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THE DUAL STRUCTURE OF BN-ALGEBRAS

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Abstract. This paper introduces the notion of a dual BN-algebra as a common generalization of KUS-algebras and *pre-Coxeter* algebras. Fundamental properties and several characterizations of dual BN-algebras are established. We investigate the subclasses of commutative, associative, and medial dual BN-algebras, deriving their structural properties and characterizations. In particular, we prove that associative dual BN-algebras are equivalent to *Coxeter* algebras and to dual BN-algebras satisfying condition (D^*). We further establish that medial dual BN-algebras are equivalent to KUS-algebras and characterize them through their relationships with CI-algebras. These results provide new characterizations and reveal close structural connections between dual BN-algebras and several existing classes of logical algebras.

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1. Introduction

In 1966, Y. Imai and K. Iséki [4] introduced the concept of a *BCK*-algebra as an algebraic system that captures certain aspects of propositional logic. In the same year, K. Iséki [3] proposed the notion of a *BCI*-algebra, which generalizes *BCK*-algebras and is closely related to *BCI*-logic. Subsequently, Hu and Li [2] introduced the concept of a *BCH*-algebra in 1983 as a common generalization of *BCK*- and *BCI*-algebras.

Motivated by these developments, several related algebraic structures have been introduced and investigated. In 1999, Ahn and Kim [1] proposed the notion of a *QS*-algebra and studied its connections with *BCK*- and *BCI*-algebras. Two years later, Neggers *et al*. [11] introduced *Q*-algebras, providing a common generalization of *BCK*-, *BCI*-, and *BCH*-algebras. Kim *et al*. [8] subsequently introduced the

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notions of *Coxeter* and *pre-Coxeter* algebras and proved that every Coxeter algebra is equivalent to a Boolean group, that is, an abelian group in which every element has order two. More recently, Kim and Kim [7] introduced *BN*-algebras and established that every *BN*-algebra is 0-commutative.

Parallel to these developments, several dual counterparts of these algebraic systems have also been studied. Meng [9] introduced the notion of a *CI*-algebra as a generalization of dual *BCK*-, dual *BCI*-, and dual *BCH*-algebras. Later, Mostafa *et al*. [10] introduced *KUS*-algebras and showed that they are equivalent to the dual of *QS*-algebras. Furthermore, Saeid [12] proved that dual *Q*-algebras are equivalent to *CI*-algebras, thereby strengthening the relationships among these dual algebraic structures.

The principle of duality plays a fundamental role in algebra by providing an alternative viewpoint for investigating algebraic systems. Through dualization, structural properties and relationships can often be translated into corresponding statements for the dual system, yielding new insights while preserving essential algebraic characteristics. Consequently, the study of dual structures has become an effective approach for extending existing theories and revealing new connections among algebraic systems.

Motivated by the absence of a dual counterpart of *BN*-algebras, this paper introduces and investigates the notion of a *dual BN*-algebra. We establish its fundamental properties and characterize its relationship with several well-known algebraic structures, including *Coxeter*, *pre-Coxeter*, *KUS*, and *CI*-algebras. In particular, we prove that the class of medial dual *BN*-algebras coincides exactly with the class of *KUS*-algebras, thereby providing a complete characterization of medial dual *BN*-algebras. We also investigate commutative and associative dual *BN*-algebras and examine their connections with other related logical algebras. These results establish dual *BN*-algebras as a natural extension of the existing family of dual logical algebras and contribute to a deeper understanding of their structural relationships.

2. Preliminaries

In this section, we recall the definitions and fundamental results concerning Coxeter algebras, pre-Coxeter algebras, BN-algebras, CI-algebras, and KUS-algebras that will be used throughout the paper. Unless otherwise stated, all concepts and results are taken from the cited references.

2.1. **Coxeter and Pre-Coxeter Algebras.**

We begin by recalling the notion of a Coxeter algebra introduced by Kim *et al.* [8].

Definition 2.1 ([8]). *A* Coxeter algebra *is a nonempty set X equipped with a binary operation $*$ and a distinguished element 0 satisfying, for all $x, y, z \in X$,*

$(C1)$**:** $x * x = 0$;

$(C2)$**:** $x * 0 = x$;

$(C3)$**:** $(x * y) * z = x * (y * z)$.

The following results will be used throughout the paper.

Proposition 2.2 ([8]). *If $(X, *, 0)$ is a Coxeter algebra, then*

$$0 * x = x$$

for every $x \in X$.

Theorem 2.3 ([8]). *A Coxeter algebra is precisely a Boolean group; that is, it is an abelian group in which every element has order two.*

Lemma 2.4 ([6]). *If $(X, *, 0)$ is a Coxeter algebra, then*

$$(y * x) * y = x$$

for all $x, y \in X$.

We next recall the notion of a pre-Coxeter algebra.

Definition 2.5 ([8]). *A* pre-Coxeter algebra *is a nonempty set X equipped with a binary operation $*$ and a distinguished element 0 satisfying, for all $x, y, z \in X$,*

$(PC1)$**:** $x * x = 0$;

$(PC2)$**:** $x * 0 = x$;

$(PC3)$**:** $x * y = 0 = y * x$ *implies* $x = y$;

$(PC4)$**:** $x * y = y * x$.

The following elementary properties are well known.

Lemma 2.6 ([5]). *Let $(X, *, 0)$ be a pre-Coxeter algebra. Then, for every $x, y \in X$,*

(i) $0 * x = x$;

(ii) $x * (0 * y) = y * (0 * x)$;

(iii) $0 * (0 * x) = x$;

(iv) $(0 * x) * (0 * y) = y * x$.

Proposition 2.7 ([8]). *Every Coxeter algebra is a pre-Coxeter algebra.*

2.2. **BN-algebras.** The concept of a BN-algebra was introduced by Kim and Kim [7].

Definition 2.8 ([7]). *A* BN-algebra *is a nonempty set X equipped with a binary operation $*$ and a distinguished element 0 satisfying, for all $x, y, z \in X$,*

$(BN1)$**:** $x * x = 0$;

$(BN2)$**:** $x * 0 = x$;

$(BN3)$**:** $(x * y) * z = (0 * z) * (y * x)$.

The following example illustrates a BN-algebra.

Example 2.9. *Let* $X = \{0, a, b\}$ *with binary operation* $*$ *given by*

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Then $(X, *, 0)$ *is a BN-algebra.*

The following properties will be used repeatedly.

