\eta,\gamma}$. Write the fourth-order Taylor expansion at x as

$$f(t) = \sum_{j=0}^{4} \frac{f^{(j)}(x)}{j!}(t-x)^j + r_x(t), \qquad r_x(t) = o(|t-x|^4) \quad (t \to x).$$

It remains to prove that $b_n^2 K_{n,\gamma}^{(\eta)}(r_x;x) \to 0$. For any $\varepsilon > 0$, choose $\delta > 0$ so that $|r_x(t)| \le \varepsilon |t-x|^4$ whenever $|t-x| \le \delta$. Thus the contribution of this region is at most

$$\varepsilon b_n^2 K_{n,\gamma}^{(\eta)}((t-x)^4;x) = O(\varepsilon).$$

Because $f \in C_B^4[0,\infty)$ and x is fixed, there is a constant $C_{x,\delta}$ such that $|r_x(t)| \le C_{x,\delta}|t-x|^4$ when $|t-x| > \delta$. Hence the complementary contribution is bounded by

$$C_{x,\delta} b_n^2 K_{n,\gamma}^{(\eta)}(|t-x|^4 \mathbf{1}_{\{|t-x|>\delta\}};x) \le \frac{C_{x,\delta} b_n^2}{\delta^2} K_{n,\gamma}^{(\eta)}((t-x)^6;x).$$

Under the representation $N_{n,x} = X + B$, the centered sixth moment is $O(b_n^{-3})$: the Poisson component satisfies $\mathbb{E}(X-m)^6 = 15m^3 + 25m^2 + m$, while B and T^γ are bounded. Therefore the last display is $O(b_n^{-1})$ and tends to zero. Letting $\varepsilon \downarrow 0$ proves the required remainder estimate and completes the expansion. □

9. Numerical Verification

The numerical examples are separated according to the mathematical statements they test. No numerical observation is used as a substitute for a theorem.

9.1. **True-error monotonicity and drift suppression.**

For the exponential specialization (25), take

$$f(x) = \sqrt{1+x^2},$$

which is smooth, nondecreasing and convex. Figure 1 displays the absolute errors for $n = 20$. Their pointwise ordering is precisely the conclusion of Theorem 6.1. Table 1 gives representative values.

Figure 2 treats γ continuously and makes the same monotonicity visible at fixed spatial points. Figure 3 and Table 2 verify the variable-parameter regime in Theorem 8.2; the faster choice $\gamma_n = n^2$ reduces the parameter contribution until the inverse-index remainder dominates.

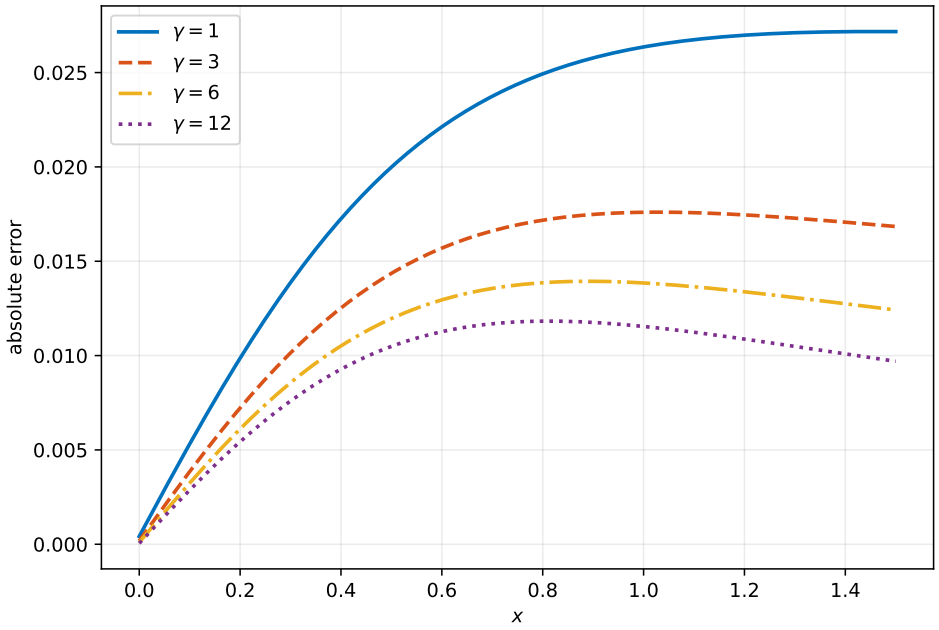


FIGURE 1. Pointwise absolute errors for the exponential specialization with $n = 20$; the curves are ordered monotonically as γ increases.

