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NEW IDENTITIES FOR CERTAIN NON-CLASSICAL GENERALIZATION OF LAGUERRE POLYNOMIALS

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Abstract. This work derives a collection of operational identities for a non-classical Laguerre sequence, referred to as Laguerre-type polynomials (LTPs). A finite power representation is obtained together with its inverse, followed by explicit conversion formulas linking the LTP and classical Laguerre bases. These formulas are then used to express derivatives of any integer order. The differentiated LTPs are represented not only in their own basis but also in the Laguerre, Hermite, Bernoulli, normalized ultraspherical, and generalized Fibonacci polynomial families. We derive repeated antiderivatives in both the LTP and Laguerre bases. The same relations also yield several ordinary and weighted definite integrals. Every coefficient is given by a finite expression, which is useful for symbolic computation as well as numerical work. These results provide tools for using the LTP family in later approximation schemes.

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1. Introduction

Special functions play an important role in mathematical physics, quantum models, and numerical computation [1–3]. Among the best-known examples are the Jacobi, Hermite, and Laguerre families. They satisfy second-order differential equations and form complete weighted L^2 systems, so finite expansions in these bases are natural tools for approximation. Many recent studies illustrate the range of applications of these polynomial bases. Recent applications cover several different settings. Laguerre-based tools were employed for space-fractional diffusion [4], whereas Laguerre wavelets were used for biological models [5]. Hermite wavelets have also been applied to parabolic Volterra partial integro-differential equations [6,7]. For fractional and diffusion–wave problems, shifted Jacobi constructions were considered in [8–10]. Related Legendre approaches occur in interface and optimal-control problems [11,12], and Chebyshev schemes appear in other differential models [13,14].

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Polynomial sequences defined by algebraic recurrences give another useful class of approximation bases. For algebraically generated bases, convolved Fibonacci connections were studied in [15]. Fibonacci coefficient polynomials have also been used in a highly accurate Galerkin treatment of modified anomalous time-fractional sub-diffusion equations [16]. Broader construction procedures were developed in [17,18], and recent operational identities for Leonardo-type polynomial bases were established in [19]. Pell-type bases have recently entered several tau schemes: Pell coefficient polynomials were used for hyperbolic equations [20], shifted Pell polynomials for third-order Rosenau–Hyman and Gilson–Pickering equations [21], and shifted Pell–Lucas polynomials for classical and fractional Rosenau–Hyman models [22]. A Lucas-based tau method for the telegraph equation was given in [23]. Other examples are the generalized q-Laguerre–Appell family [24], hybrid Legendre–Laguerre polynomials [25], new operational identities for Mott polynomials [26], generalized Fermat systems [27], and a shifted Dickson polynomial scheme for the time-fractional Huxley equation [28]. On the symbolic side, recent work includes second-kind Bernoulli polynomials [29], probabilistic poly-Bernoulli formulas [30], three-parameter non-symmetric Fibonacci polynomials and their structural identities [31], and probabilistic relations for degenerate Bernoulli and Euler families [32,33].

A further direction concerns nonclassical modifications of orthogonal polynomial families. In contrast with classical families associated with second-order operators, such modifications often arise from higher-order differential equations, added point masses, or Sobolev inner products. Classical references for Legendre-, Laguerre-, and Jacobi-type extensions include [34]. Fourth-order Legendre-type and higher-order Laguerre equations were investigated in [35,36]. Generalized ultraspherical equations and relations involving modified weights are treated in [37,38]. Operator-theoretic and Sobolev-related treatments are available in [39–41]; further examples involving Gegenbauer, Chebyshev polynomials of the second kind, and Jacobi-type families occur in [42–44]. Of particular interest here is the Laguerre-type sequence of [45]: its degree-independent differential equation has finite order exactly for nonnegative integral values of the relevant parameter.

The remainder of the paper is arranged as follows. Section 2 presents an overview of the classical, Laguerre-type, and auxiliary polynomial bases. Section 3 derives the power inversion and the two-way Laguerre connections. The next two sections develop differentiation identities in the LTP and comparison bases. Repeated antiderivatives are treated in Section 6, and Section 7 records weighted-integral consequences. Section 8 closes the paper.

2. Background on Laguerre and Auxiliary Polynomial Families

2.1. **Generalized Laguerre polynomials.** For later use, we write the generalized Laguerre polynomial in the finite form

$$\mathsf{L}_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}, \qquad n \geq 0. \tag{2.1}$$

Conversely, a monomial can be expanded in this basis as

$$x^n = n! \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \mathsf{L}_k^{(\alpha)}(x). \tag{2.2}$$

The classical Laguerre case follows by taking $\alpha = 0$:

$$\mathsf{L}_n(x) = \mathsf{L}_n^{(0)}(x). \tag{2.3}$$

Thus, the two finite identities used later become

$$\mathsf{L}_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!}, \tag{2.4}$$

$$x^n = n! \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathsf{L}_k(x). \tag{2.5}$$

2.2. **The Laguerre-type sequence.** The family $\{\mathsf{L}_n^{\alpha,N}(x)\}_{n\geq 0}$, introduced in [45], is orthogonal with respect to a Laguerre measure augmented by a point mass. Its governing differential operator is finite-order when α is a nonnegative integer and infinite-order otherwise. For $\kappa > 0$, its degree-n member obeys

$$\kappa \sum_{i=0}^{\infty} a_i(x) y^{(i)}(x) + x y''(x) + (\alpha + 1 - x) y'(x) + n y(x) = 0.$$

The continuous functions $a_i(x)$ do not depend on the degree. The corresponding orthogonality measure on $[0, \infty)$ is

$$w(x) = x^\alpha e^{-x} + \kappa \delta(x),$$

where $\alpha > -1$, $\kappa > 0$, and δ is the Dirac mass at zero. The link with the generalized Laguerre basis is

$$\widetilde{L}_n^{\alpha,\kappa}(x) = \left(1 + \kappa \binom{n+\alpha}{n-1}\right) \mathsf{L}_n^{(\alpha)}(x) + \kappa \binom{n+\alpha}{n} \frac{d}{dx} \mathsf{L}_n^{(\alpha)}(x).$$

Further details of this construction can be found in [45].

Remark 2.1. In the rest of the paper we set $\alpha = 0$. This choice retains the discrete mass modification while allowing all subsequent formulas to be expressed using the ordinary Laguerre polynomial family. Therefore

$$\widetilde{L}_n^{\kappa}(x) = \widetilde{L}_n^{0,\kappa}(x) = (1 + n\kappa)\, \mathsf{L}_n(x) + \kappa \frac{d}{dx} \mathsf{L}_n(x). \tag{2.6}$$

Remark 2.2. For this specialized family, explicit formulas for inversion, differentiation, integration, and changes of basis are still limited. The following sections derive these relations.

2.3. **Auxiliary Polynomial Bases.** Take $\{\mathsf{P}_n(x)\}_{n\geq 0}$ to be a symmetric orthogonal family and $\{\mathsf{Q}_n(x)\}_{n\geq 0}$ to be an arbitrary polynomial family, not necessarily orthogonal. Their power representations are written as follows:

$$\mathsf{P}_n(x) = \sum_{j=0}^{\lfloor n/2\rfloor} c_{j,n}\, x^{n-2j}, \tag{2.7}$$

$$\mathsf{Q}_n(x) = \sum_{j=0}^{n} e_{j,n}\, x^{n-j}, \tag{2.8}$$

where the arrays $c_{j,n}$ and $e_{j,n}$ are prescribed.

The corresponding inverse power representations are

$$x^n = \sum_{j=0}^{\lfloor n/2\rfloor} \widehat{c}_{j,n}\, \mathsf{P}_{n-2j}(x), \tag{2.9}$$

$$x^n = \sum_{j=0}^{n} \widehat{e}_{j,n}\, \mathsf{Q}_{n-j}(x), \tag{2.10}$$

with $\widehat{c}_{j,n}$ and $\widehat{e}_{j,n}$ denoting the inverse coefficient arrays.