Theorem 2.10 ([7]). *Let* $(X, *, 0)$ *be a BN-algebra. Then, for all* $x, y, z \in X$,

(i) $0 * (0 * x) = x$;

(ii) $y * x = (0 * x) * (0 * y)$;

(iii) $(0 * x) * y = (0 * y) * x$;

(iv) $x * y = 0$ *implies* $y * x = 0$;

(v) $0 * x = 0 * y$ *implies* $x = y$;

(vi) $(x * z) * (y * z) = (z * y) * (z * x)$.

2.3. **CI-algebras and KUS-algebras.** We conclude this section by recalling two algebraic systems that play an important role in establishing the main characterization results of this paper.

Definition 2.11 ([9]). *A* CI-algebra *is a nonempty set* X *equipped with a binary operation* $*$ *and a distinguished element* 1 *satisfying, for all* $x, y, z \in X$,

$(CI1)$**:** $x * x = 1$;

$(CI2)$**:** $1 * x = x$;

$(CI3)$**:** $x * (y * z) = y * (x * z)$.

Example 2.12 ([9]). *Let* $X = \{1, a, b\}$ *be a set with the binary operation* $*$ *defined by the following table:*

$*$	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

Then $(X, *, 0)$ *is a CI-algebra.*

Definition 2.13 ([10]). *A* KUS-algebra *is a nonempty set X equipped with a binary operation $*$ and a distinguished element 0 satisfying, for all $x, y, z \in X$,*

$(KUS1)$: $(z * y) * (z * x) = y * x$;

$(KUS2)$: $0 * x = x$;

$(KUS3)$: $x * x = 0$;

$(KUS4)$: $x * (y * z) = y * (x * z)$.

The following properties of KUS-algebras will be needed later.

Proposition 2.14 ([10]). *Let X be a KUS-algebra. Then, for every $x, y, z \in X$,*

(i) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$;

(ii) $x * (y * x) = y * 0$;

(iii) $x * y = 0$ *implies* $x * 0 = y * 0$;

(iv) $(x * y) * 0 = y * x$;

(v) $x * 0 = 0$ *implies* $x = 0$;

(vi) $x = 0 * (0 * x)$;

(vii) $0 * (x * y) = (0 * x) * (0 * y)$;

$(viii)$ $x * z = y * z$ *implies* $x * 0 = y * 0$.

The preceding concepts provide the algebraic framework for the remainder of the paper. We now introduce the notion of a dual BN-algebra and investigate its fundamental properties together with its relationships with Coxeter, pre-Coxeter, CI-, and KUS-algebras.

3. Main Results

In this section, we introduce the notion of a dual BN-algebra and investigate its fundamental properties. We establish several equivalent characterizations, provide illustrative examples, and examine its relationships with BN-, Coxeter, pre-Coxeter, and KUS-algebras.

3.1. Some Basic Properties of Dual BN-algebras.

We begin by defining dual BN-algebras and investigating their elementary properties. A collection of basic identities and characterizations is established, providing essential tools for the subsequent study of special classes of dual BN-algebras and their relationships with other algebraic systems.

Definition 3.1. *A dual BN-algebra is a nonempty set X with a binary operation "$\circ$" and a constant 1 satisfying the following axioms: for all $x, y, z \in X$,*

(DBN1): $x \circ x = 1$,

(DBN2): $1 \circ x = x$, *and*

(DBN3): $x \circ (y \circ z) = (z \circ y) \circ (x \circ 1)$.

Example 3.2. *Let $X = \{1, a, b\}$ be a set with the binary operation $\circ$ defined by the following table:*

$\circ$	1	a	b
1	1	a	b
a	b	1	b
b	a	a	1

Then $(X, \circ, 1)$ is a dual BN-algebra.

Example 3.3. *Let $X = \{1, a, b, c\}$ be a set with the binary operation $\circ$ defined by the following table:*

$\circ$	1	a	b	c
1	1	a	b	c
a	a	1	a	b
b	b	a	1	a
c	c	b	a	1

Then $(X, \circ, 1)$ is a dual BN-algebra.

Example 3.3 shows that a dual BN-algebra may also be a BN-algebra.

Example 3.4. *Let $X = \mathbb{R}^*$. Define the binary operation $\circ$ as $x \circ y = \frac{y}{x}$ for all $x, y \in X$. Then $(X, \circ, 1)$ is a dual BN-algebra.*

In this example, X satisfies (DBN1) since $x \circ x = \frac{x}{x} = 1$. Also, observe that $1 \circ x = \frac{x}{1} = x$. This means, X satisfies (DBN2). X also satisfies (DBN3) since $x \circ (y \circ z) = \frac{y \circ z}{x} = \frac{\frac{z}{y}}{x} = \frac{1}{x} \cdot \frac{z}{y} = \frac{\frac{1}{x}}{\frac{y}{z}} = \frac{x \circ 1}{z \circ y} = (z \circ y) \circ (x \circ 1)$.

Note that X does not satisfy (BN2) since $2 \circ 1 = \frac{1}{2} \neq 2$. Hence, $(X, \circ, 1)$ is not a BN-algebra. Thus, Example 3.4 shows that the converse of the previous observation does not hold; that is, not every dual BN-algebra is a BN-algebra.

Consider the BN-algebra $(X, *, 0)$ in Example 2.9. Note that $(X, *, 0)$ does not satisfy (DBN2) since $0 * a = b \neq a$. Hence, it is not a dual BN-algebra. Hence, Example 2.9 shows that not every BN-algebra is a dual BN-algebra.

Theorem 3.5. *A dual BN-algebra satisfying* (BN2) *is a BN-algebra.*

Proof. Let $(X, *, 0)$ be a dual BN-algebra that satisfies (BN2). Then (BN1) and (BN2) are satisfied. Let $x, y, z \in X$. By (DBN2), (BN2), and (DBN3), $(x*y)*z = (x*y)*(0*z) = (x*y)*[(0*z)*0] = (0*z)*(y*x)$. Hence, (BN3) is holds. Therefore, $(X, *, 0)$ is a BN-algebra. □

The converse of Theorem 3.5 is false in general, as shown by Example 2.9. Indeed, although Example 2.9 satisfies (BN2), it fails to satisfy (DBN2) since

$$0 * a = b \neq a.$$

Theorem 3.6. *A BN-algebra satisfying* (DBN2) *is a dual BN-algebra.*

Proof. Let $(X, \circ, 1)$ be a BN-algebra that satisfies (DBN2). Then (DBN1) and (DBN2) hold. Let $x, y, z \in X$. By using (DBN2), (BN3), and (BN2), $x \circ (y \circ z) = (1 \circ x) \circ (y \circ z) = (z \circ y) \circ x = (z \circ y) \circ (x \circ 1)$. Hence, (DBN3) is satisfied. Therefore, $(X, \circ, 1)$ is a dual BN-algebra. □

The converse of Theorem 3.6 is false in general. Example 3.2 satisfies (DBN2) but fails to satisfy (BN2) since

$$b \circ 1 = a \neq b.$$

Hence, it is not a BN-algebra.

