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ON THE NEW q-POLYNOMIALS $\Xi_n^{\alpha-n}$ VIA THE q-OPERATOR $\tilde{E}$

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Abstract. A generalized q-operator $\tilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^{\alpha} D_q)$ is introduced as a common extension of several finite and infinite q-operators used in the theory of basic hypergeometric series. By means of this operator, we define a family of polynomials $\Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q)$ that contains, as special or limiting cases, the generalized Al-Salam–Carlitz q-polynomials and the polynomial family $h_n(\omega_1, \ldots, \omega_r; b_1, \ldots, b_s; x, y \mid q)$. We establish an operational representation for $\Xi_n^{\alpha-n}$ and prove its generating function, extended generating function, Mehler-type formula, extended Mehler-type formula, Rogers-type formula, extended Rogers-type formula, and a Srivastava–Agarwal-type generating function. To strengthen the structural theory, we also derive explicit low-degree members, a lowering q-difference relation, recurrence-type identities, parameter-shift identities, a q-Appell characterization, connection-coefficient formulas, inverse connection formulas, and Jackson q-integral evaluations. Suitable parameter specializations then recover the corresponding known identities for the above two polynomial families.

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1. Introduction

Throughout this paper we assume $|q| < 1$ and use the standard notation of basic hypergeometric series. The notation and convergence conventions for q-shifted factorials, basic hypergeometric series, Cauchy's identity, and Euler's identity follow Gasper and Rahman [13]. The q-derivative and the q-Leibniz rule are used in the standard form appearing in the q-umbral and q-operator literature [1,2,5, 14,18]. We write

$$(\wp; q)_n = \begin{cases} 1, & n = 0, \\ \prod_{k=0}^{n-1} (1 - \wp q^k), & n \in \mathbb{N}, \\ \prod_{k=0}^{\infty} (1 - \wp q^k), & n = \infty, \end{cases} \tag{1.1}$$

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and

$$(\wp_1,\dots,\wp_r;q)_n=(\wp_1;q)_n\cdots(\wp_r;q)_n,\qquad n\in\mathbb{N}_0\cup\{\infty\}. \tag{1.2}$$

Definition 1.1 (Basic hypergeometric series [3,17]). *For $r,s\in\mathbb{N}_0$, the basic hypergeometric series is defined by*

$${}_r\phi_s\left(\begin{matrix}\beta_1,\dots,\beta_r\\ \gamma_1,\dots,\gamma_s\end{matrix};q,z\right)=\sum_{\kappa=0}^{\infty}\frac{(\beta_1,\dots,\beta_r;q)_\kappa}{(q,\gamma_1,\dots,\gamma_s;q)_\kappa}\left[(-1)^\kappa q^{\binom{\kappa}{2}}\right]^{1+s-r}z^\kappa, \tag{1.3}$$

where no denominator factor is zero. The usual convergence conditions are assumed whenever the series is non-terminating.

Definition 1.2 (q-binomial coefficient [4,21]). *The q-binomial coefficient is*

$$\begin{bmatrix}\mu\\ k\end{bmatrix}_q=\begin{cases}\dfrac{(q;q)_\mu}{(q;q)_k(q;q)_{\mu-k}}=\dfrac{(q^{-\mu};q)_k}{(q;q)_k}(-1)^kq^{\mu k-\binom{k}{2}}, & 0\le k\le\mu,\\ 0, & \text{otherwise.}\end{cases} \tag{1.4}$$

Cauchy's q-binomial identity is [9,20]

$$\sum_{k=0}^{\infty}\frac{(\alpha;q)_k}{(q;q)_k}z^k=\frac{(\alpha z;q)_\infty}{(z;q)_\infty},\qquad |z|<1, \tag{1.5}$$

and Euler's special case is

$$\sum_{k=0}^{\infty}\frac{z^k}{(q;q)_k}=\frac{1}{(z;q)_\infty},\qquad |z|<1. \tag{1.6}$$

Definition 1.3 (q-derivative [23]). *The q-derivative is*

$$D_q\{f(x)\}=\frac{f(x)-f(qx)}{x}. \tag{1.7}$$

Theorem 1.4 (Leibniz rule for D_q [15]). *Let D_q be defined by (1.7). Then*

$$D_q^n\{f(x)g(x)\}=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q q^{k(k-n)}D_q^k\{f(x)\}D_q^{n-k}\{g(q^kx)\}. \tag{1.8}$$

Theorem 1.5 (Elementary q-derivative identities [7]). *The following identities hold:*

$$D_q^k\{x^n\}=\frac{(q;q)_n}{(q;q)_{n-k}}x^{n-k}, \tag{1.9}$$

$$D_q^k\left\{\frac{1}{(xt;q)_\infty}\right\}=\frac{t^k}{(xt;q)_\infty},\qquad |xt|<1, \tag{1.10}$$

$$D_q^k\left\{\frac{(xv;q)_\infty}{(xt;q)_\infty}\right\}=\frac{t^k(v/t;q)_k(xvq^k;q)_\infty}{(xt;q)_\infty},\qquad |xt|<1, \tag{1.11}$$

$$D_q^k\left\{\frac{1}{(xv,xt;q)_\infty}\right\}=\frac{1}{(xv,xt;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}_q v^{k-j}t^j(xv;q)_j, \tag{1.12}$$

where the last identity is used under the restriction $\max\{|xv|,|xt|\}<1$.

Lemma 1.6. *Let $F(x)$ be one of the analytic functions occurring in the sequel, and suppose that the indicated convergence conditions are satisfied. Then the operator $\widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^{\alpha} D_q)$ may be applied term by term to the absolutely convergent expansions used below. In particular, if a numerator parameter is of the form q^{-N}, $N \in \mathbb{N}_0$, then the operator series terminates and no interchange of infinite sums is required.*

Following Srivastava and Arjika [22], the generalized Al-Salam–Carlitz q-polynomials are

$$\phi_n^{(\mathbf{a},\mathbf{b})}(x, y \mid q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a_1, a_2, \ldots, a_{s+1}; q)_k}{(b_1, b_2, \ldots, b_s; q)_k} x^k y^{n-k}. \tag{1.13}$$

Theorem 1.7 (Known identities for generalized Al-Salam–Carlitz polynomials)**.** *Let $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y \mid q)$ be defined by (1.13). Then the following identities hold.*

(1) *The generating function is*

$$\sum_{n=0}^{\infty} \phi_n^{(\mathbf{a},\mathbf{b})}(x, y \mid q) \frac{t^n}{(q; q)_n} = \frac{1}{(yt; q)_\infty} {}_{s+1}\phi_s \left(\begin{matrix} a_1, a_2, \ldots, a_{s+1} \\ b_1, b_2, \ldots, b_s \end{matrix} ; q, xt \right), \tag{1.14}$$

where $\max\{|xt|, |yt|\} < 1$.

(2) *The Rogers formula is*

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{n+m}^{(\mathbf{a},\mathbf{b})}(x, y \mid q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\ &\quad = \frac{1}{(ys, yt; q)_\infty} \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{s+1}; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} (xt)^n {}_2\phi_0 \left(\begin{matrix} q^{-n}, yt \\ - \end{matrix} ; q, \frac{sq^n}{t} \right), \end{aligned} \tag{1.15}$$

where $\max\{|ys|, |yt|\} < 1$.

(3) *The Srivastava–Agarwal-type generating function is*

$$\sum_{n=0}^{\infty} (\lambda; q)_n \phi_n^{(\mathbf{a},\mathbf{b})}(x, y \mid q) \frac{t^n}{(q; q)_n} = \frac{(\lambda yt; q)_\infty}{(yt; q)_\infty} {}_{s+2}\phi_{s+1} \left(\begin{matrix} a_1, a_2, \ldots, a_{s+1}, \lambda \\ b_1, b_2, \ldots, b_s, \lambda yt \end{matrix} ; q, xt \right). \tag{1.16}$$

Following Saad and Hassan [19], the generalized q-polynomials $h_n(\omega_1, \ldots, \omega_r; b_1, \ldots, b_s; x, y \mid q)$ are

$$h_n(\omega_1, \ldots, \omega_r; b_1, \ldots, b_s; x, y \mid q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(\omega_1, \ldots, \omega_r; q)_k}{(b_1, \ldots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} x^{n-k} y^k. \tag{1.17}$$

For compactness, write

$$W_n = \frac{(\omega_1, \ldots, \omega_r; q)_n}{(b_1, \ldots, b_s; q)_n}, \qquad \widehat{W}_n = \frac{(\widehat{\omega}_1, \ldots, \widehat{\omega}_r; q)_n}{(\widehat{b}_1, \ldots, \widehat{b}_s; q)_n}. \tag{1.18}$$

Theorem 1.8 (Known identities for h_n)**.** *Let h_n be defined by (1.17). Then the following identities hold.*

(1) *The generating function is*

$$\sum_{n=0}^{\infty} h_n(\omega_1,\ldots,\omega_r;b_1,\ldots,b_s;x,y\mid q)\frac{t^n}{(q;q)_n}=\frac{1}{(xt;q)_\infty}{}_r\phi_s\left(\begin{matrix}\omega_1,\ldots,\omega_r\\ b_1,\ldots,b_s\end{matrix};q,yt\right), \tag{1.19}$$

where $|xt|<1$.

(2) *The extended generating function is*

$$\begin{aligned}&\sum_{n=0}^{\infty} h_{n+k}(\omega_1,\ldots,\omega_r;b_1,\ldots,b_s;x,y\mid q)\frac{t^n}{(q;q)_n}\\ &\quad=\frac{x^k}{(xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\sum_{i=0}^{k}\begin{bmatrix}k\\ i\end{bmatrix}_q W_{n+i}\left[(-1)^{n+i}q^{\binom{n+i}{2}}\right]^{1+s-r}(xt;q)_i\left(\frac{y}{x}\right)^i,\end{aligned} \tag{1.20}$$

where $|xt|<1$.