Among the auxiliary families needed below, the normalized Jacobi polynomials are taken in the form [46]

$$\mathsf{J}_n^{(\alpha,\beta)}(x) = {}_2F_1\left(\begin{matrix} -n,\ n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}\,\middle|\, \frac{1-x}{2}\right). \tag{2.11}$$

Choosing equal Jacobi parameters produces the normalized ultraspherical family:

$$\mathsf{C}_n^{(\lambda)}(x) = \mathsf{J}_n^{\left(\lambda-\frac{1}{2},\lambda-\frac{1}{2}\right)}(x). \tag{2.12}$$

We also employ the symmetric generalized Fibonacci family $\{\mathsf{F}_n^{(a,b)}(x)\}_{n\geq 0}$, specified by the recurrence

$$\begin{aligned} &\mathsf{F}_n^{(a,b)}(x) = ax\,\mathsf{F}_{n-1}^{(a,b)}(x) + b\,\mathsf{F}_{n-2}^{(a,b)}(x), \qquad n\geq 2, \\ &\mathsf{F}_0^{(a,b)}(x) = 1, \qquad \mathsf{F}_1^{(a,b)}(x) = ax. \end{aligned} \tag{2.13}$$

3. Core Algebraic Relations for the LTP Basis

We begin with the algebraic identities needed later: two conversion formulas between the LTP and Laguerre bases and a pair of inverse power expansions.

3.1. **Two-way transformations between LTPs and Laguerre polynomials.** We first give the LTP-to-Laguerre conversion.

Theorem 3.1. *For each $r\geq 0$, an LTP can be expressed in the Laguerre basis as*

$$\widetilde{L}_r^{\kappa}(x) = (1+\kappa r)\,\mathsf{L}_r(x) - \kappa \sum_{i=0}^{r-1} \mathsf{L}_i(x). \tag{3.1}$$

Proof. Using the standard Laguerre derivative identity [47]

$$\frac{d}{dx}\mathsf{L}_r(x) = -\mathsf{L}^{(1)}_{r-1}(x). \tag{3.2}$$

For parameter one, this polynomial equals the finite sum

$$\mathsf{L}^{(1)}_{r-1}(x) = \sum_{i=0}^{r-1} \mathsf{L}_i(x).$$

Using the first relation inside the second one reduces the derivative to

$$\frac{d}{dx}\mathsf{L}_r(x) = -\sum_{i=0}^{r-1} \mathsf{L}_i(x).$$

Replacing the derivative term in (2.6) by the preceding finite sum proves (3.1). □

Theorem 3.2. *Conversely, each* $\mathsf{L}_r(x)$, $r \geq 0$, *satisfies*

$$\mathsf{L}_r(x) = \frac{\widetilde{L}^{\kappa}_r(x)}{\kappa r + 1} + \kappa \sum_{\ell=0}^{r-1} \frac{\widetilde{L}^{\kappa}_\ell(x)}{(\kappa\ell + 1)\big((\ell+1)\kappa + 1\big)}. \tag{3.3}$$

Proof. Define

$$\mathsf{S}_r(x) = \sum_{i=0}^{r} \mathsf{L}_i(x). \tag{3.4}$$

The definition shows that adjacent partial sums are related through

$$\mathsf{S}_r(x) = \mathsf{S}_{r-1}(x) + \mathsf{L}_r(x), \qquad r \geq 1. \tag{3.5}$$

Using (3.1) together with (3.2), we obtain

$$\widetilde{L}^{\kappa}_r(x) = (1 + \kappa r)\mathsf{L}_r(x) - \kappa \mathsf{S}_{r-1}(x), \qquad r \geq 1. \tag{3.6}$$

Rearranging this equality with respect to $\mathsf{L}_r(x)$ gives

$$\mathsf{L}_r(x) = \frac{\widetilde{L}^{\kappa}_r(x)}{1 + \kappa r} + \frac{\kappa}{1 + \kappa r}\mathsf{S}_{r-1}(x). \tag{3.7}$$

Insert the preceding expression in (3.5); after collecting the two terms one obtains

$$\mathsf{S}_r(x) = \frac{\widetilde{L}^{\kappa}_r(x)}{1 + \kappa r} + \frac{1 + \kappa(r+1)}{1 + \kappa r}\mathsf{S}_{r-1}(x). \tag{3.8}$$

Dividing the recurrence by $1 + \kappa(r+1)$:

$$\frac{\mathsf{S}_r(x)}{1 + \kappa(r+1)} = \frac{\widetilde{L}^{\kappa}_r(x)}{(1 + \kappa r)(1 + \kappa(r+1))} + \frac{\mathsf{S}_{r-1}(x)}{1 + \kappa r}. \tag{3.9}$$

With $\mathsf{T}_r = \mathsf{S}_r(x)/(1 + \kappa(r+1))$, the last identity reduces to

$$\mathsf{T}_r - \mathsf{T}_{r-1} = \frac{\widetilde{L}^{\kappa}_r(x)}{(1 + \kappa r)(1 + \kappa(r+1))}. \tag{3.10}$$

Adding these consecutive relations from $\ell = 0$ through $r-1$ gives

$$\mathsf{T}_{r-1} = \sum_{\ell=0}^{r-1} \frac{\widetilde{L}_\ell^\kappa(x)}{(1+\ell\kappa)(1+\kappa+\ell\kappa)}. \tag{3.11}$$

Consequently, this relation can be written as

$$\mathsf{S}_{r-1}(x) = (1+\kappa r) \sum_{\ell=0}^{r-1} \frac{\widetilde{L}_\ell^\kappa(x)}{(1+\ell\kappa)(1+\kappa+\ell\kappa)}. \tag{3.12}$$

Using this expression for the partial sum in (3.7) produces the desired representation

$$\mathsf{L}_r(x) = \frac{\widetilde{L}_r^\kappa(x)}{1+\kappa r} + \kappa \sum_{\ell=0}^{r-1} \frac{\widetilde{L}_\ell^\kappa(x)}{(1+\ell\kappa)(1+\kappa+\ell\kappa)}.$$

This establishes the connection in the reverse direction. □

3.2. **Direct and inverse monomial expansions.**

Theorem 3.3. *For $r \geq 0$, the following finite monomial form holds:*

$$\widetilde{L}_r^\kappa(x) = \sum_{s=0}^{r} \frac{(-1)^{r-s}\, r!\, \big((r-s+1)+\kappa(r+1)(r-s)\big)}{s!\,(r-s+1)!\,(r-s)!}\, x^{r-s}. \tag{3.13}$$

Proof. Insert (2.4) into (2.6) to get

$$\widetilde{L}_r^\kappa(x) = (1+r\kappa) \sum_{i=0}^{r} \frac{(-1)^i\, r!}{(i!)^2 (r-i)!}\, x^i + \kappa \sum_{i=1}^{r} \frac{(-1)^i\, r!}{i!(i-1)!(r-i)!}\, x^{i-1}. \tag{3.14}$$

Reindex the differentiated part by setting $i = u+1$; its contribution then takes the form

$$\widetilde{L}_r^\kappa(x) = (1+r\kappa) \sum_{u=0}^{r} \frac{(-1)^u\, r!}{(u!)^2 (r-u)!}\, x^u - \kappa \sum_{u=0}^{r-1} \frac{(-1)^u\, r!}{(u+1)!u!(r-u-1)!}\, x^u. \tag{3.15}$$

Terms carrying the same power of x may now be grouped, which gives

$$\widetilde{L}_r^\kappa(x) = \sum_{u=0}^{r} \frac{(-1)^u\, r!\, \big((u+1)+\kappa(r+1)u\big)}{(u+1)!\,u!\,(r-u)!}\, x^u. \tag{3.16}$$

Writing the coefficients in descending-power order gives the asserted form

$$\widetilde{L}_r^\kappa(x) = \sum_{s=0}^{r} \frac{(-1)^{r-s}\, r!\, \big((r-s+1)+\kappa(r+1)(r-s)\big)}{s!\,(r-s+1)!\,(r-s)!}\, x^{r-s}. \tag{3.17}$$

The displayed coefficient formula is therefore obtained. □

We now derive the inverse of (3.17). For degree zero, $x^0 = \widetilde{L}_0^\kappa(x) = 1$; the positive-degree case is stated below.