The following example establishes the logical independence of the axioms (DBN1)–(DBN3).

Example 3.7. *Let $X = \{1, a, b\}$ be a set with the binary operations $\circ_1, \circ_2, \circ_3$ defined by the following tables respectively:*

$\circ_1$	1	a	b
1	1	a	b
a	a	1	b
b	a	a	1

$\circ_2$	1	a	b
1	1	a	b
a	a	a	a
b	b	a	a

$\circ_3$	1	a	b
1	1	b	b
a	b	1	1
b	b	1	1

In $(X, \circ_1, 1)$, axioms (DBN1) and (DBN2) hold, whereas (DBN3) fails. In $(X, \circ_2, 1)$, axioms (DBN2) and (DBN3) hold, but (DBN1) fails. Finally, $(X, \circ_3, 1)$ satisfies (DBN1) and (DBN3), whereas (DBN2) fails. Hence, the axioms (DBN1)–(DBN3) are logically independent.

Lemma 3.8. *Let $(X, \circ, 1)$ be a dual BN-algebra. Then the following hold: for all $x, y, z \in X$,*

(i) $(x \circ y) \circ 1 = y \circ x$,

(ii) $(x \circ 1) \circ 1 = x$,

(iii) $x \circ y = 1$ *implies* $y \circ x = 1$,

(iv) $x \circ (y \circ 1) = y \circ (x \circ 1)$,

(v) $(x \circ 1) \circ (y \circ 1) = y \circ x$,

(vi) $(x \circ 1) \circ y = (y \circ 1) \circ x$, *and*

(vii) $x \circ 1 = y \circ 1$ *implies* $x = y$.

Proof. Suppose $(X, \circ, 1)$ is a dual BN-algebra. Let $x, y, z \in X$.

(i) By (DBN1), (DBN3), and (DBN2), $(x \circ y) \circ 1 = (x \circ y) \circ (1 \circ 1) = 1 \circ (y \circ x) = y \circ x$.

(ii) By (i) and (DBN2), $(x \circ 1) \circ 1 = 1 \circ x = x$.

(iii) Suppose $x \circ y = 1$. Then by (i) and (DBN1), $y \circ x = (x \circ y) \circ 1 = 1 \circ 1 = 1$.

(iv) By (DBN3) and (DBN2), $x \circ (y \circ 1) = (1 \circ y) \circ (x \circ 1) = y \circ (x \circ 1)$.

(v) By (iv) and (ii), $(x \circ 1) \circ (y \circ 1) = y \circ [(x \circ 1) \circ 1] = y \circ x$.

(vi) By (ii) and (v), $(x \circ 1) \circ y = (x \circ 1) \circ [(y \circ 1) \circ 1] = (y \circ 1) \circ x$.

(vii) Suppose $x \circ 1 = y \circ 1$. Then by (ii), $x = (x \circ 1) \circ 1 = (y \circ 1) \circ 1 = y$.

□

We now establish two equivalent characterizations of dual BN-algebras that will be useful in the subsequent sections.

Theorem 3.9. *Let $(X, \circ, 1)$ be any algebra of type $(2, 0)$. Then X is a dual BN-algebra if and only if for all $x, y \in X$,*

(i) $x \circ x = 1$,

(ii) $(x \circ y) \circ 1 = y \circ x$,

(iii) $(x \circ 1) \circ 1 = x$, *and*

(iv) $(x \circ 1) \circ (y \circ 1) = y \circ x$.

Proof. Suppose $(X, \circ, 1)$ is a dual BN-algebra. The necessity follows from the definition and Lemma 3.8. Conversely, let $x, y, z \in X$. Then by (ii) and (iii),
$1 \circ x = (x \circ 1) \circ 1 = x$. So, (DBN2) holds. By (iii), (iv), and (ii), we have $x \circ (y \circ z) = [(x \circ 1) \circ 1] \circ \{[(y \circ z) \circ 1] \circ 1\} = [(y \circ z) \circ 1] \circ (x \circ 1) = (z \circ y) \circ (x \circ 1)$. And so, (DBN3) is satisfied. Therefore, $(X, \circ, 1)$ is a dual BN-algebra. □

Theorem 3.10. *Let $(X, \circ, 1)$ be any algebra of type $(2, 0)$. Then X is a dual BN-algebra if and only if for all $x, y \in X$,*

(i) $x \circ x = 1$,

(ii) $1 \circ x = x$,

(iii) $(x \circ y) \circ 1 = y \circ x$, *and*

(iv) $(x \circ 1) \circ y = (y \circ 1) \circ x$.

Proof. Suppose $(X, \circ, 1)$ is a dual BN-algebra. The necessity follows from the definition and Lemma 3.8. Conversely, let $x, y, z \in X$. By using (iii), $x \circ (y \circ z) = [(y \circ z) \circ x] \circ 1 = \{[(z \circ y) \circ 1] \circ x\} \circ 1$. By (iv) and (iii), we have $\{[(z \circ y) \circ 1] \circ x\} \circ 1 = [(x \circ 1) \circ (z \circ y)] \circ 1 = (z \circ y) \circ (x \circ 1)$. Thus, $x \circ (y \circ z) = (z \circ y) \circ (x \circ 1)$. Hence, (DBN3) is satisfied. Therefore, X is a dual BN-algebra. □

Theorem 3.11. *Let $(X, \cdot, e)$ be an abelian group. Define $x \circ y = x^{-1} \cdot y$ for all $x, y \in X$. Then $(X, \circ, e)$ is a dual BN-algebra.*

Proof. Suppose $(X, \cdot, e)$ is an abelian group. Let $x \circ y = x^{-1} \cdot y$ for all $x, y \in X$. Then $x \circ x = x^{-1} \cdot x = e$ and $e \circ x = e^{-1} \cdot x = e \cdot x = x$. Hence, (DBN1) and (DBN2) hold. Also, for all $x, y, z \in X$,

$$\begin{aligned} x \circ (y \circ z) &= x \circ (y^{-1} \cdot z) \\ &= x^{-1} \cdot (y^{-1} \cdot z) \\ &= (y^{-1} \cdot z) \cdot x^{-1} \\ &= [y^{-1} \cdot (z^{-1})^{-1}] \cdot x^{-1} \\ &= (z^{-1} \cdot y)^{-1} \cdot x^{-1} \\ &= (z^{-1} \cdot y) \circ x^{-1} \\ &= (z^{-1} \cdot y) \circ (x^{-1} \cdot e) \\ &= (z \circ y) \circ (x \circ e) \end{aligned}$$

Thus, (DBN3) holds. Therefore $(X, \circ, e)$ is a dual BN-algebra. □

We next investigate the relationship between dual BN-algebras and the classes of Coxeter and pre-Coxeter algebras.

