Asia Pac. J. Math. 2026 **13**:172

AN EFFICIENT JACOBI COLLOCATION METHOD FOR NONLINEAR MICRO- AND NANOSCALE BEAM PROBLEMS

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Received Aug. 9, 2026

Abstract. An operational collocation scheme is developed for nonlinear boundary value models describing electrostatic beam actuators at micro- and nanoscales. The approximation is generated by a nonsymmetric shifted Jacobi family whose two parameters differ by two. An explicit differentiation relation is first established for this basis and subsequently assembled into operational matrices for derivatives up to fourth order. These matrices provide a direct discretization of two cantilever configurations and convert each nonlinear differential model into a finite algebraic system. An approximation estimate is also derived in a Jacobi-weighted Sobolev setting. Numerical experiments for single- and double-cantilever devices show that small residual errors can be obtained using modest polynomial degrees and confirm the computational effectiveness of the proposed construction.

2020 Mathematics Subject Classification. 33C45; 65L10; 65L60.

Key words and phrases. Jacobi polynomials; boundary value problems; operational matrices; collocation method; convergence analysis.

1. Introduction

Nonlinear boundary value problems arise in many physical, biological, and engineering models. Nonlinearity may originate from material response, complicated geometry, or interactions among different components. Higher-order examples and their applications are discussed in [1].

Orthogonal polynomials furnish stable global bases and are therefore central to approximation and computation in science and engineering. Within spectral and pseudospectral discretizations, they can approximate smooth solutions of differential equations (DEs) accurately with relatively few modes. General treatments and representative applications are given in [2–5]. Jacobi polynomials (JPs) are

DOI: 10.28924/APJM/13-172

particularly flexible because their two parameters generate several familiar classical families as special cases [6]. This flexibility has led to numerous differential-equation applications. Jacobi approximations were constructed for Hadamard fractional Klein–Gordon–Schrödinger systems [7] and ecological fractional reaction–diffusion equations [8]. Shifted Jacobi operational matrices were applied to classical and fractional DEs in [9], while Jacobi matrices were used for the fractional Bloch equation in [10]. Other examples are the Jacobi spectral projection for the first kind Fredholm integral equations [11] and the shifted Jacobi-Galerkin approach for the multi-dimensional multi-term fractional DEs [12].

Spectral methods replace the unknown solution by a truncated expansion in global functions and are effective for partial, high-order ordinary, and other DEs. For sufficiently smooth solutions, their convergence can be exponential or of high algebraic order, often reducing the number of degrees of freedom relative to local discretizations. Applications across fluid dynamics, biology, and quantum models are surveyed in [13, 14]. Collocation, Galerkin, and tau formulations are the most common variants [15, 16]. Recent collocation examples include a parameterized shifted-Chebyshev solver for a Caputo gas-dynamics equation [17], together with a Fibonacci-wavelet scheme [18]. Galerkin methods use coincident trial and test spaces [19]; in tau formulations, these spaces can instead be chosen independently [20–22].

Operational matrices translate differentiation or integration of a finite basis into matrix multiplication, thereby simplifying the algebraic implementation of spectral methods. An ultraspherical-wavelet matrix was considered in [23]. Shifted Dickson matrices of the second kind supported a scheme for anomalous time-fractional subdiffusion [24], whereas a matrix construction for nonlinear fractional Lane–Emden systems was developed in [25]. Shifted Pell–Lucas matrices were employed for fractional and classical Rosenau–Hyman models [26]; Taylor matrices were used for Emden–Fowler pantograph equations [27]; and shifted Pell matrices reduced third-order Rosenau–Hyman and Gilson–Pickering equations to algebraic systems [28]. Additional developments include multi-term variable-order fractional equations [29], a shifted Dickson collocation method for the fractional Huxley equation [30], a Chelyshkov treatment of time–space fractional reaction–diffusion [31], and an eighth-kind Chebyshev method for fractional Kawahara equations [32]. More recent related developments include robust numerical techniques for nonlinear ordinary and fractional differential problems [33], Fibonacci level-m polynomial collocation algorithms for classical and time-fractional fifth-order Korteweg–de Vries equations [34], and a shifted one-parameter Horadam polynomial framework for a time-fractional supergeneralized viscous Burgers equation [35]. Other operational-matrix approaches can be found in [36, 37].

The present work selects a shifted Jacobi family with a fixed difference between its parameters and develops the derivative identities required for computation. The resulting operational matrices are

incorporated into a collocation algorithm for beam-type boundary value problems describing micro- and nanoscale cantilever actuators.

The paper is arranged as follows. Section 2 recalls the shifted Jacobi basis and derives its differentiation formulas. Section 3 formulates the collocation algorithm for two cantilever nano-electromechanical-system (NEMS) models. Section 4 establishes an approximation bound, Section 5 reports the numerical experiments, and Section 6 concludes the study.

2. An Overview of the Shifted JPs

On $[0,1]$, the shifted normalized Jacobi sequence $\{\phi_k^{(\alpha,\theta)}(\zeta)\}_{k\geq 0}$ is defined by where $\alpha,\theta>-1$,

$$\phi_k^{(\alpha,\theta)}(\zeta)=\frac{k!\,\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}\,P_k^{(\alpha,\theta)}(2\,\zeta-1), \tag{1}$$

where $P_k^{(\alpha,\theta)}(2\zeta-1)$ denotes the corresponding shifted Jacobi polynomial. The normalized family satisfies

$$\int_0^1 \phi_m^{(\alpha,\theta)}(\zeta)\,\phi_n^{(\alpha,\theta)}(\zeta)\,w^{(\alpha,\theta)}(\zeta)d\zeta=h_n\,\delta_{n,m}, \tag{2}$$

where

$$h_n=\frac{n!\,\Gamma(\alpha+1)^2\,\Gamma(n+\theta+1)}{(2n+\lambda)\,\Gamma(n+\alpha+1)\,\Gamma(n+\lambda)},$$
$$w^{(\alpha,\theta)}(\zeta)=(1-\zeta)^\alpha\,\zeta^\theta,\quad \lambda=\alpha+\theta+1,$$

and $\delta_{n,m}$ is the Kronecker symbol. A power representation is

$$\phi_n^{(\alpha,\theta)}(\zeta)=\sum_{r=0}^{n}\frac{(-1)^{n+r}\,n!\,\Gamma(1+\alpha)\,(1+\theta)_n(1+\alpha+\theta)_{n+r}}{(n-r)!\,r!\,\Gamma(1+n+\alpha)\,(1+\theta)_r\,(1+\alpha+\theta)_n}\,\zeta^r. \tag{3}$$

Conversely, a monomial of degree r has the basis expansion

$$\zeta^r=\sum_{m=0}^{r}\frac{\binom{r}{r-m}(1+\alpha)_m(1+m+\theta)_{r-m}}{(1+m+\alpha+\theta)_m(2+2m+\alpha+\theta)_{r-m}}\,\phi_m^{(\alpha,\theta)}(\zeta). \tag{4}$$

Additional Jacobi-polynomial properties are available in [38,39].

2.1. Selected shifted Jacobi basis functions.

We specialize the shifted Jacobi basis by imposing $\theta=\alpha+2$, with $\alpha>-1$, and retain the notation $\phi_k^{(\alpha,\alpha+2)}(\zeta)$ for the selected functions.

With this parameter choice, Eqs. (3) and (4) yield the power form

$$\phi_k^{(\alpha,\alpha+2)}(\zeta)=\sum_{r=0}^{k}G_{r,k}^{\alpha}\,\zeta^r, \tag{5}$$

where

$$G_{r,k}^{\alpha}=\frac{\Gamma(\alpha+1)(\alpha+k+1)(\alpha+k+2)(-1)^{k+r}\,k!\,\Gamma(k+r+2\alpha+3)}{r!\,\Gamma(k+2\alpha+3)\,(k-r)!\,\Gamma(r+\alpha+3)}. \tag{6}$$

The corresponding inverse relation is

$$\zeta^m=\sum_{r=0}^{m}\frac{\binom{m}{m-r}(\alpha+1)_r(r+\alpha+3)_{m-r}}{(r+2\alpha+3)_r(2(r+\alpha+2))_{m-r}}\,\phi_r^{(\alpha,\alpha+2)}(\zeta). \tag{7}$$

Equations (5) and (7) will be combined to express derivatives of $\phi_k^{(\alpha,\alpha+2)}$ directly in the original basis. The following terminating-series evaluation is required to remove the hypergeometric term from the final coefficient.