Theorem 3.4. *For every positive integer r,*

$$x^r = \frac{(-1)^r\, r!}{\kappa r+1}\, \widetilde{L}_r^\kappa(x) + \sum_{s=1}^{r} \frac{(-1)^{r-s}\big(r+(r+1)\kappa(r-s)\big) r!(r-1)!}{\big(\kappa(r-s+1)+1\big)\big(\kappa r-\kappa s+1\big)(r-s)!s!}\, \widetilde{L}_{r-s}^\kappa(x). \tag{3.18}$$

Proof. Start with the standard Laguerre inversion formula

$$x^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} r!\, \mathsf{L}_i(x), \qquad r \geq 0. \tag{3.19}$$

Replace each Laguerre polynomial using the LTP expansion in (3.3):

$$x^r = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} r!\, \frac{\widetilde{L}_\ell^\kappa(x)}{1+\kappa\ell} + \kappa \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} r! \sum_{v=0}^{\ell-1} \frac{\widetilde{L}_v^\kappa(x)}{(1+v\kappa)(1+(v+1)\kappa)}. \tag{3.20}$$

Since both sums are finite, their order may be interchanged; this transforms the two finite sums into

$$x^r = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} r!\, \frac{\widetilde{L}_\ell^\kappa(x)}{1+\kappa\ell} + \sum_{\ell=0}^{r-1} \frac{\kappa r!}{(1+\ell\kappa)(1+(\ell+1)\kappa)} \left(\sum_{i=\ell+1}^{r} (-1)^i \binom{r}{i} \right) \widetilde{L}_\ell^\kappa(x). \tag{3.21}$$

For the inner alternating binomial sum we use

$$\sum_{i=\ell+1}^{r} (-1)^i \binom{r}{i} = (-1)^{\ell+1} \frac{(r-1)!}{\ell!(r-\ell-1)!}, \qquad 0 \leq \ell \leq r-1, \tag{3.22}$$

Insert this binomial evaluation into (3.21); the resulting expression is

$$x^r = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} r!\, \frac{\widetilde{L}_\ell^\kappa(x)}{1+\kappa\ell} + \sum_{\ell=0}^{r-1} \frac{(-1)^{\ell+1}\kappa r!(r-1)!}{(1+\ell\kappa)(1+(\ell+1)\kappa)\ell!(r-\ell-1)!}\, \widetilde{L}_\ell^\kappa(x), \tag{3.23}$$

Separate the last term corresponding to $\ell = r$. The remaining part becomes

$$\begin{aligned} x^r \;=\; & \frac{(-1)^r r! \widetilde{L}_r^\kappa(x)}{1+\kappa r} + \sum_{\ell=0}^{r-1} \left[(-1)^\ell \binom{r}{\ell} r!\, \frac{1}{1+\kappa\ell} \right. \\ & \left. + \frac{(-1)^{\ell+1}\kappa r!(r-1)!}{(1+\ell\kappa)(1+(\ell+1)\kappa)\ell!(r-\ell-1)!} \right] \widetilde{L}_\ell^\kappa(x). \end{aligned} \tag{3.24}$$

After combining the two rational terms over one denominator, we arrive at

$$x^r = \frac{(-1)^r r! \widetilde{L}_r^\kappa(x)}{1+\kappa r} + \sum_{\ell=0}^{r-1} (-1)^\ell \frac{\big(r+(r+1)\kappa\ell\big) r!(r-1)!}{(1+\ell\kappa)(1+(\ell+1)\kappa)\ell!(r-\ell)!}\, \widetilde{L}_\ell^\kappa(x). \tag{3.25}$$

Finally, reverse the order of the index in the last finite sum. This gives

$$x^r = \frac{(-1)^r r! \widetilde{L}_r^\kappa(x)}{1+\kappa r} + \sum_{s=1}^{r} \frac{(-1)^{r-s}\big(r+(1+r)\kappa(r-s)\big) r!(r-1)!}{\big(1+\kappa(1+r-s)\big)\big(1+r\kappa-\kappa s\big)(r-s)!s!}\, \widetilde{L}_{r-s}^\kappa(x). \tag{3.26}$$

This establishes the assertion. □

4. Differentiation in the LTP and Laguerre Bases

The differentiation results below are organized in three forms:

- higher derivatives of LTPs in the LTP basis;
- higher derivatives of LTPs in the classical Laguerre basis;
- higher derivatives of classical Laguerre polynomials in the LTP basis.

4.1. **An LTP-to-LTP differentiation formula.** Before deriving the differentiation formula, we introduce a finite identity that will be used to reduce its coefficient.

Lemma 4.1. *If $n \geq \nu \geq 1$ are integers and $0 \leq s \leq n-\nu-1$, then*

$$\begin{aligned}
&\sum_{i=0}^{s} \frac{(-1)^{s+\nu+i}(n-\nu-i-1)!}{(1+n-i)!(s-i)!i!}\big(1+n+(1+n)\kappa(n-i)-i\big) \\
&\qquad \times\big(n-\nu+\kappa(n-s-\nu)(1+n-\nu-i)-i\big) \\
&= \frac{(-1)^{\nu}\,\nu(\nu+2)_{s-2}(n-\nu-s)!}{s!\,n!} \\
&\qquad \times\Big(1+\nu+\kappa(2n-s-\nu+1)(1+\nu) \\
&\qquad\quad +\kappa^2\big(n(n-s-\nu+1)(1+\nu)+s(s+\nu)\big)\Big).
\end{aligned} \tag{4.1}$$

Proof. Let

$$\begin{aligned}
\mathsf{S}_{s,n,\nu} &= \sum_{i=0}^{s} \frac{(-1)^{s+\nu+i}}{(1+n-i)!(s-i)!i!}\Big(1+n+(1+n)\kappa(n-i)-i\Big) \\
&\qquad \times\Big(n-\nu+\kappa(n-s-\nu)(1+n-\nu-i)-i\Big)(n-\nu-i-1)!.
\end{aligned} \tag{4.2}$$

Applying Zeilberger summation to this finite term produces the following recurrence for the auxiliary quantity

$$\begin{aligned}
&(s+1)(n-\nu-s)\Big(1+\nu+\kappa(2n-s-\nu+1)(1+\nu) \\
&\qquad +\kappa^2\big(n(n-s-\nu+1)(1+\nu)+s(s+\nu)\big)\Big)\mathsf{S}_{s+1,n,\nu} \\
&= (s+\nu)\Big(1+\nu+\kappa(2n-s-\nu)(1+\nu) \\
&\qquad +\kappa^2\big(n(n-s-\nu)(1+\nu)+(s+1)(s+\nu+1)\big)\Big)\mathsf{S}_{s,n,\nu},
\end{aligned} \tag{4.3}$$

subject to

$$\mathsf{S}_{0,n,\nu} = \frac{(-1)^{\nu}\,(1+n\kappa)(1+(1+n-\nu)\kappa)\,(n-\nu)!}{n!}. \tag{4.4}$$

Its closed solution is

$$\begin{aligned}
\mathsf{S}_{s,n,\nu} &= \frac{(-1)^{\nu}\,\nu(\nu+2)_{s-2}(n-\nu-s)!}{s!\,n!} \times \\
&\Big(1+\nu+\kappa(2n-s-\nu+1)(1+\nu)+\kappa^2\big(n(n-s-\nu+1)(1+\nu)+s(s+\nu)\big)\Big),
\end{aligned}$$

which verifies the lemma. □

Theorem 4.2. *For integers $n \geq \nu \geq 1$, the derivative of order ν has the LTP expansion*

$$\begin{aligned}
D^{\nu}\widetilde{L}_n^{\kappa}(x) &= (-1)^{\nu}n!\sum_{i=0}^{n-\nu}\frac{(1+n)(1+n\kappa)-(1+\kappa+n\kappa)i}{(1+n\kappa-\kappa(\nu+i))i!(n-i+1)!}\,\widetilde{L}_{n-i-\nu}^{\kappa}(x) \\
&\quad +(-1)^{\nu}\sum_{s=1}^{n-\nu}\frac{1}{(1+n\kappa-\kappa(s+\nu))s!}\Bigg(-\frac{(1+n-s+(1+n)(n-s)\kappa)n!}{(n-s+1)!}
\end{aligned}$$

$$+\frac{\nu\,\Omega_{s,n,\nu}(s+\nu-1)!}{(1+\kappa(1+n-s-\nu))(\nu+1)!}\Bigg)\widetilde{L}^{\kappa}_{n-\nu-s}(x), \tag{4.5}$$

where

$$\Omega_{s,n,\nu} := 1+\nu+\kappa\big(1-s+n^2\kappa(1+\nu)+(s\kappa-\nu)(s+\nu)-n(1+\nu)(-2+\kappa(-1+s+\nu))\big). \tag{4.6}$$