Theorem 3.12. *Every pre-Coxeter algebra is a dual BN-algebra.*

Proof. Suppose that $(X, \circ, 1)$ is a *pre-Coxeter* algebra. Then (PC1) is (DBN1) and Lemma 2.6(i) is (DBN2). Also, by using (PC2) and (PC4) axioms, we have $x \circ (y \circ z) = (x \circ 1) \circ (y \circ z) = (x \circ 1) \circ (z \circ y) = (z \circ y) \circ (x \circ 1)$ for all $x, y, z \in X$. Hence, (DBN3) holds in X. Therefore, X is a dual BN-algebra. □

The converse of Theorem 3.12 is false in general, as demonstrated by the following example.

Example 3.13. *Let $X = \{1, a, b\}$ be a set with the binary operation $\circ$ defined by the following table:*

$\circ$	1	a	b
1	1	a	b
a	a	1	1
b	b	1	1

Then $(X, \circ, 1)$ is a dual BN-algebra. Observe that $a \circ b = 1 = b \circ a$ but $a \neq b$. This means, $(PC3)$ does not hold in X. Hence, $(X, \circ, 1)$ is not a pre-Coxeter algebra.

The following consequence is immediate from Proposition 2.7 and Theorem 3.12.

Corollary 3.14. *Every Coxeter algebra is a dual BN-algebra.*

The converse of Corollary 3.14 is false. Indeed, Example 3.2 is not a Coxeter algebra since

$$(a \circ 1) \circ b = 1 \neq b = a \circ (b \circ 1).$$

The following theorem establishes that every KUS-algebra is naturally a dual BN-algebra.

Theorem 3.15. *Every KUS-algebra is a dual BN-algebra.*

Proof. Suppose $(X, \circ, 1)$ is a KUS-algebra. Let $x, y, z \in X$. Then (DBN1) and (DBN2) hold in X by (KUS3) and (KUS2). Now, by (KUS2), $x \circ (y \circ z) = (1 \circ x) \circ [1 \circ (y \circ z)]$. By Proposition 2.14(iv), $(1 \circ x) \circ [1 \circ (y \circ z)] = [(x \circ 1) \circ 1] \circ \{[(y \circ z) \circ 1] \circ 1\}$. By applying (KUS4), $[(x \circ 1) \circ 1] \circ \{[(y \circ z) \circ 1] \circ 1\} = [(y \circ z) \circ 1] \circ \{[(x \circ 1) \circ 1] \circ 1\}$. Again, by Proposition 2.14(iv), $[(y \circ z) \circ 1] \circ \{[(x \circ 1) \circ 1] \circ 1\} = (z \circ y) \circ [1 \circ (x \circ 1)]$. By (KUS2), $(z \circ y) \circ [1 \circ (x \circ 1)] = (z \circ y) \circ (x \circ 1)$. Hence, $x \circ (y \circ z) = (z \circ y) \circ (x \circ 1)$. And so, X satisfies (DBN3). Therefore, X is a dual BN-algebra. □

The converse of Theorem 3.15 is false in general. Indeed, Example 3.2 does not satisfy (KUS4) since

$$a \circ (b \circ a) = a \circ a = 1 \neq a = b \circ 1 = b \circ (a \circ a).$$

Consequently, dual BN-algebras form a proper generalization of KUS-algebras.

3.2. **Commutative Dual BN-algebras.** We now investigate the commutativity of dual BN-algebras and establish several characterizations together with their relationships with pre-Coxeter, Coxeter, and BN-algebras.

Definition 3.16. *Let $(X, \circ, 1)$ be a dual BN-algebra. Then $(X, \circ, 1)$ is said to be* commutative *if*

$$x \circ y = y \circ x$$

for all $x, y \in X$.

Example 3.17. *Let $X = \{1, a, b\}$ be a set with binary operation $\circ$ defined by*

$\circ$	1	a	b
1	1	a	b
a	a	1	a
b	b	a	1

Then $(X, \circ, 1)$ is a commutative dual BN-algebra.

In contrast, the dual BN-algebra in Example 3.2 is not commutative since

$$a \circ 1 = b \neq a = 1 \circ a.$$

The following theorem provides a simple characterization of commutative dual BN-algebras.

Theorem 3.18. *Let $(X, \circ, 1)$ be a dual BN-algebra. Then X is commutative if and only if*

$$x \circ 1 = x$$

for every $x \in X$.

Proof. Suppose that $(X, \circ, 1)$ is commutative. Then, by (DBN2),

$$x \circ 1 = 1 \circ x = x$$

for every $x \in X$.

Conversely, suppose that $x \circ 1 = x$ for every $x \in X$. Let $x, y \in X$. By Lemma 3.8(iv),

$$x \circ y = x \circ (y \circ 1) = y \circ (x \circ 1) = y \circ x.$$

Hence, $(X, \circ, 1)$ is commutative. □

The following consequence is immediate from Theorem 3.12 together with axiom (PC4).

Corollary 3.19. *Every pre-Coxeter algebra is a commutative dual BN-algebra.*

The converse of Corollary 3.19 is false. Indeed, Example 3.13 is a commutative dual BN-algebra that is not a pre-Coxeter algebra.

Combining Proposition 2.7 with Corollary 3.19 immediately yields the following result.

Corollary 3.20. *Every Coxeter algebra is a commutative dual BN-algebra.*

The converse of Corollary 3.20 does not hold. Indeed, Example 3.13 is not a Coxeter algebra since

$$a \circ (a \circ b) = a \circ 1 = a \neq b = 1 \circ b = (a \circ a) \circ b.$$

We next characterize commutative dual BN-algebras in terms of BN-algebras.

Theorem 3.21. *A dual BN-algebra is commutative if and only if it is a BN-algebra satisfying* (DBN2).

