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ORDERED 1- AND 2-ALMOST IDEALS IN ORDERED b-SEMIRINGS: STRUCTURAL CHARACTERIZATIONS, REGULARITY, AND HOMOMORPHISMS

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Abstract. In this paper, we introduce and investigate ordered 1- and 2-almost ideals in ordered b-semirings, together with their bi-almost, quasi-almost, and bi-quasi-almost forms. The definitions retain the closure requirements of the corresponding nonordered notions while incorporating the lower-set condition induced by the partial order. We show that, for $i \in \{1, 2\}$, the ordered i-bi-almost, ordered i-quasi-almost, and ordered i-bi-quasi-almost ideals determine the same class, namely the class of nonempty lower sub-b-semirings. Thus,

$$\text{ordered } i\text{-BQAId} \iff \text{ordered } i\text{-BAId} \iff \text{ordered } i\text{-QAId} \implies \text{ordered } i\text{-AId},$$

and the final implication is not reversible in general. A finite example with a nontrivial partial order shows that the lower closure can play an essential role in the almost ideal property. We further prove that the common class is closed under nonempty intersections whenever the intersection is nonempty and under unions of chains, and we introduce the associated lower sub-b-semiring hull. Under operation-specific regularity, several ordinary ordered ideal classes admit fixed-point characterizations, including intersection–product relations for one-sided ideals. We also establish preservation results for ordered almost ideals under order-preserving b-semiring homomorphisms and obtain isomorphism invariance. Finally, the ordered 2-theory is derived systematically from the ordered 1-theory through operation duality.

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1. Introduction

Semirings form an important class of algebraic structures equipped with two associative binary operations connected by distributive laws. The notion of a semiring goes back to Vandiver [15].

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Since additive inverses are not required, semirings provide a natural framework for studying algebraic systems that are not covered by ring theory. The theory of semirings has developed in several directions, including ideal theory, ordered structures, regularity, and structural decompositions. For example, Adhikari and Das [1] studied structure spaces associated with semirings, providing a useful viewpoint for examining families of ideals and related substructures.

An ordered semiring is a semiring equipped with a partial order that is compatible with its operations. The presence of the order leads to a natural modification of ordinary ideal theory: in addition to an algebraic absorption condition, an ordered ideal-like subset is usually required to be a lower subset. Gan and Jiang [2] studied fundamental properties of ordered ideals in ordered semirings, while Han et al. [5] investigated semiring orders and their related properties. Regularity is another important structural condition in this setting. Mandal [7] considered fuzzy ideals in ordered semirings, and Palakawong na Ayutthaya and Pibaljommee [8,9] obtained characterizations of regular ordered semirings in terms of ordered quasi-ideals and related substructures.

Besides one-sided and two-sided ideals, weaker ideal-like structures such as bi-ideals and quasi-ideals have been studied extensively. Bi-ideals were considered in semigroup theory by Good and Hughes [4], while Kemprasit [6] investigated relationships among quasi-ideals, bi-ideals, and one-sided ideals in semigroups and rings. Such structures are useful because they weaken the ordinary absorption requirements while still retaining enough algebraic information to describe important parts of the ambient structure.

Chinram [3] introduced the notion of a b-semiring, which consists of two semigroup operations satisfying four distributive-type identities. An important feature of a b-semiring is the symmetry of its axioms under interchange of the two operations. Consequently, two parallel ideal structures arise naturally. In the first structure, $\uplus$ provides the internal closure while $\oslash$ occurs in the ideal conditions. In the second structure, the roles of $\uplus$ and $\oslash$ are interchanged.

The ideal theory of b-semirings has recently been developed in several directions. Palanikumar et al. [11] investigated various classes of one-sided ideals in b-semirings. Suguna et al. [12] studied several types of bi-quasi-ideals and their extensions, while Suguna et al. [13] considered generalized m-bi-ideals and obtained related characterizations. In another direction, Suguna et al. [14] investigated different types of almost ideals in b-semirings. These studies indicate that one-sided ideals, bi-ideals, quasi-ideals, bi-quasi-ideals, and almost ideals form a closely connected family of substructures in the two-operation setting.

Among these notions, almost ideals are central to the present work. Palanikumar and Arulmozhi [10] introduced several types of almost ideals in semirings. In contrast with ordinary ideals, which are defined by absorption inclusions, almost ideals replace the corresponding inclusion by a nonempty-intersection condition. Thus, almost ideals provide weaker ideal-like structures. When such notions

are transferred to an ordered setting, however, two issues must be treated carefully. First, the closure assumptions appearing in the original nonordered definitions should be retained in order to preserve consistency with the underlying notions. Second, the lower-set condition may interact with the nonempty-intersection condition in ways that have no analogue in the equality order. Therefore, a meaningful ordered theory should clarify both the effect of the closure assumptions and the genuine role played by the partial order.

Motivated by these observations, we introduce ordered 1- and 2-almost ideals in ordered b-semirings, together with their bi-almost, quasi-almost, and bi-quasi-almost forms. The definitions retain the closure requirements of the corresponding nonordered notions and incorporate the lower-set condition through lower closures of the relevant products. A main structural consequence is that these closure requirements force several apparently different almost ideal classes to coincide. More precisely, for $i \in \{1, 2\}$,

$$\text{ordered } i\text{-BQAId} \iff \text{ordered } i\text{-BAId} \iff \text{ordered } i\text{-QAId} \implies \text{ordered } i\text{-AId},$$

where the first three notions are considered within the class of nonempty lower sub-b-semirings. The converse of the final implication fails in general. In particular, we provide a finite example with a nontrivial partial order showing that the lower closure itself may be essential for obtaining an almost ideal property.

We further identify the common class determined by the ordered bi-almost, quasi-almost, and bi-quasi-almost conditions as precisely the class of nonempty lower sub-b-semirings. This structural description leads to natural closure properties: the common class is closed under nonempty intersections whenever the intersection is nonempty and under unions of chains. We also introduce the lower sub-b-semiring hull of a nonempty subset and show that it gives the smallest member of the common class containing that subset.

To connect the new notions with the structure of the ambient ordered b-semiring, we introduce operation-specific regularity conditions motivated by the standard regularity of ordered semirings [9]. Under ordered 1-regularity, ordinary ordered right ideals, left ideals, bi-ideals, and quasi-ideals admit fixed-point descriptions involving lower products. In particular, if R is an ordered 1-right ideal and L is an ordered 1-left ideal, then

$$R \cap L = (R \oslash L].$$

The corresponding ordered 2-results are obtained through the symmetry of the two operations.

We also investigate preservation properties under order-preserving b-semiring homomorphisms. We show that the lower closure of a homomorphic image preserves the ordered right, left, and two-sided almost ideal properties. For the bi-almost, quasi-almost, and bi-quasi-almost classes, the common lower sub-b-semiring characterization yields a unified preservation result. The lower sub-b-semiring hull

is also compatible with homomorphic images, and the ordered almost ideal properties are invariant under ordered b-semiring isomorphisms.