Proof. Differentiate the finite LTP power series term by term to obtain

$$D^{\nu}\widetilde{L}^{\kappa}_{n}(x) = n!\sum_{i=0}^{n-\nu}\frac{(-1)^{n-i}\big(1+n+(1+n)\kappa(n-i)-i\big)(1+n-\nu-i)_{\nu}}{(n-i)!(1+n-i)!i!}\,x^{n-i-\nu}. \tag{4.7}$$

Express each remaining monomial using

$$x^{n-i-\nu} = \sum_{v=0}^{n-i-\nu} C_{v}^{(n,i,\nu)}\,\widetilde{L}^{\kappa}_{n-i-\nu-v}(x), \tag{4.8}$$

where

$$\begin{aligned} C_{v}^{(n,i,\nu)} = (-1)^{n-\nu-i-v}&\frac{n-\nu-i+\kappa(1+n-\nu-i)(n-\nu-i-v)}{(1+\kappa(1+n-\nu-i-v))(1+\kappa(n-\nu-i)-\kappa v)} \\ &\times\frac{(n-\nu-i-1)!(n-\nu-i)!}{(n-\nu-i-v)!v!}. \end{aligned} \tag{4.9}$$

After this substitution, the derivative gives

$$\begin{aligned} D^{\nu}\widetilde{L}^{\kappa}_{n}(x) = \quad &\sum_{i=0}^{n-\nu}\frac{(-1)^{\nu}n!}{(1+n\kappa-\kappa(\nu+i))(1+n-i)!i!}\big(1+n+(1+n)\kappa(n-i)-i\big) \\ &\times\widetilde{L}^{\kappa}_{n-i-\nu}(x) \\ &+\sum_{i=0}^{n-\nu}\frac{(-1)^{n-i}n!}{(n-i)!(1+n-i)!i!}\big(1+n+(1+n)\kappa(n-i)-i\big) \\ &\times(1+n-\nu-i)_{\nu}\sum_{v=1}^{n-i-\nu} C_{v}^{(n,i,\nu)}\widetilde{L}^{\kappa}_{n-i-\nu-v}(x). \end{aligned} \tag{4.10}$$

Set $s=i+v$ and interchange the finite summations:

$$\begin{aligned} D^{\nu}\widetilde{L}^{\kappa}_{n}(x) = \quad &\sum_{i=0}^{n-\nu}\frac{(-1)^{\nu}n!}{(1+n\kappa-\kappa(\nu+i))(1+n-i)!i!} \\ &\times\big(1+n+(1+n)\kappa(n-i)-i\big)\widetilde{L}^{\kappa}_{n-i-\nu}(x) \\ &+\sum_{s=1}^{n-\nu}\Bigg(\sum_{i=0}^{s-1}\frac{(-1)^{s+\nu+i}}{(1+\kappa(1+n-s-\nu))(1+n\kappa-s\kappa-\kappa\nu)} \\ &\times\big(1+n+(1+n)\kappa(n-i)-i\big) \\ &\times\big(n-\nu+\kappa(n-s-\nu)(1+n-\nu-i)-i\big) \\ &\times\frac{n!(n-\nu-i-1)!}{(n-s-\nu)!(1+n-i)!(s-i)!i!}\Bigg)\widetilde{L}^{\kappa}_{n-\nu-s}(x). \end{aligned} \tag{4.11}$$

Lemma 4.1 reduces the inner coefficient to

$$\begin{aligned}
&\sum_{i=0}^{s-1}\frac{(-1)^{s+\nu+i}(n-\nu-i-1)!}{(1+n-i)!(s-i)!i!}\big(1+n+(1+n)\kappa(n-i)-i\big)\\
&\qquad\times\big(n-\nu+\kappa(n-s-\nu)(1+n-\nu-i)-i\big)\\
&-\frac{(-1)^{\nu}(n-s-\nu)!}{n!s!}\\
&\qquad\times\Big(-\frac{(1+n-s+(1+n)(n-s)\kappa)n!}{(n-s+1)!}\big(1+\kappa(1+n-s-\nu)\big)\\
&\qquad+\frac{\nu\,\Omega_{s,n,\nu}(s+\nu-1)!}{(\nu+1)!}\Big).
\end{aligned}\tag{4.12}$$

Inserting the simplified coefficient into the preceding expansion yields (4.5). □

Theorem 4.3. *If $1\le\nu\le n$, the same derivative in the Laguerre basis is*

$$\frac{d^{\nu}}{dx^{\nu}}\widetilde{L}_n^{\kappa}(x)=\sum_{s=0}^{n-\nu}\frac{(-1)^{\nu}}{s!}\,(\nu+1)_{s-1}\,\big(\nu+\kappa(n\nu-s)\big)\,\mathsf{L}_{n-\nu-s}(x).\tag{4.13}$$

Proof. Differentiate each monomial in the finite power representation separately. This leads to

$$\frac{d^{\nu}}{dx^{\nu}}\widetilde{L}_n^{\kappa}(x)=\sum_{i=0}^{n-\nu}\frac{(-1)^{n-i}\,n!\,\big(1+(n-i)(1+\kappa(n+1))\big)}{(n-i+1)!\,i!\,(n-i-\nu)!}\,x^{\,n-i-\nu}.\tag{4.14}$$

Next use the monomial inversion in the Laguerre basis

$$x^{\,n-i-\nu}=(n-i-\nu)!\sum_{v=0}^{n-i-\nu}(-1)^{\,n-i-\nu-v}\binom{n-i-\nu}{v}\,\mathsf{L}_{\,n-i-\nu-v}(x).\tag{4.15}$$

which changes (4.14) into

$$\begin{aligned}
\frac{d^{\nu}}{dx^{\nu}}\widetilde{L}_n^{\kappa}(x)\;&=\;\sum_{i=0}^{n-\nu}\frac{(-1)^{n-i}\,n!\,(n-\nu-i)!}{(1+n-i)!\,i!}\big(1+(n-i)(1+\kappa(n+1))\big)\\
&\qquad\times\sum_{v=0}^{n-i-\nu}\frac{(-1)^{n-\nu-i-v}}{(n-\nu-i-v)!\,v!}\,\mathsf{L}_{\,n-i-\nu-v}(x).
\end{aligned}\tag{4.16}$$

After setting $s=i+v$ and reversing the summations,

$$\begin{aligned}
\frac{d^{\nu}}{dx^{\nu}}\widetilde{L}_n^{\kappa}(x)\;&=\;\sum_{s=0}^{n-\nu}\frac{1}{(n-\nu-s)!}\Bigg(\sum_{i=0}^{s}\frac{(-1)^{s+\nu+i}\,n!\,(n-\nu-i)!}{(n-i+1)!\,(s-i)!\,i!}\\
&\qquad\times\big((n-i+1)+\kappa(n+1)(n-i)\big)\Bigg)\\
&\qquad\times\mathsf{L}_{\,n-\nu-s}(x).
\end{aligned}\tag{4.17}$$

Denote the inner finite coefficient by

$$R_s=\sum_{i=0}^{s}\frac{(-1)^{s+\nu+i}\,n!\,(n-\nu-i)!\,\big((n-i+1)+\kappa(n+1)(n-i)\big)}{(n-i+1)!\,(s-i)!\,i!}.\tag{4.18}$$