Lemma 1. *For $p \in \mathbb{N}_0$, $q \in \mathbb{N}$, and parameter values for which all denominator factors are defined, the terminating series satisfies*

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} -p,\ -3-2k+p+2q-2\alpha,\ -2-k-\alpha \\ -2-2k-2\alpha,\ -2-k+q-\alpha \end{matrix}\middle| 1\right) \\
&= \frac{2^{3+2k+2\alpha}\Gamma(1+k+\alpha)}{\pi(q-1)!\Gamma(3+k-q+\alpha)\Gamma(3+2k+2\alpha)}\times \\
&\begin{cases}
\frac{1}{4}\left(2k^2+p^2-2k(-3+p+q-2\alpha)+p(-3+2q-2\alpha)-2(-2+q-\alpha)(1+\alpha)\right)\Gamma\left(\frac{1+p}{2}\right) & \\
\Gamma\left(\frac{p}{2}+q\right)\Gamma\left(\frac{3}{2}+k-\frac{p}{2}+\alpha\right)\Gamma\left(2+k-\frac{p}{2}-q+\alpha\right), & p\ even, \\
\Gamma\left(1+\frac{p}{2}\right)\Gamma\left(\frac{1+p}{2}+q\right)\Gamma\left(2+k-\frac{p}{2}+\alpha\right)\Gamma\left(\frac{5}{2}+k-\frac{p}{2}-q+\alpha\right), & p\ odd.
\end{cases}
\end{aligned}
\tag{8}
$$

Proof. Introduce the abbreviation

$$
H_{p,k,q} = {}_3F_2\left(\begin{matrix} -p,\ -3-2k+p+2q-2\alpha,\ -2-k-\alpha \\ -2-2k-2\alpha,\ -2-k+q-\alpha \end{matrix}\middle| 1\right),
$$

and apply Zeilberger's algorithm [40]. It produces the recurrence

$$
\begin{aligned}
&-(p-1)(p+2q-2)(-p+2+k-q+\alpha)^2 H_{p-2,k,q} - (5+2k-2p-2q+2\alpha)\times \\
&\left(2\alpha p+2\alpha q+2kp+2kq-p^2-2pq-2q^2-2\alpha-2k+5p+6q-4\right) H_{p-1,k,q} \\
&+(4-p+2k-2q+2\alpha)(2\alpha+2k-p+3)(\alpha+k-p-q+3)^2\, H_{p,k,q} = 0,
\end{aligned}
\tag{9}
$$

with the initial conditions:

$$
H_{0,k,q} = 1, \quad H_{1,k,q} = \frac{q}{(1+k+\alpha)(2+k-q+\alpha)}.
$$

Solving this recurrence subject to the two initial values gives

$$
\begin{aligned}
&H_{p,k,q} \\
&= \frac{2^{3+2k+2\alpha}\Gamma(1+k+\alpha)}{\pi(q-1)!\Gamma(3+k-q+\alpha)\Gamma(3+2k+2\alpha)}\times \\
&\begin{cases}
\frac{1}{4}\left(2k^2+p^2-2k(-3+p+q-2\alpha)+p(-3+2q-2\alpha)-2(-2+q-\alpha)(1+\alpha)\right)\Gamma\left(\frac{1+p}{2}\right) & \\
\Gamma\left(\frac{p}{2}+q\right)\Gamma\left(\frac{3}{2}+k-\frac{p}{2}+\alpha\right)\Gamma\left(2+k-\frac{p}{2}-q+\alpha\right), & p\ even, \\
\Gamma\left(1+\frac{p}{2}\right)\Gamma\left(\frac{1+p}{2}+q\right)\Gamma\left(2+k-\frac{p}{2}+\alpha\right)\Gamma\left(\frac{5}{2}+k-\frac{p}{2}-q+\alpha\right), & p\ odd.
\end{cases}
\end{aligned}
\tag{10}
$$

This is precisely the claimed evaluation. □

Theorem 1. *If $k, q \in \mathbb{Z}^+$ and $k \geq q$, then*

$$\frac{d^q\,\phi_k^{(\alpha,\alpha+2)}(\zeta)}{d\,\zeta^q} = \sum_{p=0}^{k-q} F_{p,k}^q\,\phi_p^{(\alpha,\alpha+2)}(\zeta), \tag{11}$$

where $F_{p,k}^q$ are given by

$$\begin{aligned} F_{p,k}^q =& \frac{4^q\Gamma(k+1)(2\alpha+2p+3)\Gamma(p+2\alpha+3)}{(\alpha+p+1)(\alpha+p+2)\Gamma(p+1)\Gamma(k+2\alpha+3)} \\ &\times \begin{cases} \frac{(q)_{\frac{1}{2}(k-p-q)}\left(2\alpha^2+k^2+2\alpha(k+p+3)+3k+p^2+3p-q^2+q+4\right)\Gamma\left(\frac{1}{2}(k+p+q+3)+\alpha\right)}{4\left(\frac{1}{2}(k-p-q)\right)!\Gamma\left(\frac{1}{2}(k+p-q+5)+\alpha\right)}, & (k-q-p)\ \textit{even}, \\ \frac{\Gamma\left(\frac{1}{2}(k-p+q+1)\right)\Gamma\left(\frac{1}{2}(k+p+q+4)+\alpha\right)}{(q-1)!\ \Gamma\left(\frac{1}{2}(k-p-q+1)\right)\Gamma\left(\frac{1}{2}(k+p-q+4)+\alpha\right)}, & (k-p-q)\ \textit{odd}. \end{cases} \end{aligned} \tag{12}$$

Proof. Termwise differentiation of the finite expansion (5) gives

$$\begin{aligned} \frac{d^q\,\phi_k^{(\alpha,\alpha+2)}(\zeta)}{d\,\zeta^q} =& \frac{(1+k+\alpha)(2+k+\alpha)k!\Gamma(1+\alpha)}{\Gamma(3+k+2\alpha)}\times \\ & \sum_{\ell=0}^{k-q}\frac{(-1)^\ell\Gamma(1+\alpha)\Gamma(3+2k-\ell+2\alpha)}{\ell!(k-\ell-q)!\Gamma(3+k-\ell+\alpha)}\zeta^{k-\ell-q}. \end{aligned} \tag{13}$$

Substituting the inverse expansion (7) into this expression yields

$$\begin{aligned} \frac{d^q\,\phi_k^{(\alpha,\alpha+2)}(\zeta)}{d\,\zeta^q} =& \frac{(1+k+\alpha)(2+k+\alpha)k!\Gamma(1+\alpha)^2}{\Gamma(3+k+2\alpha)}\sum_{\ell=0}^{k-q}\frac{(-1)^\ell\Gamma(3+2k-\ell+2\alpha)}{\ell!(k-\ell-q)!\Gamma(3+k-\ell+\alpha)}\times \\ & \sum_{s=0}^{k-\ell-q}\frac{\binom{k-\ell-q}{s}(1+\alpha)_{k-\ell-q-s}(3+k-\ell-q-s+\alpha)_s}{(2(2+k-\ell-q-s+\alpha))_s(3+k-\ell-q-s+2\alpha)_{k-\ell-q-s}}\,\phi_{k-\ell-q-s}^{(\alpha,\alpha+2)}(\zeta). \end{aligned} \tag{14}$$

Reindexing the two finite sums gives

$$\begin{aligned} \frac{d^q\,\phi_k^{(\alpha,\alpha+2)}(\zeta)}{d\,\zeta^q} =& \frac{(1+k+\alpha)(2+k+\alpha)}{\Gamma(3+k+2\alpha)}\sum_{p=0}^{k-q}\frac{(3+2k-2p-2q+2\alpha)k!\Gamma(3+k-p-q+2\alpha)}{(1+k-p-q+\alpha)(2+k-p-q+\alpha)(k-p-q)!}\times \\ & \sum_{\ell=0}^{p}\frac{(-1)^\ell\Gamma(3+k-\ell-q+\alpha)\Gamma(3+2k-\ell+2\alpha)}{\ell!(p-\ell)!\Gamma(3+k-\ell+\alpha)\Gamma(4+2k-\ell-p-2q+2\alpha)}\,\phi_{k-q-p}^{(\alpha,\alpha+2)}(\zeta). \end{aligned} \tag{15}$$

