GLOBAL EXISTENCE RESULTS FOR GIERER-MEINHARDT SYSTEMS ON TIME EVOLVING DOMAINS

ABDELOUAHAB GOUADRIA

Department of Mathematics, University, Batna 2, Algeria
abdelouahab.gouadria@univ-batna2.dz

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Abstract. Global solutions to a Gierer-Meinhardt model of two substances defined by reaction-diffusion equations are shown in this article. By employing Lyapunov functionals and investigating the regularizing properties inherent to parabolic equations, we rigorously establish the existence and asymptotic behavior of solutions under appropriate assumptions. Numerical simulations are used to corroborate the analytical findings. This research differs from previous work because it relies on spatial domains that vary over time, rather than being static.
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1. Introduction

Reaction-diffusion equation systems have garnered a lot of interest recently since they are frequently used to simulate chemical and biological processes. One of the important systems among these is the Gierer-Meinhardt. because in many fields (like biomathematics), the spatial domains are made up of living objects (cells).The evolution of the spatial domain plays a crucial role in shaping the system's dynamical behavior. As a result, a decent representation of a reaction-diffusion model must include the spatial domain transformation.

For reaction-diffusion systems, the behavior of solutions and their global existence with two equations on a class of time-varying spatial domains are unresolved issues addressed in part by this work.

2. Results from the Past

The standard norms in spaces $L^p(\Omega), L^\infty(\Omega)$ and $C(\bar{\Omega})$ are each represented by

$$\|u\|_p^p = \frac{1}{|\Omega|}\int_\Omega |u(x)|^p \, dx \tag{2.1}$$

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$$\|u\|_{\infty} = \max_{x\in\Omega} |u(x)| \tag{2.2}$$

$$\|u\|_{c(\bar{\Omega})} = \max_{x\in\bar{\Omega}} |u(x)| \tag{2.3}$$

The discovery of the regenerating hydra was made by Trembley in 1744 [21] , while the corresponding mathematical model was later proposed by Gierer and Meinhardt [11] in 1972, in response to Turing's brilliant notion [22]. The reaction-diffusion equations in this system expressed as :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \alpha_1 \Delta u = \sigma - \mu u + \dfrac{u^p}{v^q} \\ \dfrac{\partial v}{\partial t} - \alpha_2 \Delta v = -\nu v + \dfrac{u^r}{v^s} \end{cases} \quad \text{for any } x \in \Omega, t > 0 \tag{2.4}$$

utilizing Neummann boundary conditions

$$\frac{\partial u}{\partial \eta} = 0, \text{ and } \frac{\partial v}{\partial \eta} = 0, x \in \partial\Omega, t > 0 \tag{2.5}$$

as well as the starting parameters

$$\begin{cases} u(x,0) = \phi_1(t) > 0 \\ v(x,0) = \phi_2(t) > 0 \end{cases} , x \in \Omega \tag{2.6}$$

where $\Omega \subset \mathbb{R}^N$ is a limited area having a smooth border. $\partial\Omega$, $\alpha_1, \alpha_2 > 0, \mu, \upsilon, \sigma > 0, p, q, r,$ and s are non negative with $p > 1$. Rothe demonstrated the global existence of solutions in $(0, \infty)$ in 1984 [20] under specific circumstances when $p = 2, q = 1, r = 2, s = 0$ and $N = 3$.

A broader framework was examined by Masuda and Takahashi [18] for (u, v):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \alpha_1 \Delta u = \sigma_1(x) - \mu u + \rho_1(x,u)\dfrac{u^p}{v^q} \\ \dfrac{\partial v}{\partial t} - \alpha_2 \Delta v = \sigma_2(x) - \nu v + \rho_2(x,u)\dfrac{u^r}{v^s} \end{cases} \tag{2.7}$$

with $\sigma_1, \sigma_2 \in C^1(\bar{\Omega}), \sigma_1 \geq 0, \sigma_2 \geq 0, \rho_1, \rho_2 \in C^1(\bar{\Omega} \times \bar{\mathbb{R}}_+^2) \cap L^\infty(\bar{\Omega} \times \bar{\mathbb{R}}_+^2)$. where $\rho_1 \geq 0, \rho_2 > 0$ and p, q, r, s denote nonnegative constants with $\dfrac{p-1}{r} < \dfrac{q}{s+1}$, (2.6) is a special case of system (2.7)

Abdelmalek, Louafi, and Youkana [3] demonstrated the global existence of solutions to the three-component phyllotaxis Gierer–Meinhardt system (2.8) through the use of a Lyapunov functional.

$$\begin{cases} u_t - a_1 \Delta u = \sigma_1 - b_1 u + \dfrac{u^{p_1}}{v^{q_1}(w^{r_1} + c)} \\ v_t - a_2 \Delta v = \sigma_2 - b_2 v + \dfrac{u^{p_2}}{v^{q_2} w^{r_2}} \\ w_t - a_3 \Delta w = \sigma_3 - b_3 w + \dfrac{u^{p_3}}{v^{q_3} w^{r_3}} \end{cases} \tag{2.8}$$

for $\sigma > 0, c \geq 0$, and

$$0 < p_1 - 1 < \max\left\{p_2 \min\left(\frac{q_1}{q_2+1}, \frac{r_1}{r_2}, 1\right), p_3 \min\left(\frac{r_1}{r_3+1}, \frac{q_1}{q_3}, 1\right)\right\} \tag{2.9}$$

The study was later extended to the m-component Gierer–Meinhardt system by Gouadria and Abdelmalek (2013), see [1].