(3) *The extended generating function for the generalized Al-Salam–Carlitz q-polynomials is*

$$\begin{aligned}&\sum_{n=0}^{\infty}\phi_{n+k}^{(\mathbf{a},\mathbf{b})}(x,y\mid q)\frac{t^n}{(q;q)_n}\\ &\quad=\frac{1}{(yt;q)_\infty}\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\sum_{i=0}^{k}\begin{bmatrix}k\\ i\end{bmatrix}_q\frac{(a_1,\ldots,a_{s+1};q)_{n+i}}{(b_1,\ldots,b_s;q)_{n+i}}(yt;q)_i x^i y^{k-i},\end{aligned} \tag{1.21}$$

where $|yt|<1$.

(4) *The Mehler formula for* h_n *is*

$$\begin{aligned}&\sum_{n=0}^{\infty} h_n(\omega_1,\ldots,\omega_r;b_1,\ldots,b_s;x,y\mid q)h_n(\widehat{\omega}_1,\ldots,\widehat{\omega}_r;\widehat{b}_1,\ldots,\widehat{b}_s;z,c\mid q)\frac{t^n}{(q;q)_n}\\ &=\frac{1}{(xzt;q)_\infty}\sum_{m=0}^{\infty}\widehat{W}_m\frac{(xct)^m}{(q;q)_m}\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-r}\sum_{k=0}^{\infty}\frac{(yzt)^k}{(q;q)_k}\sum_{i=0}^{m}\begin{bmatrix}m\\ i\end{bmatrix}_q\\ &\quad\times W_{k+i}\left[(-1)^{k+i}q^{\binom{k+i}{2}}\right]^{1+s-r}(xzt;q)_i\left(\frac{y}{x}\right)^i,\end{aligned} \tag{1.22}$$

where $|xzt|<1$.

(5) *The Mehler formula for the generalized Al-Salam–Carlitz q-polynomials is*

$$\begin{aligned}&\sum_{n=0}^{\infty}\phi_n^{(\mathbf{a},\mathbf{b})}(x,y\mid q)\phi_n^{(\widehat{\mathbf{a}},\widehat{\mathbf{b}})}(z,c\mid q)\frac{t^n}{(q;q)_n}\\ &=\frac{1}{(yct;q)_\infty}\sum_{m=0}^{\infty}\frac{(\widehat{a}_1,\ldots,\widehat{a}_{s+1};q)_m}{(\widehat{b}_1,\ldots,\widehat{b}_s;q)_m}\frac{(yzt)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(xct)^k}{(q;q)_k}\\ &\quad\times\sum_{i=0}^{m}\begin{bmatrix}m\\ i\end{bmatrix}_q\frac{(a_1,\ldots,a_{s+1};q)_{k+i}}{(b_1,\ldots,b_s;q)_{k+i}}(yct;q)_i\left(\frac{x}{y}\right)^i,\end{aligned} \tag{1.23}$$

where $|yct|<1$.

(6) *The extended Mehler formula for h_n is*

$$\sum_{n=0}^{\infty} h_n(\omega_1,\dots,\omega_r;b_1,\dots,b_s;x,y\mid q)h_{n+k}(\widehat{\omega}_1,\dots,\widehat{\omega}_r;\widehat{b}_1,\dots,\widehat{b}_s;z,c\mid q)\frac{t^n}{(q;q)_n}$$
$$=\frac{1}{(xzt;q)_\infty}\sum_{j=0}^{\infty}\frac{(xct)^j}{(q;q)_j}\sum_{i=0}^{k}\begin{bmatrix}k\\ i\end{bmatrix}_q \widehat{W}_{i+j}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{1+s-r}(xzt;q)_i y^{k-i}c^i$$
$$\times\sum_{n=0}^{\infty}\frac{(yztq^i)^n}{(q;q)_n}\sum_{\ell=0}^{j}\begin{bmatrix}j\\ \ell\end{bmatrix}_q W_{n+\ell}\left[(-1)^{n+\ell}q^{\binom{n+\ell}{2}}\right]^{1+s-r}(xztq^i;q)_\ell\left(\frac{y}{x}\right)^\ell, \tag{1.24}$$

where $|xzt|<1$.

(7) *The extended Mehler formula for the generalized Al-Salam–Carlitz q-polynomials is*

$$\sum_{n=0}^{\infty}\phi_n^{(\mathbf{a},\mathbf{b})}(x,y\mid q)\phi_{n+k}^{(\widehat{\mathbf{a}},\widehat{\mathbf{b}})}(z,c\mid q)\frac{t^n}{(q;q)_n}$$
$$=\frac{1}{(yct;q)_\infty}\sum_{j=0}^{\infty}\frac{(yzt)^j}{(q;q)_j}\sum_{i=0}^{k}\begin{bmatrix}k\\ i\end{bmatrix}_q\frac{(\widehat{a}_1,\dots,\widehat{a}_{s+1};q)_{i+j}}{(\widehat{b}_1,\dots,\widehat{b}_s;q)_{i+j}}(yct;q)_i x^{k-i}z^i$$
$$\times\sum_{n=0}^{\infty}\frac{(xctq^i)^n}{(q;q)_n}\sum_{\ell=0}^{j}\begin{bmatrix}j\\ \ell\end{bmatrix}_q\frac{(a_1,\dots,a_{s+1};q)_{n+\ell}}{(b_1,\dots,b_s;q)_{n+\ell}}(yctq^i;q)_\ell, \tag{1.25}$$

where $|yct|<1$.

(8) *The Rogers formula for h_n is*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_{n+m}(\omega_1,\dots,\omega_r;b_1,\dots,b_s;x,y\mid q)\frac{t^n}{(q;q)_n}\frac{w^m}{(q;q)_m}$$
$$=\frac{1}{(xw,xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(yw)^n}{(q;q)_n}\sum_{k=0}^{\infty}W_{n+k}\frac{(yt)^k}{(q;q)_k}\left[(-1)^{n+k}q^{\binom{n+k}{2}}\right]^{1+s-r}(xw;q)_k, \tag{1.26}$$

where $\max\{|xw|,|xt|\}<1$.

(9) *The extended Rogers formula for h_n is*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_{n+m+k}(\omega_1,\dots,\omega_r;b_1,\dots,b_s;x,y\mid q)\frac{t^n}{(q;q)_n}\frac{w^m}{(q;q)_m}$$
$$=\frac{1}{(xw,xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\sum_{i=0}^{k}\begin{bmatrix}k\\ i\end{bmatrix}_q W_{n+i}\left[(-1)^{n+i}q^{\binom{n+i}{2}}\right]^{1+s-r}(xw,xt;q)_i y^i x^{k-i}$$
$$\times\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_q(xtq^i;q)_j\left(\frac{w}{t}\right)^j, \tag{1.27}$$

where $\max\{|xw|,|xt|\}<1$.

(10) *The extended Rogers formula for the generalized Al-Salam–Carlitz q-polynomials is*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi_{n+m+k}^{(\mathbf{a},\mathbf{b})}(x,y\mid q)\frac{t^n}{(q;q)_n}\frac{w^m}{(q;q)_m}$$

$$=\frac{1}{(yw,yt;q)_\infty}\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\sum_{i=0}^{k}\begin{bmatrix}k\\i\end{bmatrix}_q\frac{(a_1,\dots,a_{s+1};q)_{n+i}}{(b_1,\dots,b_s;q)_{n+i}}(yw,yt;q)_i x^i y^{k-i} \tag{1.28}$$

$$\times\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q(ytq^i;q)_j\left(\frac{w}{t}\right)^j, \tag{1.29}$$

where $\max\{|yw|,|yt|\}<1$.

Novelty and contribution. The operator and polynomial family studied in this paper are motivated by the need for a single finite-parameter framework that contains both the generalized Al-Salam–Carlitz polynomials of Srivastava and Arjika [22] and the generalized q-polynomials studied by Saad and Hassan [19]. The main novelty is not merely the introduction of an additional parameter, but the simultaneous preservation of operational, generating, Mehler-type, Rogers-type, lowering, connection, and q-integral structures. Table 1 summarizes the relation between the present work and the closest previous results.

Reference or family	Main object	Present extension
Srivastava–Arjika [22]	Generalized Al-Salam–Carlitz polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y\mid q)$	Obtained as a specialization of $\Xi_n^{\alpha-n}$ when $\alpha\to-\infty$, $r=s+1$, and the variables are interchanged.
Saad–Hassan [19]	Polynomials $h_n(\omega_1,\dots,\omega_r;b_1,\dots,b_s;x,y\mid q)$	Obtained as a normalized limiting case of the same polynomial family.
Fang, Chen–Liu, Chen–Gu and related operator work [6, 10–12]	Finite and infinite q-exponential or homogeneous operators	Recovered as special or limiting forms of $\widetilde{E}(q^{-\alpha},\boldsymbol{\tau},\boldsymbol{\xi};q,cq^{\alpha}D_q)$.
Present paper	Finite-parameter family $\Xi_n^{\alpha-n}$ generated by $\widetilde{E}$	Provides unified generating, extended generating, Mehler-type, Rogers-type, Srivastava–Agarwal-type, lowering, connection, inverse connection, and Jackson-integral formulas.

The remainder of the paper is organized as follows. Section 2 introduces the generalized q-operator $\widetilde{E}(q^{-\alpha},\boldsymbol{\tau},\boldsymbol{\xi};q,cq^{\alpha}D_q)$ and proves operator identities needed later. Section 3 defines the new polynomials $\Xi_n^{\alpha-n}$ and derives their generating function and extended generating function. Sections 4 and 5 establish

Mehler-type and Rogers-type identities and their extensions. Section 6 gives a Srivastava–Agarwal-type generating function, and Section 7 records the principal specializations.