Zeilberger summation applied to the inner finite sum provides the recurrence

$$(s+1)(n-\nu-s)\big(\nu+\kappa(n\nu-s)\big)\,R_{s+1,n,\nu}=(s+\nu)\big(\nu+\kappa(n\nu-s-1)\big)\,R_{s,n,\nu}, \tag{4.19}$$

together with the initial value

$$R_{0,n,\nu}=(-1)^{\nu}(1+n\kappa)(n-\nu)!.$$

The recurrence, together with its starting value, has the closed solution

$$R_s=\frac{(-1)^{\nu}}{s!}\,(\nu+1)_{s-1}\,(n-\nu-s)!\,\big(\nu+\kappa(n\nu-s)\big). \tag{4.20}$$

Replacing the auxiliary coefficient in (4.17) by its closed form gives

$$\frac{d^{\nu}}{dx^{\nu}}\widetilde{L}_n^{\kappa}(x)=\sum_{s=0}^{n-\nu}\frac{(-1)^{\nu}\,(\nu+1)_{s-1}\,\big(\nu+\kappa(n\nu-s)\big)}{s!}\,\mathsf{L}_{n-\nu-s}(x). \tag{4.21}$$

This proves the Laguerre-basis formula. □

We now express the same derivative through several comparison bases; only the basis-conversion step changes from one case to another.

5. Derivative Connections with Other Polynomial Bases

Theorem 5.1. *For $n,\nu\in\mathbb{N}$ with $1\le\nu\le n$, the Bernoulli expansion is*

$$\begin{aligned}\frac{d^{\nu}}{dx^{\nu}}\widetilde{L}_n^{\kappa}(x)=n!\sum_{s=0}^{n-\nu}&\frac{(-1)^{n-s+1}\,(s-n-1)\big(s+\kappa-n^2\kappa+s\kappa+n(-1+s\kappa)\big)}{(s+1)\,s!\,(n-s+1)!\,(n-s-\nu)!}\\&\times\left[-1+{}_2F_2\left(\begin{matrix}-1-s,\;n-s+\dfrac{1}{1+\kappa+n\kappa}\\ 1+n-s,\;n-s-1+\dfrac{1}{1+\kappa+n\kappa}\end{matrix}\,\middle|\,1\right)\right]\mathsf{B}_{n-\nu-s}(x).\end{aligned} \tag{5.1}$$

Proof. The formula is obtained by combining the series representation of the LTPs in (3.17) and the inverse power expansion for the Bernoulli polynomial family. □

Theorem 5.2. *Under the same restrictions, the normalized Jacobi representation is*

$$\frac{d^{\nu}}{dx^{\nu}}\widetilde{L}_n^{\kappa}(x)=\sum_{s=0}^{n-\nu}\mathsf{A}_{s,n,\nu}^{(\alpha,\beta)}\,\mathsf{J}_{n-\nu-s}^{(\alpha,\beta)}(x), \tag{5.2}$$

where

$$\mathsf{A}_{s,n,\nu}^{(\alpha,\beta)}=\sum_{i=0}^{s}\frac{(-1)^{n-i}n!\big(1+(n-i)(1+\kappa(n+1))\big)}{i!(n-i+1)!(n-i-\nu)!}\,\mathcal{J}_{n-i-\nu,n-\nu-s}^{(\alpha,\beta)},$$

and, for $0\le k\le m$,

$$\begin{aligned}\mathcal{J}_{m,k}^{(\alpha,\beta)}=2^k\binom{m}{k}(2k+\alpha+\beta+1)\frac{\Gamma(k+\alpha+\beta+1)\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)\Gamma(k+\beta+1)}\\\times\sum_{j=0}^{m-k}(-1)^{m-k-j}2^j\binom{m-k}{j}\frac{\Gamma(k+\beta+j+1)}{\Gamma(2k+\alpha+\beta+j+2)}.\end{aligned}$$

Proof. The formula is obtained by combining the series representation of the LTPs in (3.17) and the inverse power expansion for the normalized Jacobi polynomial family. □

Theorem 5.3. *For $0 \le \nu \le n$, the Hermite-basis form is*

$$\begin{aligned}\frac{d^{\nu}}{dx^{\nu}}\widetilde{L}_n^{\kappa}(x) &= \sum_{s=0}^{\lfloor\frac{n-\nu}{2}\rfloor} \mathsf{a}_{s,n,\nu}\,\mathsf{H}_{n-\nu-2s}(x) \\ &\quad + \sum_{s=0}^{\lfloor\frac{n-\nu-1}{2}\rfloor} \mathsf{b}_{s,n,\nu}\,\mathsf{H}_{n-\nu-2s-1}(x),\end{aligned} \tag{5.3}$$

where the even-parity coefficient is

$$\mathsf{a}_{s,n,\nu} = \frac{(-1)^n\,2^{-n+\nu}\,(1+n\kappa)}{s!\,(n-2s-\nu)!}\,{}_4F_2\left(\left.\begin{matrix}-\frac{1}{2}-\frac{n}{2},\ -\frac{n}{2},\ -s,\ 1-\frac{n}{2}-\frac{1}{2(1+\kappa+n\kappa)} \\ \frac{1}{2},\ -\frac{n}{2}-\frac{1}{2(1+\kappa+n\kappa)}\end{matrix}\right| -4\right), \tag{5.4}$$

whereas the odd-parity coefficient is

$$\mathsf{b}_{s,n,\nu} = \frac{(-1)^{n+1}\,2^{1-n+\nu}\,(n-\kappa+n^2\kappa)}{s!\,(n-2s-\nu-1)!}\,{}_4F_2\left(\left.\begin{matrix}\frac{1}{2}-\frac{n}{2},\ -\frac{n}{2},\ -s,\ 1+\frac{\kappa-n(1+n\kappa)}{2(1+\kappa+n\kappa)} \\ \frac{3}{2},\ \frac{\kappa-n(1+n\kappa)}{2(1+\kappa+n\kappa)}\end{matrix}\right| -4\right). \tag{5.5}$$

Proof. The formula is obtained by combining the series representation of the LTPs in (3.17) and the inverse power expansion for the Hermite polynomial family. □

Theorem 5.4. *For $0 \le \nu \le n$, the expansion of the ν-th derivative of $\widetilde{L}_n^{\kappa}(x)$ in the normalized ultraspherical basis is*

$$\begin{aligned}\frac{d^{\nu}}{dx^{\nu}}\widetilde{L}_n^{\kappa}(x) &= \sum_{s=0}^{\lfloor\frac{n-\nu}{2}\rfloor} \mathsf{u}_s^{(n,\nu)}\,\mathsf{C}_{n-\nu-2s}^{(\lambda)}(x) \\ &\quad + \sum_{s=0}^{\lfloor\frac{n-\nu-1}{2}\rfloor} \mathsf{v}_s^{(n,\nu)}\,\mathsf{C}_{n-\nu-2s-1}^{(\lambda)}(x),\end{aligned} \tag{5.6}$$

where, on putting

$$\Delta_n = 1+\kappa(n+1),$$

the even-parity coefficient is

$$\begin{aligned}\mathsf{u}_s^{(n,\nu)} &= \frac{(-1)^n\,2^{-n+\nu}(1+n\kappa)\Gamma(\lambda)\big(n-\nu-2s+\lambda\big)(2\lambda)_{n-\nu-2s}}{s!\,(n-\nu-2s)!\,\Gamma(n-\nu-s+\lambda+1)} \\ &\quad \times{}_5F_2\left(\left.\begin{matrix}-s,\ -\frac{n}{2},\ -\frac{n+1}{2},\ s+\nu-n-\lambda,\ 1-\frac{n}{2}-\frac{1}{2\Delta_n} \\ \frac{1}{2},\ -\frac{n}{2}-\frac{1}{2\Delta_n}\end{matrix}\right| 4\right),\end{aligned} \tag{5.7}$$

and the odd-parity coefficient is

$$\mathsf{v}_s^{(n,\nu)} = \frac{(-1)^{n-1}\,2^{-n+\nu+1}\big(n+\kappa(n^2-1)\big)\Gamma(\lambda)\big(n-\nu-2s-1+\lambda\big)(2\lambda)_{n-\nu-2s-1}}{s!\,(n-\nu-2s-1)!\,\Gamma(n-\nu-s+\lambda)}$$

$$\times {}_5F_2\left(\left.\begin{matrix}-s,\ -\frac{n}{2},\ \frac{1-n}{2},\ 1+s+\nu-n-\lambda,\ \frac{3-n}{2}-\frac{1}{2\Delta_n}\\ \frac{3}{2},\ \frac{1-n}{2}-\frac{1}{2\Delta_n}\end{matrix}\right|\ 4\right). \tag{5.8}$$

Both hypergeometric series terminate because of the upper parameter $-s$.