The inner finite sum may equivalently be collected into a terminating hypergeometric series:

$$\begin{aligned} \frac{d^q\,\phi_k^{(\alpha,\alpha+2)}(\zeta)}{d\,\zeta^q} =& \frac{k!\Gamma(3+k-q+\alpha)\Gamma(3+2k+2\alpha)}{\Gamma(1+k+\alpha)\Gamma(3+k+2\alpha)}\times \\ & \sum_{p=0}^{k-q}\frac{(3+2k-2p-2q+2\alpha)\Gamma(1+k-p-q+\alpha)\Gamma(3+k-p-q+2\alpha)}{p!(k-p-q)!\Gamma(3+k-p-q+\alpha)\Gamma(4+2k-p-2q+2\alpha)}\times \\ & {}_3F_2\left(\begin{matrix} -p,\ -3-2k+p+2q-2\alpha,\ -2-k-\alpha \\ -2-2k-2\alpha,\ -2-k+q-\alpha \end{matrix}\,\middle|\,1\right)\phi_{k-q-p}^{(\alpha,\alpha+2)}(\zeta). \end{aligned} \tag{16}$$

Applying Lemma 1 to this ${}_3F_2(1)$ term separates the even and odd contributions:

$$\frac{d^q\,\phi_k^{(\alpha,\alpha+2)}(\zeta)}{d\,\zeta^q} = \sum_{p=0}^{\left\lfloor\frac{k-q}{2}\right\rfloor} R_{p,k,q}\,\phi_{k-q-2p}^{(\alpha,\alpha+2)}(\zeta) + \sum_{p=0}^{\left\lfloor\frac{1}{2}(k-q-1)\right\rfloor}\bar{R}_{p,k,q}\,\phi_{k-q-2p-1}^{(\alpha,\alpha+2)}(\zeta), \tag{17}$$

where

$$R_{p,k,q} = \frac{2^{-1+2q}(-3-2k+2q+4p-2\alpha)k!\Gamma\left(\frac{3}{2}+k-p+\alpha\right)\Gamma(3+k-q-2p+2\alpha)(q)_p}{(1+k-q-2p+\alpha)(2+k-q-2p+\alpha)p!(k-q-2p)!\Gamma\left(\frac{5}{2}+k-q-p+\alpha\right)\Gamma(3+k+2\alpha)}\times \tag{18}$$
$$\left(-2-k^2+q+3p-2p(q+p)+k(-3+q+2p-2\alpha)-3\alpha+(q+2p)\alpha-\alpha^2\right),$$

$$\bar{R}_{p,k,q} = -\frac{2^{2q}q(-1-2k+2q+4p-2\alpha)k!\Gamma\left(\frac{3}{2}+k-p+\alpha\right)\Gamma(2+k-q-2p+2\alpha)(1+q)_p}{(k-q-2p+\alpha)(1+k-q-2p+\alpha)p!(k-q-2p-1)!\Gamma\left(\frac{3}{2}+k-q-p+\alpha\right)\Gamma(3+k+2\alpha)}. \tag{19}$$

Relabeling the resulting basis indices therefore gives

$$\frac{d^q\,\phi_k^{(\alpha,\alpha+2)}(\zeta)}{d\,\zeta^q} = \sum_{p=0}^{k-q} F_{p,k}^q\,\phi_p^{(\alpha,\alpha+2)}(\zeta), \tag{20}$$

with the coefficients $F_{p,k}^q$ given as in (12). □

Corollary 1. *The first four derivatives of $\phi_k^{(\alpha,\alpha+2)}(\zeta)$ consequently have the expansions*

$$\frac{d\,\phi_k^{(\alpha,\alpha+2)}(\zeta)}{d\,\zeta} = \sum_{p=0}^{k-1} F_{p,k}^1\,\phi_p^{(\alpha,\alpha+2)}(\zeta), \tag{21}$$

$$\frac{d^2\,\phi_k^{(\alpha,\alpha+2)}(\zeta)}{d\,\zeta^2} = \sum_{p=0}^{k-2} F_{p,k}^2\,\phi_p^{(\alpha,\alpha+2)}(\zeta), \tag{22}$$

$$\frac{d^3\,\phi_k^{(\alpha,\alpha+2)}(\zeta)}{d\,\zeta^3} = \sum_{p=0}^{k-3} F_{p,k}^3\,\phi_p^{(\alpha,\alpha+2)}(\zeta), \tag{23}$$

$$\frac{d^4\,\phi_k^{(\alpha,\alpha+2)}(\zeta)}{d\,\zeta^4} = \sum_{p=0}^{k-4} F_{p,k}^4\,\phi_p^{(\alpha,\alpha+2)}(\zeta), \tag{24}$$

where the coefficients $F_{p,k}^q$ are defined in (12).

Proof. The proof of this corollary can be easily obtained after putting respectively $q = 1, 2, 3, 4$ in Theorem 1. □

Remark 1. *Let $\boldsymbol{\phi}(\zeta) = [\phi_0^{(\alpha,\alpha+2)}(\zeta), \phi_1^{(\alpha,\alpha+2)}(\zeta), \ldots, \phi_{\mathcal{N}}^{(\alpha,\alpha+2)}(\zeta)]^T$. Corollary 1 gives the following matrix representations for its first four derivatives:*

$$\frac{d\boldsymbol{\phi}(\zeta)}{d\zeta} = \boldsymbol{\mathcal{A}}\,\boldsymbol{\phi}(\zeta), \tag{25}$$

$$\frac{d^2\boldsymbol{\phi}(\zeta)}{d\zeta^2} = \boldsymbol{\mathcal{B}}\,\boldsymbol{\phi}(\zeta), \tag{26}$$

$$\frac{d^3\boldsymbol{\phi}(\zeta)}{d\zeta^3} = \boldsymbol{\mathcal{C}}\,\boldsymbol{\phi}(\zeta), \tag{27}$$

$$\frac{d^4\boldsymbol{\phi}(\zeta)}{d\zeta^4} = \boldsymbol{\mathcal{Z}}\,\boldsymbol{\phi}(\zeta), \tag{28}$$

where $\boldsymbol{\mathcal{A}} = (F_{p,k}^1)$, $\boldsymbol{\mathcal{B}} = (F_{p,k}^2)$, $\boldsymbol{\mathcal{C}} = (F_{p,k}^3)$ and $\boldsymbol{\mathcal{Z}} = (F_{p,k}^4)$ are the operational matrices of derivatives each of order $(\mathcal{N}+1)\times(\mathcal{N}+1)$ whose entries can be calculated from (12).

For illustration, setting $\mathcal{N} = 6$ *and* $\alpha = \frac{3}{2}$ *produces the matrices*

$$\boldsymbol{\mathcal{A}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{14}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{128}{245} & \frac{288}{49} & 0 & 0 & 0 & 0 & 0 \\ \frac{267}{490} & \frac{160}{147} & \frac{55}{6} & 0 & 0 & 0 & 0 \\ \frac{32}{147} & \frac{688}{441} & \frac{160}{99} & \frac{416}{33} & 0 & 0 & 0 \\ \frac{29}{147} & \frac{256}{441} & \frac{295}{99} & \frac{896}{429} & \frac{210}{13} & 0 & 0 \\ \frac{288}{2695} & \frac{3088}{4851} & \frac{1120}{1089} & \frac{7456}{1573} & \frac{1792}{715} & \frac{1088}{55} & 0 \end{pmatrix}, \tag{29}$$

$$\boldsymbol{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{576}{35} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{384}{49} & \frac{2640}{49} & 0 & 0 & 0 & 0 & 0 \\ \frac{592}{49} & \frac{10240}{441} & \frac{1040}{9} & 0 & 0 & 0 & 0 \\ \frac{384}{49} & \frac{19840}{441} & \frac{4480}{99} & \frac{2240}{11} & 0 & 0 & 0 \\ \frac{720}{77} & \frac{2048}{77} & \frac{12880}{121} & \frac{114688}{1573} & \frac{45696}{143} & 0 & 0 \end{pmatrix}, \tag{30}$$

$$\boldsymbol{\mathcal{C}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1056}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{6144}{49} & \frac{33280}{49} & 0 & 0 & 0 & 0 & 0 \\ \frac{1824}{7} & \frac{10240}{21} & \frac{5600}{3} & 0 & 0 & 0 & 0 \\ \frac{18432}{77} & \frac{92672}{77} & \frac{143360}{121} & \frac{487424}{121} & 0 & 0 & 0 \end{pmatrix}, \tag{31}$$

and

$$\boldsymbol{\mathcal{Z}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{13312}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{16384}{7} & \frac{76800}{7} & 0 & 0 & 0 & 0 & 0 \\ \frac{476160}{77} & \frac{2621440}{231} & \frac{1218560}{33} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