2. The Identities of the Operator $\widetilde{E}$

Definition 2.1. *Let $\boldsymbol{\tau} = (\tau_1, \dots, \tau_r)$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_s)$. The generalized q-operator is defined by*

$$\widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^{\alpha} D_q) = \sum_{k=0}^{\infty} \frac{(q^{-\alpha}, \boldsymbol{\tau}; q)_k}{(q, \boldsymbol{\xi}; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (cq^{\alpha} D_q)^k. \tag{2.1}$$

The following specializations show that (2.1) contains several previously studied operators. Each item below is now explicitly supported by the reference in which the corresponding operator, or the corresponding operator framework, was introduced or used. The reductions are understood up to the standard harmless normalization of the scalar argument, for example replacing c by cq^{α} before taking a limiting value of α.

(1) Setting $\alpha \to -\infty$, $r = 1$, $s = 0$, and $\tau_1 = 0$ gives the q-exponential operator $T(cD_q)$ introduced by Chen and Liu [7].

(2) Setting $\alpha \to -\infty$, $r = 1$, and $s = 0$ gives the Cauchy operator $T(\tau, c; D_q)$ introduced by Chen and Gu [6]; the closely related operator ${}_1\Phi_0(\tau; -; q, -cD_q)$ was used in Fang's extension of the q-Chu–Vandermonde identity [10].

(3) Taking $r = s = 0$ gives the finite q-operator ${}_1\Phi_0(q^{-\alpha}; -; q, cq^{\alpha} D_q)$, which belongs to the finite q-differential-operator framework developed by Fang [11].

(4) Taking $r = s = 1$ gives the finite q-exponential operator ${}_2\Phi_1(q^{-\alpha}, \tau_1; \xi_1; q, cq^{\alpha} D_q)$; finite two-parameter q-exponential operators of this type were studied by Fang [11] and by Zhang and Yang [24].

(5) Setting $\alpha \to -\infty$ and $r = s + 1$ gives the generalized q-operator ${}_{s+1}\phi_s(\tau_1, \dots, \tau_{s+1}; \xi_1, \dots, \xi_s; q, cD_q)$ considered in generalized q-difference-operator work by Fang [12]; the associated homogeneous q-difference operator used for generalized Al-Salam–Carlitz polynomials is the one used by Srivastava and Arjika [22].

(6) Setting $\alpha \to -\infty$, $r = 2$, and $s = 1$ gives the generalized exponential operator with three parameters studied by Li and Tan [16].

(7) Setting $\alpha \to -\infty$ in the general case gives the generalized q-exponential operator ${}_r\Phi_s$ appearing in the operator approach to q-polynomials developed by Saad and Hassan [19].

For clarity, Table 1 records the same reference-supported correspondences in a compact form.

Remark 2.2. *All formulas are interpreted under the restrictions that no denominator factor vanishes and that all non-terminating products and series converge absolutely. The terminating case is covered automatically when one of the numerator parameters is a non-positive integral power of q.*

TABLE 1. Previously used operators recovered from (2.1).

Specialization in (2.1)	Recovered operator	Source
$\alpha \to -\infty$, $r = 1$, $s = 0$, $\tau_1 = 0$	$T(cD_q)$	Chen and Liu [7]
$\alpha \to -\infty$, $r = 1$, $s = 0$	$T(\tau, c; D_q)$ and ${}_1\Phi_0(\tau; -; q, -cD_q)$	Chen and Gu [6]; Fang [10]
$r = s = 0$	finite ${}_1\Phi_0(q^{-\alpha}; -; q, cq^{\alpha}D_q)$	Fang [11]
$r = s = 1$	finite ${}_2\Phi_1(q^{-\alpha}, \tau_1; \xi_1; q, cq^{\alpha}D_q)$	Fang [11]; Zhang and Yang [24]
$\alpha \to -\infty$, $r = s + 1$	generalized ${}_{s+1}\phi_s$ and homogeneous q-difference operators	Fang [12]; Srivastava and Arjika [22]
$\alpha \to -\infty$, $r = 2$, $s = 1$	three-parameter generalized exponential operator	Li and Tan [16]
$\alpha \to -\infty$, general r, s	generalized ${}_r\Phi_s$ operator	Saad and Hassan [19]

Theorem 2.3. *For $\kappa \in \mathbb{N}_0$,*

$$\begin{aligned}
&\widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^{\alpha}D_q)\left\{\frac{\wp^{\kappa}}{(\wp\nu, \wp v; q)_{\infty}}\right\} \\
&= \frac{1}{(\wp\nu, \wp v; q)_{\infty}} \sum_{j=0}^{\kappa}\sum_{i=0}^{\infty}\sum_{\ell=0}^{i} (\wp\nu, \wp v; q)_j \, q^{\alpha(i+j)} \frac{(q^{-\alpha}, \boldsymbol{\tau}; q)_{i+j}}{(\boldsymbol{\xi}; q)_{i+j}} \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{s-r} \frac{c^{i+j}}{(q; q)_i} \begin{bmatrix} \kappa \\ j \end{bmatrix}_q \\
&\quad \times \wp^{\kappa-j} \begin{bmatrix} i \\ \ell \end{bmatrix}_q v^{i-\ell}\nu^{\ell}(\wp v q^j; q)_{\ell}. \qquad (2.2)
\end{aligned}$$

Proof. Put

$$G(\wp) = \frac{1}{(\wp\nu, \wp v; q)_{\infty}}.$$

By Definition 2.1,

$$\begin{aligned}
&\widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^{\alpha}D_q)\left\{\frac{\wp^{\kappa}}{(\wp\nu, \wp v; q)_{\infty}}\right\} \\
&\quad = \sum_{m=0}^{\infty} \frac{(q^{-\alpha}, \boldsymbol{\tau}; q)_m}{(q, \boldsymbol{\xi}; q)_m} \left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-r} c^m q^{\alpha m} D_q^m\{\wp^{\kappa} G(\wp)\}. \qquad (2.3)
\end{aligned}$$

The q-Leibniz rule (1.8) gives

$$\begin{aligned}
D_q^m\{\wp^{\kappa} G(\wp)\} &= \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q q^{j(j-m)} D_q^j\{\wp^{\kappa}\} D_q^{m-j}\{G(q^j\wp)\} \\
&= \sum_{j=0}^{\min\{m,\kappa\}} \begin{bmatrix} m \\ j \end{bmatrix}_q q^{j(j-m)} \frac{(q; q)_{\kappa}}{(q; q)_{\kappa-j}} \wp^{\kappa-j} D_q^{m-j}\left\{\frac{1}{(\wp\nu q^j, \wp v q^j; q)_{\infty}}\right\}. \qquad (2.4)
\end{aligned}$$

Using (1.12) with $x=\wp$, v replaced by νq^j, and t replaced by vq^j, we obtain

$$D_q^{m-j}\left\{\frac{1}{(\wp\nu q^j,\wp vq^j;q)_\infty}\right\}=\frac{1}{(\wp\nu q^j,\wp vq^j;q)_\infty}\sum_{\ell=0}^{m-j}\begin{bmatrix}m-j\\ \ell\end{bmatrix}_q v^{m-j-\ell}\nu^\ell q^{j(m-j)}(\wp vq^j;q)_\ell. \tag{2.5}$$

Since

$$\frac{1}{(\wp\nu q^j,\wp vq^j;q)_\infty}=\frac{(\wp\nu,\wp v;q)_j}{(\wp\nu,\wp v;q)_\infty},$$

substitution of (2.4) and (2.5) into (2.3), followed by the change of index $m=i+j$, yields

$$\frac{1}{(\wp\nu,\wp v;q)_\infty}\sum_{j=0}^{\kappa}\sum_{i=0}^{\infty}\sum_{\ell=0}^{i}(\wp\nu,\wp v;q)_j\frac{(q^{-\alpha},\boldsymbol{\tau};q)_{i+j}}{(q,\boldsymbol{\xi};q)_{i+j}}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{1+s-r}c^{i+j}q^{\alpha(i+j)}$$
$$\times\begin{bmatrix}i+j\\ j\end{bmatrix}_q q^{-ij}\frac{(q;q)_\kappa}{(q;q)_{\kappa-j}}\wp^{\kappa-j}\begin{bmatrix}i\\ \ell\end{bmatrix}_q v^{i-\ell}\nu^\ell(\wp vq^j;q)_\ell.$$

Finally, the elementary identity

$$\frac{1}{(q;q)_{i+j}}\begin{bmatrix}i+j\\ j\end{bmatrix}_q\frac{(q;q)_\kappa}{(q;q)_{\kappa-j}}=\frac{1}{(q;q)_i}\begin{bmatrix}\kappa\\ j\end{bmatrix}_q$$

and the cancellation of the powers of q inside the standard q-binomial normalization reduce the preceding expression exactly to (2.2). This proves the theorem. □

Corollary 2.4. *Setting $\nu=0$ in (2.2) gives*

$$\widetilde{E}(q^{-\alpha},\boldsymbol{\tau},\boldsymbol{\xi};q,cq^\alpha D_q)\left\{\frac{\wp^\kappa}{(\wp v;q)_\infty}\right\}$$
$$=\frac{1}{(\wp v;q)_\infty}\sum_{j=0}^{\kappa}\sum_{i=0}^{\infty}(\wp v;q)_j q^{\alpha(i+j)}\frac{(q^{-\alpha},\boldsymbol{\tau};q)_{i+j}}{(\boldsymbol{\xi};q)_{i+j}}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{s-r}\frac{c^{i+j}}{(q;q)_i}\begin{bmatrix}\kappa\\ j\end{bmatrix}_q\wp^{\kappa-j}v^i. \tag{2.6}$$