Finally, instead of developing a second parallel collection of proofs, we use the operation symmetry of a b-semiring to construct an operation-dual ordered b-semiring. This duality transfers the ordered 1-theory systematically to the ordered 2-setting and thereby avoids unnecessary repetition.

The remainder of the paper is organized as follows. Section 2 recalls the basic definitions and notation concerning b-semirings, ordered b-semirings, ordered ideals, and almost ideals. Section 3 introduces ordered 1-almost ideals, establishes their basic relationships, analyzes the collapse caused by the closure requirements, and provides nontrivial ordered examples. Section 4 develops the ordered 2-theory through operation duality. Section 5 studies the common lower sub-b-semiring class, its generated hull, operation-specific regularity, and homomorphism preservation properties. The final section summarizes the main results.

2. Preliminaries

In this section, we recall the basic concepts and notation used throughout the paper. We first give the definition of a b-semiring and introduce the order compatibility required for an ordered b-semiring. We then recall several classes of ordered ideals and almost ideals. Throughout this paper, $\mathcal{Z}$ denotes an ordered b-semiring unless otherwise stated.

Definition 2.1. *An algebra $(\mathcal{S}, \uplus, \oslash)$ is called a b-semiring* [3] *if $(\mathcal{S}, \uplus)$ and $(\mathcal{S}, \oslash)$ are semigroups and the following identities hold for all $\zeta, \tau, \upsilon \in \mathcal{S}$:*

$$\zeta \uplus (\tau \oslash \upsilon) = (\zeta \uplus \tau) \oslash (\zeta \uplus \upsilon),$$

$$(\zeta \oslash \tau) \uplus \upsilon = (\zeta \uplus \upsilon) \oslash (\tau \uplus \upsilon),$$

$$\zeta \oslash (\tau \uplus \upsilon) = (\zeta \oslash \tau) \uplus (\zeta \oslash \upsilon),$$

$$(\zeta \uplus \tau) \oslash \upsilon = (\zeta \oslash \upsilon) \uplus (\tau \oslash \upsilon).$$

Definition 2.2. *A quadruple*

$$(\mathcal{Z}, \uplus, \oslash, \preccurlyeq)$$

is called an ordered b-semiring *if $(\mathcal{Z}, \uplus, \oslash)$ is a b-semiring, $(\mathcal{Z}, \preccurlyeq)$ is a partially ordered set, and the order is compatible with both operations. More precisely, for all $x, y, z \in \mathcal{Z}$, $x \preccurlyeq y$ implies*

$$x \uplus z \preccurlyeq y \uplus z, \qquad z \uplus x \preccurlyeq z \uplus y,$$

$$x \oslash z \preccurlyeq y \oslash z, \qquad z \oslash x \preccurlyeq z \oslash y.$$

Let A and B be nonempty subsets of $\mathcal{Z}$. We use the following notation:

$$A \uplus B = \{a \uplus b \mid a \in A,\ b \in B\}$$

and

$$A \oslash B = \{a \oslash b \mid a \in A,\ b \in B\}.$$

When $A = \{a\}$ or $B = \{b\}$, the braces are omitted. For example, $a \oslash B$ denotes $\{a\} \oslash B$.

For a nonempty subset A of $\mathcal{Z}$, define

$$(A] = \{x \in \mathcal{Z} \mid x \preccurlyeq a \text{ for some } a \in A\}.$$

A subset A is called a *lower set* if $A = (A]$. A nonempty subset T of $\mathcal{Z}$ is called a *sub-b-semiring* of $\mathcal{Z}$ if T is closed under both $\uplus$ and $\oslash$.

Remark 2.1. *Let A and B be nonempty subsets of an ordered b-semiring $\mathcal{Z}$. The following properties hold* [8]*:*

(1)
$$A \subseteq (A] \quad \textit{and} \quad ((A]] = (A].$$

(2) *If $A \subseteq B$, then*
$$(A] \subseteq (B].$$

(3) *For each $\star \in \{\uplus, \oslash\}$,*
$$A \star (B] \subseteq (A] \star (B] \subseteq (A \star B]$$
and
$$(A] \star B \subseteq (A] \star (B] \subseteq (A \star B].$$

Following the notions of ordered quasi-ideals and ordered bi-ideals in ordered semirings [9], we use the following terminology for the first ideal structure of an ordered b-semiring.

Definition 2.3. *Let I, B, and Q be nonempty subsets of an ordered b-semiring $\mathcal{Z}$.*

(1) *The set I is called an* ordered 1-right ideal *of $\mathcal{Z}$ if $(I, \uplus)$ is a subsemigroup of $(\mathcal{Z}, \uplus)$,*
$$(I \oslash \mathcal{Z}] \subseteq I,$$
and $I = (I]$.

Similarly, I is called an ordered 1-left ideal *of $\mathcal{Z}$ if $(I, \uplus)$ is a subsemigroup of $(\mathcal{Z}, \uplus)$,*
$$(\mathcal{Z} \oslash I] \subseteq I,$$
and $I = (I]$.

The set I is called an ordered 1-ideal*, abbreviated as an ordered 1-Id, if it is both an ordered 1-right ideal and an ordered 1-left ideal.*

(2) *The set B is called an* ordered 1-bi-ideal*, abbreviated as an ordered 1-BId, if $(B, \uplus)$ is a subsemigroup of $(\mathcal{Z}, \uplus)$,*
$$(B \oslash \mathcal{Z} \oslash B] \subseteq B,$$
and $B = (B]$.

(3) *The set Q is called an* ordered 1-quasi-ideal, *abbreviated as an ordered* 1-*QId, if $(Q, \uplus)$ is a subsemigroup of $(\mathcal{Z}, \uplus)$,*

$$(Q \oslash \mathcal{Z}] \cap (\mathcal{Z} \oslash Q] \subseteq Q,$$

and $Q = (Q]$.

The second ideal structure is obtained by interchanging the roles of $\uplus$ and $\oslash$.

Definition 2.4. *Let I, B, and Q be nonempty subsets of an ordered b-semiring $\mathcal{Z}$.*

(1) *The set I is called an* ordered 2-right ideal *of $\mathcal{Z}$ if $(I, \oslash)$ is a subsemigroup of $(\mathcal{Z}, \oslash)$,*

$$(I \uplus \mathcal{Z}] \subseteq I,$$

and $I = (I]$.

Similarly, I is called an ordered 2-left ideal *of $\mathcal{Z}$ if $(I, \oslash)$ is a subsemigroup of $(\mathcal{Z}, \oslash)$,*

$$(\mathcal{Z} \uplus I] \subseteq I,$$

and $I = (I]$.

The set I is called an ordered 2-ideal, *abbreviated as an ordered 2-Id, if it is both an ordered 2-right ideal and an ordered 2-left ideal.*

(2) *The set B is called an* ordered 2-bi-ideal, *abbreviated as an ordered 2-BId, if $(B, \oslash)$ is a subsemigroup of $(\mathcal{Z}, \oslash)$,*

$$(B \uplus \mathcal{Z} \uplus B] \subseteq B,$$

and $B = (B]$.