Proof. Using (3.17), differentiate termwise and expand the resulting monomials in the normalized ultraspherical basis. Separating the even and odd powers and simplifying the corresponding finite sums yields the two terminating ${}_5F_2$ coefficients in (5.7) and (5.8), respectively. □

Theorem 5.5. *For $0 \le \nu \le n$, the generalized Fibonacci representation is*

$$\begin{aligned}\frac{d^\nu}{dx^\nu}\widetilde{L}_n^\kappa(x) &= \sum_{s=0}^{\lfloor\frac{n-\nu}{2}\rfloor} \mathsf{f}_s^{(n,\nu)}\,\mathsf{F}_{n-\nu-2s}^{(a,b)}(x)\\ &\quad+\sum_{s=0}^{\lfloor\frac{n-\nu-1}{2}\rfloor} \mathsf{g}_s^{(n,\nu)}\,\mathsf{F}_{n-\nu-2s-1}^{(a,b)}(x),\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}\mathsf{f}_s^{(n,\nu)} &= \frac{(-1)^{n+s}\,a^{-n+\nu}\,b^s\,(1+n\kappa)\,(1+n-2s-\nu)}{s!\,(n-s-\nu+1)!}\\ &\quad\times {}_5F_2\left(\left.\begin{matrix}-\frac{1}{2}-\frac{n}{2},\ -\frac{n}{2},\ -s,\ 1-\frac{n}{2}-\frac{1}{2(1+\kappa+n\kappa)},\ -1-n+s+\nu\\ \frac{1}{2},\ -\frac{n}{2}-\frac{1}{2(1+\kappa+n\kappa)}\end{matrix}\right|\ -\frac{a^2}{b}\right),\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\mathsf{g}_s^{(n,\nu)} &= \frac{(-1)^{n+s}\,a^{\,1-n+\nu}\,b^s\,(1+\kappa+n\kappa)\,\sqrt{\pi}\,(n-2s-\nu)}{s!\,(n-s-\nu)!}\\ &\quad\times\Gamma\left(\frac{3-n-\dfrac{1}{1+\kappa+n\kappa}}{2}\right)\\ &\quad\times {}_5\tilde{F}_2\left(\left.\begin{matrix}\frac{1-n}{2},\ -\frac{n}{2},\ -s,\ \dfrac{3-n-\dfrac{1}{1+\kappa+n\kappa}}{2},\ -n+s+\nu\\ \frac{3}{2},\ \dfrac{\kappa-n(1+n\kappa)}{2(1+\kappa+n\kappa)}\end{matrix}\right|\ -\frac{a^2}{b}\right).\end{aligned} \tag{5.11}$$

Proof. The formula is obtained by combining the series representation of the LTPs in (3.17) and the inverse power expansion for the generalized Fibonacci polynomial family. □

6. Repeated Antiderivatives in the Laguerre and LTP Bases

This section contains two principal results for repeated antiderivatives of the Laguerre-type polynomials: the first gives their representation in the classical Laguerre basis, while the second expresses them in their original LTP basis.

Theorem 6.1. *We first express the repeated antiderivative in the classical Laguerre basis. For $n, \nu \in \mathbb{N}$, denote a ν-fold primitive of $\widetilde{L}_n^{\kappa}(x)$ by $\mathsf{S}_n^{\nu}(x)$, namely*

$$\mathsf{S}_n^{\nu}(x) \quad = \quad \underbrace{\int\int\cdots\int}_{\nu \text{ times}} \widetilde{L}_n^{\kappa}(x)\,dx. \tag{6.1}$$

Its representation in ordinary Laguerre polynomials is

$$\mathsf{S}_n^{\nu}(x) = \sum_{\ell=0}^{n} \frac{(-1)^{\ell+\nu}\,(\nu-\ell+1)_{\ell-1}\,\big(\nu+\kappa(\ell+n\nu)\big)}{\ell!}\,\mathsf{L}_{n+\nu-\ell}(x) + \rho_{\nu-1}(x), \tag{6.2}$$

where $\rho_{\nu-1}$ is an arbitrary polynomial of degree no greater than $\nu-1$.

Proof. Apply ν antiderivatives separately to every monomial in (3.17). The polynomial part is then

$$\mathsf{S}_n^{\nu}(x) = n!\sum_{i=0}^{n} \frac{(-1)^{n-i}\,\big(1+n+(1+n)\kappa(n-i)-i\big)}{(n-i)!\,(n-i+1)!\,i!\,(n-i+1)_{\nu}}\,x^{n-i+\nu} + \rho_{\nu-1}(x), \tag{6.3}$$

where $\deg \rho_{\nu-1} \le \nu-1$. Each resulting monomial is converted to the Laguerre basis by (2.5), leading to

$$\begin{aligned}\mathsf{S}_n^{\nu}(x) \quad &= \quad n!\sum_{i=0}^{n} \frac{(-1)^{n-i}\,\big(1+n+(1+n)\kappa(n-i)-i\big)}{(n-i)!\,(n-i+1)!\,i!\,(n-i+1)_{\nu}} \\ &\quad \times \sum_{v=0}^{n+\nu-i} \frac{(-1)^{n+\nu-i-v}\,\big((n+\nu-i)!\big)^2}{(n+\nu-i-v)!\,v!}\,\mathsf{L}_{n-i+\nu-v}(x) + \rho_{\nu-1}(x).\end{aligned} \tag{6.4}$$

Put $\ell = i+v$ and interchange the summations:

$$\begin{aligned}\mathsf{S}_n^{\nu}(x) \quad &= \quad n!\sum_{\ell=0}^{n} \frac{1}{(n-\ell+\nu)!}\left(\sum_{i=0}^{\ell} \frac{(-1)^{\ell+\nu+i}(n+\nu-i)!}{(n-i+1)!\,(\ell-i)!\,i!}\big(1+n+(1+n)\kappa(n-i)-i\big)\right) \\ &\quad \times \mathsf{L}_{n+\nu-\ell}(x) \\ &\quad +\rho_{\nu-1}(x).\end{aligned} \tag{6.5}$$

Define the finite coefficient

$$\mathcal{S}_{\ell,n,\nu} = \sum_{i=0}^{\ell} \frac{(-1)^{\ell+\nu+i}\,\big(1+n+(1+n)\kappa(n-i)-i\big)\,(n+\nu-i)!}{(n-i+1)!\,(\ell-i)!\,i!}. \tag{6.6}$$

With this notation, (6.5) becomes

$$\mathsf{S}_n^{\nu}(x) = n!\sum_{\ell=0}^{n} \frac{\mathcal{S}_{\ell,n,\nu}}{(n-\ell+\nu)!}\,\mathsf{L}_{n+\nu-\ell}(x) + \rho_{\nu-1}(x). \tag{6.7}$$

The finite coefficient can be evaluated by Zeilberger summation, and its closed form is

$$\mathcal{S}_{\ell,n,\nu} = \frac{(-1)^{\ell+\nu}\,(\nu-\ell+1)_{\ell-1}\,\big(\nu+\kappa(\ell+n\nu)\big)\,(n+\nu-\ell)!}{\ell!\,n!}. \tag{6.8}$$

Replacing the auxiliary sum by this closed value produces the announced expansion:

$$\mathsf{S}_n^{\nu}(x) = \sum_{\ell=0}^{n} \frac{(-1)^{\ell+\nu}\,(\nu-\ell+1)_{\ell-1}\,\big(\nu+\kappa(\ell+n\nu)\big)}{\ell!}\,\mathsf{L}_{n+\nu-\ell}(x) + \rho_{\nu-1}(x).$$

The stated identity follows. □

To derive the repeated-antiderivative formula in the original LTP basis, we first establish the following finite identity, which is used to simplify the coefficients arising in that representation.