3. Collocation Algorithm for the Two Cantilever NEMS Models

This section constructs the discretization for a cantilever NEMS boundary value problem containing negative powers of the unknown function [41]:

$$\frac{d^4\chi}{d\zeta^4} + \frac{\nu_K}{\chi^K} + \frac{\mu}{\chi^2} + \frac{\gamma}{\chi} = 0, \qquad 0 < \zeta < 1, \qquad K = 3, 4. \tag{33}$$

For a single cantilever, the model is supplemented by

$$\chi(0) = 1, \quad \chi'(0) = 0, \quad \chi''(1) = 0, \quad \chi'''(1) = 0, \tag{34}$$

whereas the double-cantilever configuration is subject to

$$\chi(0) = 1, \quad \chi'(0) = 0, \quad \chi(1) = 1, \quad \chi'(1) = 0. \tag{35}$$

Remark 2. *Setting $\nu_K = 0$ in (33) recovers the corresponding micro-electromechanical model.*

Since $\{\phi_k^{(\alpha,\alpha+2)}(\zeta)\}$ is an orthogonal basis of $L^2_{w^{(\alpha,\alpha+2)}(\zeta)}(0,1)$, a function $\chi \in L^2_{w^{(\alpha,\alpha+2)}(\zeta)}(0,1)$ admits the expansion

$$\chi = \sum_{i=0}^{\infty} \hat{\chi}_i \, \phi_i^{(\alpha,\alpha+2)}(\zeta). \tag{36}$$

Truncation after degree $\mathcal{N}$ gives

$$\chi \approx \chi_{\mathcal{N}} = \sum_{i=0}^{\mathcal{N}} \hat{\chi}_i \, \phi_i^{(\alpha,\alpha+2)}(\zeta) = \hat{\boldsymbol{\chi}}^T \boldsymbol{\phi}(\zeta), \tag{37}$$

where $\hat{\boldsymbol{\chi}}^T = [\hat{\chi}_0, \hat{\chi}_1, \ldots, \hat{\chi}_{\mathcal{N}}]$ and $\boldsymbol{\phi}(\zeta)$ is the vector in Remark 1. Substitution into (33) defines the residual

$$\begin{aligned} \mathcal{R}_{\mathcal{N}}(\zeta) &= \frac{d^4\chi_{\mathcal{N}}}{d\zeta^4} + \frac{\nu_K}{\chi_{\mathcal{N}}^K} + \frac{\mu}{\chi_{\mathcal{N}}^2} + \frac{\gamma}{\chi_{\mathcal{N}}} \\ &= \frac{d^4\chi_{\mathcal{N}}}{d\zeta^4} + \nu_K \, \chi_{\mathcal{N}}^{-K} + \mu \, \chi_{\mathcal{N}}^{-2} + \gamma \, \chi_{\mathcal{N}}^{-1} \\ &= \hat{\boldsymbol{\chi}}^T \boldsymbol{\mathcal{Z}} \, \boldsymbol{\phi}(\zeta) + \nu_K \, (\hat{\boldsymbol{\chi}}^T \boldsymbol{\phi}(\zeta))^{-K} + \mu \, (\hat{\boldsymbol{\chi}}^T \boldsymbol{\phi}(\zeta))^{-2} + \gamma \, (\hat{\boldsymbol{\chi}}^T \boldsymbol{\phi}(\zeta))^{-1}. \end{aligned} \tag{38}$$

Enforcing this residual at the selected nodes yields

$$\hat{\boldsymbol{\chi}}^T \boldsymbol{\mathcal{Z}} \, \boldsymbol{\phi}(\zeta_i) + \nu_K \, (\hat{\boldsymbol{\chi}}^T \boldsymbol{\phi}(\zeta_i))^{-K} + \mu \, (\hat{\boldsymbol{\chi}}^T \boldsymbol{\phi}(\zeta_i))^{-2} + \gamma \, (\hat{\boldsymbol{\chi}}^T \boldsymbol{\phi}(\zeta_i))^{-1} = 0, \quad i = 1, 2, \ldots, \mathcal{N} - 3, \tag{39}$$

where ζ_i are the first $\mathcal{N} - 3$ selected Jacobi collocation nodes. The conditions (34) and (35) become, respectively,

$$\hat{\boldsymbol{\chi}}^T \boldsymbol{\phi}(0) = 1, \quad \hat{\boldsymbol{\chi}}^T \boldsymbol{\mathcal{A}} \, \boldsymbol{\phi}(0) = 0, \quad \hat{\boldsymbol{\chi}}^T \boldsymbol{\mathcal{B}} \, \boldsymbol{\phi}(1) = 0, \quad \hat{\boldsymbol{\chi}}^T \boldsymbol{\mathcal{C}} \, \boldsymbol{\phi}(1) = 0, \tag{40}$$

and

$$\hat{\boldsymbol{\chi}}^T \boldsymbol{\phi}(0) = 1, \quad \hat{\boldsymbol{\chi}}^T \boldsymbol{\mathcal{A}} \, \boldsymbol{\phi}(0) = 0, \quad \hat{\boldsymbol{\chi}}^T \boldsymbol{\phi}(1) = 1, \quad \hat{\boldsymbol{\chi}}^T \boldsymbol{\mathcal{A}} \, \boldsymbol{\phi}(1) = 0. \tag{41}$$

Together, (39) and the appropriate set of algebraic conditions, (40) or (41), form $(\mathcal{N} + 1)$, nonlinear equations. Newton iteration is used to compute the coefficient vector.

4. Error Bound

We next estimate $\|\chi - \chi_{\mathcal{N}}\|_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}}$ in a Jacobi-weighted Sobolev framework. Define

$$\mathbf{H}^m_{w^{(\alpha,\alpha+2)}(\zeta)}(0,1) = \{\chi : D^k_\zeta \chi \in L^2_{w^{(\alpha,\alpha+2)}(\zeta)}(0,1),\ 0 \le k \le m\}, \tag{42}$$

with inner product, norm, and seminorm

$$\begin{aligned} (\chi, v)_{\mathbf{H}^m_{w^{(\alpha,\alpha+2)}(\zeta)}} &= \sum_{k=0}^{m} (D^k_\zeta \chi, D^k_\zeta v)_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}}, \\ ||\chi||^2_{\mathbf{H}^m_{w^{(\alpha,\alpha+2)}(\zeta)}} &= (\chi,\chi)_{\mathbf{H}^m_{w^{(\alpha,\alpha+2)}(\zeta)}}, \quad |\chi|_{\mathbf{H}^m_{w^{(\alpha,\alpha+2)}(\zeta)}} = ||D^m_\zeta \chi||_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}}, \end{aligned} \tag{43}$$

where $m \in \mathbb{N}$.

Lemma 2 ([42]). *Let $n \ge 1$, $n + r > 1$, and $n + s > 1$, where r and s are fixed constants. Then*

$$\frac{\Gamma(n+r)}{\Gamma(n+s)} \le \mathbf{o}^{r,s}_n\, n^{r-s}, \tag{44}$$

where

$$\mathbf{o}^{r,s}_n = exp\left(\frac{r-s}{2\,(n+s-1)} + \frac{1}{12\,(n+r-1)} + \frac{(r-s)^2}{n}\right). \tag{45}$$

Remark 3. *For fixed r and s, the factor $\mathbf{o}^{r,s}_n$ satisfies*

$$\mathbf{o}^{r,s}_n = 1 + O(n^{-1}).$$

Theorem 2. *Let $\chi_{\mathcal{N}} = \sum_{i=0}^{\mathcal{N}} \hat{\chi}_i \phi^{(\alpha,\alpha+2)}_i(\zeta)$, be the degree-$\mathcal{N}$ truncation of $\chi = \sum_{i=0}^{\infty} \hat{\chi}_i \phi^{(\alpha,\alpha+2)}_i(\zeta) \in \mathbf{H}^m_{w^{(\alpha,\alpha+2)}(\zeta)}(0,1)$. For $0 \le k \le m \le \mathcal{N}+1$, the following estimate holds:*

$$||D^k_\zeta (\chi - \chi_{\mathcal{N}})||_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}} \lesssim \mathcal{N}^{-\frac{1}{2}(m-k)\,(\alpha+2)}\ ||D^m_\zeta \chi||_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}}. \tag{46}$$

Here, $a \lesssim b$ means that $a \le c\,b$ for a constant c independent of the truncation degree.