(3) *The set Q is called an* ordered 2-quasi-ideal, *abbreviated as an ordered 2-QId, if $(Q, \oslash)$ is a subsemigroup of $(\mathcal{Z}, \oslash)$,*

$$(Q \uplus \mathcal{Z}] \cap (\mathcal{Z} \uplus Q] \subseteq Q,$$

and $Q = (Q]$.

For subsets A and B of a semiring $(S, +, \cdot)$, we write

$$AB = \{ab \mid a \in A,\ b \in B\}.$$

We next recall the almost ideal notions that motivate the ordered versions studied in the following sections.

Definition 2.5. *Let A, B, and Q be nonempty subsets of a semiring $(S, +, \cdot)$* [10].

(1) *The set A is called a* 1-right almost ideal, *abbreviated as a* 1-*RAId, if*

$$AS \cap A \neq \emptyset.$$

It is called a 1-left almost ideal, *abbreviated as a* 1-*LAId, if*

$$SA \cap A \neq \emptyset.$$

The set A is called a 1-almost ideal, *abbreviated as a* 1-*AId, if it is both a* 1-*RAId and a* 1-*LAId.*

(2) *A subsemiring B of S is called a* 1-bi-almost ideal, *abbreviated as a* 1-*BAId, if*

$$BSB \cap B \neq \emptyset.$$

(3) *A subsemiring Q of S is called a* 1-quasi-almost ideal, *abbreviated as a* 1-*QAId, if*

$$(QS \cap SQ) \cap Q \neq \emptyset.$$

(4) *A subsemiring Q of S is called a* 1-right bi-quasi-almost ideal, *abbreviated as a* 1-*RBQAId, if*

$$(QS \cap QSQ) \cap Q \neq \emptyset.$$

It is called a 1-left bi-quasi-almost ideal, *abbreviated as a* 1-*LBQAId, if*

$$(SQ \cap QSQ) \cap Q \neq \emptyset.$$

(5) *A subsemiring Q of S is called a* 1-bi-quasi-almost ideal, *abbreviated as a* 1-*BQAId, if it is both a* 1-*RBQAId and a* 1-*LBQAId.*

3. Ordered 1-Almost Ideals

Throughout this section, $\mathcal{Z}$ denotes an ordered b-semiring. We introduce ordered versions of 1-almost ideals by combining the corresponding nonempty-intersection conditions with the lower-set condition. In accordance with Definition 2.5, ordered 1-right and 1-left almost ideals are not required to satisfy additional closure conditions, whereas ordered 1-bi-almost, 1-quasi-almost, and 1-bi-quasi-almost ideals are required to be sub-b-semirings of $\mathcal{Z}$.

Definition 3.1. *Let A be a nonempty subset of $\mathcal{Z}$.*

(1) *The set A is called an* ordered 1-right almost ideal, *abbreviated as an ordered* 1-*RAId, if*

$$(A \oslash \mathcal{Z}] \cap A \neq \emptyset$$

and $A = (A]$.

(2) *The set A is called an* ordered 1-left almost ideal, *abbreviated as an ordered* 1-*LAId, if*

$$(\mathcal{Z} \oslash A] \cap A \neq \emptyset$$

and $A = (A]$.

(3) *The set A is called an* ordered 1-almost ideal, *abbreviated as an ordered* 1*-AId, if it is both an ordered* 1*-RAId and an ordered* 1*-LAId.*

Definition 3.2. *Let B and Q be nonempty subsets of $\mathcal{Z}$.*

(1) *A sub-b-semiring B of $\mathcal{Z}$ is called an* ordered 1-bi-almost ideal, *abbreviated as an ordered* 1*-BAId, if*

$$(B \oslash \mathcal{Z} \oslash B] \cap B \neq \emptyset$$

and

$$B = (B].$$

(2) *A sub-b-semiring Q of $\mathcal{Z}$ is called an* ordered 1-quasi-almost ideal, *abbreviated as an ordered* 1*-QAId, if*

$$\big((Q \oslash \mathcal{Z}] \cap (\mathcal{Z} \oslash Q]\big) \cap Q \neq \emptyset$$

and

$$Q = (Q].$$

(3) *A sub-b-semiring Q of $\mathcal{Z}$ is called an* ordered 1-right bi-quasi-almost ideal, *abbreviated as an ordered* 1*-RBQAId, if*

$$\big((Q \oslash \mathcal{Z}] \cap (Q \oslash \mathcal{Z} \oslash Q]\big) \cap Q \neq \emptyset$$

and

$$Q = (Q].$$

(4) *A sub-b-semiring Q of $\mathcal{Z}$ is called an* ordered 1-left bi-quasi-almost ideal, *abbreviated as an ordered* 1*-LBQAId, if*

$$\big((\mathcal{Z} \oslash Q] \cap (Q \oslash \mathcal{Z} \oslash Q]\big) \cap Q \neq \emptyset$$

and

$$Q = (Q].$$

(5) *A sub-b-semiring Q of $\mathcal{Z}$ is called an* ordered 1-bi-quasi-almost ideal, *abbreviated as an ordered* 1*-BQAId, if it is both an ordered* 1*-RBQAId and an ordered* 1*-LBQAId.*

Remark 3.1. *The closure assumptions in Definition 3.2 are retained from the corresponding nonordered notions in Definition 2.5. Thus, the ordered versions differ from their nonordered counterparts only through the lower-set condition and the use of lower closures of the relevant products.*

In particular, if $\preccurlyeq$ is the equality order on $\mathcal{Z}$, then

$$(A] = A$$

for every nonempty subset A of $\mathcal{Z}$. Hence the definitions of ordered 1*-BAIds,* 1*-QAIds, and* 1*-BQAIds reduce to the same closure and nonempty-intersection conditions as their corresponding nonordered counterparts, with $\uplus$ and $\oslash$ playing the roles of the two operations of the b-semiring.*

For comparison with ordered bi-ideals, we introduce the corresponding bi-quasi-ideal terminology.