Proof. Suppose that $(X, \circ, 1)$ is a commutative dual BN-algebra. Clearly, (DBN1) is precisely (BN1). By commutativity and (DBN2),

$$x \circ 1 = 1 \circ x = x,$$

so (BN2) holds. Furthermore, for all $x, y, z \in X$,

$$(x \circ y) \circ z = z \circ (x \circ y) = z \circ (y \circ x) = (1 \circ z) \circ (y \circ x),$$

where the last equality follows from (DBN2). Hence,

$$(x \circ y) \circ z = (1 \circ z) \circ (y \circ x),$$

which is (BN3). Therefore, $(X, \circ, 1)$ is a BN-algebra satisfying (DBN2).

Conversely, suppose that $(X, \circ, 1)$ is a BN-algebra satisfying (DBN2). Then (DBN1) and (DBN2) are immediate. Moreover,

$$x \circ (y \circ z) = (1 \circ x) \circ (y \circ z).$$

Using Theorem 2.10(iii),

$$(1 \circ x) \circ (y \circ z) = [1 \circ (y \circ z)] \circ x.$$

Applying (BN1), (BN3), and (BN2),

$$[1 \circ (y \circ z)] \circ x = [(1 \circ 1) \circ (y \circ z)] \circ x = [(z \circ y) \circ 1] \circ x = (z \circ y) \circ (x \circ 1).$$

Hence,

$$x \circ (y \circ z) = (z \circ y) \circ (x \circ 1),$$

so (DBN3) holds. Therefore, $(X, \circ, 1)$ is a dual BN-algebra. Since (DBN2) implies

$$x \circ 1 = x$$

for every $x \in X$, Theorem 3.18 shows that $(X, \circ, 1)$ is commutative. □

3.3. **Associative Dual BN-algebras.** In this subsection, we investigate the associativity of dual BN-algebras. We first establish several basic properties of associative dual BN-algebras and then characterize them in terms of Coxeter algebras, Boolean groups, and a natural identity denoted by (D^*).

Definition 3.22. *Let $(X, \circ, 1)$ be a dual BN-algebra. Then $(X, \circ, 1)$ is said to be* associative *if*

$$(x \circ y) \circ z = x \circ (y \circ z)$$

for all $x, y, z \in X$.

Example 3.23. *Let $X = \{1, a, b, c\}$ be a set equipped with the binary operation $\circ$ defined by*

$\circ$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Then $(X, \circ, 1)$ is an associative dual BN-algebra.

The dual BN-algebra in Example 3.2 is not associative since

$$(a \circ b) \circ 1 = b \circ 1 = a \neq 1 = a \circ a = a \circ (b \circ 1).$$

Theorem 3.24. *Every associative dual BN-algebra is commutative.*

Proof. Let $(X, \circ, 1)$ be an associative dual BN-algebra, and let $x, y \in X$. By Lemma 3.8(i),

$$x \circ y = (y \circ x) \circ 1.$$

Using associativity,

$$(y \circ x) \circ 1 = y \circ (x \circ 1).$$

Applying Lemma 3.8(iv),

$$y \circ (x \circ 1) = x \circ (y \circ 1).$$

Finally, associativity together with Lemma 3.8(i) yields

$$x \circ (y \circ 1) = (x \circ y) \circ 1 = y \circ x.$$

Hence,

$$x \circ y = y \circ x,$$

showing that X is commutative. □

The converse of Theorem 3.24 does not hold in general. Indeed, the commutative dual BN-algebra in Example 3.13 is not associative since

$$(a \circ b) \circ b = 1 \circ b = b \neq a = a \circ 1 = a \circ (b \circ b).$$

Proposition 3.25. *Let $(X, \circ, 1)$ be an associative dual BN-algebra. Then, for all $x, y \in X$,*

(i) $y \circ (y \circ x) = x$;

(ii) $(x \circ y) \circ x = y$; *and*

(iii) $(x \circ y) \circ y = x$.

Proof. Let $x, y \in X$.

(i) By (DBN2) and (DBN1),

$$x = 1 \circ x = (y \circ y) \circ x.$$

Since X is associative,

$$(y \circ y) \circ x = y \circ (y \circ x).$$

Hence,

$$y \circ (y \circ x) = x.$$

(ii) By Theorem 3.24,

$$(x \circ y) \circ x = (y \circ x) \circ x.$$

Since X is associative,

$$(y \circ x) \circ x = y \circ (x \circ x).$$

Using (DBN1), (DBN2), and Theorem 3.24,

$$y \circ (x \circ x) = y \circ 1 = 1 \circ y = y.$$

Therefore,

$$(x \circ y) \circ x = y.$$

(iii) By associativity,

$$(x \circ y) \circ y = x \circ (y \circ y).$$

Applying (DBN1), (DBN2), and Theorem 3.24,

$$x \circ (y \circ y) = x \circ 1 = 1 \circ x = x.$$

Hence,

$$(x \circ y) \circ y = x.$$

□

Theorem 3.26. *Every associative dual BN-algebra is an abelian group.*

Proof. Let $(X, \circ, 1)$ be an associative dual BN-algebra.

Associativity holds by assumption. Moreover, by Theorem 3.24,

$$x \circ y = y \circ x$$

for all $x, y \in X$, so the operation is commutative.

By (DBN2) and Theorem 3.24,

$$1 \circ x = x = x \circ 1,$$

showing that 1 is the identity element of X. Furthermore, (DBN1) gives

$$x \circ x = 1$$

for every $x \in X$, so each element is its own inverse.

Therefore, $(X, \circ, 1)$ is an abelian group. □

Theorem 3.27. *A dual BN-algebra is associative if and only if it is a Coxeter algebra.*

Proof. Suppose $(X, *, 0)$ is an associative dual BN-algebra. Then (C1) coincides with (DBN1). By Theorem 3.24 and (DBN2),

$$x * 0 = 0 * x = x$$

for all $x \in X$, so (C2) holds. Since X is associative,

$$x * (y * z) = (x * y) * z$$

for all $x, y, z \in X$, establishing (C3). Hence, X is a Coxeter algebra.

Conversely, suppose $(X, *, 0)$ is a Coxeter algebra. By Corollary 3.14, X is a dual BN-algebra. Since every Coxeter algebra satisfies

$$x * (y * z) = (x * y) * z,$$

associativity holds in X. Therefore, X is an associative dual BN-algebra. □

The following corollary is an immediate consequence of Theorems 2.3 and 3.27.

Corollary 3.28. *A dual BN-algebra is associative if and only if it is a Boolean group.*

The next result introduces a condition that characterizes associativity in dual BN-algebras.

Definition 3.29. *Let $(X, \circ, 1)$ be a dual BN-algebra. Then $(X, \circ, 1)$ is said to satisfy* condition (D^*) *if*

$$x \circ (y \circ z) = (y \circ x) \circ z$$

for all $x, y, z \in X$.