Proof. Subtracting the truncated expansion from the full series and differentiating k times gives

$$D^k_\zeta (\chi - \chi_{\mathcal{N}}) = \sum_{n=\mathcal{N}+1}^{\infty} \hat{\chi}_n\, D^k_\zeta\, \phi^{(\alpha,\alpha+2)}_n(\zeta) = \sum_{n=\mathcal{N}+1}^{\infty} \hat{\chi}_n \sum_{r=k}^{n} \frac{G^\alpha_{r,n}\,\Gamma(r+1)}{\Gamma(r-k+1)}\,\zeta^{r-k}. \tag{47}$$

Taking the squared weighted L^2 norm on both sides yields

$$||D^k_\zeta (\chi - \chi_{\mathcal{N}})||^2_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}} = \sum_{n=\mathcal{N}+1}^{\infty} |\hat{\chi}_n|^2 \sum_{r=k}^{n} \frac{\left(G^\alpha_{r,n}\right)^2\,\Gamma^2(r+1)\,\Gamma\left(2\,(r-k)+\alpha+3\right)\,\Gamma\left(\alpha+1\right)}{\Gamma^2(r-k+1)\,\Gamma(2\,(r-k+\alpha+2))}. \tag{48}$$

Similarly, the mth derivative of the full expansion satisfies

$$||D^m_\zeta \chi||^2_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}} = \sum_{n=m}^{\infty} |\hat{\chi}_n|^2 \sum_{r=m}^{n} \frac{\left(G^\alpha_{r,n}\right)^2\,\Gamma^2(r+1)\,\Gamma\left(2\,(r-m)+\alpha+3\right)\,\Gamma\left(\alpha+1\right)}{\Gamma^2(r-m+1)\,\Gamma(2\,(r-m+\alpha+2))}. \tag{49}$$

Multiplying and dividing the summand in (48) by the corresponding mth-derivative quantity gives

$$\begin{aligned}||D_\zeta^k\left(\chi-\chi_{\mathcal{N}}\right)||^2_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}} &= \sum_{n=\mathcal{N}+1}^{\infty}|\hat{\chi}_n|^2\frac{\sum_{r=k}^{n}\frac{\left(G_{r,n}^{\alpha}\right)^2\Gamma^2(r+1)\,\Gamma\left(2\,(r-k)+\alpha+3\right)\Gamma(\alpha+1)}{\Gamma^2(r-k+1)\,\Gamma\left(2\,(r-k+\alpha+2)\right)}}{\sum_{r=m}^{n}\frac{\left(G_{r,n}^{\alpha}\right)^2\Gamma^2(r+1)\,\Gamma\left(2\,(r-m)+\alpha+3\right)\Gamma(\alpha+1)}{\Gamma^2(r-m+1)\,\Gamma\left(2\,(r-m+\alpha+2)\right)}}\\ &\quad\times\sum_{r=m}^{n}\frac{\left(G_{r,n}^{\alpha}\right)^2\,\Gamma^2(r+1)\,\Gamma\left(2\,(r-m)+\alpha+3\right)\,\Gamma\left(\alpha+1\right)}{\Gamma^2(r-m+1)\,\Gamma\left(2\,(r-m+\alpha+2)\right)}.\end{aligned}\tag{50}$$

Stirling's formula gives the two estimates

$$\begin{aligned}\frac{\Gamma^2(r+1)\,\Gamma\left(2\,(r-k)+\alpha+3\right)}{\Gamma^2(r-k+1)\,\Gamma\left(2\,(r-k+\alpha+2)\right)}&\lesssim r^{2\,k}\,\left(2\,(r-k)\right)^{-(\alpha+1)},\\ \frac{\Gamma^2(r+1)\,\Gamma\left(2\,(r-m)+\alpha+3\right)}{\Gamma^2(r-m+1)\,\Gamma\left(2\,(r-m+\alpha+2)\right)}&\lesssim r^{2\,m}\,\left(2\,(r-m)\right)^{-(\alpha+1)}.\end{aligned}\tag{51}$$

Introduce $\lambda_1^*=\max_{k\le r\le n}\{(G_{r,n}^{\alpha})^2\}$ and $\lambda_2^*=\max_{m\le r\le n}\{(G_{r,n}^{\alpha})^2\}$. The quotient is then bounded by

$$\begin{aligned}\frac{\sum_{r=k}^{n}\frac{\left(G_{r,n}^{\alpha}\right)^2\Gamma^2(r+1)\,\Gamma\left(2\,(r-k)+\alpha+3\right)\Gamma(\alpha+1)}{\Gamma^2(r-k+1)\,\Gamma\left(2\,(r-k+\alpha+2)\right)}}{\sum_{r=m}^{n}\frac{\left(G_{r,n}^{\alpha}\right)^2\Gamma^2(r+1)\,\Gamma\left(2\,(r-m)+\alpha+3\right)\Gamma(\alpha+1)}{\Gamma^2(r-m+1)\,\Gamma\left(2\,(r-m+\alpha+2)\right)}}&\lesssim\frac{n^{2\,k}\,\left(2\,(n-k)\right)^{-(\alpha+1)}\,(n-k+1)\,\lambda_1^*}{n^{2\,m}\,\left(2\,(n-m)\right)^{-(\alpha+1)}\,(n-m+1)\,\lambda_2^*}\\ &\lesssim\frac{n^{2\,k}\,(n-k)^{-(\alpha+1)}\,\Gamma(n-k+2)\,\Gamma(n-m+1)}{n^{2\,m}\,(n-m)^{-(\alpha+1)}\,\Gamma(n-m+2)\,\Gamma(n-k+1)}\\ &\lesssim n^{2\,(k-m)}\,\left(\frac{n-k}{n-m}\right)^{-\alpha}.\end{aligned}\tag{52}$$

Another application of the same asymptotic estimate gives

$$\left(\frac{n-k}{n-m}\right)^{-\alpha}\lesssim n^{-\alpha\,(m-k)},\tag{53}$$

and hence (52) reduces to

$$\frac{\sum_{r=k}^{n}\frac{\left(G_{r,n}^{\alpha}\right)^2\Gamma^2(r+1)\,\Gamma\left(2\,(r-k)+\alpha+3\right)\Gamma(\alpha+1)}{\Gamma^2(r-k+1)\,\Gamma\left(2\,(r-k+\alpha+2)\right)}}{\sum_{r=m}^{n}\frac{\left(G_{r,n}^{\alpha}\right)^2\Gamma^2(r+1)\,\Gamma\left(2\,(r-m)+\alpha+3\right)\Gamma(\alpha+1)}{\Gamma^2(r-m+1)\,\Gamma\left(2\,(r-m+\alpha+2)\right)}}\lesssim n^{(k-m)\,(\alpha+2)}.\tag{54}$$

Substitution of (54) into (50) now yields

$$\begin{aligned}||D_\zeta^k\left(\chi-\chi_{\mathcal{N}}\right)||^2_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}}&\lesssim(\mathcal{N}+1)^{(k-m)\,(\alpha+2)}\times\\ &\quad\sum_{n=\mathcal{N}+1}^{\infty}|\hat{\chi}_n|^2\sum_{r=m}^{n}\frac{\left(G_{r,n}^{\alpha}\right)^2\,\Gamma^2(r+1)\,\Gamma\left(2\,(r-m)+\alpha+3\right)\,\Gamma\left(\alpha+1\right)}{\Gamma^2(r-m+1)\,\Gamma\left(2\,(r-m+\alpha+2)\right)}\\ &\le(\mathcal{N}+1)^{(k-m)\,(\alpha+2)}\,||D_\zeta^m\,\chi||^2_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}}\\ &=\left(\frac{\Gamma(\mathcal{N}+2)}{\Gamma(\mathcal{N}+1)}\right)^{(k-m)\,(\alpha+2)}\,||D_\zeta^m\,\chi||^2_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}}\\ &\lesssim\mathcal{N}^{(k-m)\,(\alpha+2)}\,||D_\zeta^m\,\chi||^2_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}}.\end{aligned}\tag{55}$$

Taking square roots completes the proof. □

Corollary 2. *Suppose $\chi_{\mathcal{N}}$ is the approximate solution of $\chi \in \mathbf{H}^{m}_{w^{(\alpha,\alpha+2)}(\zeta)}(0,1)$. Then, for $0 \leq m \leq \mathcal{N}+1$, we have*

$$||\chi - \chi_{\mathcal{N}}||_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}} \lesssim \mathcal{N}^{-\frac{m}{2}(\alpha+2)} \, ||D^m_{\zeta}\chi||_{L^2_{w^{(\alpha,\alpha+2)}(\zeta)}}. \tag{56}$$

Proof. Setting $k = 0$ in Theorem 2 yields the desired result. □

5. Illustrative Examples

Because an exact solution of (33) is unavailable, accuracy is assessed through the absolute residual

$$\mathcal{E} = \left| \frac{d^4 \chi_{\mathcal{N}}}{d\zeta^4} + \frac{\nu_K}{\chi^K_{\mathcal{N}}} + \frac{\mu}{\chi^2_{\mathcal{N}}} + \frac{\gamma}{\chi_{\mathcal{N}}} \right|. \tag{57}$$

5.1. **For Cantilever NEMS Model Subject to (34).** For the single-cantilever conditions (34), we examine $(K = 3, \nu_3 = 0.2, \mu = 0.5, \gamma = 0.25)$ and $(K = 4, \nu_4 = 1, \mu = 1.5, \gamma = 0.5)$.