Definition 3.3. *Let Q be a nonempty subset of $\mathcal{Z}$ such that $(Q, \uplus)$ is a subsemigroup of $(\mathcal{Z}, \uplus)$ and $Q = (Q]$.*

(1) *The set Q is called an* ordered 1-right bi-quasi-ideal, *abbreviated as an ordered 1-RBQId, if*

$$(Q \oslash \mathcal{Z}] \cap (Q \oslash \mathcal{Z} \oslash Q] \subseteq Q.$$

(2) *The set Q is called an* ordered 1-left bi-quasi-ideal, *abbreviated as an ordered 1-LBQId, if*

$$(\mathcal{Z} \oslash Q] \cap (Q \oslash \mathcal{Z} \oslash Q] \subseteq Q.$$

(3) *The set Q is called an* ordered 1-bi-quasi-ideal, *abbreviated as an ordered 1-BQId, if it is both an ordered 1-RBQId and an ordered 1-LBQId.*

Proposition 3.1. *For every nonempty subset A of $\mathcal{Z}$,*

$$(A \oslash \mathcal{Z} \oslash A] \subseteq (A \oslash \mathcal{Z}] \cap (\mathcal{Z} \oslash A].$$

Proof. Since $\mathcal{Z} \oslash A \subseteq \mathcal{Z}$, associativity of $\oslash$ gives

$$A \oslash \mathcal{Z} \oslash A = A \oslash (\mathcal{Z} \oslash A) \subseteq A \oslash \mathcal{Z}.$$

Similarly, since $A \oslash \mathcal{Z} \subseteq \mathcal{Z}$, we have

$$A \oslash \mathcal{Z} \oslash A = (A \oslash \mathcal{Z}) \oslash A \subseteq \mathcal{Z} \oslash A.$$

The required inclusion now follows from Remark 2.1. □

Proposition 3.2. *Let A be a nonempty lower subset of $\mathcal{Z}$.*

(1) *If $(A, \uplus)$ is a subsemigroup of $(\mathcal{Z}, \uplus)$, then A is an ordered 1-BQId if and only if it is an ordered 1-BId.*

(2) *If A is a sub-b-semiring of $\mathcal{Z}$, then A is an ordered 1-BQAId if and only if it is an ordered 1-BAId.*

Proof. By Proposition 3.1,

$$(A \oslash \mathcal{Z}] \cap (A \oslash \mathcal{Z} \oslash A] = (A \oslash \mathcal{Z} \oslash A]$$

and

$$(\mathcal{Z} \oslash A] \cap (A \oslash \mathcal{Z} \oslash A] = (A \oslash \mathcal{Z} \oslash A].$$

Consequently, the conditions defining an ordered 1-RBQId and an ordered 1-LBQId both reduce to

$$(A \oslash \mathcal{Z} \oslash A] \subseteq A,$$

which is precisely the defining absorption condition for an ordered 1-BId. This proves (1).

Similarly, the conditions defining an ordered 1-RBQAId and an ordered 1-LBQAId both reduce to

$$(A \oslash \mathcal{Z} \oslash A] \cap A \neq \emptyset.$$

Hence, an ordered 1-BQAId is exactly an ordered 1-BAId. □

Lemma 3.1. *Let A be a nonempty subset of $\mathcal{Z}$. Then:*

(1) $(A \oslash \mathcal{Z}]$ *is an ordered* 1*-RAId of* $\mathcal{Z}$.

(2) $(\mathcal{Z} \oslash A]$ *is an ordered* 1*-LAId of* $\mathcal{Z}$.

(3) $(\mathcal{Z} \oslash A \oslash \mathcal{Z}]$ *is an ordered* 1*-AId of* $\mathcal{Z}$.

Proof. Set

$$R = (A \oslash \mathcal{Z}].$$

By Remark 2.1, $R = (R]$. Moreover,

$$R \oslash \mathcal{Z} \subseteq (A \oslash \mathcal{Z} \oslash \mathcal{Z}] \subseteq (A \oslash \mathcal{Z}] = R,$$

because $\mathcal{Z} \oslash \mathcal{Z} \subseteq \mathcal{Z}$. Since $R \oslash \mathcal{Z} \neq \emptyset$, it follows that

$$(R \oslash \mathcal{Z}] \cap R \neq \emptyset.$$

Thus, R is an ordered 1-RAId. This proves (1).

The proof of (2) is analogous. For (3), put

$$T = (\mathcal{Z} \oslash A \oslash \mathcal{Z}].$$

Then $T = (T]$, and

$$T \oslash \mathcal{Z} \subseteq (\mathcal{Z} \oslash A \oslash \mathcal{Z} \oslash \mathcal{Z}] \subseteq T.$$

Similarly,

$$\mathcal{Z} \oslash T \subseteq (\mathcal{Z} \oslash \mathcal{Z} \oslash A \oslash \mathcal{Z}] \subseteq T.$$

Both sets on the left are nonempty. Therefore,

$$(T \oslash \mathcal{Z}] \cap T \neq \emptyset$$

and

$$(\mathcal{Z} \oslash T] \cap T \neq \emptyset.$$

Hence, T is an ordered 1-AId. □

Theorem 3.1. *Every ordered* 1*-Id is an ordered* 1*-AId.*

Proof. Let I be an ordered 1-Id of $\mathcal{Z}$. By Definition 2.3, $I = (I]$ and

$$(I \oslash \mathcal{Z}] \subseteq I \quad \text{and} \quad (\mathcal{Z} \oslash I] \subseteq I.$$

Since I and $\mathcal{Z}$ are nonempty, both $I \oslash \mathcal{Z}$ and $\mathcal{Z} \oslash I$ are nonempty. Hence,

$$(I \oslash \mathcal{Z}] \neq \emptyset \quad \text{and} \quad (\mathcal{Z} \oslash I] \neq \emptyset.$$

Therefore,

$$(I \oslash \mathcal{Z}] \cap I = (I \oslash \mathcal{Z}] \neq \emptyset$$

and

$$(\mathcal{Z} \oslash I] \cap I = (\mathcal{Z} \oslash I] \neq \emptyset.$$

Thus, I is both an ordered 1-RAId and an ordered 1-LAId. Hence, I is an ordered 1-AId. □

Theorem 3.2. *Every ordered* 1*-QId is an ordered* 1*-BQId. Moreover, every ordered* 1*-BId is an ordered* 1*-BQId.*

Proof. Let Q be an ordered 1-QId. By Proposition 3.1,

$$(Q \oslash \mathcal{Z} \oslash Q] \subseteq (Q \oslash \mathcal{Z}] \cap (\mathcal{Z} \oslash Q] \subseteq Q.$$

Hence, Q is an ordered 1-BId and therefore an ordered 1-BQId by Proposition 3.2.

The second assertion follows directly from Proposition 3.2. □

The closure conditions in Definition 3.2 lead to an important structural consequence. Namely, the bi-almost, quasi-almost, and bi-quasi-almost conditions do not determine distinct subclasses among nonempty lower sub-b-semirings.

Proposition 3.3. *Let A be a nonempty lower sub-b-semiring of $\mathcal{Z}$. Then A is simultaneously an ordered* 1*-BAId, an ordered* 1*-QAId, and an ordered* 1*-BQAId. Consequently,*

$$\textit{ordered } 1\textit{-BQAId} \iff \textit{ordered } 1\textit{-BAId} \iff \textit{ordered } 1\textit{-QAId}$$

within the class of nonempty lower sub-b-semirings of $\mathcal{Z}$.