In 2019, Gouadria and Abdelmalek [10] studied the nature of the solutions of a Gierer-Meinhardt type system with two activators and two inhibitors, using a modified Lyapunov functional :

$$\begin{cases} \partial_t u_1 - a_1 \Delta u_1 = f_1(u_1, u_2, v_1, v_2) = \sigma_1 - b_1 u_1 + \dfrac{u_1^{p_1} u_2^{q_1}}{v_1^{r_1} v_2^{s_1}} \\ \partial_t u_2 - a_2 \Delta u_2 = f_2(u_1, u_2, v_1, v_2) = \sigma_2 - b_2 u_2 + \dfrac{u_1^{p_2} u_2^{q_2}}{v_1^{r_2} v_2^{s_2}} \\ \partial_t v_1 - a_3 \Delta v_1 = g_1(u_1, u_2, v_1, v_2) = -b_3 v_1 + \dfrac{u_1^{p_3} u_2^{q_3}}{v_1^{r_3} v_2^{s_3}} \\ \partial_t v_2 - a_4 \Delta v_2 = g_2(u_1, u_2, v_1, v_2) = -b_4 v_2 + \dfrac{u_1^{p_4} u_2^{q_4}}{v_1^{r_4} v_2^{s_4}} \end{cases} \quad (x \in \Omega, t > 0) \tag{2.10}$$

with Neumann boundary conditions

$$\frac{\partial u_1}{\partial \eta} = \frac{\partial u_2}{\partial \eta} = \frac{\partial v_1}{\partial \eta} = \frac{\partial v_2}{\partial \eta} = 0 \text{ in } \partial\Omega \times \{t > 0\}, \tag{2.11}$$

and the initial data

$$\begin{cases} u_1(0, x) = \varphi_1(x) > 0 \\ u_2(0, x) = \varphi_2(x) > 0 \\ v_1(0, x) = \varphi_3(x) > 0 \\ v_2(0, x) = \varphi_4(x) > 0 \end{cases} \text{in } \Omega, \tag{2.12}$$

and $\varphi_i \in C\left(\overline{\Omega}\right)$ for all $i = 1, 2, 3, 4$.where Ω is an open bounded domain of class C^1 in $\mathbb{R}^N$, with the boundary $\partial\Omega$; $\partial/\partial\eta$ denotes the external normal derivative on $\partial\Omega$.

Previous studies have primarily considered fixed spatial domains. The question of global existence for reaction–diffusion systems on time-evolving domains has been partially addressed by Douaifia and Abdelmalek [7,8], Crampin [5], Madzvamuse [17], among others. More recently, The shadow system of a singular Gierer–Meinhardt model on an evolving domain was analyzed by Kavallaris in 2021, revealing that Turing instability and pattern formation can significantly differ as the domain evolves.

3. Preliminary Observations and Notations

Consider a confined, simply linked domain that is time dependent $\Omega_t \in \mathbb{R}^N (N \geq 1)$ with a shifting border $\partial\Omega_t$ that is smooth $t \in [0, t], t > 0$. Using a C^k-diffeomorphism, for $(k \geq 2)$,The time-dependent domain Ω_t is mapped onto a fixed reference domain Ω_0.

$$\varphi_t : \Omega_0 \to \Omega_t$$

Given any $x := x(t) \in \Omega_t$, there exists a corresponding $y \in \Omega_0$

We examine a two-component reaction-diffusion system :

$$\begin{cases} \dfrac{\partial a}{\partial t} + \nabla(\theta a) - d_1 \nabla a = \sigma_1 - \mu_1 a + \rho_1(a,b)\dfrac{a^{p_1}}{b^{q_1}} \text{ ,} in\ \Omega_t \times (0,T] \\ \dfrac{\partial b}{\partial t} + \nabla(\theta b) - d_2 \nabla b = \sigma_2 - \mu_2 b + \rho_2(a,b)\dfrac{a^{p_2}}{b^{q_2}} \text{ ,} in\ \Omega_t \times (0,T] \end{cases} \tag{3.1}$$

utilizing Neummann boundary conditions

$$\frac{\partial a}{\partial \eta}(x,t) = \frac{\partial b}{\partial \eta}(x,t) = 0,\ on\ \ \partial\Omega_t \times (0,T], \tag{3.2}$$

and preliminary information

$$\begin{cases} a(y,0) = a_0(y), \\ b(y,0) = b_0(y), \end{cases} \quad \text{on } \bar{\Omega}_0 \tag{3.3}$$

With ($N \in \mathbb{N}$), let Ω_t be a subset of $\mathbb{R}^N$, where it has a moving border, is finite, time-varying, and simply connected, among other features $\partial\Omega_t$ has some smoothness. Through to a suitable C^l-diffeomorfism $(l \geq 2)$

$$\varphi_t : \Omega_0 \to \Omega_t,$$

the domain Ω_t can be traced to a static domain Ω_0. Additionally,regarding the variable t, we think the diffeomorfism φ_t should be a C^2 map .Due to the flow velocity $v(x,t)$, induced by the time variation of the domain $\Omega(t)$, two additional terms $(\vartheta.\Delta)u$ and $u(\vartheta.\Delta)$ appear. They are added to the reaction equations of the traditional diffusion model as dilution and advection terms, respectively.

Under the following fundamental presumptions, we address particular classes of semilinear parabolic equations on a class of evolving domains in this section:

$(Asm1)$ we investigate the flow velocity provided by :

$$\vartheta = \frac{dx}{dt} \tag{3.4}$$

$(Asm2)$ Deformation of the isotropic domain φ_t , satisfies (for $T > 0$)

$$\varphi_t(y) = x = \chi(t)y, y = (y_{1,......,}y_N) \in \Omega_0, t \in [0,T] \tag{3.5}$$

with $\chi(t) \in (\mathbb{R}_+, \mathbb{R}_+^*)$, morever $\chi(0) = 1$.