- At $K = 3$, $\nu_3 = 0.2$, $\mu = 0.5$, $\gamma = 0.25$:
  Table 1 compares the $\mathcal{E}$ obtained with $\mathcal{N} = 18$ against the value reported in [41] for $m = 15$. Figure 1 plots $\mathcal{E}$ for $\alpha = 1$ and illustrates the accuracy achieved at low degrees. The $\mathcal{E}$ and computed deflection for $\mathcal{N} = 18$ are displayed in Figure 2, while Table 2 records the effect of $\mathcal{N}$ and α.
- At $K = 4$, $\nu_4 = 1$, $\mu = 1.5$, $\gamma = 0.5$:
  Table 3 compares the present result at $\mathcal{N} = 21$ with the $m = 7$ result of [41]. Figure 3 shows that the $\mathcal{E}$ remains small over the interval when $\alpha = 0.5$. Additional values for different $\mathcal{N}$ and α appear in Table 4, and Figure 4 presents both the $\mathcal{E}$ and the approximation for $\mathcal{N} = 18$ and $\alpha = 1.5$.

Table 1. Comparison of the $\mathcal{E}$.

	Proposed method at $\mathcal{N} = 18$		
Method in [41] at $m = 15$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$
6.95319×10^{-10}	9.77107×10^{-13}	1.01569×10^{-12}	1.68837×10^{-12}

5.2. **For double Cantilever NEMS Model subject to (35).** For the double-cantilever configuration, computations are performed for $(K = 3, \nu_3 = 0.2, \mu = 0.5, \gamma = 0.25)$ and $(K = 4, \nu_4 = 1, \mu = 1.5, \gamma = 0.5)$.

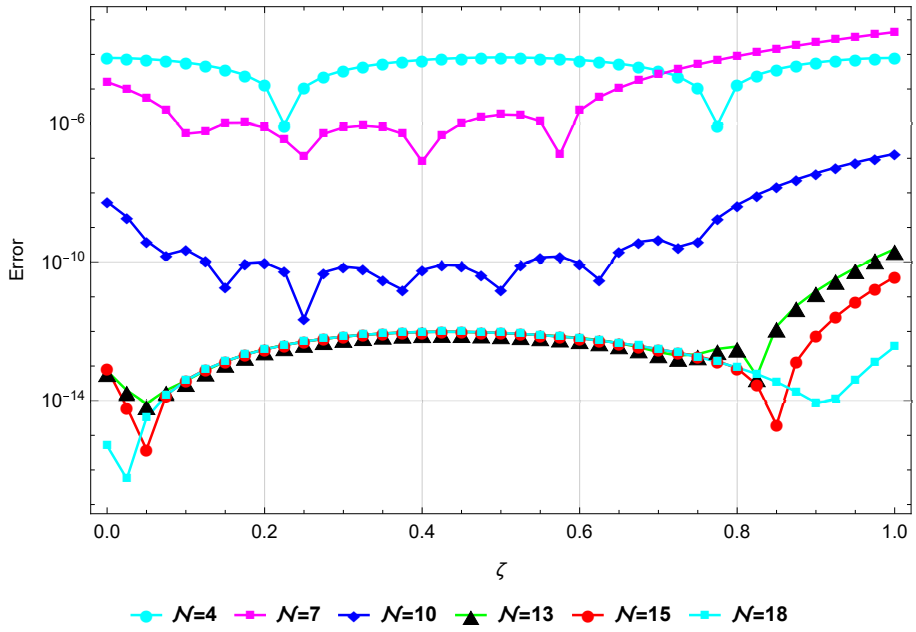


FIGURE 1. The $\mathcal{E}$ at $\alpha = 1$.

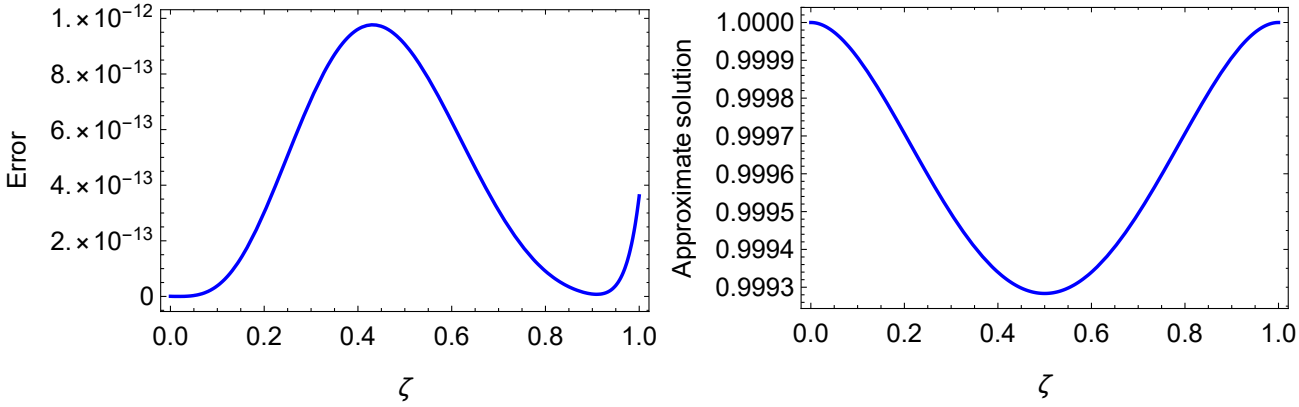


FIGURE 2. The $\mathcal{E}$ (left) and approximate solution (right) at $\mathcal{N} = 18$ and $\alpha = 1$.

TABLE 2. The $\mathcal{E}$ at various values of $\mathcal{N}$ and α.

$\mathcal{N}$	6	9	12	15	18
$\alpha = 0.5$	9.79338×10^{-5}	5.89878×10^{-7}	2.28372×10^{-10}	3.39863×10^{-11}	2.25095×10^{-12}
$\alpha = 2.5$	6.74534×10^{-5}	6.44518×10^{-7}	2.61771×10^{-10}	4.87164×10^{-11}	1.97484×10^{-12}

TABLE 3. Comparison of the $\mathcal{E}$.

	Proposed method at $\mathcal{N} = 21$		
Method in [41] at $m = 7$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$
4.9544×10^{-9}	6.91258×10^{-12}	3.04412×10^{-12}	5.42771×10^{-11}

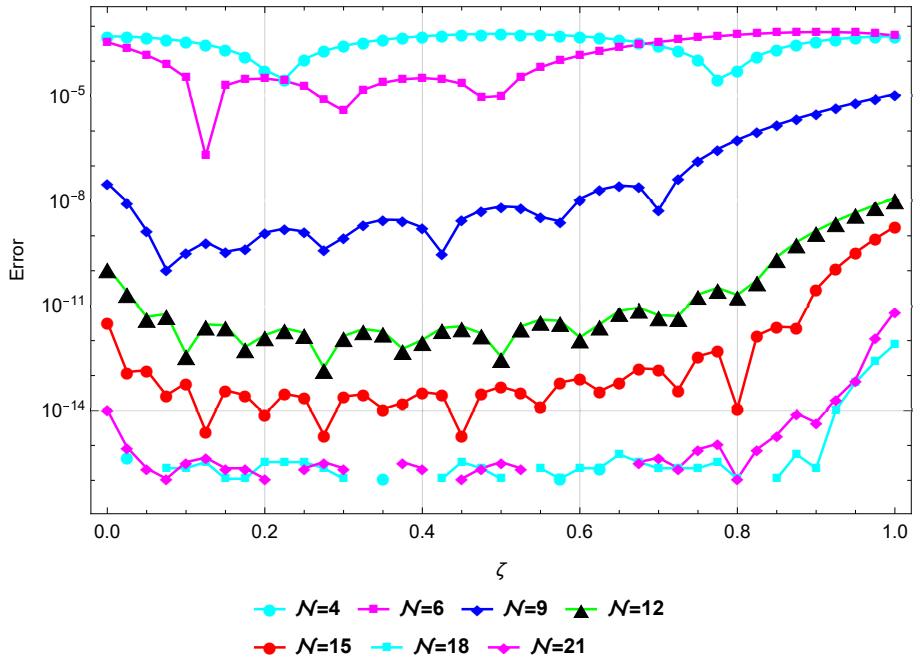


FIGURE 3. The $\mathcal{E}$ at $\alpha = 0.5$.