Proof. Let A be a nonempty lower sub-b-semiring of $\mathcal{Z}$, and choose $a \in A$. Since A is closed under $\oslash$, we have

$$a \oslash a \in A \quad \text{and} \quad (a \oslash a) \oslash a \in A.$$

Since $a \in A \subseteq \mathcal{Z}$,

$$a \oslash a \in A \oslash \mathcal{Z} \quad \text{and} \quad a \oslash a \in \mathcal{Z} \oslash A.$$

Hence,

$$a \oslash a \in \big((A \oslash \mathcal{Z}] \cap (\mathcal{Z} \oslash A]\big) \cap A,$$

and therefore

$$\big((A \oslash \mathcal{Z}] \cap (\mathcal{Z} \oslash A]\big) \cap A \neq \emptyset.$$

Thus, A is an ordered 1-QAId.

Similarly, associativity of $\oslash$ gives

$$(a \oslash a) \oslash a \in A \oslash \mathcal{Z} \oslash A.$$

Since $(a \oslash a) \oslash a \in A$, we obtain

$$(A \oslash \mathcal{Z} \oslash A] \cap A \neq \emptyset.$$

Hence, A is an ordered 1-BAId.

Finally, by Proposition 3.2, every ordered 1-BAId that is a sub-b-semiring is an ordered 1-BQAId. Therefore, A is simultaneously an ordered 1-BAId, an ordered 1-QAId, and an ordered 1-BQAId. □

Theorem 3.3. *Every ordered* 1*-QAId is an ordered* 1*-AId.*

Proof. Let Q be an ordered 1-QAId. Then

$$\big((Q \oslash \mathcal{Z}] \cap (\mathcal{Z} \oslash Q]\big) \cap Q \neq \emptyset.$$

Since

$$\big((Q \oslash \mathcal{Z}] \cap (\mathcal{Z} \oslash Q]\big) \cap Q \subseteq (Q \oslash \mathcal{Z}] \cap Q$$

and

$$\big((Q \oslash \mathcal{Z}] \cap (\mathcal{Z} \oslash Q]\big) \cap Q \subseteq (\mathcal{Z} \oslash Q] \cap Q,$$

both intersections on the right are nonempty. Therefore, Q is both an ordered 1-RAId and an ordered 1-LAId. Hence, it is an ordered 1-AId. □

The converse of Theorem 3.3 does not hold. The following example also shows that the partial order may play an essential role in producing the almost ideal property.

Example 3.1. *Let*

$$\mathcal{W} = \{0, a, b, a^2, ab, ba, b^2\}.$$

The nonzero elements of $\mathcal{W}$ are regarded as words of length one or two over the alphabet $\{a, b\}$. Define

$$x \uplus y = 0$$

for all $x, y \in \mathcal{W}$, and define

$$u \oslash v = \begin{cases} uv, & u, v \neq 0 \text{ and } |u| + |v| \leq 2, \\ 0, & \text{otherwise}. \end{cases}$$

The operation $\oslash$ is truncated concatenation and is associative, with 0 as an absorbing element. The operation $\uplus$ is also associative. Since every expression involving $\uplus$ reduces to 0, the four identities in Definition 2.1 are satisfied. Hence,

$$(\mathcal{W}, \uplus, \oslash)$$

is a b-semiring.

Define a partial order $\preccurlyeq$ on $\mathcal{W}$ by reflexivity together with

$$a^2 \prec ab,$$

and no other comparisons between distinct elements. This order is compatible with $\uplus$, since

$$x \uplus z = y \uplus z = 0 \quad \text{and} \quad z \uplus x = z \uplus y = 0$$

for all $x, y, z \in \mathcal{W}$.

It remains to verify compatibility with $\oslash$. *Since the only nontrivial order relation is*

$$a^2 \preccurlyeq ab,$$

it is enough to consider this pair. Let $z \in \mathcal{W}$. *If* $z = 0$, *then*

$$a^2 \oslash 0 = ab \oslash 0 = 0$$

and

$$0 \oslash a^2 = 0 \oslash ab = 0.$$

If $z \neq 0$, *then* $|z| \geq 1$, *whereas both* a^2 *and* ab *have length two. Hence the truncated concatenation rule gives*

$$a^2 \oslash z = 0 = ab \oslash z$$

and

$$z \oslash a^2 = 0 = z \oslash ab.$$

Therefore,

$$a^2 \oslash z \preccurlyeq ab \oslash z \quad \textit{and} \quad z \oslash a^2 \preccurlyeq z \oslash ab$$

for every $z \in \mathcal{W}$. *Thus,* $\preccurlyeq$ *is compatible with both operations, and consequently*

$$(\mathcal{W}, \uplus, \oslash, \preccurlyeq)$$

is an ordered b*-semiring.*

Consider

$$A = \{b, a^2, ba\}.$$

Since the only nontrivial order relation is $a^2 \prec ab$ *and* $ab \notin A$, *we have*

$$A = (A].$$

A direct calculation gives

$$A \oslash \mathcal{W} = \{0, ba, b^2\}.$$

Hence,

$$(A \oslash \mathcal{W}] = \{0, ba, b^2\},$$

and therefore

$$(A \oslash \mathcal{W}] \cap A = \{ba\} \neq \emptyset.$$

On the other hand,

$$\mathcal{W} \oslash A = \{0, ab, b^2\}.$$

Since $a^2 \prec ab$, *its lower closure is*

$$(\mathcal{W} \oslash A] = \{0, a^2, ab, b^2\}.$$

Consequently,

$$(\mathcal{W} \oslash A] \cap A = \{a^2\} \neq \emptyset.$$

Thus, A is both an ordered 1-RAId and an ordered 1-LAId. Hence, A is an ordered 1-AId.

However,

$$(A \oslash \mathcal{W}] \cap (\mathcal{W} \oslash A] = \{0, b^2\},$$

and hence

$$\big((A \oslash \mathcal{W}] \cap (\mathcal{W} \oslash A]\big) \cap A = \emptyset.$$

Therefore, A is not an ordered 1-QAId.

It is worth observing that the partial order plays an essential role in this example. Indeed,

$$(\mathcal{W} \oslash A) \cap A = \emptyset,$$

whereas

$$(\mathcal{W} \oslash A] \cap A = \{a^2\} \neq \emptyset$$

because $a^2 \prec ab$. Thus, the ordered 1-left almost ideal property of A arises genuinely from the lower closure induced by the nontrivial partial order.

Although every nonempty lower sub-b-semiring is an ordered 1-BQAId, such a subset need not be an ordered 1-BQId, as the following example shows.

Example 3.2. *Let*

$$\mathcal{W} = \{0\} \cup \{w \in \{a, b\}^+ \mid 1 \leq |w| \leq 3\}.$$

Thus, the nonzero elements of $\mathcal{W}$ are all words of length at most three over the alphabet $\{a, b\}$. Define

$$x \uplus y = 0$$

for all $x, y \in \mathcal{W}$, and define

$$u \oslash v = \begin{cases} uv, & u, v \neq 0 \text{ and } |u| + |v| \leq 3, \\ 0, & \text{otherwise}. \end{cases}$$

The operation $\oslash$ is truncated concatenation. It is associative, and 0 is an absorbing element. Since $\uplus$ is the constant-zero operation, it is associative and the four identities in Definition 2.1 are satisfied. Hence,

$$(\mathcal{W}, \uplus, \oslash)$$

is a b-semiring.