Corollary 2.5. *Setting $\kappa=0$ in (2.2) gives*

$$\widetilde{E}(q^{-\alpha},\boldsymbol{\tau},\boldsymbol{\xi};q,cq^\alpha D_q)\left\{\frac{1}{(\wp\nu,\wp v;q)_\infty}\right\}$$
$$=\frac{1}{(\wp\nu,\wp v;q)_\infty}\sum_{i=0}^{\infty}\sum_{\ell=0}^{i}q^{\alpha i}\frac{(q^{-\alpha},\boldsymbol{\tau};q)_i}{(\boldsymbol{\xi};q)_i}\left[(-1)^i q^{\binom{i}{2}}\right]^{s-r}\frac{c^i}{(q;q)_i}\begin{bmatrix}i\\ \ell\end{bmatrix}_q v^{i-\ell}\nu^\ell(\wp v;q)_\ell$$
$$=\frac{1}{(\wp\nu,\wp v;q)_\infty}\sum_{\ell=0}^{\infty}\sum_{i=0}^{\infty}(\wp v;q)_\ell\frac{(q^{-\alpha},\boldsymbol{\tau};q)_{i+\ell}}{(\boldsymbol{\xi};q)_{i+\ell}}\left[(-1)^{i+\ell}q^{\binom{i+\ell}{2}}\right]^{s-r}\frac{(cvq^\alpha)^i}{(q;q)_i}\frac{(c\nu q^\alpha)^\ell}{(q;q)_\ell}. \tag{2.7}$$

3. The Generating Function for $\Xi_n^{\alpha-n}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)$

Definition 3.1. *The polynomials $\Xi_n^{\alpha-n}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)$ are defined by*

$$\Xi_n^{\alpha-n}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q\frac{(q^{-\alpha},\boldsymbol{\tau};q)_k}{(\boldsymbol{\xi};q)_k}q^{\alpha k}c^k\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\wp^{n-k}. \tag{3.1}$$

Theorem 3.2. *For $n \in \mathbb{N}_0$,*

$$\widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^{\alpha} D_q)\{\wp^n\} = \Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q). \tag{3.2}$$

Proof. Starting from Definition 2.1, we have

$$\widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^{\alpha} D_q)\{\wp^n\} = \sum_{k=0}^{\infty} \frac{(q^{-\alpha}, \boldsymbol{\tau}; q)_k}{(q, \boldsymbol{\xi}; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} c^k q^{\alpha k} D_q^k\{\wp^n\}.$$

The derivative $D_q^k\{\wp^n\}$ is zero for $k > n$. Therefore the infinite operator sum truncates at $k = n$. By (1.9),

$$\begin{aligned}\widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^{\alpha} D_q)\{\wp^n\} &= \sum_{k=0}^{n} \frac{(q^{-\alpha}, \boldsymbol{\tau}; q)_k}{(q, \boldsymbol{\xi}; q)_k} q^{\alpha k} c^k \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} \frac{(q; q)_n}{(q; q)_{n-k}} \wp^{n-k} \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{-\alpha}, \boldsymbol{\tau}; q)_k}{(\boldsymbol{\xi}; q)_k} q^{\alpha k} c^k \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} \wp^{n-k},\end{aligned}$$

where we used $(q, \boldsymbol{\xi}; q)_k = (q; q)_k (\boldsymbol{\xi}; q)_k$. The last expression is exactly the defining sum (3.1). □

The limiting case $\alpha \to -\infty$ gives the polynomial family (1.17); after additionally taking $r = s + 1$ and interchanging the two polynomial variables, one obtains (1.13).

Theorem 3.3 (Generating function)**.** *For $|\wp t| < 1$,*

$$\sum_{n=0}^{\infty} \Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) \frac{t^n}{(q; q)_n} = \frac{1}{(\wp t; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(q^{-\alpha}, \boldsymbol{\tau}; q)_k}{(q, \boldsymbol{\xi}; q)_k} q^{\alpha k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (ct)^k. \tag{3.3}$$

Proof. By (3.2),

$$\sum_{n=0}^{\infty} \Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) \frac{t^n}{(q; q)_n} = \sum_{n=0}^{\infty} \widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^{\alpha} D_q)\{\wp^n\} \frac{t^n}{(q; q)_n}.$$

Lemma 1.6 permits moving the operator outside the absolutely convergent series. Hence

$$\begin{aligned}\sum_{n=0}^{\infty} \Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) \frac{t^n}{(q; q)_n} &= \widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^{\alpha} D_q) \left\{ \sum_{n=0}^{\infty} \frac{(\wp t)^n}{(q; q)_n} \right\} \\ &= \widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^{\alpha} D_q) \left\{ \frac{1}{(\wp t; q)_{\infty}} \right\}.\end{aligned}$$

Using (1.10),

$$(cq^{\alpha} D_q)^k \left\{ \frac{1}{(\wp t; q)_{\infty}} \right\} = c^k q^{\alpha k} \frac{t^k}{(\wp t; q)_{\infty}}.$$

Substitution into Definition 2.1 gives (3.3). □

Theorem 3.4 (Extended generating function)**.** *Let $m \in \mathbb{N}_0$ and $|\wp t| < 1$. Then*

$$\sum_{n=0}^{\infty} \Xi_{n+m}^{\alpha-n-m}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) \frac{t^n}{(q; q)_n}$$

$$= \frac{1}{(\wp t;q)_\infty} \sum_{j=0}^{m} \sum_{i=0}^{\infty} \begin{bmatrix} m \\ j \end{bmatrix}_q \frac{(q^{-\alpha}, \boldsymbol{\tau}; q)_{i+j}}{(\boldsymbol{\xi}; q)_{i+j}} \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{s-r} (\wp t; q)_j q^{\alpha(i+j)} \frac{(ct)^i c^j}{(q;q)_i} \wp^{m-j}. \tag{3.4}$$

Proof. From (3.2) and Euler's identity,

$$\begin{aligned} \sum_{n=0}^{\infty} \Xi_{n+m}^{\alpha-n-m}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) \frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty} \widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^\alpha D_q)\{\wp^{n+m}\} \frac{t^n}{(q;q)_n} \\ &= \widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^\alpha D_q) \left\{ \wp^m \sum_{n=0}^{\infty} \frac{(\wp t)^n}{(q;q)_n} \right\} \\ &= \widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^\alpha D_q) \left\{ \frac{\wp^m}{(\wp t;q)_\infty} \right\}. \end{aligned}$$

Now apply Corollary 2.4 with $\kappa = m$ and $v = t$. The resulting expression is exactly (3.4). □

4. Mehler's Formula for $\Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q)$

Theorem 4.1 (Mehler-type formula)**.** *For* $|\wp z t| < 1$,

$$\begin{aligned} &\sum_{n=0}^{\infty} \Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) \Xi_n^{\beta-n}(z, \boldsymbol{\tau}'; \boldsymbol{\xi}', f \mid q) \frac{t^n}{(q;q)_n} \\ &= \frac{1}{(\wp z t;q)_\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{i=0}^{\infty} \frac{(q^{-\beta}, \boldsymbol{\tau}'; q)_k}{(q, \boldsymbol{\xi}'; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{s-r} \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{s-r} (ftq^\beta)^k \frac{(q^{-\alpha}, \boldsymbol{\tau}; q)_{i+j}}{(\boldsymbol{\xi}; q)_{i+j}} \\ &\quad \times (\wp z t; q)_j q^{\alpha(i+j)} \frac{c^{i+j}}{(q;q)_i} \begin{bmatrix} k \\ j \end{bmatrix}_q \wp^{k-j} (zt)^i. \end{aligned} \tag{4.1}$$

Proof. By the operational representation (3.2), the first polynomial in the left-hand side can be written as

$$\Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) = \widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^\alpha D_q)\{\wp^n\}.$$

Therefore, using Lemma 1.6,

$$\begin{aligned} &\sum_{n=0}^{\infty} \Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) \Xi_n^{\beta-n}(z, \boldsymbol{\tau}'; \boldsymbol{\xi}', f \mid q) \frac{t^n}{(q;q)_n} \\ &\quad = \widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^\alpha D_q) \left\{ \sum_{n=0}^{\infty} \Xi_n^{\beta-n}(z, \boldsymbol{\tau}'; \boldsymbol{\xi}', f \mid q) \frac{(\wp t)^n}{(q;q)_n} \right\}. \end{aligned}$$

Now apply the generating function (3.3) to the second polynomial, with the replacements

$$(\alpha, \boldsymbol{\tau}, \boldsymbol{\xi}, c, \wp, t) \mapsto (\beta, \boldsymbol{\tau}', \boldsymbol{\xi}', f, z, \wp t).$$

This gives

$$\sum_{n=0}^{\infty} \Xi_n^{\beta-n}(z, \boldsymbol{\tau}'; \boldsymbol{\xi}', f \mid q) \frac{(\wp t)^n}{(q;q)_n} = \frac{1}{(z\wp t;q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-\beta}, \boldsymbol{\tau}'; q)_k}{(q, \boldsymbol{\xi}'; q)_k} q^{\beta k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (f\wp t)^k.$$

The scalar factors independent of the operator may be pulled outside, and it remains to evaluate

$$\widetilde{E}(q^{-\alpha},\boldsymbol{\tau},\boldsymbol{\xi};q,cq^{\alpha}D_q)\left\{\frac{\wp^k}{(z\wp t;q)_\infty}\right\}.$$

Corollary 2.4 applies with $\kappa=k$ and $v=zt$. Hence

$$\begin{aligned}&\widetilde{E}(q^{-\alpha},\boldsymbol{\tau},\boldsymbol{\xi};q,cq^{\alpha}D_q)\left\{\frac{\wp^k}{(z\wp t;q)_\infty}\right\}\\&=\frac{1}{(\wp zt;q)_\infty}\sum_{j=0}^{k}\sum_{i=0}^{\infty}(\wp zt;q)_j q^{\alpha(i+j)}\frac{(q^{-\alpha},\boldsymbol{\tau};q)_{i+j}}{(\boldsymbol{\xi};q)_{i+j}}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{s-r}\frac{c^{i+j}}{(q;q)_i}\begin{bmatrix}k\\j\end{bmatrix}_q\wp^{k-j}(zt)^i.\end{aligned}$$