TABLE 4. The $\mathcal{E}$ at various values of $\mathcal{N}$ and α.

$\mathcal{N}$	6	9	12	15	18
$\alpha = 1.3$	5.83217×10^{-4}	1.19006×10^{-5}	1.30689×10^{-8}	2.12696×10^{-9}	4.38116×10^{-12}
$\alpha = 3.4$	4.07048×10^{-4}	1.27243×10^{-5}	1.45748×10^{-8}	2.77071×10^{-9}	2.85938×10^{-12}

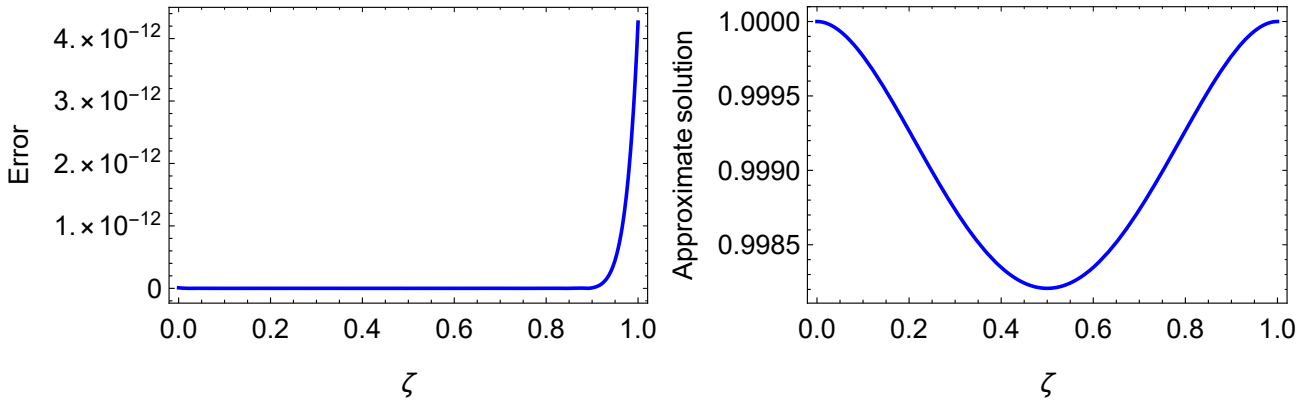


FIGURE 4. The $\mathcal{E}$ (left) and approximate solution (right) at $\mathcal{N} = 18$ and $\alpha = 1.5$.

- At $K = 3$, $\nu_3 = 0.2$, $\mu = 0.5$, $\gamma = 0.25$:

  Table 5 compares the $\mathcal{E}$ at $\mathcal{N} = 21$ with the result of [41] for $m = 15$ and $c = 0.1$. Figure 5 displays $\mathcal{E}$ for $\alpha = 3$, and Figure 6 tracks its maximum as α varies. Both results demonstrate the stability of the spectral approximation.

- At $K = 4$, $\nu_4 = 1$, $\mu = 1.5$, $\gamma = 0.5$:

  Table 6 reports $\mathcal{E}$ for several values of $\mathcal{N}$ and α. Figure 7 shows the $\mathcal{E}$ and approximate deflection for $\mathcal{N} = 18$ and $\alpha = 6.5$.

TABLE 5. Comparison of the $\mathcal{E}$.

	Proposed method at $\mathcal{N} = 21$		
Method in [41] at $m = 15$ and $c = 0.1$	$\alpha = 0.7$	$\alpha = 1.7$	$\alpha = 2.7$
6.95319×10^{-10}	6.2029×10^{-12}	4.56463×10^{-12}	2.19807×10^{-11}

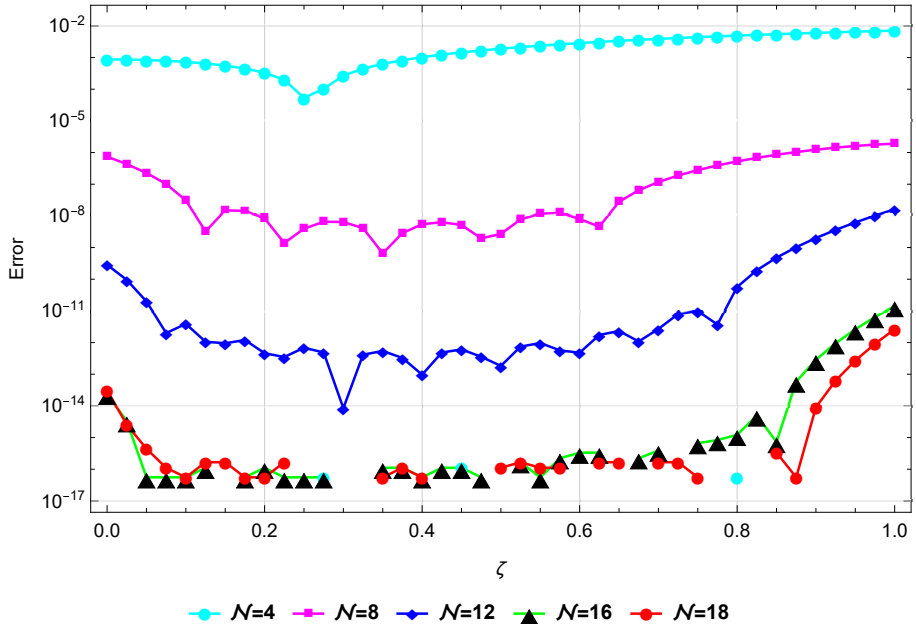


FIGURE 5. The $\mathcal{E}$ at $\alpha = 3$.

TABLE 6. The $\mathcal{E}$ at various values of $\mathcal{N}$ and α.

$\mathcal{N}$	6	9	12	15	18	21
$\alpha = 1$	1.10556×10^{-2}	1.58149×10^{-4}	8.35685×10^{-7}	5.27454×10^{-9}	2.75626×10^{-11}	2.35849×10^{-11}
$\alpha = 2.5$	1.03711×10^{-2}	1.57535×10^{-4}	9.10583×10^{-7}	6.43889×10^{-9}	3.68761×10^{-11}	3.26811×10^{-13}

6. CONCLUDING REMARKS

An operational collocation method based on a nonsymmetric shifted Jacobi basis was developed for two nonlinear beam-type boundary value models. Closed differentiation formulas supplied the first

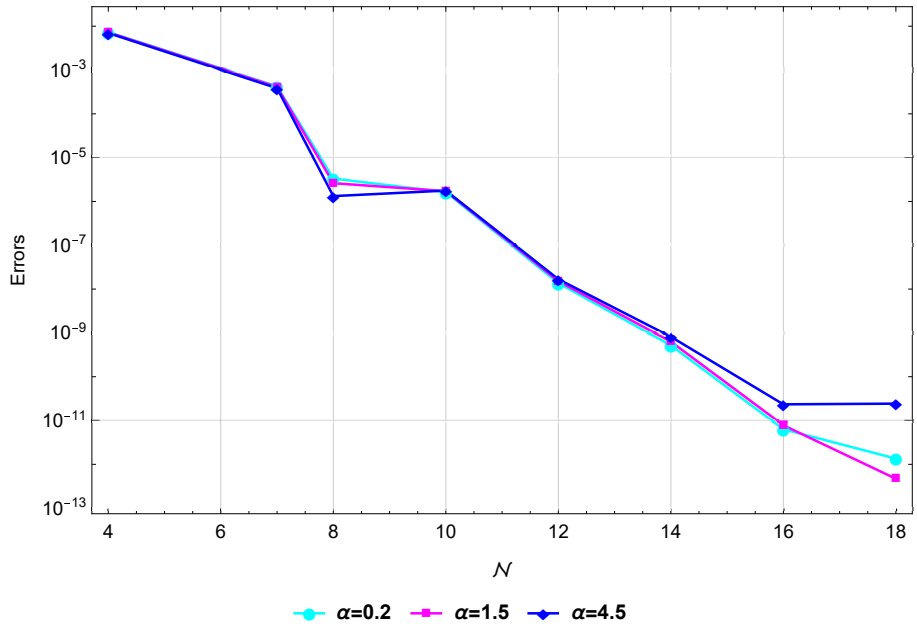


FIGURE 6. The maximum $\mathcal{E}$ at different values of α.