Example 3.30. *Let $X = \{1, a\}$ be a set equipped with the binary operation $\circ$ defined by*

$\circ$	1	a
1	1	a
a	a	1

Then $(X, \circ, 1)$ is a dual BN-algebra satisfying condition (D^*).

The dual BN-algebra in Example 3.2 does not satisfy condition (D^*) since

$$a \circ (b \circ 1) = a \circ a = 1 \neq b = a \circ 1 = (b \circ a) \circ 1.$$

Proposition 3.31. *Let $(X, \circ, 1)$ be a dual BN-algebra satisfying condition (D^*). Then, for all $x, y \in X$,*

(i) $x \circ 1 = x$;

(ii) *X is commutative; and*

(iii) *if $y \circ x = 1$, then $x = y$.*

Proof. Let $x, y \in X$.

(i) By (DBN1), condition (D^*), (DBN1), and (DBN2),

$$x \circ 1 = x \circ (x \circ x) = (x \circ x) \circ x = 1 \circ x = x.$$

(ii) This follows immediately from (i) and Theorem 3.18.

(iii) Suppose that $y \circ x = 1$. By (i) and (DBN1),

$$x = x \circ 1 = x \circ (y \circ y).$$

Applying condition (D^*),

$$x \circ (y \circ y) = (y \circ x) \circ y.$$

Since $y \circ x = 1$, (DBN2) yields

$$(y \circ x) \circ y = 1 \circ y = y.$$

Hence, $x = y$.

□

Theorem 3.32. *A dual BN-algebra is associative if and only if it satisfies condition (D^*).*

Proof. Let $(X, \circ, 1)$ be a dual BN-algebra.

Suppose first that X is associative. By Theorem 3.24,

$$x \circ (y \circ z) = (x \circ y) \circ z = (y \circ x) \circ z$$

for all $x, y, z \in X$. Hence, X satisfies condition (D^*).

Conversely, suppose that X satisfies condition (D^*). By Proposition 3.31(ii), X is commutative. Therefore,

$$x \circ (y \circ z) = (y \circ x) \circ z = (x \circ y) \circ z$$

for all $x, y, z \in X$, showing that X is associative. □

3.4. **Medial dual BN-algebras.** This subsection introduces medial dual BN-algebras and investigates their fundamental properties. Several equivalent characterizations are established, together with their relationships to commutative and associative dual BN-algebras.

Definition 3.33. *Let $(X, \circ, 1)$ be a dual BN-algebra. Then $(X, \circ, 1)$ is said to be* medial *if*

$$(x \circ y) \circ (z \circ u) = (x \circ z) \circ (y \circ u)$$

for all $x, y, z, u \in X$.

Example 3.34. *Let $X = \{1, a, b\}$ be a set equipped with the binary operation $\circ$ defined by*

$\circ$	1	a	b
1	1	a	b
a	b	1	a
b	a	b	1

Then $(X, \circ, 1)$ is a medial dual BN-algebra.

The dual BN-algebra in Example 3.2 is not medial since

$$(a \circ b) \circ (1 \circ b) = b \circ b = 1 \neq a = b \circ 1 = (a \circ 1) \circ (b \circ b).$$

Proposition 3.35. *Let $(X, \circ, 1)$ be a medial dual BN-algebra. Then, for all $x, y, z \in X$,*

(i) $(x \circ y) \circ y = x$;

(ii) $x \circ (y \circ x) = y \circ 1$;

(iii) $(x \circ y) \circ 1 = (x \circ z) \circ (y \circ z)$;

(iv) *if $y \circ x = 1$, then $(x \circ y) \circ x = x$; and*

(v) *if $x \circ y = 1$, then $x = y$.*

Proof. Let $x, y, z \in X$.

(i) By (DBN2),

$$(x \circ y) \circ y = (x \circ y) \circ (1 \circ y).$$

Since X is medial,

$$(x \circ y) \circ (1 \circ y) = (x \circ 1) \circ (y \circ y).$$

Using (DBN1) and Lemma 3.8(ii),

$$(x \circ 1) \circ (y \circ y) = (x \circ 1) \circ 1 = x.$$

Hence,

$$(x \circ y) \circ y = x.$$

(ii) By Lemma 3.8(i) and part (i),

$$x \circ (y \circ x) = [(y \circ x) \circ x] \circ 1 = y \circ 1.$$

(iii) Since $z \circ z = 1$ by (DBN1),

$$(x \circ y) \circ 1 = (x \circ y) \circ (z \circ z).$$

Applying the medial identity,

$$(x \circ y) \circ (z \circ z) = (x \circ z) \circ (y \circ z),$$

which proves the desired identity.

(iv) Suppose that $y \circ x = 1$. By (DBN2),

$$(x \circ y) \circ x = (x \circ y) \circ (1 \circ x).$$

Using the medial identity,

$$(x \circ y) \circ (1 \circ x) = (x \circ 1) \circ (y \circ x).$$

Since $y \circ x = 1$, Lemma 3.8(ii) yields

$$(x \circ 1) \circ (y \circ x) = (x \circ 1) \circ 1 = x.$$

Hence,

$$(x \circ y) \circ x = x.$$

(v) Suppose that $x \circ y = 1$. Then, by part (i) and (DBN2),

$$x = (x \circ y) \circ y = 1 \circ y = y.$$

□

Proposition 3.36. *Let $(X, \circ, 1)$ be a medial dual BN-algebra. Then the following statements are equivalent:*

(i) $(x \circ y) \circ x = y$ *for all* $x, y \in X$;

(ii) X *is commutative; and*

(iii) X *is associative.*

Proof. Let $(X, \circ, 1)$ be a medial dual BN-algebra.

(i)⇒(ii). Assume that

$$(x \circ y) \circ x = y$$

for all $x, y \in X$. Then

$$x \circ y = x \circ [(x \circ y) \circ x].$$

By Proposition 3.35(ii) and Lemma 3.8(i),

$$x \circ [(x \circ y) \circ x] = (x \circ y) \circ 1 = y \circ x.$$

Hence,

$$x \circ y = y \circ x,$$

and therefore X is commutative.

(ii)⇒(iii). Assume that X is commutative. By (DBN2),

$$(x \circ y) \circ z = (x \circ y) \circ (1 \circ z).$$

Since X is medial,

$$(x \circ y) \circ (1 \circ z) = (x \circ 1) \circ (y \circ z).$$

By Theorem 3.18,

$$x \circ 1 = x,$$

so that

$$(x \circ 1) \circ (y \circ z) = x \circ (y \circ z).$$

Consequently,

$$(x \circ y) \circ z = x \circ (y \circ z),$$

showing that X is associative.