Remark 3.1. *Due to the presumptions $(Asm1) - (Asm2)$, the flow velocity's explicit form ϑ is as follows :*

$$\vartheta(x,t) = \frac{\dot{\chi}(t)}{\chi(t)}x, x \in \Omega_t, t \in [0,T] \tag{3.6}$$

where $\dot{\chi}(t) := \dfrac{d\chi(t)}{\chi(t)}$Consequently, the flow velocity ϑ's divergence is provided by

$$\nabla\vartheta = N\frac{\dot{\chi}(t)}{\chi(t)} \tag{3.7}$$

Through the use of difeomorphism ϑ_t , Each function $a_i(i=1,...,m)$ in the system :

$$\begin{cases} \dfrac{\partial a_i}{\partial t} + \nabla.(a_i, \vartheta) - d_i \Delta a_i = f_i(x,t,a) \text{ in } \Omega_t \times \mathbb{R}_+^*, i = 1..m \\ \dfrac{\partial a_i}{\partial \nu} = 0 \\ a_i(x,0) = a_{0i}(y) \end{cases}$$

is associated with the function $\bar{a}_i$ defined below

$$\bar{a}_i(y,t) := a_i(\rho_t(y),t) = a_i(x,t), i = 1....., = m \tag{3.8}$$

where

$$(\bar{a}_i)_{i=1}^m := \bar{a}$$

Then, for each

$$i = 1....m$$

$$\frac{\partial a_i}{\partial t} = \frac{\partial \bar{a}_i}{\partial t} + \nabla \bar{a}.\frac{\partial \varphi_t^{-1}(x)}{\partial t} = \frac{\partial \bar{a}_i}{\partial t} - \vartheta.\nabla a_i \tag{3.9}$$

$$\nabla.(\vartheta a_i) = \vartheta.\nabla a_i + a_i(\nabla.\vartheta) = \vartheta.\nabla a_i + NN\frac{\dot{\chi}(t)}{\chi(t)}\bar{a}_i \tag{3.10}$$

$$\nabla a_i = \sum_{j=1}^n H_y \bar{a}_i \frac{\partial \varphi_t^{-1}(x)}{\partial x_j}.\frac{\partial \varphi_t^{-1}(x)}{\partial x_j} + \nabla \bar{a}_i \frac{\partial^2 \varphi_t^{-1}(x)}{\partial x_j^2} = \frac{1}{\chi^2(t)}\Delta \bar{a}_i \tag{3.11}$$

where φ_t^{-1} represents the opposite of φ_t concerning the spatial variable, and $H_y\bar{a}_i$ indicates the Hessian matrix of a_i $(for\ i = 1,...,m)$. Consequently, the system

$$\begin{cases} \dfrac{\partial a_i}{\partial t} + \nabla(a_i \vartheta) - d_i \nabla a_i = f_i(a,x,t)\ , for, i = 1....m \\ \dfrac{\partial a_i}{\partial \eta} = 0\ , \qquad on\ \partial\Omega_0 \times \mathbb{R}_+^*,\ i = 1....m \\ a_i(x,0) = a_{0i}(y), \qquad on\ \bar{\Omega}_0, i = 1....m \end{cases} \tag{3.12}$$

Let us assume that, within the fixed reference domain $f(x,t,a) = f(a) := (f_i(a))_{i=1}^m)$, the system can be equivalently reformulated as the auxiliary reaction–diffusion system Ω_0 :

$$\begin{cases} \dfrac{\partial \bar{a}_i}{\partial t} - \dfrac{d_i}{\chi^2}\bar{a}_i = f_i(\bar{a}) - N\dfrac{\dot{\chi}}{\chi}\bar{a}_i\ , in\ \Omega_0 \times (0,T], i = 1....m \\ \dfrac{\partial \bar{a}_i}{\partial \eta} = 0\ , \qquad on\ \partial\Omega_0 \times \{t > 0\}\ , i = 1....m \\ \bar{a}_i(y,0) = \bar{a}_{0i}(y), \qquad on\ \bar{\Omega}_0, i = 1....m \end{cases} \tag{3.13}$$

By applying the subsequent variable change (see [16]):

$$\rho(t) := \int_0^t \frac{ds}{\chi^2(s)} \tag{3.14}$$

and $\hat{a}_i(y,\rho) := \bar{a}_i(y,t)$, in the system (3.13). Over the static reference domain Ω_0 , the system (3.12) can be equivalently represented by the reaction–diffusion system given below :

$$\begin{cases} \dfrac{\partial \hat{a}_i}{\partial t} - d_i \Delta \hat{a}_i = \chi^2 f_i(\hat{a}) - N\dot{\chi}\chi\hat{a}_i \, , in \ \Omega_0 \times (0, \bar{T}], i = 1....m \\ \dfrac{\partial \hat{a}_i}{\partial \eta} - d_i \Delta \hat{a}_i = 0 \, , \qquad on \ \partial\Omega_0 \times \{t > 0\} \, , i = 1....m \\ \hat{a}_i(y,0) = \hat{a}_{0i}(y), \qquad on \ \bar{\Omega}_0, i = 1....m \end{cases} \tag{3.15}$$

where $(a_i)_{i=1}^m =: \hat{a}, \bar{T} = \rho(T)$, and the fact $\rho(t) = t$ was invoked unambiguously in our analysis.