TABLE 1. Pointwise absolute errors for $f(x) = \sqrt{1 + x^2}$ and $n = 20$.

x	$\gamma = 1$	$\gamma = 3$	$\gamma = 6$	$\gamma = 12$
0.00	0.00041651	0.00017851	0.00009612	0.00004998
0.25	0.01194182	0.00874786	0.00739652	0.00656900
0.50	0.01998838	0.01435968	0.01196114	0.01048832
0.75	0.02438499	0.01692963	0.01374453	0.01178681
1.00	0.02635893	0.01759702	0.01384903	0.01154423
1.25	0.02705732	0.01737588	0.01323170	0.01068260
1.50	0.02717138	0.01684082	0.01241699	0.00969546

TABLE 2. Maximum normalized residual for two variable-parameter regimes.

n	$\gamma_n = n$	$\gamma_n = n^2$
10	0.07905034	0.01273173
20	0.04171365	0.00447097
50	0.01724858	0.00131818
100	0.00872149	0.00068596
200	0.00438537	0.00035774

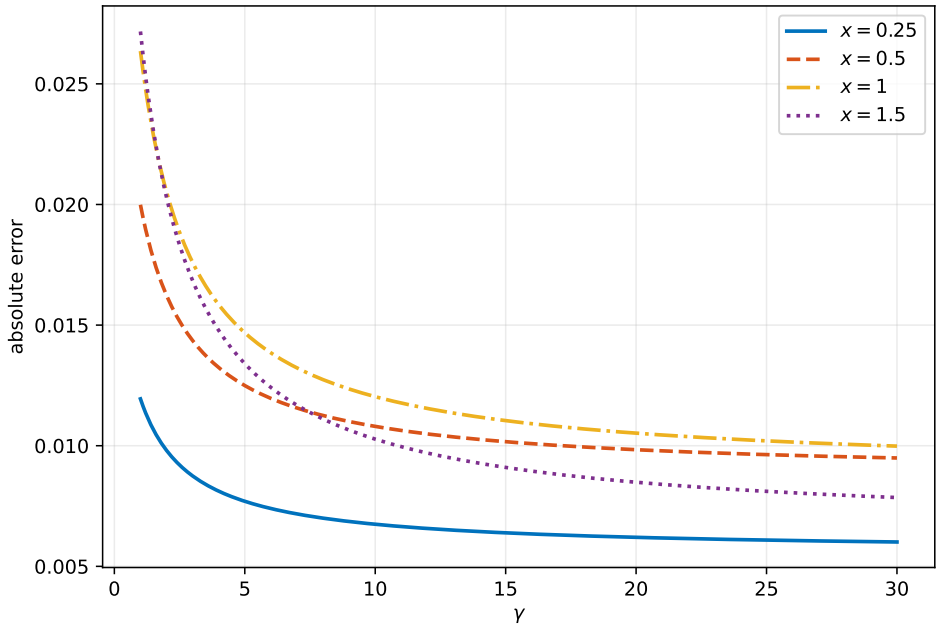


FIGURE 2. Absolute error as a function of γ at fixed points in the exponential specialization.

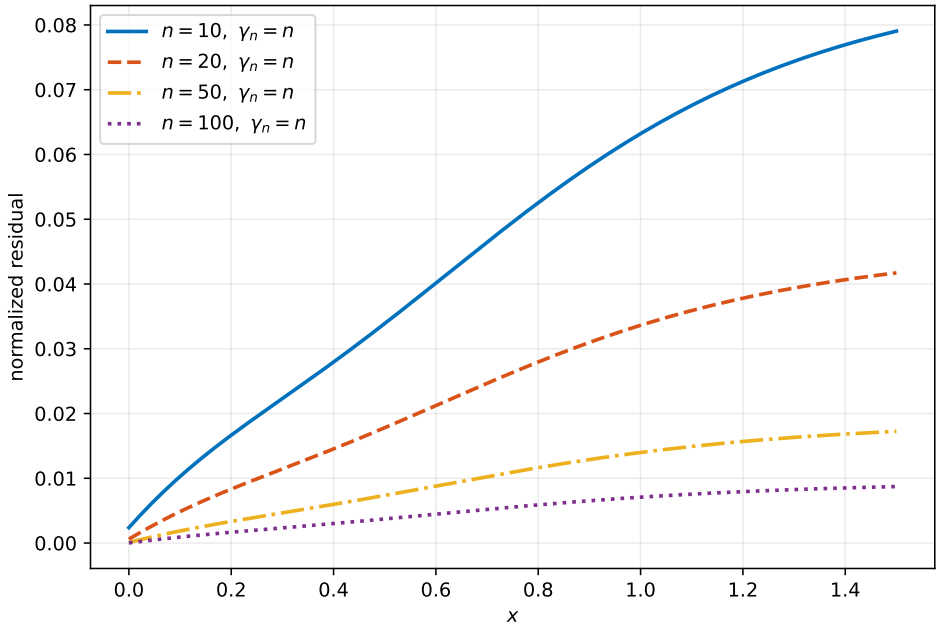


FIGURE 3. Normalized residual after subtracting the drift-free limit in the regime $\gamma_n = n$.