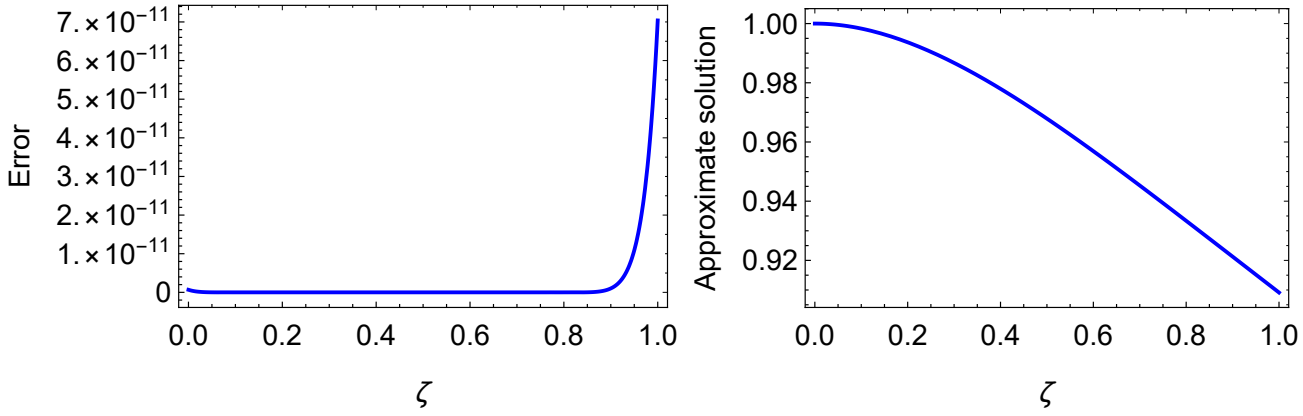


FIGURE 7. The $\mathcal{E}$ (left) and approximate solution (right) at $\mathcal{N} = 18$ and $\alpha = 6.5$.

four derivative matrices and enabled a compact algebraic treatment of the single- and double-cantilever configurations. A weighted Sobolev estimate quantified the decay of the truncation error, while the numerical tests produced small residuals with moderate polynomial degrees and improved upon the cited comparison values. The same construction may be extended to other nonlinear boundary value systems, and related Jacobi parameter choices may provide useful bases for further applied models.

7. APPENDIX

This appendix records representative polynomial approximations associated with the computations in Section 5.

7.1. **For Cantilever NEMS Model subject to** (34)**.**

- At $K = 3$, $\nu_3 = 0.2$, $\mu = 0.5$, $\gamma = 0.25$ and $\alpha = 1$:

$$\begin{aligned}\chi_{18} = {} & 1.4699346766628793\times10^{-13}\zeta^{18} - 1.1363751870994975\times10^{-12}\zeta^{17} + 1.0132391171549881\times10^{-13}\zeta^{16} \\ & + 2.2708072645257543\times10^{-11}\zeta^{15} - 9.385240084365552\times10^{-11}\zeta^{14} + 1.8348450492568057\times10^{-10}\zeta^{13} \\ & - 2.024255326744748\times10^{-9}\zeta^{12} + 1.1069995967352228\times10^{-8}\zeta^{11} - 2.5760644271918167\times10^{-8}\zeta^{10} \\ & + 2.8238977405041403\times10^{-8}\zeta^{9} - 1.5473072886777265\times10^{-6}\zeta^{8} + 6.140312676457934\times10^{-6}\zeta^{7} \\ & - 7.164338194198936\times10^{-6}\zeta^{6} - 5.003279028260078\times10^{-16}\zeta^{5} - 0.0114583\zeta^{4} + 0.0229238\zeta^{3} \\ & - 0.0114629\zeta^{2} - 1.2537925511287161\times10^{-20}\zeta + 1.\end{aligned} \tag{58}$$

- At $K = 4$, $\nu_4 = 1$, $\mu = 1.5$, $\gamma = 0.5$ and $\alpha = 1.5$:

$$\begin{aligned}\chi_{18} = {} & 1.1036997781736625\times10^{-12}\zeta^{18} - 1.280824565166019\times10^{-11}\zeta^{17} - 1.4992207995243877\times10^{-10}\zeta^{16} \\ & + 1.5243600216785256\times10^{-9}\zeta^{15} - 5.459046405515413\times10^{-9}\zeta^{14} + 1.0302524078049404\times10^{-8}\zeta^{13} \\ & - 4.621124281469219\times10^{-8}\zeta^{12} + 2.1737472651511653\times10^{-7}\zeta^{11} - 4.970877858594983\times10^{-7}\zeta^{10} \\ & + 5.437841990130826\times10^{-7}\zeta^{9} - 0.0000109017\zeta^{8} + 0.0000426648\zeta^{7} - 0.000049788\zeta^{6} \\ & - 6.147163737779161\times10^{-15}\zeta^{5} - 0.0286458\zeta^{4} + 0.0573415\zeta^{3} - 0.0286779\zeta^{2} \\ & + 6.810222444107002\times10^{-19}\zeta + 1.\end{aligned} \tag{59}$$

7.2. **For Double Cantilever NEMS Model Subject to (35).**

- At $K = 3$, $\nu_3 = 0.2$, $\mu = 0.5$, $\gamma = 0.25$ and $\alpha = 0.7$:

$$\begin{aligned}\chi_{21} = {} & 4.8190765230852815\times10^{-11}\zeta^{21} - 4.5415158123783503\times10^{-10}\zeta^{20} + 1.979818619434358\times10^{-9}\zeta^{19} \\ & - 5.297940469268769\times10^{-9}\zeta^{18} + 9.740513089383888\times10^{-9}\zeta^{17} - 1.3060086234223809\times10^{-8}\zeta^{16} \\ & + 1.328817632732132\times10^{-8}\zeta^{15} - 1.0839503070453524\times10^{-8}\zeta^{14} + 8.460994111420299\times10^{-9}\zeta^{13} \\ & - 1.0614403566183882\times10^{-8}\zeta^{12} + 3.259049421027811\times10^{-8}\zeta^{11} - 1.306359713050125\times10^{-7}\zeta^{10} \\ & + 3.4844069803012235\times10^{-7}\zeta^{9} - 2.00733816265014\times10^{-6}\zeta^{8} + 0.0000124198\zeta^{7} - 0.0000436932\zeta^{6} \\ & + 6.8547194621433504\times10^{-15}\zeta^{5} - 0.0114583\zeta^{4} + 0.0463673\zeta^{3} - 0.0699092\zeta^{2} \\ & - 6.84327947328559\times10^{-18}\zeta + 1.\end{aligned} \tag{60}$$

- At $K = 4$, $\nu_4 = 1$, $\mu = 1.5$, $\gamma = 0.5$ and $\alpha = 6.5$:

$$\begin{aligned}\chi_{18} = {} & 9.109637421005977\times10^{-12}\zeta^{18} - 6.94979606226183\times10^{-11}\zeta^{17} + 2.3673947319016526\times10^{-10}\zeta^{16} \\ & - 4.379039325484592\times10^{-10}\zeta^{15} + 2.3164751205212906\times10^{-10}\zeta^{14} + 1.3528364474059845\times10^{-9}\zeta^{13} \\ & - 6.929818432218372\times10^{-9}\zeta^{12} + 3.10317928326883\times10^{-8}\zeta^{11} - 1.3009402208603777\times10^{-7}\zeta^{10} \\ & + 3.482855138093635\times10^{-7}\zeta^{9} - 2.0073017511100918\times10^{-6}\zeta^{8} + 0.0000124198\zeta^{7} \\ & - 0.0000436932\zeta^{6} - 1.1144581533625278\times10^{-13}\zeta^{5} - 0.0114583\zeta^{4} + 0.0463673\zeta^{3} \\ & - 0.0699092\zeta^{2} - 6.8564025418385534\times10^{-18}\zeta + 1.\end{aligned} \tag{61}$$

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Funding. This research work was funded by Umm Al-Qura University, Saudi Arabia, under grant number: 26UQU4331287GSSR09.

Acknowledgments. The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia, for funding this research work through grant number: 26UQU4331287GSSR09.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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