Addressing the open question of whether the solution for the Gierer-Meinhardt system on a spatially linear isotropically developing domain is global, unique, and uniformly bounded is the goal of this section. The response takes the form :

$$\begin{cases} \dfrac{\partial a_i}{\partial t} + \nabla(\vartheta a_i) - d_i \nabla a_i = \sigma_1 - \mu_i a_i + \rho_i(a_1, a_2)\dfrac{a_1^{p_i}}{a_2^{q_i}} \, , in \ \Omega_t \times (0, T] \\ \dfrac{\partial a_i}{\partial \eta}(x,t) = 0, \ on \ \partial\Omega_t \times (0,T], i = 1..2 \\ a_i(y,0) = a_{i0}(y), on \ \bar{\Omega}_0 \end{cases} \tag{3.16}$$

where $T > 0, a_i := a_i(x,t)$, with $x := x(t) = (x_1(t), ..., x_N(t))$, where η denotes the unit normal vector directed outward from the boundary of $\partial\Omega_t$,with $p_1 > 1$, $q_i, r_i, \sigma_i, \mu_i$ and d_i are positive , $\rho_i \in C^1(\mathbb{R}_+^2, \mathbb{R}_+)$ for $i = 1, 2$.

Throughout this subsection, we shall employ the following assumptions :

$(Asm3)$ The flow velocity $\vartheta(x,t)$ is assumed to coincide with the velocity of the evolving domain , i.e, $\vartheta = \dfrac{dx}{dt}$.

$(Asm4)$ Deformation of the isotropic domain φ_t satisfies

$$x = \varphi_t(y) = \chi(t)y, y \in \Omega_0, t \in [0,T] \tag{3.17}$$

where $\chi \in c^2\ (\mathbb{R}_+^2, \mathbb{R}_+^*)$ and $\chi(0) = 1$.

$(Asm5)$ There exist two positive constants $\zeta_1, \zeta_2 > 0$ verify the following relation

$$\zeta_1 \leq \Upsilon_i(t) := \mu_i \chi^2(t) + N\chi(t)\frac{d\chi(t)}{dt} \leq \zeta_2, \forall t \in [0,\tau], \tau > 0 \tag{3.18}$$

and

$$\chi(t) \geq \zeta_3, \forall t > 0 \tag{3.19}$$

$(Asm6)$ $\dfrac{p-1}{r} < \min(\dfrac{q}{s+1}, \dfrac{m}{n}, 1)$

$(Asm7)$ there exist $\rho_{-i}, \bar{\rho}_i > 0 (i = 1, 2)$, such that

$$\rho_{-i} \leq \rho_i(w_1, w_2) \leq \bar{\rho}_i, w_i \geq 0, (i = 1, 2) \tag{3.20}$$

Remark 3.2. *For any domain growth function χ with a positive derivative (such as the logistic function, which is possible in biology), $\mu_1,\mu_2 \in \mathbb{R}_+^*$ satisfy the assmption $(Asm5)$.*

Through the use of difeomorphism φ_t , a_i for $i = 1..2$ can be mapped as a new functions, which are defined as follows:

$$\bar{a}_i(y, t) := a_i(\varphi_t(y), t) = a_i(x, t) \tag{3.21}$$

Then, following the same procedure as in $(3.8) - (3.11)$, the system (3.16) can be equivalently reformulated as an auxiliary reaction–diffusion system defined on the static reference domain Ω_0:

$$\begin{cases} \dfrac{\partial \bar{a}_1}{\partial t} - \dfrac{d_1}{\chi^2(t)} \nabla \bar{a} = \sigma_1 - \left(\mu_1 + N\dfrac{\dot{\chi}(t)}{\chi(t)} \right) \bar{a}_1 + \rho_1(\bar{a}_1, \bar{a}_2) \dfrac{\bar{a}_1^{p_1}}{\bar{a}_2^{q_1}} \, ,in \; \Omega_0 \times (0, T] \\ \dfrac{\partial \bar{a}_i}{\partial \eta}(x, t) = 0, i = 1..2 \; on \;\; \partial\Omega_0 \times \{t > 0\} \\ \bar{a}_i(y, 0) = a_{i0}(y) \; , on \; \bar{\Omega}_0 \end{cases} \tag{3.22}$$

With the aid of the variable change (3.14), and

$$\hat{a}_i(y, \rho) := \bar{a}_i(y, t), i = 1..2 \tag{3.23}$$

in system (3.22). Consequently, system (3.16) can be expressed alongside the following reaction–diffusion system defined on the fixed reference domain Ω_0 and for $i = 1..2$

$$\begin{cases} \dfrac{\partial \hat{a}_i}{\partial t} - d_i \nabla \hat{a}_i = \sigma_i \chi^2(t) + \chi^2(t) \rho_i(\hat{a}_1, \hat{a}_2) \dfrac{\hat{a}_1^{p_i}}{\hat{a}_2^{q_i}} - \Upsilon_i(t)\hat{a}_i \; = F_i(\hat{a}_1, \hat{a}_2), in \; \Omega_0 \times (0, \bar{T}], \\ \dfrac{\partial \hat{a}_i}{\partial \eta}(x, t) = 0, on \;\; \partial\Omega_0 \times \{t > 0\} \\ \hat{a}_i(y, 0) = a_{i0}(y), on \; \bar{\Omega}_0 \end{cases} \tag{3.24}$$

where $\bar{T} = \rho(T)$, and we have utilized the fact without any doubt. $t := \rho$.