9.2. **Non-Poisson finite-scale comparison and second-order asymptotics.** Now take the genuinely non-Poisson generator

$$\varphi_\eta(y) = e^y(1 + \eta y), \qquad \eta = 0.5,$$

with $b_n = n$. Then

$$R_{j,n}(x) = 1 + \frac{j\eta}{1+\eta n x}, \qquad j \geq 1,$$

so $R_{1,n}(x) \geq 1$, and Theorem 5.1 guarantees monotone decrease of the second central moment as γ increases. Figure 4 shows this exact finite-scale effect.

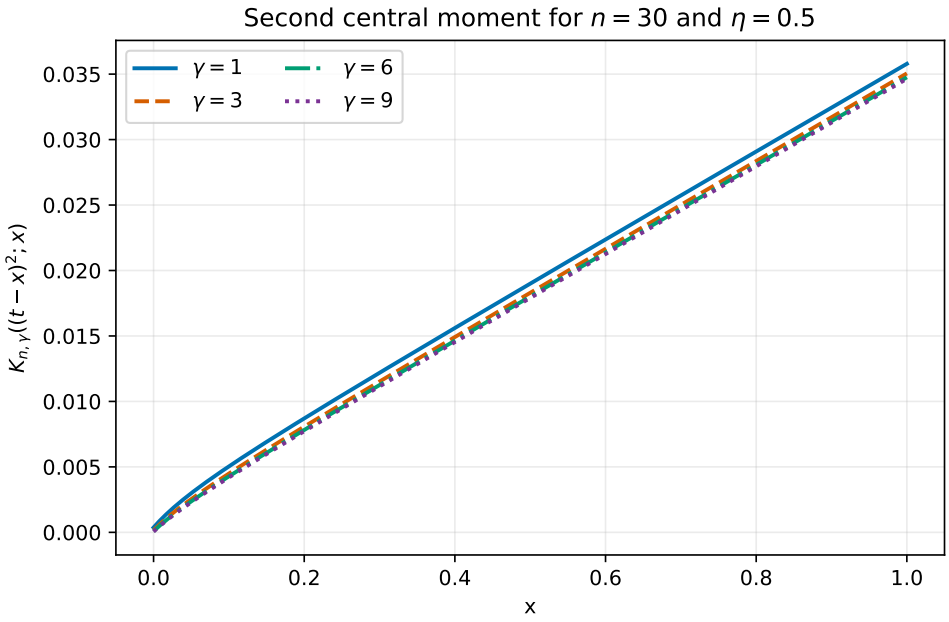


FIGURE 4. Second central moments for $n = 30$, $\eta = 0.5$, and several values of γ.

TABLE 3. Exact second central moments for $b_n = n$, $n = 30$, and $\eta = 0.5$.

x	$\gamma = 1$	$\gamma = 3$	$\gamma = 6$	$\gamma = 9$
0.00	0.00037037	0.00015873	0.00008547	0.00005848
0.05	0.00298942	0.00253968	0.00236438	0.00229657
0.10	0.00503704	0.00449206	0.00427595	0.00419181
0.20	0.00870370	0.00807540	0.00782357	0.00772515
0.40	0.01560847	0.01492063	0.01464329	0.01453467
0.60	0.02237037	0.02165873	0.02137118	0.02125848
0.80	0.02908832	0.02836386	0.02807082	0.02795592
1.00	0.03578704	0.03505456	0.03475809	0.03464181

To show why this moment comparison must not be promoted to unrestricted true-error monotonicity, take the smooth bounded function

$$g(x) = e^{-x} + \frac{1}{2}\sin(\pi x).$$

Figure 5 shows crossings of the actual absolute-error curves. This does not contradict Theorem 6.1, because g is not nondecreasing and convex on the interval.