Multiplying this expression by the outer coefficient

$$\frac{(q^{-\beta},\boldsymbol{\tau}';q)_k}{(q,\boldsymbol{\xi}';q)_k}q^{\beta k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(ft)^k$$

and using $q^{\beta k}(ft)^k=(ftq^{\beta})^k$ gives exactly (4.1). □

Theorem 4.2 (Extended Mehler-type formula)**.** *Let $m\in\mathbb{N}_0$ and $|\wp zt|<1$. Then*

$$\begin{aligned}&\sum_{n=0}^{\infty}\Xi_n^{\alpha-n}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)\Xi_{n+m}^{\beta-n-m}(z,\boldsymbol{\tau}';\boldsymbol{\xi}',f\mid q)\frac{t^n}{(q;q)_n}\\&=\sum_{j=0}^{m}\sum_{i=0}^{\infty}\begin{bmatrix}m\\j\end{bmatrix}_q\frac{(q^{-\beta},\boldsymbol{\tau}';q)_{i+j}}{(\boldsymbol{\xi}';q)_{i+j}}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{s-r}q^{\beta(i+j)}\frac{(ft)^i(fq^i)^j}{(q;q)_i}z^{m-j}\\&\quad\times\frac{1}{(z\wp tq^j;q)_\infty}\sum_{a=0}^{i}\sum_{b=0}^{\infty}(z\wp tq^j;q)_a q^{\alpha(a+b)}\frac{(q^{-\alpha},\boldsymbol{\tau};q)_{a+b}}{(\boldsymbol{\xi};q)_{a+b}}\left[(-1)^{a+b}q^{\binom{a+b}{2}}\right]^{s-r}\frac{c^{a+b}}{(q;q)_b}\begin{bmatrix}i\\a\end{bmatrix}_q\\&\quad\times\wp^{i-a}(ztq^j)^b.\end{aligned}\tag{4.2}$$

Proof. By (3.2), the left-hand side can be written as

$$\widetilde{E}(q^{-\alpha},\boldsymbol{\tau},\boldsymbol{\xi};q,cq^{\alpha}D_q)\left\{\sum_{n=0}^{\infty}\Xi_{n+m}^{\beta-n-m}(z,\boldsymbol{\tau}';\boldsymbol{\xi}',f\mid q)\frac{(\wp t)^n}{(q;q)_n}\right\}.$$

The extended generating function (3.4), applied to the second polynomial and with t replaced by $\wp t$, gives

$$\begin{aligned}&\sum_{n=0}^{\infty}\Xi_{n+m}^{\beta-n-m}(z,\boldsymbol{\tau}';\boldsymbol{\xi}',f\mid q)\frac{(\wp t)^n}{(q;q)_n}\\&=\sum_{j=0}^{m}\sum_{i=0}^{\infty}\begin{bmatrix}m\\j\end{bmatrix}_q\frac{(q^{-\beta},\boldsymbol{\tau}';q)_{i+j}}{(\boldsymbol{\xi}';q)_{i+j}}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{s-r}q^{\beta(i+j)}\frac{(f\wp t)^i f^j}{(q;q)_i}(z\wp t;q)_j z^{m-j}\frac{1}{(z\wp t;q)_\infty}.\end{aligned}$$

Since $(z\wp t;q)_j/(z\wp t;q)_\infty=1/(z\wp tq^j;q)_\infty$, each summand requires the operator to act on

$$\frac{\wp^i}{(z\wp tq^j;q)_\infty}.$$

Corollary 2.4, with $\kappa = i$ and $v = ztq^j$, gives the inner double sum over a and b. Collecting the scalar factors outside the operator yields (4.2). □

5. Rogers' Formula for $\Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q)$

Theorem 5.1 (Rogers-type formula). *If* $\max\{|\wp t|, |\wp s|\} < 1$, *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Xi_{n+m}^{\alpha-n-m}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}$$
$$= \frac{1}{(\wp t, \wp s; q)_\infty} \sum_{\ell=0}^{\infty}\sum_{i=0}^{\infty} (\wp s; q)_\ell \frac{(q^{-\alpha}, \boldsymbol{\tau}; q)_{i+\ell}}{(\boldsymbol{\xi}; q)_{i+\ell}} \left[(-1)^{i+\ell} q^{\binom{i+\ell}{2}}\right]^{s-r} \frac{(csq^\alpha)^i}{(q;q)_i} \frac{(ctq^\alpha)^\ell}{(q;q)_\ell}. \tag{5.1}$$

Proof. Using (3.2) and Euler's identity twice,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Xi_{n+m}^{\alpha-n-m}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}$$
$$= \widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^\alpha D_q) \left\{ \sum_{n=0}^{\infty} \frac{(\wp t)^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(\wp s)^m}{(q;q)_m} \right\}$$
$$= \widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^\alpha D_q) \left\{ \frac{1}{(\wp t, \wp s; q)_\infty} \right\}.$$

The restriction $\max\{|\wp t|, |\wp s|\} < 1$ guarantees the required convergence. Applying Corollary 2.5 with $\nu = t$ and $v = s$ gives (5.1). □

Theorem 5.2 (Extended Rogers-type formula). *Let* $\kappa \in \mathbb{N}_0$ *and* $\max\{|\wp\nu|, |\wp v|\} < 1$. *Then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Xi_{n+m+\kappa}^{\alpha-n-m-\kappa}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) \frac{\nu^n}{(q;q)_n} \frac{v^m}{(q;q)_m}$$
$$= \frac{1}{(\wp\nu, \wp v; q)_\infty} \sum_{j=0}^{\kappa}\sum_{i=0}^{\infty}\sum_{\ell=0}^{i} (\wp\nu, \wp v; q)_j q^{\alpha(i+j)} \frac{(q^{-\alpha}, \boldsymbol{\tau}; q)_{i+j}}{(\boldsymbol{\xi}; q)_{i+j}} \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{s-r} \frac{c^{i+j}}{(q;q)_i} \begin{bmatrix} \kappa \\ j \end{bmatrix}_q$$
$$\times \wp^{\kappa-j} \begin{bmatrix} i \\ \ell \end{bmatrix}_q v^{i-\ell} \nu^\ell (\wp v q^j; q)_\ell. \tag{5.2}$$

Proof. By the operational representation,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Xi_{n+m+\kappa}^{\alpha-n-m-\kappa}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) \frac{\nu^n}{(q;q)_n} \frac{v^m}{(q;q)_m}$$
$$= \widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^\alpha D_q) \left\{ \wp^\kappa \sum_{n=0}^{\infty} \frac{(\wp\nu)^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(\wp v)^m}{(q;q)_m} \right\}$$
$$= \widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^\alpha D_q) \left\{ \frac{\wp^\kappa}{(\wp\nu, \wp v; q)_\infty} \right\}.$$

Theorem 2.3 is exactly the evaluation of this last expression, and therefore (5.2) follows. □

6. Srivastava–Agarwal-Type Generating Function for $\Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q)$

Theorem 6.1. *For $|\wp t| < 1$,*

$$\sum_{n=0}^{\infty} \Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q)(\lambda; q)_n \frac{t^n}{(q; q)_n}$$
$$= \frac{(\lambda\wp t; q)_\infty}{(\wp t; q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-\alpha}, \lambda, \boldsymbol{\tau}; q)_k}{(q, \lambda\wp t, \boldsymbol{\xi}; q)_k} q^{\alpha k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (ct)^k. \tag{6.1}$$

Equivalently, the right-hand side is

$$\frac{(\lambda\wp t; q)_\infty}{(\wp t; q)_\infty} \widetilde{E}(q^{-\alpha}, \lambda, \boldsymbol{\tau}, \lambda\wp t, \boldsymbol{\xi}; q, ct),$$

where the latter notation means the corresponding basic hypergeometric series in the scalar argument ct.

Proof. By the operator representation (3.2),

$$\sum_{n=0}^{\infty} \Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q)(\lambda; q)_n \frac{t^n}{(q; q)_n} = \widetilde{E}(q^{-\alpha}, \boldsymbol{\tau}, \boldsymbol{\xi}; q, cq^\alpha D_q) \left\{ \sum_{n=0}^{\infty} (\lambda; q)_n \frac{(\wp t)^n}{(q; q)_n} \right\}.$$

Cauchy's identity (1.5) gives

$$\sum_{n=0}^{\infty} (\lambda; q)_n \frac{(\wp t)^n}{(q; q)_n} = \frac{(\lambda\wp t; q)_\infty}{(\wp t; q)_\infty}.$$

Using (1.11) with $v = \lambda t$ gives

$$D_q^k \left\{ \frac{(\lambda\wp t; q)_\infty}{(\wp t; q)_\infty} \right\} = t^k (\lambda; q)_k \frac{(\lambda\wp t q^k; q)_\infty}{(\wp t; q)_\infty} = t^k \frac{(\lambda; q)_k}{(\lambda\wp t; q)_k} \frac{(\lambda\wp t; q)_\infty}{(\wp t; q)_\infty}.$$

Substitution into Definition 2.1 yields

$$\frac{(\lambda\wp t; q)_\infty}{(\wp t; q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-\alpha}, \boldsymbol{\tau}; q)_k}{(q, \boldsymbol{\xi}; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} c^k q^{\alpha k} t^k \frac{(\lambda; q)_k}{(\lambda\wp t; q)_k},$$

which is exactly (6.1). □

7. Further Structural Identities and Examples

The preceding sections give the main generating, Mehler-type, Rogers-type, and Srivastava–Agarwal-type identities. In this section we record additional structural consequences of Definition 3.1. These results are useful for comparing the family $\Xi_n^{\alpha-n}$ with more classical q-polynomial systems, and they also provide simple checks on the parameters appearing in the previous identities.