4. The Lower-Bounded Solution's Uniqueness and Local Existence

Given that the nonlinearity (F_1, F_2) is continuously differentiable on $\mathbb{R}_+ \times \mathbb{R}_+^*$ and by suppose that $a_{10}, a_{20} \in L^\infty(\Omega_0)$. Demonstrating the existence of a unique local nonnegative classical solution is a fundamental problem in the analysis of such systems. For system (3.24) on $[0, \bar{T}_{\max})$, where $\bar{T}_{\max}$ denotes the maximal existence (or blow-up) time in $L^\infty(\Omega_0)$ (see, e.g., [14,20], the equivalence between systems (3.16) and (3.24) yields the following result :

Theorem 4.1. *Suppose that a_{10}, $a_{20} \in L^{\infty}(\Omega_0)$, and $(Asm1)-(Asm2)$ are satisfied. Then the system (3.12) admits a unique classical solution (a_1, a_2) on $\Omega_t \times [0, Tmax)$, where $0 < Tmax \leq \infty$. Furthermore if*

$$T_{\max} < \infty, \text{ then } \lim_{t \to T_{\max}} (\|a(.t)\|_{L^{\infty}(\Omega_t)} + \|b(.t)\|_{L^{\infty}(\Omega_t)}) = +\infty \tag{4.1}$$

The comparison principle leads to the following conclusion.

Corollary 4.1. *Additionally, with the identical presumptions as stated in Theorem 4.1, $(Asm4)$ and a_{10}, $a_{20} > 0$ wait. Next , there is $\zeta > 0$, such that*

$$\hat{a}_i(y,t) \geq \zeta, \forall y \in \bar{\Omega}_0, \forall t \in (0, \bar{T}_{max}), i = 1..2 \tag{4.2}$$

Remark 4.1. *For the solution of system (3.16), we obtain the same conclusion as Corollary 4.1 since the systems (3.16) and (3.24) are equivalent.*

5. Existence of Global Solution

Proving the global solution for system (3.24) guarantees the global existence of the solution for system (3.16). Our objective is therefore to create a consistent boundary of $\|\hat{a}_1(.,t)\|_{\infty}$, and $\|\hat{a}_2(.,t)\|_{\infty}$ on $[0, \bar{T}_{\max})$ .In order to achieve this, it suffices to establish a uniform estimate for $\left\| \dfrac{\hat{a}_1^p}{\hat{a}_2^q} \right\|_{L^{\tau}(\Omega_0)}$ on $[0, \bar{T}_{\max})$ for some $\tau > \dfrac{N}{2}$. For this purpose, we employ the potential Lyapunov functional listed below :

$$L(t) = \int_{\Omega_0} \frac{\hat{a}_1^{\alpha}}{\hat{a}_2^{\beta}} dy \tag{5.1}$$

where

$(Asm8)$: α, and β are positive constants in the sense that

$$\alpha \geq 2max\left(1, \frac{\zeta_1}{\zeta_2}\right), \text{ and } \frac{2d_1 d_2}{(d_1 + d_2)^2} \geq \beta \tag{5.2}$$

After some preparation, the proof will be given later, but for now, we are prepared to present the basic finding.

Theorem 5.1. *We assume that the conditions $(Asm1)-(Asm7)$ are assumed to be true, in addition $a_{10}; a_{20} \in L^{\infty}(\Omega_0)$; and a_{10}, a_{20} then the solution of the system (3.16) is global and uniformly bounded.*

The following findings serve as the foundation for the demonstration of Theorem 5.1.

Lemma 5.2. *Let p, q, r, s the same parameters of system (3.16) satisfy $(Asm6)$.For $\alpha, \beta, \gamma, \varepsilon > 0$ there exist :$\kappa = \kappa(\alpha, \beta, \gamma), \delta > 0$ and $\theta := \theta(\alpha) \in (0,1)$ such that*

$$\alpha \frac{\hat{a}_1^{p_1+\alpha-1}}{\hat{a}_2^{q_1+\beta}} \leq \varepsilon\beta \frac{\hat{a}_1^{p_2+\alpha}}{\hat{a}_2^{q_2+\beta+1}} + \varepsilon^{-\delta} \left(\frac{\hat{a}_1^{\alpha}}{\hat{a}_2^{\beta}}\right)^{\theta}, a_1 \geq 0, a_2 \geq \gamma \tag{5.3}$$

Proof. In accordance with the proof of [19], Lemma 33.11, we substitute ε-Young's inequality for Young's inequality. □

Proposition 5.1. *Let $(\hat{a},\hat{b})$ the solution of (3.24) on $\Omega_0\times[0,\bar{T}_{\max})$. Given the assumption that requirements $(Asm1)$ through $(Asm8)$ are met, there is a positive constant $\hat{C}$ such that the functional L*

$$L(t)\le\hat{C},\forall t\in[0,\bar{T}_{max}) \tag{5.4}$$

Proof. Let $\bar{T}^*\in[0,\bar{T}_{\max})$ Using Green's formula and the boundary's homogeneous Neumann conditions, we obtain

$$\begin{aligned}\frac{d}{dt}L(t) &= I+J \\ &= j_1+j_2+\int_{\Omega_0}Q\left(\hat{b}\nabla\hat{a},\hat{a}\nabla\hat{b}\right)\frac{\hat{a}^{\alpha-2}}{\hat{b}^{\beta+2}}dy\end{aligned} \tag{5.5}$$