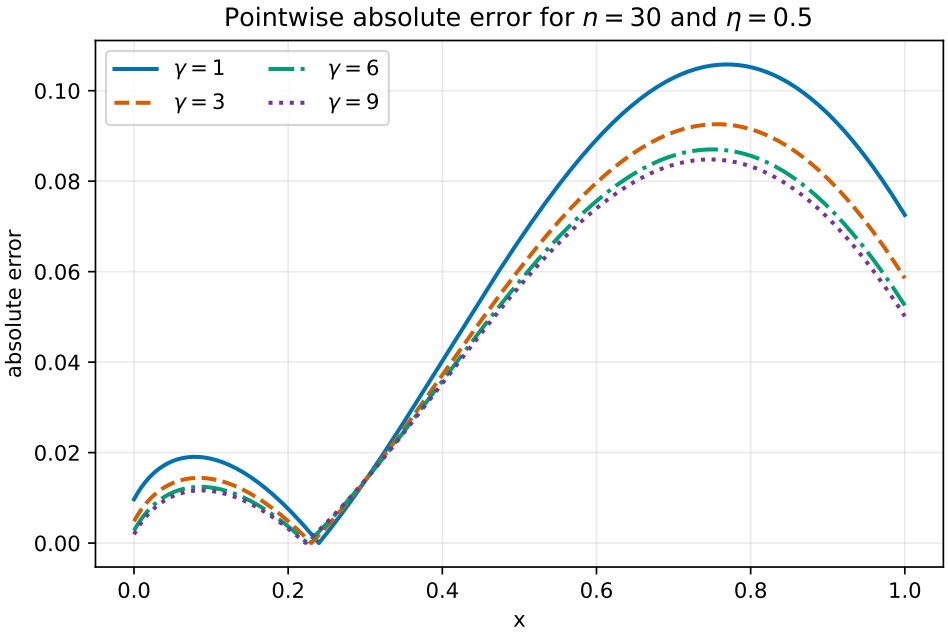


FIGURE 5. Crossing of pointwise absolute-error curves for the nonmonotone target g, illustrating the distinction between moment improvement and unrestricted true-error ordering.

Finally, Figures 6 and 7 verify the first- and second-order limits for $\eta = 0.5$ and $\gamma = 3$. The corresponding tables report representative residuals.

TABLE 4. First-order asymptotic verification at $x = 0.4$, $\gamma = 3$, and $\eta = 0.5$.

n	$n(K_{n,3}f(x) - f(x))$	limit	absolute difference
20	-1.13884706	-1.03573776	0.10310930
40	-1.09964603	-1.03573776	0.06390828
80	-1.07130834	-1.03573776	0.03557059
160	-1.05450585	-1.03573776	0.01876809
320	-1.04537831	-1.03573776	0.00964055
640	-1.04062358	-1.03573776	0.00488582

10. CONCLUSION

The power node is a known ingredient; the contribution of this paper is the comparison structure and the asymptotic information that can be extracted from it. For the full analytic-weight family, the

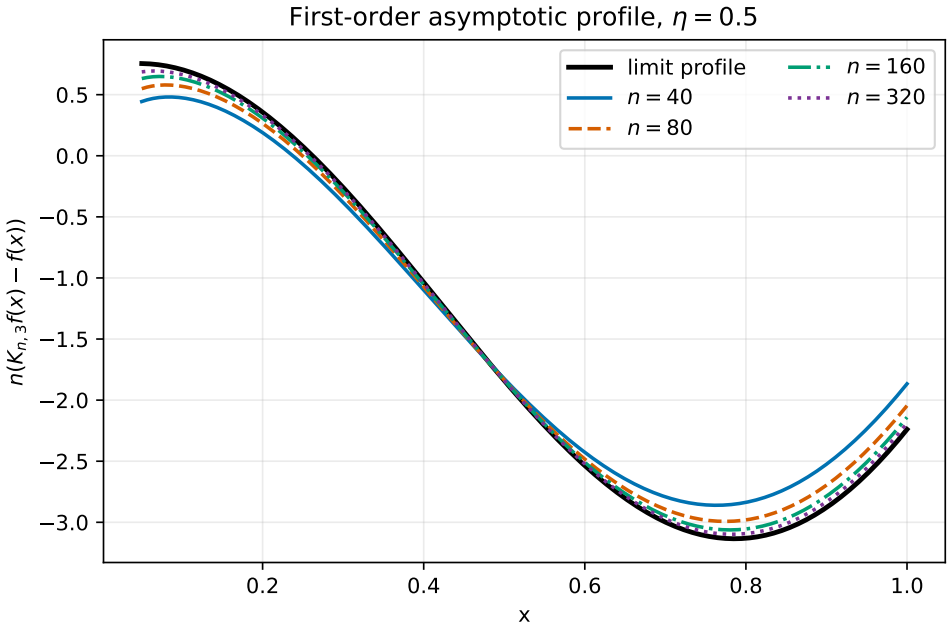


FIGURE 6. First-order generator-sensitive Voronovskaja profile for $\eta = 0.5$ and $\gamma = 3$.