For convenience, put

$$A_k(\alpha, \boldsymbol{\tau}, \boldsymbol{\xi}, c) = \frac{(q^{-\alpha}, \boldsymbol{\tau}; q)_k}{(\boldsymbol{\xi}; q)_k} q^{\alpha k} c^k \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}, \qquad k \in \mathbb{N}_0. \tag{7.1}$$

Then Definition 3.1 can be written in the compact form

$$\Xi_n^{\alpha-n}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q A_k(\alpha,\boldsymbol{\tau},\boldsymbol{\xi},c)\wp^{n-k}. \tag{7.2}$$

The coefficient sequence $\{A_k\}_{k\geq 0}$ is the basic structural datum of the family. It separates the polynomial variable $\wp$ from the parameter arrays $\boldsymbol{\tau}$ and $\boldsymbol{\xi}$, and this separation is responsible for the q-Appell-type properties below.

Theorem 7.1 (q-Appell characterization). *Let $\{P_n(\wp)\}_{n\geq 0}$ be a polynomial sequence with $\deg P_n = n$ and $P_0 = 1$. Suppose that its exponential generating function has the form*

$$\sum_{n=0}^{\infty}P_n(\wp)\frac{t^n}{(q;q)_n}=\frac{F(t)}{(\wp t;q)_\infty},\qquad F(0)=1. \tag{7.3}$$

Then

$$D_q\{P_n(\wp)\}=(1-q^n)P_{n-1}(\wp),\qquad n\geq 1. \tag{7.4}$$

Conversely, if a sequence satisfies (7.4) and has an absolutely convergent generating function in a neighbourhood of $t=0$, then its generating function is of the form (7.3). In particular, $\Xi_n^{\alpha-n}$ is a q-Appell sequence with

$$F_{\alpha,\tau,\xi,c}(t)=\sum_{k=0}^{\infty}\frac{(q^{-\alpha},\tau;q)_k}{(q,\xi;q)_k}q^{\alpha k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(ct)^k. \tag{7.5}$$

Proof. Apply D_q with respect to $\wp$ to both sides of (7.3). Since

$$D_q\left\{\frac{1}{(\wp t;q)_\infty}\right\}=\frac{t}{(\wp t;q)_\infty},$$

we obtain

$$\sum_{n=0}^{\infty}D_q\{P_n(\wp)\}\frac{t^n}{(q;q)_n}=t\sum_{n=0}^{\infty}P_n(\wp)\frac{t^n}{(q;q)_n}.$$

The right-hand side is

$$\sum_{n=1}^{\infty}(1-q^n)P_{n-1}(\wp)\frac{t^n}{(q;q)_n},$$

because $(q;q)_n=(1-q^n)(q;q)_{n-1}$. Equating coefficients gives (7.4). Conversely, if (7.4) holds, then applying D_q to the generating function $G(\wp,t)$ gives $D_qG=tG$. The analytic solutions of this first-order q-difference equation in $\wp$ are precisely $G(\wp,t)=F(t)/(\wp t;q)_\infty$, where F is independent of $\wp$. For $\Xi_n^{\alpha-n}$, Theorem 3.3 gives the stated F. □

Theorem 7.2 (Connection-coefficient formula). *Let $\Xi_n^A(\wp)$ and $\Xi_n^B(\wp)$ denote two polynomial families of the form (7.2), with coefficient-generating functions*

$$F_A(t)=\sum_{k=0}^{\infty}A_k\frac{t^k}{(q;q)_k},\qquad F_B(t)=\sum_{k=0}^{\infty}B_k\frac{t^k}{(q;q)_k}, \tag{7.6}$$

where $F_B(0) = 1$. Define the connection coefficients $C_m(A, B)$ by

$$\frac{F_A(t)}{F_B(t)} = \sum_{m=0}^{\infty} C_m(A, B) \frac{t^m}{(q; q)_m}. \tag{7.7}$$

Then, for every $n \in \mathbb{N}_0$,

$$\Xi_n^A(\wp) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q C_m(A, B) \Xi_{n-m}^B(\wp). \tag{7.8}$$

Proof. By the generating functions,

$$\sum_{n=0}^{\infty} \Xi_n^A(\wp) \frac{t^n}{(q; q)_n} = \frac{F_A(t)}{(\wp t; q)_\infty} = \left(\frac{F_A(t)}{F_B(t)}\right) \left(\frac{F_B(t)}{(\wp t; q)_\infty}\right).$$

Using (7.7) and the generating function of Ξ_n^B, the product becomes

$$\left(\sum_{m=0}^{\infty} C_m(A, B) \frac{t^m}{(q; q)_m}\right) \left(\sum_{j=0}^{\infty} \Xi_j^B(\wp) \frac{t^j}{(q; q)_j}\right).$$

The Cauchy product of these two q-exponential series gives

$$\sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q C_m(A, B) \Xi_{n-m}^B(\wp)\right) \frac{t^n}{(q; q)_n}.$$

Equating coefficients proves (7.8). □

Corollary 7.3 (Inverse connection formula)**.** *With the notation of Theorem 7.2, define $D_m(A, B)$ by*

$$\frac{F_B(t)}{F_A(t)} = \sum_{m=0}^{\infty} D_m(A, B) \frac{t^m}{(q; q)_m}. \tag{7.9}$$

Then

$$\Xi_n^B(\wp) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q D_m(A, B) \Xi_{n-m}^A(\wp). \tag{7.10}$$

Moreover, the two coefficient arrays are inverse in the sense that

$$\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q C_m(A, B) D_{n-m}(A, B) = \delta_{n,0}. \tag{7.11}$$

Proof. Formula (7.10) follows from Theorem 7.2 after interchanging A and B. Multiplying the two quotient series F_A/F_B and F_B/F_A gives 1. Comparing the coefficient of $t^n/(q; q)_n$ in the product gives (7.11). □

Corollary 7.4 (Expansion in monomials and inverse monomial expansion)**.** *Taking $B_m = 0$ for $m \geq 1$ gives $\Xi_n^B(\wp) = \wp^n$ and hence*

$$\Xi_n^{\alpha-n}(\wp, \tau; \xi, c \mid q) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q A_m(\alpha, \tau, \xi, c) \wp^{n-m}. \tag{7.12}$$

If R_m is defined by

$$\frac{1}{F_{\alpha,\tau,\xi,c}(t)} = \sum_{m=0}^{\infty} R_m \frac{t^m}{(q;q)_m}, \tag{7.13}$$

then the inverse expansion is

$$\wp^n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q R_m \Xi_{n-m}^{\alpha-n+m}(\wp, \tau; \xi, c \mid q). \tag{7.14}$$

Proof. The first formula is exactly (7.2). The inverse formula follows from Corollary 7.3 with B equal to the monomial sequence. The shift in the superscript reflects the fact that the degree of the polynomial in the last factor is $n-m$. □

Proposition 7.5 (First members of the family)**.** *The first members of the polynomial sequence are*

$$\Xi_0^{\alpha}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) = 1, \tag{7.15}$$

$$\Xi_1^{\alpha-1}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) = \wp + A_1, \tag{7.16}$$

$$\Xi_2^{\alpha-2}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) = \wp^2 + (1+q)A_1\wp + A_2, \tag{7.17}$$

$$\Xi_3^{\alpha-3}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) = \wp^3 + (1+q+q^2)A_1\wp^2 + (1+q+q^2)A_2\wp + A_3, \tag{7.18}$$

$$\begin{aligned}\Xi_4^{\alpha-4}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) = {} & \wp^4 + (1+q+q^2+q^3)A_1\wp^3 + (1+q+2q^2+q^3+q^4)A_2\wp^2 \\ & + (1+q+q^2+q^3)A_3\wp + A_4,\end{aligned} \tag{7.19}$$

where $A_k = A_k(\alpha, \boldsymbol{\tau}, \boldsymbol{\xi}, c)$ is given by (7.1).

Proof. These formulas follow by substituting $n = 0, 1, 2, 3, 4$ in (7.2) and using the elementary values

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = 1+q, \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = 1+q+q^2,$$

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q = \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q = 1+q+q^2+q^3, \quad \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1+q+2q^2+q^3+q^4.$$

No limiting or convergence argument is needed, since each polynomial is a finite sum. □

Theorem 7.6 (Lowering q-difference relation)**.** *For $n \in \mathbb{N}$,*

$$D_q\left\{\Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q)\right\} = (1-q^n)\Xi_{n-1}^{\alpha-n+1}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q). \tag{7.20}$$

Equivalently,

$$\frac{\Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q) - \Xi_n^{\alpha-n}(q\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q)}{\wp} = (1-q^n)\Xi_{n-1}^{\alpha-n+1}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q). \tag{7.21}$$

Proof. Using (7.2) and the monomial derivative (1.9), we get

$$D_q\left\{\Xi_n^{\alpha-n}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)\right\}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q A_k D_q\{\wp^{n-k}\}$$
$$=\sum_{k=0}^{n-1}\begin{bmatrix}n\\k\end{bmatrix}_q A_k(1-q^{n-k})\wp^{n-k-1}.$$

Since

$$\begin{bmatrix}n\\k\end{bmatrix}_q(1-q^{n-k})=(1-q^n)\begin{bmatrix}n-1\\k\end{bmatrix}_q,$$

we obtain

$$D_q\left\{\Xi_n^{\alpha-n}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)\right\}=(1-q^n)\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_q A_k\wp^{n-1-k}$$
$$=(1-q^n)\Xi_{n-1}^{\alpha-n+1}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q),$$

which proves (7.20). The equivalent form (7.21) follows directly from the definition of D_q. □

Corollary 7.7 (Higher lowering relation)**.** *If* $0\le m\le n$*, then*

$$D_q^m\left\{\Xi_n^{\alpha-n}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)\right\}=\frac{(q;q)_n}{(q;q)_{n-m}}\,\Xi_{n-m}^{\alpha-n+m}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q).\tag{7.22}$$

In particular, $D_q^n\{\Xi_n^{\alpha-n}\}=(q;q)_n$.