□

where the notations above represent

$$J_1=-\alpha\Upsilon_1(t)\int_{\Omega_0}\frac{\hat{a}^{\alpha}}{\hat{b}^{\beta}}+\alpha\sigma_1\chi^2(t)\int_{\Omega_0}\frac{\hat{a}^{\alpha-1}}{\hat{b}^{\beta}}+\alpha\chi^2(t)\int_{\Omega_0}\rho_1(\hat{a},\hat{b})\frac{\hat{a}^{\alpha-1+p}}{\hat{b}^{\beta+q}}$$
$$J_2=+\beta\Upsilon_2(t)\int_{\Omega_0}\frac{\hat{a}^{\alpha}}{\hat{b}^{\beta}}-\beta\sigma_2\chi^2(t)\int_{\Omega_0}\frac{\hat{a}^{\alpha}}{\hat{b}^{\beta+1}}-\beta\chi^2(t)\int_{\Omega_0}\rho_2(\hat{a},\hat{b})\frac{\hat{a}^{\alpha+r}}{\hat{b}^{\beta+1+s}}$$

$$Q(\hat{b}\nabla\hat{a},\hat{a}\nabla\hat{b})=\alpha(\alpha-1)d_1\hat{b}^2\left|\nabla\hat{a}\right|^2-\alpha\beta(d_1+d_2)\hat{b}\Delta\hat{a}\cdot\hat{a}\nabla\hat{b}+\beta(\beta+1)d_2\hat{a}^2\left|\Delta\hat{b}\right|^2$$

Q is a quadratic form in relation to $\hat{b}\nabla\hat{a}$ and $\hat{a}\nabla\hat{b}$, in the light of assumption $(Asm8)$ $Q\left(\hat{b}\nabla\hat{a},\hat{a}\nabla\hat{b}\right)$ is nonpositive, thus we obtain

$$\frac{d}{dt}L(t)\le j_1+j_2=J \tag{5.6}$$

By virtue of assumptions $(Asm5)$ and $(Asm7)$, we have

$$J\le(-\alpha\zeta_1+\beta\zeta_2)\,L(t)+\alpha\sigma_1\bar{\chi}\int\frac{\hat{a}^{\alpha-1}}{\hat{b}^{\beta}}+\int\Re(\hat{a},\hat{b})dy \tag{5.7}$$

where $\bar{\chi}:=\max_{t\in[0,\bar{T}^*]}\chi^2(t)$, and

$$\begin{aligned}\Re(\hat{a},\hat{b})\quad:&=\bar{\chi}\alpha\bar{\rho}_1\left[\frac{\hat{a}^{p+\alpha-1}}{\hat{b}^{q+\beta}}\right]-\beta\bar{\chi}\rho_{\_2}\left[\frac{\hat{a}^{\alpha+r}}{\hat{b}^{\beta+1+s}}\right] \\ &\le\bar{\chi}\alpha\bar{\rho}_1\frac{\hat{a}^{p+\alpha-1}}{\hat{b}^{q+\beta}}-\beta\zeta_3^2\rho_{\_2}\left[\frac{\hat{a}^{\alpha+r}}{\hat{b}^{\beta+1+s}}\right]\end{aligned} \tag{5.8}$$

Applying Corollary 4.1 and Lemma 5.2 together with $\varepsilon=\dfrac{\zeta_3^2\rho_{-2}}{\bar{\rho}_1\bar{\chi}}$, we get

$$\Re(\hat{a},\hat{b})\le\bar{\rho}_1\bar{\chi}\varepsilon^{-\delta}\kappa(\frac{\hat{a}^{\alpha}}{\hat{b}^{\beta}})^{\theta} \tag{5.9}$$

where the constants $\delta,\kappa>0$ and $\theta\in(0,1)$ are referenced in Lemma 5.2. Assessment

(5.7)-(5.9) can be used to express (5.6) as follows:

$$\frac{d}{dt}L(t) \leq (-\alpha\zeta_1 + \beta\zeta_2)L(t) + \alpha\sigma_1\bar{\chi}\int_{\Omega_0}\left(\frac{\hat{a}^\alpha}{\hat{b}^\beta}\right)^{\dfrac{\alpha-1}{\alpha}}\left(\frac{1}{\hat{b}^{\dfrac{\beta}{\alpha}}}\right)dy + \bar{\rho}_1\bar{\chi}\varepsilon^{-\delta}\kappa\int_{\Omega_0}\left(\frac{\hat{a}^\alpha}{\hat{b}^\beta}\right)^\theta dy \tag{5.10}$$

By applying Hölder's inequality and Corollary 4.1, we get

$$\frac{d}{dt}L(t) \leq (-\alpha\zeta_1 + \beta\zeta_2)L(t) + \zeta_3\left(Lt)\right)^{\dfrac{\alpha-1}{\alpha}} + \zeta_4\left(L(t)\right)^\theta \tag{5.11}$$

where $\zeta_3 := \dfrac{\alpha\sigma_1\bar{\chi}\,|\Omega_0|^{\dfrac{1}{\alpha}}}{\zeta^{\dfrac{\beta}{\alpha}}}$ and $\zeta_4 := \bar{\rho}_1\bar{\chi}\left(\dfrac{\bar{\rho}_1\bar{\chi}}{\zeta_3^2\rho_{-2}}\right)^\delta\kappa\,|\Omega_0|\,1-\theta$ With the help of the assumption $(Asm8)$, we have $(-\alpha\zeta_1+\beta\zeta_2) < 0$, on the other hand, since $\zeta_3, \zeta_4 > 0$ and $\theta, \dfrac{\alpha-1}{\alpha} \in (0,1)$ then according to [8], Lemma 2.2 .The required inequality (5.4), is satisfied by a positive constant $\hat{C}$.