Define a partial order $\preccurlyeq$ on $\mathcal{W}$ by

$$x \preccurlyeq y \quad \Longleftrightarrow \quad x = y \text{ or } x = 0.$$

Thus, 0 *is the least element of* $\mathcal{W}$*, while distinct nonzero elements are incomparable. The order is compatible with* $\uplus$ *because every sum is* 0*. It is also compatible with* $\oslash$*: if* $0 \preccurlyeq y$*, then for every* $z \in \mathcal{W}$*,*

$$0 \oslash z = 0 \preccurlyeq y \oslash z \quad \textit{and} \quad z \oslash 0 = 0 \preccurlyeq z \oslash y.$$

Therefore,

$$(\mathcal{W}, \uplus, \oslash, \preccurlyeq)$$

is an ordered b-semiring.

Consider

$$B = \{0, a, a^2, a^3\}.$$

Clearly, $B = (B]$*. Moreover,* B *is closed under both* $\uplus$ *and* $\oslash$*; hence* B *is a sub-b-semiring of* $\mathcal{W}$*.*

Since

$$a \oslash a \oslash a = a^3 \in B,$$

we have

$$a^3 \in (B \oslash \mathcal{W} \oslash B] \cap B.$$

Therefore,

$$(B \oslash \mathcal{W} \oslash B] \cap B \neq \emptyset,$$

and B *is an ordered* 1*-BAId. By Proposition 3.2,* B *is also an ordered* 1*-BQAId.*

On the other hand,

$$a \oslash b \oslash a = aba.$$

Hence,

$$aba \in B \oslash \mathcal{W} \oslash B \subseteq (B \oslash \mathcal{W} \oslash B].$$

Since $aba \notin B$*, it follows that*

$$(B \oslash \mathcal{W} \oslash B] \not\subseteq B.$$

Thus, B *is not an ordered* 1*-BId and consequently is not an ordered* 1*-BQId.*

Remark 3.2. *Proposition 3.3 and Theorem 3.3 show that the ordered* 1*-almost ideal classes considered above satisfy*

$$\textit{ordered } 1\textit{-BQAId} \iff \textit{ordered } 1\textit{-BAId} \iff \textit{ordered } 1\textit{-QAId} \implies \textit{ordered } 1\textit{-AId},$$

where the first three notions are considered within the class of nonempty lower sub-b-semirings. The converse of the final implication fails in general, as shown in Example 3.1.

Thus, retaining the sub-b-semiring condition from the corresponding nonordered definitions causes the three apparently different almost ideal notions to collapse into the same class. This observation clarifies the effect of the closure assumption and shows that the genuinely broader notion in this hierarchy is the ordered 1*-almost ideal.*

Moreover,

$$\textit{ordered } 1\textit{-BQId} \iff \textit{ordered } 1\textit{-BId}$$

for subsets satisfying the closure condition in Definition 3.3, by Proposition 3.2.

4. Operation Duality and Ordered 2-Almost Ideals

The axioms of an ordered b-semiring are symmetric with respect to the two operations. We therefore develop the ordered 2-theory through operation duality rather than repeating the arguments of Section 3. This approach also makes clear that the ordered 1- and 2-structures are two manifestations of the same underlying theory.

Definition 4.1. *Let A be a nonempty subset of an ordered b-semiring $\mathcal{Z}$.*

(1) *The set A is called an* ordered 2-right almost ideal, *abbreviated as an ordered 2-RAId, if*

$$(A \uplus \mathcal{Z}] \cap A \neq \emptyset$$

and $A = (A]$.

(2) *The set A is called an* ordered 2-left almost ideal, *abbreviated as an ordered 2-LAId, if*

$$(\mathcal{Z} \uplus A] \cap A \neq \emptyset$$

and $A = (A]$.

(3) *The set A is called an* ordered 2-almost ideal, *abbreviated as an ordered 2-AId, if it is both an ordered 2-RAId and an ordered 2-LAId.*

In agreement with Definition 3.2, the closure assumptions from the corresponding nonordered notions are retained in the following definitions.

Definition 4.2. *Let B and Q be nonempty subsets of $\mathcal{Z}$.*

(1) *A sub-b-semiring B of $\mathcal{Z}$ is called an* ordered 2-bi-almost ideal, *abbreviated as an ordered 2-BAId, if*

$$(B \uplus \mathcal{Z} \uplus B] \cap B \neq \emptyset$$

and

$$B = (B].$$

(2) *A sub-b-semiring Q of $\mathcal{Z}$ is called an* ordered 2-quasi-almost ideal, *abbreviated as an ordered 2-QAId, if*

$$\big((Q \uplus \mathcal{Z}] \cap (\mathcal{Z} \uplus Q]\big) \cap Q \neq \emptyset$$

and

$$Q = (Q].$$

(3) *A sub-b-semiring Q of $\mathcal{Z}$ is called an* ordered 2-right bi-quasi-almost ideal, *abbreviated as an ordered 2-RBQAId, if*

$$\big((Q \uplus \mathcal{Z}] \cap (Q \uplus \mathcal{Z} \uplus Q]\big) \cap Q \neq \emptyset$$

and

$$Q = (Q].$$

(4) *A sub-b-semiring Q of $\mathcal{Z}$ is called an* ordered 2-left bi-quasi-almost ideal, *abbreviated as an ordered 2-LBQAId, if*

$$\big((\mathcal{Z} \uplus Q] \cap (Q \uplus \mathcal{Z} \uplus Q]\big) \cap Q \neq \emptyset$$

and

$$Q = (Q].$$

(5) *A sub-b-semiring Q of $\mathcal{Z}$ is called an* ordered 2-bi-quasi-almost ideal, *abbreviated as an ordered 2-BQAId, if it is both an ordered 2-RBQAId and an ordered 2-LBQAId.*

Definition 4.3. *Let Q be a nonempty subset of $\mathcal{Z}$ such that $(Q, \oslash)$ is a subsemigroup of $(\mathcal{Z}, \oslash)$ and $Q = (Q]$.*

(1) *The set Q is called an* ordered 2-right bi-quasi-ideal, *abbreviated as an ordered 2-RBQId, if*

$$(Q \uplus \mathcal{Z}] \cap (Q \uplus \mathcal{Z} \uplus Q] \subseteq Q.$$

(2) *The set Q is called an* ordered 2-left bi-quasi-ideal, *abbreviated as an ordered 2-LBQId, if*

$$(\mathcal{Z} \uplus Q] \cap (Q \uplus \mathcal{Z} \uplus Q] \subseteq Q.$$

(3) *The set Q is called an* ordered 2-bi-quasi-ideal, *abbreviated as an ordered 2-BQId, if it is both an ordered 2-RBQId and an ordered 2-LBQId.*

We now formalize the operation symmetry that transfers the results of Section 3 to the ordered 2-setting.