Proof. Apply Theorem 7.6 successively m times. The scalar factors multiply as

$$(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})=\frac{(q;q)_n}{(q;q)_{n-m}}.$$

The remaining polynomial has degree $n-m$, giving (7.22). The case $m=n$ leaves $\Xi_0^{\alpha}=1$. □

Theorem 7.8 (Raising recurrence in the degree)**.** *For* $n\in\mathbb{N}_0$*,*

$$\Xi_{n+1}^{\alpha-n-1}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)=\wp\,\Xi_n^{\alpha-n}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)$$
$$+\sum_{j=0}^{n}q^{n-j}\begin{bmatrix}n\\j\end{bmatrix}_q A_{j+1}(\alpha,\boldsymbol{\tau},\boldsymbol{\xi},c)\wp^{n-j}.\tag{7.23}$$

Equivalently, since

$$\frac{A_{j+1}(\alpha,\boldsymbol{\tau},\boldsymbol{\xi},c)}{A_j(\alpha,\boldsymbol{\tau},\boldsymbol{\xi},c)}=q^{\alpha}c\,\frac{(1-q^{-\alpha+j})(1-\tau_1q^j)\cdots(1-\tau_rq^j)}{(1-\xi_1q^j)\cdots(1-\xi_sq^j)}\left((-1)^jq^j\right)^{1+s-r},\tag{7.24}$$

formula (7.23) *can be written entirely in terms of the coefficient sequence* A_j.

Proof. Start from (7.2) with n replaced by $n+1$:

$$\Xi_{n+1}^{\alpha-n-1} = \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q A_k \wp^{n+1-k}.$$

Use the standard q-Pascal relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

The part containing $\begin{bmatrix} n \\ k \end{bmatrix}_q$ gives

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A_k \wp^{n+1-k} = \wp \Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q).$$

For the second part put $j = k-1$. Then

$$\sum_{k=1}^{n+1} q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q A_k \wp^{n+1-k} = \sum_{j=0}^{n} q^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_q A_{j+1} \wp^{n-j},$$

which proves (7.23). Formula (7.24) follows immediately from the definition (7.1) by dividing A_{j+1} by A_j. □

Remark 7.9. *The recurrence (7.23) is a general raising relation for the full parameter family. A simple universal three-term recurrence with coefficients depending only on n is not expected for arbitrary parameter arrays, because the quotient $(q^{-\alpha}, \boldsymbol{\tau}; q)_k/(\boldsymbol{\xi}; q)_k$ is completely general. Under special choices of $\boldsymbol{\tau}$ and $\boldsymbol{\xi}$, however, the raising relation may simplify to shorter recurrences.*

Definition 7.10 (Jackson q-integral)**.** *For $0 < |q| < 1$ and a function f for which the series converges, the Jackson q-integral over $[0, a]$ is*

$$\int_0^a f(x)\, d_q x = (1-q)a \sum_{m=0}^{\infty} q^m f(aq^m). \tag{7.25}$$

In particular, for $N \in \mathbb{N}_0$,

$$\int_0^a x^N\, d_q x = \frac{(1-q)a^{N+1}}{1-q^{N+1}}. \tag{7.26}$$

Theorem 7.11 (Jackson q-integral of $\Xi_n^{\alpha-n}$)**.** *For $n \in \mathbb{N}_0$,*

$$\int_0^a \Xi_n^{\alpha-n}(x, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q)\, d_q x = (1-q) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A_k(\alpha, \boldsymbol{\tau}, \boldsymbol{\xi}, c) \frac{a^{n-k+1}}{1-q^{n-k+1}}. \tag{7.27}$$

Proof. The polynomial is a finite sum, so the Jackson integral may be applied term by term. Substituting (7.2) gives

$$\int_0^a \Xi_n^{\alpha-n}(x,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)\,d_qx=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q A_k\int_0^a x^{n-k}\,d_qx.$$

Using (7.26) with $N=n-k$ proves (7.27). □

Corollary 7.12 (Generating function for the Jackson integrals)**.** *If $|at|<1$ and the relevant non-terminating series converges, then*

$$\sum_{n=0}^{\infty}\left(\int_0^a \Xi_n^{\alpha-n}(x,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)\,d_qx\right)\frac{t^n}{(q;q)_n}=\int_0^a\frac{1}{(xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^{-\alpha},\boldsymbol{\tau};q)_k}{(q,\boldsymbol{\xi};q)_k}q^{\alpha k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\times(ct)^k\,d_qx. \tag{7.28}$$

Proof. This follows by integrating the generating function (3.3) term by term. The convergence condition $|at|<1$ guarantees the convergence of the reciprocal product expansion on the Jackson lattice $x=aq^m$. □

8. Orthogonality Test, Second-Order q-Difference Consequence, and a Concrete Specialization

The preceding section shows that $\Xi_n^{\alpha-n}$ is a q-Appell-type sequence. In the theory of orthogonal polynomials, a monic orthogonal sequence must satisfy a three-term recurrence by Favard's theorem; see, for example, Chihara's monograph [8]. Because the present family contains arbitrary numerator and denominator parameter arrays, one should not expect a universal three-term recurrence for all choices of parameters. The following result gives a precise test for when such a recurrence can begin to occur. It is useful both as a structural result and as a practical check on possible orthogonal specializations.

Theorem 8.1 (Favard-type coefficient test)**.** *Let*

$$P_n(\wp)=\Xi_n^{\alpha-n}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)$$

be written in the form (7.2). Suppose that, for a fixed admissible choice of the parameters, the monic sequence $\{P_n\}_{n\geq0}$ satisfies a three-term recurrence of the form

$$P_{n+1}(\wp)=(\wp+u_n)P_n(\wp)+v_nP_{n-1}(\wp),\qquad n\geq1, \tag{8.1}$$

with $P_{-1}=0$ and $P_0=1$. Then necessarily

$$u_n=q^nA_1, \tag{8.2}$$

$$v_n=[n]_q\left(q^{n-1}A_2-q^nA_1^2\right), \tag{8.3}$$

where $[n]_q = (1-q^n)/(1-q)$*, and the coefficients* $A_k = A_k(\alpha, \boldsymbol{\tau}, \boldsymbol{\xi}, c)$ *must satisfy, for every* $n \geq 2$*,*

$$\left(\begin{bmatrix} n+1 \\ 3 \end{bmatrix}_q - \begin{bmatrix} n \\ 3 \end{bmatrix}_q \right) A_3 = q^n A_1 \begin{bmatrix} n \\ 2 \end{bmatrix}_q A_2 + [n-1]_q A_1 [n]_q \left(q^{n-1} A_2 - q^n A_1^2 \right). \tag{8.4}$$

Consequently, a parameter choice for which (8.4) *fails for some* $n \geq 2$ *cannot yield an orthogonal polynomial sequence with respect to a quasi-definite moment functional.*

Proof. Since P_n is monic, comparison of the coefficient of $\wp^n$ in (8.1) gives

$$\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q A_1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q A_1 + u_n.$$

Because $\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} n \\ 1 \end{bmatrix}_q = q^n$, this proves (8.2). Next compare the coefficient of $\wp^{n-1}$. The left-hand side contributes $\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q A_2$, while the right-hand side contributes

$$\begin{bmatrix} n \\ 2 \end{bmatrix}_q A_2 + u_n \begin{bmatrix} n \\ 1 \end{bmatrix}_q A_1 + v_n.$$

Using (8.2) and the identity

$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q - \begin{bmatrix} n \\ 2 \end{bmatrix}_q = q^{n-1}[n]_q,$$

we obtain (8.3). Finally, compare the coefficient of $\wp^{n-2}$. The left-hand side gives $\begin{bmatrix} n+1 \\ 3 \end{bmatrix}_q A_3$. The three terms on the right-hand side give respectively

$$\begin{bmatrix} n \\ 3 \end{bmatrix}_q A_3, \qquad u_n \begin{bmatrix} n \\ 2 \end{bmatrix}_q A_2, \qquad v_n \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q A_1.$$

Substitution of (8.2) and (8.3) gives (8.4). Favard's theorem implies that any orthogonal monic sequence must satisfy a recurrence of the form (8.1); hence failure of the displayed necessary condition rules out orthogonality. □

Corollary 8.2 (Conditional second-order q-difference equation)**.** *Assume that a specialization of* $\Xi_n^{\alpha-n}$ *satisfies the three-term recurrence* (8.1)*. Then* $P_n(\wp) = \Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q)$ *satisfies the second-order q-difference equation*

$$\frac{v_n}{1-q^n} D_q^2 P_n(\wp) + (q\wp + u_n) D_q P_n(\wp) + q^{n+1} P_n(\wp) = 0, \qquad n \geq 1, \tag{8.5}$$

where u_n *and* v_n *are given by* (8.2) *and* (8.3)*.*

Proof. Apply D_q to (8.1). The q-product rule gives

$$D_q\{\wp P_n(\wp)\} = P_n(\wp) + q\wp D_q P_n(\wp).$$

Using the lowering relation (7.20), we have

$$D_q P_{n+1} = (1-q^{n+1})P_n, \qquad D_q P_{n-1} = \frac{1}{1-q^n} D_q^2 P_n.$$

Therefore

$$(1-q^{n+1})P_n = P_n + (q\wp + u_n)D_q P_n + \frac{v_n}{1-q^n} D_q^2 P_n.$$

Rearranging gives (8.5). □

Remark 8.3. *Corollary 8.2 is intentionally stated conditionally. For arbitrary arrays $\boldsymbol{\tau}$ and $\boldsymbol{\xi}$, the family is too general to be universally orthogonal. The theorem identifies the exact coefficient obstruction that must first be overcome before one can search for positive measures or quasi-definite moment functionals.*

This example is useful because it separates the genuinely finite parameter α from the limiting cases that recover the older polynomial families.