Lemma 5.3. *Let $(\hat{a}, \hat{b})$ the solution of system (3.24) on $\Omega_0 \times [0, \bar{T}_{\max})$ then for $\tau \in [1, +\infty)$ we have :*

$$\frac{\hat{a}^p}{\hat{b}^q} \in L^\infty((0, T_{max}); L^\tau(\Omega_0)) \tag{5.12}$$

Proof. Assuming α is sufficiently large, and using Young's inequality in combination with Corollary (4.1), this yields the following estimate:

$$\begin{aligned}\int_{\Omega_0}\frac{\hat{a}^{p\tau}}{\hat{b}^{q\tau}}dy &= \int_{\Omega_0}\left(\frac{\hat{a}^{p\tau}}{\hat{b}^{q\tau}}\right)\hat{b}^{\dfrac{\beta\rho\tau}{\alpha}-q\tau}dy \\ &\leq L(t) + \int_{\Omega_0}\frac{1}{\hat{b}^{(\alpha q-\beta p)\tau(\alpha-p\tau)^{-1}}}dy \\ &\leq \hat{C} + \frac{|\Omega_0|}{\zeta^{(\alpha q-\beta p)\tau(\alpha-p\tau)^{-1}}}\end{aligned} \tag{5.13}$$

□

On $(0, T_{\max})$ hence, the desired result is established.

Proof. (of Theorem 5.1) With the use of transformations (3.21) and (3.23), it is sufficient to demonstrate that the solution $(\hat{a}, \hat{b})$ of system (3.24) meets the following estimation:

$$\forall t \in (0, T_{max}), \|\hat{a}(.,t)\|_{L^\infty(\Omega_0)} + \left\|\hat{b}(.,t)\right\|_{L^\infty(\Omega_0)} \leq \xi(t) \tag{5.14}$$

□

Where $\xi \in C(\mathbb{R}_+;\mathbb{R}_+)$, is as defined above, Lemma 5.3 and the L^P-regularity theory for the heat operator, in fact, allow us to obtain

$$\zeta_5 := sup_{t\in(0,\bar{T}_{max})}\|\hat{a}(.,t)\|_{L^\infty(\Omega_0)} < \infty \tag{5.15}$$

Let $\bar{T}^* \in (0, \bar{T}_{\max})$, the solution $\hat{A}_i$ to the initial-boundary value problem below is established to be bounded above by $\hat{a}_i$ via the comparison principle :

$$\begin{cases} \dfrac{\partial \bar{A}}{\partial t} - d_1 \nabla \hat{A} = \sigma_2 \bar{\chi}(t) + \rho_2 \bar{\chi}(t) \zeta_{\hat{a},\hat{b}} - \zeta_1 \hat{A} \;, in\ \Omega_0 \times (0, \bar{T}^*] \\ \\ \dfrac{\partial \hat{A}}{\partial \nu}(y,t) = 0, \; on \;\; \partial\Omega_0 \times \{t > 0\} \\ \\ \hat{A}(y,0) = \hat{a}_0(y) \;, \; on \; \bar{\Omega}_0 \end{cases} \tag{5.16}$$

where $\zeta_{\hat{a},\hat{b}} := \zeta_{\hat{a},\hat{b}}(\zeta, \zeta_5) > 0$. Applying the L^p-regularity theory to the heat operator once more, we obtain

$$sup_{t\in(0,\bar{T}^*]} \|\hat{a}(.,t)\|_{L^\infty(\Omega_0)} \le sup_{t\in(0,\bar{T}^*]} \left\|\hat{A}(.,t)\right\|_{L^\infty(\Omega_0)} < \infty \tag{5.17}$$

In view of (5.15) and (5.17), assertion (5.14) is verified, which completes the proof.

6. Examples and Simulation Results

This section presents specific examples of system (3.16). Numerical computations and MATLAB simulations are carried out to illustrate and confirm the theoretical results established in the previous section

Example 6.1. *In the numerical simulations of system (3.16) the parameters are selected as:*

$$\begin{Bmatrix} \sigma_1 = 3 & \mu_1 = 5 & p = 2 & q = 1 \\ \sigma_2 = 2 & \mu_2 = 3 & r = 3 & s = 0 \end{Bmatrix} \tag{6.1}$$

and

$$\rho_1(a,b) = \frac{10}{1 + 10^{-4} \times u^2}, \rho_2(a,b) = \frac{1}{2}, \forall a, b \in \mathbb{R}_+^* \tag{6.2}$$

A logistic-type function is used to describe the evolution of the domain is

$$\chi(t) = \frac{5}{1 + e^{(ln(4) - t)}}, \forall t \in \mathbb{R}_+^* \tag{6.3}$$

with the following initial values (for $y \in \Omega_0 := (1,2) \subset \mathbb{R}$)

$$\begin{cases} a_0(y) = 2.8 - 0.03cos(y) \\ b_0(y) = 2.53 - 0.03cos(y) \end{cases} \tag{6.4}$$

It is not diffcult to verify the parameters (6.1) and the functions (6.2) $-$ (6.4) satisfy the assumptions stated in Theorem 5.1.