Lemma 6.2. *For every integer $\ell \geq 1$, define*

$$\begin{aligned}\mathcal{Q}^{(\kappa)}_{\ell,n,\nu} &= \sum_{i=0}^{\ell} \frac{(-1)^{\ell+\nu+i}(n+\nu-i-1)!}{(\ell-i)!\,i!\,(n-i+1)!}\big(1+n+(1+n)\kappa(n-i)-i\big)\\ &\quad\times\big(n+\nu+\kappa(n-\ell+\nu)(1+n+\nu-i)-i\big).\end{aligned} \tag{6.9}$$

Then

$$\mathcal{Q}^{(\kappa)}_{\ell,n,\nu} = \begin{cases} (-1)^{\nu+1}\nu\dfrac{(n+\nu-1)!}{n!}\Big[1+\kappa(2n+\nu)+\kappa^2(n^2+n\nu+1)\Big], & \ell=1,\\[2ex] \dfrac{(-1)^{\ell+\nu}\nu(n+\nu-\ell)!(\nu-\ell+1)_{\ell-2}}{\ell!\,n!}\,\mathcal{B}^{(\kappa)}_{\ell,n,\nu}, & \ell\geq 2,\end{cases}$$

where

$$\mathcal{B}^{(\kappa)}_{\ell,n,\nu} = -1+\nu+\kappa(\nu-1)(-\ell+2n+\nu+1)+\kappa^2\Big[\ell(-\ell+\nu)+n(\nu-1)(-\ell+n+\nu+1)\Big].$$

Proof. Applying Zeilberger's algorithm to the summand in (6.9) yields the first-order recurrence

$$\begin{aligned}&(\ell+1)(n+\nu-\ell)\mathcal{B}^{(\kappa)}_{\ell,n,\nu}\mathcal{Q}^{(\kappa)}_{\ell+1,n,\nu}\\ &\quad+(\nu-\ell)\mathcal{B}^{(\kappa)}_{\ell+1,n,\nu}\mathcal{Q}^{(\kappa)}_{\ell,n,\nu}=0,\end{aligned}$$

with the initial value

$$\mathcal{Q}^{(\kappa)}_{1,n,\nu} = (-1)^{\nu+1}\nu\frac{(n+\nu-1)!}{n!}\Big[1+\kappa(2n+\nu)+\kappa^2(n^2+n\nu+1)\Big].$$

Solving this recurrence gives

$$\mathcal{Q}^{(\kappa)}_{\ell,n,\nu} = \frac{(-1)^{\ell+\nu}\nu(n+\nu-\ell)!(\nu-\ell+1)_{\ell-2}}{\ell!\,n!}\,\mathcal{B}^{(\kappa)}_{\ell,n,\nu},\qquad \ell\geq 2,$$

which, together with the initial value, proves the result. □

Theorem 6.3. *For $n,\nu\in\mathbb{N}$, the repeated antiderivative of $\widetilde{L}^{\kappa}_{n}(x)$ admits the following representation in the original LTP basis:*

$$\begin{aligned}\mathsf{S}^{\nu}_{n}(x) &= \sum_{i=0}^{n}\frac{(-1)^{\nu}n!(n+\nu-i)!}{\big(1+\kappa(n+\nu-i)\big)(n-i)!(n-i+1)!\,i!\,(n-i+1)_{\nu}}\\ &\quad\times\big(1+n+(1+n)\kappa(n-i)-i\big)\widetilde{L}^{\kappa}_{n-i+\nu}(x)\\ &\quad+\sum_{\ell=1}^{n}\frac{n!}{\big(1+\kappa(n-\ell+\nu)\big)\big(1+\kappa(1+n-\ell+\nu)\big)(n-\ell+\nu)!}\\ &\quad\times\Bigg[\mathcal{Q}^{(\kappa)}_{\ell,n,\nu}-\frac{(-1)^{\nu}(n+\nu-\ell)!}{\ell!(n-\ell+1)!}\\ &\qquad\times\Big((1+n)(1+n\kappa)-(1+\kappa+n\kappa)\ell\Big)\Big(1+\kappa(1+n-\ell+\nu)\Big)\Bigg]\widetilde{L}^{\kappa}_{n+\nu-\ell}(x)\\ &\quad+\rho_{\nu-1}(x),\end{aligned} \tag{6.10}$$

where $\rho_{\nu-1}(x)$ is an arbitrary polynomial of degree not exceeding $\nu-1$, and $\mathcal{Q}^{(\kappa)}_{\ell,n,\nu}$ is given explicitly in Lemma 6.2.

Proof. Integrating the finite power expansion (3.17) successively ν times gives

$$\begin{aligned}\mathsf{S}^{\nu}_{n}(x) &= n!\sum_{i=0}^{n}\frac{(-1)^{n-i}\big(1+n+(1+n)\kappa(n-i)-i\big)}{(n-i)!(n-i+1)!\,i!\,(n-i+1)_{\nu}}x^{n-i+\nu}\\ &\quad+\rho_{\nu-1}(x),\end{aligned}\tag{6.11}$$

where

$$\deg\rho_{\nu-1}\le\nu-1.$$

Expanding every monomial in (6.11) by means of (3.18) gives

$$\begin{aligned}\mathsf{S}^{\nu}_{n}(x) &= \sum_{i=0}^{n}\frac{(-1)^{n-i}\big(1+n+(1+n)\kappa(n-i)-i\big)n!}{(n-i)!(n-i+1)!\,i!\,(n-i+1)_{\nu}}\\ &\quad\times\Bigg(\frac{(-1)^{n+\nu-i}(n+\nu-i)!}{1+\kappa(n+\nu-i)}\widetilde{L}^{\kappa}_{n-i+\nu}(x)\\ &\quad+\sum_{\ell=1}^{n-i+\nu}\frac{(-1)^{n-\ell+\nu-i}\big(n+\nu+\kappa(1+n+\nu-i)(n-\ell+\nu-i)-i\big)}{\big(1-\kappa\ell+\kappa(n+\nu-i)\big)\big(1+\kappa(1+n-\ell+\nu-i)\big)}\\ &\quad\times\frac{(n+\nu-i-1)!(n+\nu-i)!}{\ell!(n-\ell+\nu-i)!}\widetilde{L}^{\kappa}_{n-i+\nu-\ell}(x)\Bigg)\\ &\quad+\rho_{\nu-1}(x).\end{aligned}\tag{6.12}$$

Collecting the coefficients of the same LTP basis elements, we obtain

$$\begin{aligned}\mathsf{S}^{\nu}_{n}(x) &= \sum_{i=0}^{n}\frac{(-1)^{\nu}\big(1+n+(1+n)\kappa(n-i)-i\big)n!(n+\nu-i)!}{\big(1+\kappa(n+\nu-i)\big)(n-i)!(n-i+1)!\,i!\,(n-i+1)_{\nu}}\widetilde{L}^{\kappa}_{n-i+\nu}(x)\\ &\quad+\sum_{\ell=1}^{n}\frac{n!}{\big(1+\kappa(n-\ell+\nu)\big)\big(1+\kappa(1+n-\ell+\nu)\big)(n-\ell+\nu)!}\\ &\quad\times\Bigg(\sum_{i=0}^{\ell-1}\frac{(-1)^{\ell+\nu+i}(n+\nu-i-1)!}{(\ell-i)!\,i!\,(n-i+1)!}\big(1+n+(1+n)\kappa(n-i)-i\big)\\ &\qquad\times\big(n+\nu+\kappa(n-\ell+\nu)(1+n+\nu-i)-i\big)\Bigg)\widetilde{L}^{\kappa}_{n+\nu-\ell}(x)\\ &\quad+\rho_{\nu-1}(x).\end{aligned}\tag{6.13}$$