(iii)⇒(i). Assume that X is associative. By Theorem 3.24, X is commutative. Therefore,

$$(x \circ y) \circ x = (y \circ x) \circ x.$$

Applying Proposition 3.35(i),

$$(y \circ x) \circ x = y.$$

Hence,

$$(x \circ y) \circ x = y.$$

This completes the proof. □

Theorem 3.37. *Let $(X, \circ, 1)$ be a medial dual BN-algebra. Define a binary operation $+$ on X by*

$$x + y = (x \circ 1) \circ y, \qquad x, y \in X.$$

Then $(X, +, 1)$ is an abelian group.

Proof. Let $x, y, z \in X$.

First, we verify associativity:

$$\begin{aligned}
(x + y) + z &= [(x \circ 1) \circ y] + z \\
&= \{[(x \circ 1) \circ y] \circ 1\} \circ z \\
&= [y \circ (x \circ 1)] \circ z \qquad \text{(Lemma 3.8(i))} \\
&= [x \circ (y \circ 1)] \circ (1 \circ z) \qquad \text{(Lemma 3.8(iv) and (DBN2))} \\
&= (x \circ 1) \circ [(y \circ 1) \circ z] \qquad \text{(mediality)} \\
&= x + (y + z).
\end{aligned}$$

Hence, $+$ is associative.

Next, by Lemma 3.8(ii),

$$x + 1 = (x \circ 1) \circ 1 = x,$$

while (DBN1) and (DBN2) give

$$1 + x = (1 \circ 1) \circ x = 1 \circ x = x.$$

Thus, 1 is the identity element.

Moreover,

$$x + (x \circ 1) = (x \circ 1) \circ (x \circ 1) = 1$$

by (DBN1), and

$$(x \circ 1) + x = [(x \circ 1) \circ 1] \circ x = x \circ x = 1$$

by Lemma 3.8(ii) and (DBN1). Hence, every element x has inverse $x \circ 1$.

Finally, Lemma 3.8(vi) yields

$$x + y = (x \circ 1) \circ y = (y \circ 1) \circ x = y + x.$$

Therefore, $(X, +, 1)$ is an abelian group. □

Lemma 3.38. *A dual BN-algebra $(X, \circ, 1)$ is medial if and only if*

$$x \circ (y \circ z) = y \circ (x \circ z)$$

for all $x, y, z \in X$.

Proof. Assume that X is medial. Then

$$x \circ (y \circ z) = (1 \circ x) \circ (y \circ z) = (1 \circ y) \circ (x \circ z) = y \circ (x \circ z),$$

where the first and last equalities follow from (DBN2).

Conversely, assume that

$$x \circ (y \circ z) = y \circ (x \circ z)$$

for all $x, y, z \in X$. Let $x, y, z, u \in X$. Then

$$\begin{aligned}
(x \circ y) \circ (z \circ u) &= z \circ [(x \circ y) \circ u] \\
&= [u \circ (x \circ y)] \circ (z \circ 1) && \text{(DBN3)} \\
&= [x \circ (u \circ y)] \circ (z \circ 1) && \text{(assumption)} \\
&= z \circ \{[x \circ (u \circ y)] \circ 1\} && \text{(DBN3)} \\
&= z \circ [(u \circ y) \circ x] && \text{(Lemma 3.8(i))} \\
&= (u \circ y) \circ (z \circ x) && \text{(DBN3)} \\
&= (x \circ z) \circ [(u \circ y) \circ 1] && \text{(Lemma 3.8(i))} \\
&= (x \circ z) \circ (y \circ u),
\end{aligned}$$

proving that X is medial. □

Proposition 3.39. *If $(X, \circ, 1)$ is a medial dual BN-algebra, then*

$$(x \circ y) \circ z = (z \circ y) \circ x$$

for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$. By Proposition 3.35(i),

$$(x \circ y) \circ z = (x \circ y) \circ [(z \circ y) \circ y].$$

Applying Lemma 3.38,

$$(x \circ y) \circ [(z \circ y) \circ y] = (z \circ y) \circ [(x \circ y) \circ y].$$

Using Proposition 3.35(i) once more,

$$(z \circ y) \circ [(x \circ y) \circ y] = (z \circ y) \circ x.$$

Hence,

$$(x \circ y) \circ z = (z \circ y) \circ x.$$

□

Theorem 3.40. *If $(X, \circ, 1)$ is a medial dual BN-algebra, then both the left and right cancellation laws hold. That is, for all $a, b, x \in X$,*

$$a \circ x = b \circ x \implies a = b,$$

and

$$x \circ a = x \circ b \implies a = b.$$

Proof. Let $a, b, x \in X$.

Suppose that $a \circ x = b \circ x$. By Proposition 3.35(i),

$$a = (a \circ x) \circ x = (b \circ x) \circ x = b.$$

Hence, the right cancellation law holds.

Now suppose that $x \circ a = x \circ b$. Then

$$a = (a \circ x) \circ x = [(x \circ a) \circ 1] \circ x = [(x \circ b) \circ 1] \circ x,$$

by Proposition 3.35(i) and Lemma 3.8(i). Again applying Lemma 3.8(i) together with Proposition 3.35(i),

$$[(x \circ b) \circ 1] \circ x = (b \circ x) \circ x = b.$$

Therefore, $a = b$, establishing the left cancellation law. □

Proposition 3.41. *Every associative dual BN-algebra is medial.*

Proof. Let $(X, \circ, 1)$ be an associative dual BN-algebra and let $x, y, z, u \in X$. Then

$$(x \circ y) \circ (z \circ u) = [(x \circ y) \circ z] \circ u = [x \circ (y \circ z)] \circ u.$$

By Theorem 3.24,

$$[x \circ (y \circ z)] \circ u = [x \circ (z \circ y)] \circ u.$$

Using associativity once again,

$$[x \circ (z \circ y)] \circ u = [(x \circ z) \circ y] \circ u = (x \circ z) \circ (y \circ u).$$

Hence,

$$(x \circ y) \circ (z \circ u) = (x \circ z) \circ (y \circ u),$$

showing that X is medial. □

The converse of Proposition 3.41 does not hold in general. Indeed, the medial dual BN-algebra in Example 3.34 is not associative since

$$a \circ (b \circ 1) = a \circ a = 1 \neq b = a \circ 1 = (a \circ b) \circ 1.$$

The following corollary is an immediate consequence of Theorem 3.27 and Proposition 3.41.