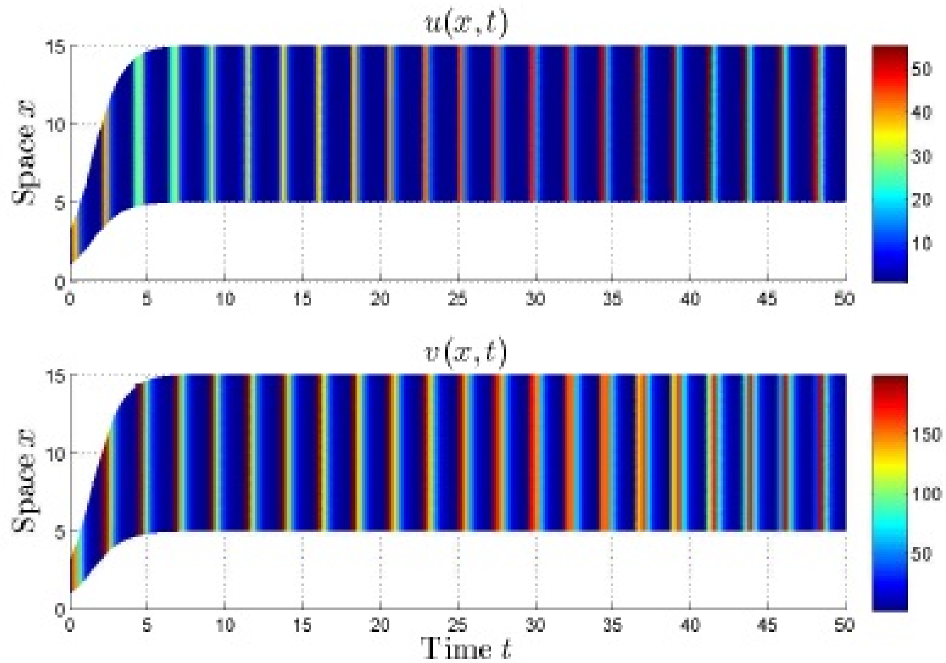

FIGURE 6.1. Numerical simulations of system (3.16) on the evolving spatial domain $\Omega_t = (\chi(t), 3\chi(t))$, subject to the parameters of Example 6.1.

Figure 6 : Approximate solution of system (3.16) over the evolving domain $\Omega_t = (\chi(t), 3\chi(t))$, using the parameter set from Example 6.1

Figure 6 presents the numerical solution of system (3.16) on a logistic-growth evolving domain according to (6.1) − (6.4), validating the theoretical existence and uniform boundedness, while highlighting distinct vertical structures.

Example 6.2. *In system* (3.16) *the parameters are selected as:*

$$\left\{ \begin{array}{llll} \sigma_1 = 1 & \mu_1 = 9 & p = 2 & q = 2 \\ \sigma_2 = 0 & \mu_2 = 10 & r = 2 & s = 1 \end{array} \right\} \tag{6.5}$$

and

$$\rho_1(a,b) = 3, \rho_2(a,b) = 2, \forall a,b \in \mathbb{R}_+^* \tag{6.6}$$

The domain evolution is assumed to follow an exponential growth function

$$\chi(t) = e^{0.03t}, \forall t \in \mathbb{R}_+^* \tag{6.7}$$

together with the initial data conditions (for $y \in \Omega_0 := (0,4) \subset \mathbb{R}$):

$$\begin{aligned} u_0(y) &= 0.38 - 0.03cos(y) \\ v_0(y) &= 0.53 - 0.03cos(y) \end{aligned} \tag{6.8}$$

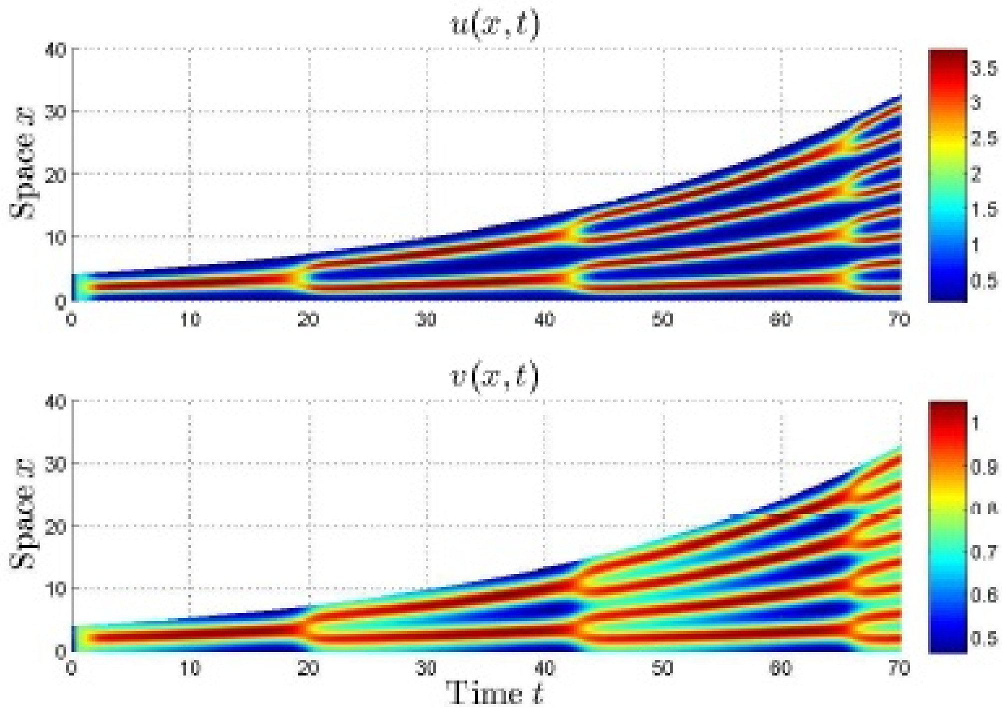

FIGURE 6.2. Numerical simulation of system (5.12) on the evolving domain $\Omega_t = (0, 4\chi(t))$, subject to the parameters of Example 6.2.

It is evident that the parameters presented in (6.5) and the functions defined in $(6.6) - (6.8)$ satisfy the assumptions of Theorem 5.1. Figure 6.2 illustrates the approximate solution of system (3.16) on an evolving domain exhibiting exponential growth, corresponding to the input data $(6.5) - (6.8)$. These numerical results validate the theoretical predictions concerning existence and uniform boundedness, and furthermore, reveal the emergence of interesting horizontal patterns.

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