By (6.9), the finite sum in (6.13) is the complete sum $\mathcal{Q}^{(\kappa)}_{\ell,n,\nu}$ with its terminal $i=\ell$ term removed. Therefore,

$$\begin{aligned}&\sum_{i=0}^{\ell-1}\frac{(-1)^{\ell+\nu+i}(n+\nu-i-1)!}{(\ell-i)!\,i!\,(n-i+1)!}\big(1+n+(1+n)\kappa(n-i)-i\big)\\ &\qquad\times\big(n+\nu+\kappa(n-\ell+\nu)(1+n+\nu-i)-i\big)\\ &= \mathcal{Q}^{(\kappa)}_{\ell,n,\nu}-\frac{(-1)^{\nu}(n+\nu-\ell)!}{\ell!(n-\ell+1)!}\\ &\qquad\times\Big((1+n)(1+n\kappa)-(1+\kappa+n\kappa)\ell\Big)\Big(1+\kappa(1+n-\ell+\nu)\Big).\end{aligned}\tag{6.14}$$

Substituting (6.14) into (6.13) and using the closed form for $\mathcal{Q}^{(\kappa)}_{\ell,n,\nu}$ obtained in Lemma 6.2 yields (6.10). □

7. Weighted-Integral Consequences

Corollary 7.1. *Let $n, \nu \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$, with $1 \le \nu \le n$ and $0 \le \ell \le n-\nu$. Then the Hermite projection of the derivative is*

$$\int_{-\infty}^{\infty} \frac{d^\nu \widetilde{L}_n^\kappa(x)}{dx^\nu}\, \mathsf{H}_\ell(x) e^{-x^2}\, dx = Z_{n,\nu,\ell}, \tag{7.1}$$

where

$$Z_{n,\nu,\ell} = \begin{cases} h_\ell\, \mathsf{a}_{\frac{n-\nu-\ell}{2},\,n,\,\nu}, & \text{if } n-\nu-\ell \text{ is even}, \\ h_\ell\, \mathsf{b}_{\frac{n-\nu-\ell-1}{2},\,n,\,\nu}, & \text{if } n-\nu-\ell \text{ is odd}, \end{cases} \tag{7.2}$$

and

$$h_\ell = 2^\ell \ell! \sqrt{\pi}. \tag{7.3}$$

More precisely, the Hermite orthogonality relation is

$$\int_{-\infty}^{\infty} \mathsf{H}_n(x)\mathsf{H}_\ell(x) e^{-x^2}\, dx = 2^\ell \ell! \sqrt{\pi}\, \delta_{n\ell} = h_\ell\, \delta_{n\ell}. \tag{7.4}$$

Proof. Start with (5.3):

$$\frac{d^\nu \widetilde{L}_n^\kappa(x)}{dx^\nu} = \sum_{s=0}^{\lfloor \frac{n-\nu}{2} \rfloor} \mathsf{a}_{s,n,\nu}\, \mathsf{H}_{n-\nu-2s}(x) + \sum_{s=0}^{\lfloor \frac{n-\nu-1}{2} \rfloor} \mathsf{b}_{s,n,\nu}\, \mathsf{H}_{n-\nu-2s-1}(x), \tag{7.5}$$

where $\mathsf{a}_{s,n,\nu}$ and $\mathsf{b}_{s,n,\nu}$ are defined in (5.4) and (5.5). Multiply by $\mathsf{H}_\ell(x)e^{-x^2}$ and then integrate over $\mathbb{R}$:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d^\nu \widetilde{L}_n^\kappa(x)}{dx^\nu}\, \mathsf{H}_\ell(x)\, e^{-x^2}\, dx \;=\; & \sum_{s=0}^{\lfloor \frac{n-\nu}{2} \rfloor} \mathsf{a}_{s,n,\nu} \int_{-\infty}^{\infty} \mathsf{H}_{n-\nu-2s}(x)\mathsf{H}_\ell(x) e^{-x^2}\, dx \\ & + \sum_{s=0}^{\lfloor \frac{n-\nu-1}{2} \rfloor} \mathsf{b}_{s,n,\nu} \int_{-\infty}^{\infty} \mathsf{H}_{n-\nu-2s-1}(x)\mathsf{H}_\ell(x) e^{-x^2}\, dx. \end{aligned} \tag{7.6}$$

Hermite orthogonality gives

$$\int_{-\infty}^{\infty} \mathsf{H}_n(x)\mathsf{H}_\ell(x) e^{-x^2}\, dx = h_\ell\, \delta_{n\ell}, \tag{7.7}$$

hence

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d^\nu \widetilde{L}_n^\kappa(x)}{dx^\nu}\, \mathsf{H}_\ell(x)\, e^{-x^2}\, dx \;=\; & \sum_{s=0}^{\lfloor \frac{n-\nu}{2} \rfloor} \mathsf{a}_{s,n,\nu}\, h_\ell\, \delta_{n-\nu-2s,\ell} \\ & + \sum_{s=0}^{\lfloor \frac{n-\nu-1}{2} \rfloor} \mathsf{b}_{s,n,\nu}\, h_\ell\, \delta_{n-\nu-2s-1,\ell}. \end{aligned} \tag{7.8}$$

For even $n-\nu-\ell$, we have

$$\int_{-\infty}^{\infty} \frac{d^{\nu}\widetilde{L}_n^{\kappa}(x)}{dx^{\nu}}\,\mathsf{H}_{\ell}(x)\,e^{-x^2}\,dx = h_{\ell}\,\mathsf{a}_{\frac{n-\nu-\ell}{2},\,n,\,\nu}. \tag{7.9}$$

For odd $n-\nu-\ell$, we have

$$\int_{-\infty}^{\infty} \frac{d^{\nu}\widetilde{L}_n^{\kappa}(x)}{dx^{\nu}}\,\mathsf{H}_{\ell}(x)\,e^{-x^2}\,dx = h_{\ell}\,\mathsf{b}_{\frac{n-\nu-\ell-1}{2},\,n,\,\nu}. \tag{7.10}$$

The two cases are summarized by

$$\int_{-\infty}^{\infty} \frac{d^{\nu}\widetilde{L}_n^{\kappa}(x)}{dx^{\nu}}\,\mathsf{H}_{\ell}(x)\,e^{-x^2}\,dx = Z_{n,\nu,\ell}, \tag{7.11}$$

which proves the assertion. □

Corollary 7.2. *Let $n,\nu,s\in\mathbb{N}$, $1\le\nu\le n$, and $0\le s\le n-\nu$. Then the Laguerre projection is*

$$\int_0^{\infty} e^{-x}\,\frac{d^{\nu}\widetilde{L}_n^{\kappa}(x)}{dx^{\nu}}\,\mathsf{L}_s(x)\,dx = \frac{(-1)^{\nu}\big(\nu+\kappa(-n+\nu+n\nu+s)\big)(n-s-1)!}{\nu!\,(n-\nu-s)!}. \tag{7.12}$$

Proof. Multiply (4.13) by $e^{-x}\mathsf{L}_s(x)$ and integrate. Laguerre orthogonality eliminates every term except the degree-s coefficient. □

8. Concluding Remarks

A finite operational framework has been established for the Laguerre-type sequence associated with a Laguerre measure containing a discrete mass. Starting from its defining relation, we obtained both directions of the connection with the ordinary Laguerre family and derived mutually inverse finite power representations. These identities were then used to express derivatives of arbitrary integer order in the LTP, Laguerre, Hermite, Bernoulli, normalized ultraspherical, and generalized Fibonacci bases. Repeated antiderivatives were treated in both the Laguerre and LTP bases, with the arising finite sums reduced to explicit coefficients. The final weighted-integral consequences follow by combining the derivative expansions with the relevant orthogonality relations. Since every coefficient appearing in these formulas is finite, the results can be used directly in symbolic calculations and can also support later spectral or operational-matrix constructions based on this mass-modified Laguerre family.

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