Proposition 4.1. *Let*

$$\mathcal{Z} = (\mathcal{Z}, \uplus, \oslash, \preccurlyeq)$$

be an ordered b-semiring, and define

$$\mathcal{Z}^{\partial} = (\mathcal{Z}, \oslash, \uplus, \preccurlyeq).$$

Then $\mathcal{Z}^{\partial}$ is an ordered b-semiring.

Moreover, for every nonempty subset A of $\mathcal{Z}$, each ordered 2-structure of A in $\mathcal{Z}$ is equivalent to the corresponding ordered 1-structure of A in $\mathcal{Z}^{\partial}$. In particular, this correspondence holds for ordered right ideals, left ideals, ideals, bi-ideals, quasi-ideals, bi-quasi-ideals, and all of their almost counterparts.

Proof. For clarity, write

$$x \uplus^{\partial} y := x \oslash y \quad \text{and} \quad x \oslash^{\partial} y := x \uplus y$$

for all $x, y \in \mathcal{Z}$. Thus,

$$\mathcal{Z}^{\partial} = (\mathcal{Z}, \uplus^{\partial}, \oslash^{\partial}, \preccurlyeq).$$

Since both $(\mathcal{Z}, \uplus)$ and $(\mathcal{Z}, \oslash)$ are semigroups, it follows immediately that both

$$(\mathcal{Z}, \uplus^{\partial}) \quad \text{and} \quad (\mathcal{Z}, \oslash^{\partial})$$

are semigroups.

We verify the four identities of a b-semiring. For all $x, y, z \in \mathcal{Z}$, we have

$$\begin{aligned} x \uplus^{\partial} (y \oslash^{\partial} z) &= x \oslash (y \uplus z) \\ &= (x \oslash y) \uplus (x \oslash z) \\ &= (x \uplus^{\partial} y) \oslash^{\partial} (x \uplus^{\partial} z), \end{aligned}$$

where the second equality follows from the third identity in Definition 2.1. Similarly,

$$\begin{aligned} (x \oslash^{\partial} y) \uplus^{\partial} z &= (x \uplus y) \oslash z \\ &= (x \oslash z) \uplus (y \oslash z) \\ &= (x \uplus^{\partial} z) \oslash^{\partial} (y \uplus^{\partial} z), \end{aligned}$$

by the fourth identity in Definition 2.1. Moreover,

$$\begin{aligned} x \oslash^{\partial} (y \uplus^{\partial} z) &= x \uplus (y \oslash z) \\ &= (x \uplus y) \oslash (x \uplus z) \\ &= (x \oslash^{\partial} y) \uplus^{\partial} (x \oslash^{\partial} z), \end{aligned}$$

by the first identity, and

$$\begin{aligned} (x \uplus^{\partial} y) \oslash^{\partial} z &= (x \oslash y) \uplus z \\ &= (x \uplus z) \oslash (y \uplus z) \\ &= (x \oslash^{\partial} z) \uplus^{\partial} (y \oslash^{\partial} z), \end{aligned}$$

by the second identity. Hence,

$$(\mathcal{Z}, \uplus^{\partial}, \oslash^{\partial})$$

is a b-semiring.

Next, let $x \preccurlyeq y$. Since the order on $\mathcal{Z}$ is compatible with both original operations, for every $z \in \mathcal{Z}$,

$$x \oslash z \preccurlyeq y \oslash z, \qquad z \oslash x \preccurlyeq z \oslash y,$$

and

$$x \uplus z \preccurlyeq y \uplus z, \qquad z \uplus x \preccurlyeq z \uplus y.$$

Equivalently,

$$x \uplus^{\partial} z \preccurlyeq y \uplus^{\partial} z, \qquad z \uplus^{\partial} x \preccurlyeq z \uplus^{\partial} y,$$

and

$$x \oslash^{\partial} z \preccurlyeq y \oslash^{\partial} z, \qquad z \oslash^{\partial} x \preccurlyeq z \oslash^{\partial} y.$$

Therefore,

$$\mathcal{Z}^{\partial} = (\mathcal{Z}, \uplus^{\partial}, \oslash^{\partial}, \preccurlyeq)$$

is an ordered b-semiring.

It remains to verify the correspondence between the ordered 2-structures of $\mathcal{Z}$ and the ordered 1-structures of $\mathcal{Z}^{\partial}$. Since the partial order is unchanged, a subset A is a lower subset of $\mathcal{Z}$ if and only if it is a lower subset of $\mathcal{Z}^{\partial}$. Also,

$$(A, \oslash) \text{ is a subsemigroup of } (\mathcal{Z}, \oslash)$$

if and only if

$$(A, \uplus^{\partial}) \text{ is a subsemigroup of } (\mathcal{Z}, \uplus^{\partial}).$$

Moreover, A is a sub-b-semiring of $\mathcal{Z}$ if and only if it is a sub-b-semiring of $\mathcal{Z}^{\partial}$, because this condition requires closure under both operations.

For example,

$$(A \uplus \mathcal{Z}] \cap A \neq \emptyset$$

is equivalent to

$$(A \oslash^{\partial} \mathcal{Z}] \cap A \neq \emptyset.$$

Thus, A is an ordered 2-RAId of $\mathcal{Z}$ if and only if it is an ordered 1-RAId of $\mathcal{Z}^{\partial}$. Similarly,

$$(\mathcal{Z} \uplus A] \cap A \neq \emptyset$$

is equivalent to

$$(\mathcal{Z} \oslash^{\partial} A] \cap A \neq \emptyset,$$

so ordered 2-LAIds correspond exactly to ordered 1-LAIds.

Likewise,

$$(A \uplus \mathcal{Z} \uplus A] \cap A \neq \emptyset$$

is precisely

$$(A \oslash^{\partial} \mathcal{Z} \oslash^{\partial} A] \cap A \neq \emptyset,$$

and

$$\big((A \uplus \mathcal{Z}] \cap (\mathcal{Z} \uplus A]\big) \cap A \neq \emptyset$$

is precisely

$$\big((A \oslash^{\partial} \mathcal{Z}] \cap (\mathcal{Z} \oslash^{\partial} A]\big) \cap A \neq \emptyset.$$

Hence, the ordered 2-BAId and ordered 2-QAId conditions in $\mathcal{Z}$ are exactly the ordered 1-BAId and ordered 1-QAId conditions in $\mathcal{Z}^{\partial}$.

The same substitution of

$$\uplus = \oslash^{\partial} \quad \text{and} \quad \oslash = \uplus^{\partial}$$

in the remaining definitions gives the corresponding equivalences for ordered ideals, bi-ideals, quasi-ideals, bi-quasi-ideals, RBQAIds, LBQAIds, BQAIds, and their almost forms. Therefore, all the asserted correspondences hold. □

The following consequences collect the basic ordered 2-results without repeating their ordered 1-proofs.