Definition 8.4 (Finite one-parameter specialization)**.** *Let $N \in \mathbb{N}_0$ and take $\alpha = N$, $r = s = 0$. Define*

$$\mathcal{P}_n^{(N)}(\wp; c \mid q) = \Xi_n^{N-n}(\wp, -; -, c \mid q) = \sum_{k=0}^{\min\{n,N\}} \begin{bmatrix} n \\ k \end{bmatrix}_q (q^{-N};q)_k q^{Nk} c^k (-1)^k q^{\binom{k}{2}} \wp^{n-k}. \tag{8.6}$$

Theorem 8.5 (Generating function and terminating transformation)**.** *For $N \in \mathbb{N}_0$ and $|\wp t| < 1$,*

$$\sum_{n=0}^{\infty} \mathcal{P}_n^{(N)}(\wp; c \mid q) \frac{t^n}{(q;q)_n} = \frac{1}{(\wp t;q)_\infty} \sum_{k=0}^{N} \frac{(q^{-N};q)_k}{(q;q)_k} (-1)^k q^{Nk+\binom{k}{2}} (ct)^k. \tag{8.7}$$

Equivalently,

$$\sum_{n=0}^{\infty} \mathcal{P}_n^{(N)}(\wp; c \mid q) \frac{t^n}{(q;q)_n} = \frac{1}{(\wp t;q)_\infty} {}_1\phi_0 \left(\begin{matrix} q^{-N} \\ - \end{matrix} ; q, cq^N t \right). \tag{8.8}$$

Consequently, for every $M \in \mathbb{N}_0$,

$$\sum_{n=0}^{M} \begin{bmatrix} M \\ n \end{bmatrix}_q \mathcal{P}_n^{(N)}(\wp; c \mid q) y^{M-n} = \sum_{k=0}^{\min\{M,N\}} \begin{bmatrix} M \\ k \end{bmatrix}_q (q^{-N};q)_k q^{Nk} c^k (-1)^k q^{\binom{k}{2}} (\wp + y)_q^{M-k}, \tag{8.9}$$

where

$$(\wp + y)_q^m := \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q \wp^j y^{m-j}$$

is the standard homogeneous Rogers–Szegő type q-sum.

Proof. Equations (8.7) and (8.8) are obtained from Theorem 3.3 after setting $\alpha = N$ and $r = s = 0$. Since $(q^{-N};q)_k = 0$ for $k > N$, the series in the operator parameter terminates. To prove (8.9), multiply (8.7) by Euler's expansion

$$\frac{1}{(yt;q)_\infty} = \sum_{m=0}^{\infty} \frac{(yt)^m}{(q;q)_m}.$$

On the left, the coefficient of $t^M/(q;q)_M$ is

$$\sum_{n=0}^{M} \begin{bmatrix} M \\ n \end{bmatrix}_q \mathcal{P}_n^{(N)}(\wp; c \mid q) y^{M-n}.$$

On the right, expanding $1/((\wp t;q)_\infty (yt;q)_\infty)$ gives the coefficient $(\wp + y)_q^{M-k}$ after the terminating coefficient indexed by k is chosen from the finite hypergeometric factor. Equating coefficients proves (8.9). □

Corollary 8.6 (Evaluation at the origin)**.** *For* $0 \le n \le N$,

$$\mathcal{P}_n^{(N)}(0; c \mid q) = (q^{-N};q)_n q^{Nn} c^n (-1)^n q^{\binom{n}{2}}, \tag{8.10}$$

and $\mathcal{P}_n^{(N)}(0; c \mid q) = 0$ *for* $n > N$.

Proof. Set $\wp = 0$ in (8.6). Only the term $k = n$ can survive. If $n > N$, that term is zero because $(q^{-N};q)_n = 0$. □

9. Specializations and Conclusion

The preceding theorems contain the identities listed in Theorems 1.7 and 1.8. The formulas in Sections 2–6 have been rechecked against the original sequence of results: the operator identity, its two corollaries, the operational representation, the generating and extended generating functions, the Mehler-type and extended Mehler-type formulas, the Rogers-type and extended Rogers-type formulas, and the Srivastava–Agarwal-type generating function. We record the reductions explicitly, because they are also useful checks on the consistency of the parameters.

Corollary 9.1 (Reduction to h_n)**.** *Under the normalized limiting process* $\alpha \to -\infty$ *and the parameter identification*

$$\boldsymbol{\tau} = (\omega_1, \dots, \omega_r), \qquad \boldsymbol{\xi} = (b_1, \dots, b_s), \qquad c = y,$$

together with the replacement $\wp = x$*, the polynomial* $\Xi_n^{\alpha-n}(\wp, \boldsymbol{\tau}; \boldsymbol{\xi}, c \mid q)$ *reduces to the family* $h_n(\omega_1, \dots, \omega_r; b_1, \dots, b_s; x, y \mid q)$ *in* (1.17).

Proof. In the normalized limit, the finite parameter factor involving $q^{-\alpha}$ supplies precisely the limiting coefficient that remains in the defining sum of h_n. After the identifications above, the summand in (3.1) becomes

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(\omega_1, \ldots, \omega_r; q)_k}{(b_1, \ldots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} x^{n-k} y^k,$$

which is the kth summand of (1.17). Summing over $0 \le k \le n$ proves the assertion. □

Corollary 9.2 (Reduction to generalized Al-Salam–Carlitz polynomials)**.** *If in Corollary 9.1 one further takes $r = s + 1$ and interchanges the two polynomial variables in the standard way, then $\Xi_n^{\alpha-n}$ reduces to $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y \mid q)$ in (1.13).*

Proof. For $r = s+1$ the exponent $1+s-r$ is zero, so the factor $\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}$ disappears. Interchanging the variables so that the parameter-dependent part multiplies x^k and the remaining monomial is y^{n-k} gives

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a_1, \ldots, a_{s+1}; q)_k}{(b_1, \ldots, b_s; q)_k} x^k y^{n-k},$$

which is the defining summand of (1.13). □

Corollary 9.3 (Specialized generating functions)**.** *The generating function (3.3) yields (1.19) under the reduction in Corollary 9.1, and yields (1.14) under the reduction in Corollary 9.2.*

Proof. Apply the same parameter identifications to both sides of (3.3). The product factor $1/(\wp t; q)_\infty$ becomes $1/(xt; q)_\infty$ in the h_n case, and the remaining sum becomes the basic hypergeometric series ${}_r\phi_s$ in (1.19). For the Al-Salam–Carlitz case, the interchange of variables changes the product factor to $1/(yt; q)_\infty$, and the remaining series becomes ${}_{s+1}\phi_s$ with argument xt, which is (1.14). □

Corollary 9.4 (Specialized extended generating functions)**.** *The extended generating function (3.4) gives (1.20). Under the additional specialization $r = s + 1$ and the corresponding interchange of variables, it gives (1.21).*

Proof. Set $\wp = x$, $c = y$, $\boldsymbol{\tau} = \boldsymbol{\omega}$, and $\boldsymbol{\xi} = \mathbf{b}$ in (3.4), and then apply the normalized limit $\alpha \to -\infty$. The coefficient array becomes W_{n+i} and the monomial factors combine as $x^k (y/x)^i$, giving (1.20). When $r = s+1$, the sign-power factor is one; after the variable interchange, the same expression is rearranged as (1.21). □

Corollary 9.5 (Specialized Mehler-type identities)**.** *The Mehler-type formulas (4.1) and (4.2) reduce respectively to (1.22), (1.24), (1.23), and (1.25) under the parameter identifications described above.*

Proof. Use Corollaries 9.1 and 9.2 for each of the two polynomial factors in the left-hand sides of (4.1) and (4.2). On the right-hand sides, the parameter ratios reduce to W_n and $\widehat{W}_n$ in the h_n case. In the Al-Salam–Carlitz case the exponent $1+s-r$ vanishes, and the variable interchange transforms the products $(\wp zt;q)_\infty$ and $(z\wp tq^j;q)_\infty$ into the corresponding products $(yct;q)_\infty$ and $(yctq^j;q)_\infty$. Term-by-term comparison then gives the listed formulas. □

Corollary 9.6 (Specialized Rogers-type identities and Srivastava–Agarwal formula)**.** *The Rogers-type identities* (5.1) *and* (5.2) *reduce to* (1.26), (1.27), (1.15), *and* (1.28). *The Srivastava–Agarwal-type formula* (6.1) *reduces to* (1.16) *in the generalized Al-Salam–Carlitz specialization.*

Proof. For Rogers-type identities, begin with (5.1) or (5.2) and use the same limiting and variable substitutions as in Corollaries 9.1 and 9.2. The double Euler product on the left becomes the double generating product for the corresponding polynomial family, while the coefficient ratios on the right become W_{n+k} or the Al-Salam–Carlitz parameter quotient. For (6.1), the additional factor $(\lambda;q)_n$ is unaffected by the specialization; when $r=s+1$ and the variables are interchanged, the right-hand side becomes the series in (1.16). □

In conclusion, the generalized finite q-operator $\widetilde{E}(q^{-\alpha},\boldsymbol{\tau},\boldsymbol{\xi};q,cq^{\alpha}D_q)$ provides an operational framework for the polynomial family $\Xi_n^{\alpha-n}(\wp,\boldsymbol{\tau};\boldsymbol{\xi},c\mid q)$. This framework produces the principal generating, extended generating, Mehler-type, extended Mehler-type, Rogers-type, extended Rogers-type, and Srivastava–Agarwal-type identities. The additional structural results in Sections 7 and 8 give explicit low-degree examples, lowering and recurrence-type identities, a q-Appell characterization, connection and inverse connection formulas, Jackson q-integral evaluations, an orthogonality test, a conditional second-order q-difference equation, and a finite terminating specialization. The explicit reductions above show that the construction unifies several known q-polynomial identities while retaining the finite parameter α as an additional degree of generality.

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