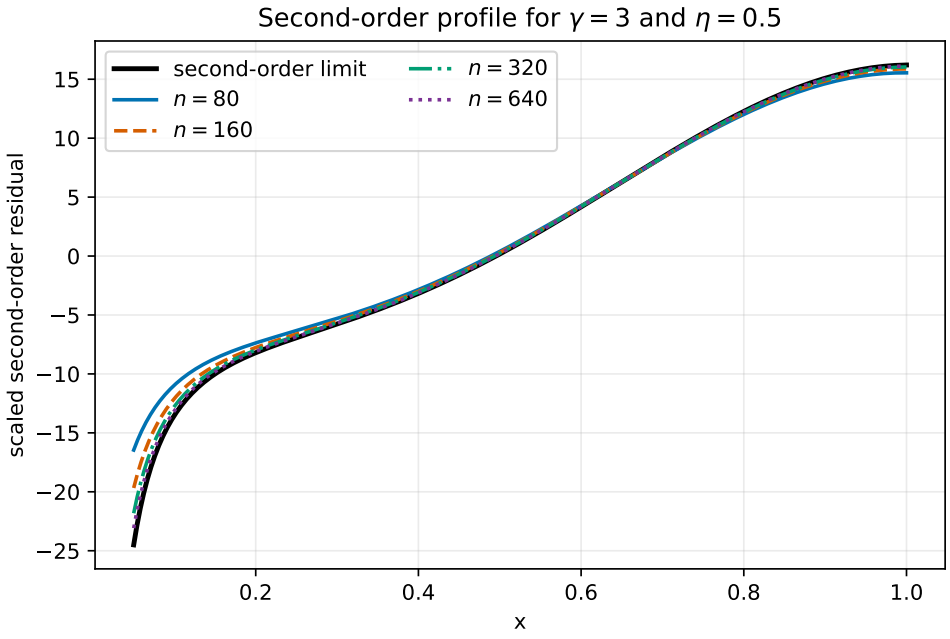


FIGURE 7. Second-order asymptotic residual for $\eta = 0.5$ and $\gamma = 3$.

second central moment admits an exact finite-scale difference identity and a necessary-and-sufficient improvement criterion. For the exponential specialization, a coupling constructed from the same Poisson and uniform variables upgrades the moment comparison on a natural shape class: if the target is nondecreasing and convex, the approximation error decreases as γ increases, and the limiting operator as the parameter diverges is the discrete Szász operator.

TABLE 5. Second-order asymptotic verification at $x = 0.4$, $\gamma = 3$, and $\eta = 0.5$.

n	scaled residual	limit	absolute difference
40	-2.55633101	-3.16949598	0.61316497
80	-2.84564706	-3.16949598	0.32384892
160	-3.00289488	-3.16949598	0.16660110
320	-3.08497654	-3.16949598	0.08451944
640	-3.12692499	-3.16949598	0.04257099
1280	-3.14813184	-3.16949598	0.02136414

The same parameter enters the asymptotic behavior in a different manner. In the general Voronovskaja limit, the contribution of the analytic generator can be separated from the displacement caused by Kantorovich averaging; in the exponential case, choosing $\gamma_n \to \infty$ eliminates this displacement at first order. The perturbed exponential generator $e^y(1 + \eta y)$ supplies a complementary non-Poisson regime. Its exact Poisson–Bernoulli representation exposes the generator contribution at the probabilistic level and leads to the stated second-order formula. These conclusions bring four effects that are often treated separately—finite-scale dispersion, shape-constrained error ordering, parameter-dependent drift removal, and a non-Poisson second-order correction—under the same analytic-weight model. The computations in Section 9 are consistent with the corresponding theoretical regimes and also show the limitation of the moment criterion: a smaller second central moment need not yield a smaller pointwise error for an arbitrary target function.

Authors' Contributions. Both authors contributed to the theoretical development, verification of the derivations and numerical calculations, and revision of the manuscript. Both authors read and approved the final version.

Data Availability. No external data set was used in this study. The numerical quantities reported here are determined by the formulas and parameter values specified in the manuscript.

Disclosure. The authors take responsibility for the final manuscript, including the mathematical derivations, numerical calculations, and bibliographic information.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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