Corollary 3.42. *Every Coxeter algebra is a medial dual BN-algebra.*

Theorem 3.43. *Every medial dual BN-algebra is a CI-algebra.*

Proof. Let $(X, *, 1)$ be a medial dual BN-algebra. By (DBN1) and (DBN2), X satisfies (CI1) and (CI2). Moreover, by Lemma 3.38,

$$x * (y * z) = y * (x * z)$$

for all $x, y, z \in X$, which is precisely (CI3). Hence, $(X, *, 1)$ is a CI-algebra. □

The converse of Theorem 3.43 does not hold in general. Consider the CI-algebra in Example 2.12. Since

$$1 * (1 * a) = 1 * a = a \neq 1 = 1 * 1 = (a * 1) * (1 * 1),$$

axiom (DBN3) fails. Consequently, X is not a medial dual BN-algebra.

Theorem 3.44. *Every CI-algebra satisfying $(BN2)$ is a medial dual BN-algebra.*

Proof. Let $(X, \circ, 1)$ be a CI-algebra satisfying (BN2). Then (DBN1) and (DBN2) hold. Let $x, y, z \in X$. Using (BN2) and (CI3),

$$x \circ (y \circ z) = x \circ [(y \circ z) \circ 1] = (y \circ z) \circ (x \circ 1).$$

Applying (BN2) and (CI3) again,

$$(y \circ z) \circ (x \circ 1) = [y \circ (z \circ 1)] \circ (x \circ 1) = [z \circ (y \circ 1)] \circ (x \circ 1) = (z \circ y) \circ (x \circ 1).$$

Hence,

$$x \circ (y \circ z) = (z \circ y) \circ (x \circ 1),$$

so (DBN3) is satisfied. Therefore, $(X, \circ, 1)$ is a dual BN-algebra.

Since (CI3) gives

$$x \circ (y \circ z) = y \circ (x \circ z),$$

Lemma 3.38 implies that X is medial. Thus, $(X, \circ, 1)$ is a medial dual BN-algebra. □

The converse of Theorem 3.44 does not hold in general. Indeed, the medial dual BN-algebra in Example 3.34 is a CI-algebra but does not satisfy (BN2), since

$$a \circ 1 = b \neq a.$$

Theorem 3.45. *A dual system $(X, \circ, 1)$ is a medial dual BN-algebra if and only if it is a CI-algebra satisfying*

$$x \circ y = (y \circ x) \circ 1$$

for all $x, y \in X$.

Proof. Suppose that $(X, \circ, 1)$ is a medial dual BN-algebra. By Theorem 3.43, X is a CI-algebra. Moreover, Lemma 3.8(i) yields

$$x \circ y = (y \circ x) \circ 1$$

for all $x, y \in X$.

Conversely, suppose that $(X, \circ, 1)$ is a CI-algebra satisfying

$$x \circ y = (y \circ x) \circ 1$$

for all $x, y \in X$. By (CI1) and (CI2), axioms (DBN1) and (DBN2) hold. Let $x, y, z \in X$. Using the assumed identity and (CI3),

$$x \circ (y \circ z) = x \circ [(z \circ y) \circ 1] = (z \circ y) \circ (x \circ 1).$$

Thus, (DBN3) is satisfied, and so $(X, \circ, 1)$ is a dual BN-algebra.

Finally, (CI3) gives

$$x \circ (y \circ z) = y \circ (x \circ z),$$

which, by Lemma 3.38, implies that X is medial. □

Theorem 3.46. *A dual BN-algebra is medial if and only if it is a KUS-algebra.*

Proof. Suppose that $(X, \circ, 1)$ is a medial dual BN-algebra. By (DBN2) and (DBN1), axioms (KUS2) and (KUS3) hold. Moreover, using the medial law, (DBN1), and (DBN2),

$$(z \circ y) \circ (z \circ x) = (z \circ z) \circ (y \circ x) = 1 \circ (y \circ x) = y \circ x,$$

establishing (KUS1). Furthermore, Lemma 3.38 yields

$$x \circ (y \circ z) = y \circ (x \circ z),$$

which is exactly (KUS4). Hence, $(X, \circ, 1)$ is a KUS-algebra.

Conversely, suppose that $(X, \circ, 1)$ is a KUS-algebra. By Theorem 3.15, X is a dual BN-algebra. Since (KUS4) asserts that

$$x \circ (y \circ z) = y \circ (x \circ z)$$

for all $x, y, z \in X$, Lemma 3.38 implies that X is medial. □

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References

[1] S.S. Ahn, H.S. Kim, On QS-Algebras, J. Chungcheong Math. Soc. 12 (1999), 33–41.

[2] Q.P. Hu, X. Li, On BCH-Algebras, Math. Semin. Notes, Kobe Univ. 11 (1983), 313–320.

[3] K. Iséki, An Algebra Related with a Propositional Calculus, Proc. Japan Acad. 42 (1966), 26–29. https://doi.org/10.3792/pja/1195522171.

[4] K. Iséki, S. Tanaka, An Introduction to the Theory of BCK-Algebras, Math. Japon. 23 (1978), 1–26.

[5] C.B. Kim, H.S. Kim, (Pre-)Coxeter Algebras and Related Topics, Math. Slovaca 60 (2010), 33–42. https://doi.org/10.2478/s12175-009-0165-9.

[6] C.B. Kim, H.S. Kim, On BM-Algebras, Sci. Math. Jpn. 63 (2006), 421–427.

[7] C.B. Kim, H.S. Kim, On BN-Algebras, Kyungpook Math. J. 53 (2013), 175–184. https://doi.org/10.5666/KMJ.2013.53.2.175.

[8] H.S. Kim, Y.H. Kim, J. Neggers, Coxeter Algebras and Pre-Coxeter Algebras in Smarandache Setting, Honam Math. J. 26 (2004), 471–481.

[9] B.L. Meng, CI-Algebras, Sci. Math. Jpn. 71 (2010), 11–17. https://doi.org/10.32219/isms.71.1_11.

[10] S.M. Mostafa, M.A. Abdel Naby, F.A. Halim, A.T. Hameed, On KUS-Algebras, Int. J. Algebra 7 (2013), 131–144. https://doi.org/10.12988/ija.2013.13013.

[11] J. Neggers, S.S. Ahn, H.S. Kim, On Q-Algebras, Int. J. Math. Math. Sci. 27 (2001), 749–757. https://doi.org/10.1155/S0161171201006627.

[12] A. Borumand Saeid, CI-Algebra Is Equivalent to Dual Q-Algebra, J. Egypt. Math. Soc. 21 (2013), 1–2. https://doi.org/10.1016/j.joems.2012.08.021.