Proposition 4.2. *Let A be a nonempty subset of $\mathcal{Z}$.*

(1)

$$(A \uplus \mathcal{Z} \uplus A] \subseteq (A \uplus \mathcal{Z}] \cap (\mathcal{Z} \uplus A].$$

(2) *If $A = (A]$ and $(A, \oslash)$ is a subsemigroup of $(\mathcal{Z}, \oslash)$, then*

$$A \text{ is an ordered 2-BQId} \iff A \text{ is an ordered 2-BId}.$$

(3) *Every ordered 2-QId is an ordered 2-BQId, and every ordered 2-BId is an ordered 2-BQId.*

Proof. Apply Proposition 4.1 together with Propositions 3.1 and 3.2, and Theorem 3.2, to the operation-dual ordered b-semiring $\mathcal{Z}^{\partial}$. □

The lower-product constructions also transfer directly.

Lemma 4.1. *Let A be a nonempty subset of $\mathcal{Z}$. Then:*

(1) *$(A \uplus \mathcal{Z}]$ is an ordered 2-RAId of $\mathcal{Z}$.*

(2) *$(\mathcal{Z} \uplus A]$ is an ordered 2-LAId of $\mathcal{Z}$.*

(3) *$(\mathcal{Z} \uplus A \uplus \mathcal{Z}]$ is an ordered 2-AId of $\mathcal{Z}$.*

Proof. Apply Lemma 3.1 to $\mathcal{Z}^{\partial}$ and use Proposition 4.1. □

The next result is the ordered 2-counterpart of the structural collapse obtained in Section 3.

Proposition 4.3. *Let A be a nonempty lower sub-b-semiring of $\mathcal{Z}$. Then A is simultaneously an ordered 2-BAId, an ordered 2-QAId, and an ordered 2-BQAId. Consequently,*

$$\text{ordered 2-BQAId} \iff \text{ordered 2-BAId} \iff \text{ordered 2-QAId}$$

within the class of nonempty lower sub-b-semirings of $\mathcal{Z}$.

Proof. This is Proposition 3.3 applied to $\mathcal{Z}^{\partial}$. Notice that the class of sub-b-semirings is unchanged when the two operations are interchanged. □

For completeness, ordinary ordered 2-ideals still imply their almost counterpart.

Theorem 4.1. *Every ordered 2-Id is an ordered 2-AId.*

Proof. Apply Theorem 3.1 to the operation-dual ordered b-semiring $\mathcal{Z}^{\partial}$, and then use Proposition 4.1. □

Theorem 4.2. *Every ordered 2-QAId is an ordered 2-AId.*

Proof. Apply Theorem 3.3 to $\mathcal{Z}^{\partial}$. □

The converse of Theorem 4.2 does not hold, and this failure may depend essentially on the partial order rather than occurring already in the unordered structure.

Example 4.1. *Let*

$$(\mathcal{W}, \uplus, \oslash, \preccurlyeq)$$

and

$$A = \{b, a^2, ba\}$$

be as in Example 3.1. Consider the operation-dual ordered b-semiring

$$\mathcal{W}^{\partial} = (\mathcal{W}, \oslash, \uplus, \preccurlyeq).$$

By Example 3.1 and Proposition 4.1, A is an ordered 2-AId of $\mathcal{W}^{\partial}$, but it is not an ordered 2-QAId.

Moreover, the same nontrivial relation

$$a^2 \prec ab$$

is responsible for the lower-closure phenomenon in the dual structure. Thus, this example also demonstrates a genuinely order-dependent ordered 2-almost ideal.

Although every nonempty lower sub-b-semiring is an ordered 2-BQAId, it need not be an ordered 2-BQId.

Example 4.2. *Let*

$$(\mathcal{W}, \uplus, \oslash, \preccurlyeq)$$

and

$$B = \{0, a, a^2, a^3\}$$

be as in Example 3.2. Then, in the operation-dual ordered b-semiring

$$\mathcal{W}^{\partial} = (\mathcal{W}, \oslash, \uplus, \preccurlyeq),$$

Proposition 4.1 implies that B is an ordered 2-BQAId but not an ordered 2-BQId.

Remark 4.1. *The operation-dual principle shows that no independent parallel theory is required for the ordered* 2*-case. In particular, Proposition 4.3 and Theorem 4.2 give*

$$\textit{ordered } 2\textit{-BQAId} \iff \textit{ordered } 2\textit{-BAId} \iff \textit{ordered } 2\textit{-QAId} \implies \textit{ordered } 2\textit{-AId},$$

where the first three notions are considered within the class of nonempty lower sub-b-semirings. The converse of the final implication fails by Example 4.1.

Furthermore,

$$\textit{ordered } 2\textit{-BQId} \iff \textit{ordered } 2\textit{-BId}$$

for lower subsets that are subsemigroups of $(\mathcal{Z}, \oslash)$.

5. Structural Consequences, Regularity, and Homomorphisms

The results of Sections 3 and 4 show that the closure assumptions in the definitions of bi-almost, quasi-almost, and bi-quasi-almost ideals have stronger consequences than the corresponding nonempty-intersection conditions alone suggest. In this section, we develop several structural consequences of this observation. We first identify the common class determined by these notions and study its closure properties. We then relate ordinary ordered ideals to operation-specific regularity and finally investigate the behavior of ordered almost ideals under homomorphisms.

5.1. **The common lower sub-b-semiring class.** Let

$$\mathfrak{L}_b(\mathcal{Z}) = \{A \subseteq \mathcal{Z} \mid A \neq \emptyset,\ A = (A],\ \text{and } A \text{ is a sub-}b\text{-semiring of } \mathcal{Z}\}.$$

Thus, $\mathfrak{L}_b(\mathcal{Z})$ is the family of all nonempty lower sub-b-semirings of $\mathcal{Z}$.

The collapse results obtained earlier admit the following unified form.

Theorem 5.1. *Let* A *be a nonempty subset of an ordered b-semiring* $\mathcal{Z}$*. The following statements are equivalent:*

(1) $A \in \mathfrak{L}_b(\mathcal{Z})$.
(2) *A is an ordered* 1*-BAId.*
(3) *A is an ordered* 1*-QAId.*
(4) *A is an ordered* 1*-BQAId.*
(5) *A is an ordered* 2*-BAId.*
(6) *A is an ordered* 2*-QAId.*
(7) *A is an ordered* 2*-BQAId.*

Proof. By Definition 3.2, each of the ordered 1-BAId, 1-QAId, and 1-BQAId conditions requires A to be a nonempty lower sub-b-semiring. Conversely, Proposition 3.3 shows that every member of $\mathfrak{L}_b(\mathcal{Z})$ satisfies all three ordered 1-almost conditions.

Similarly, Definition 4.2 and Proposition 4.3 give the equivalence of (1) with (5)–(7). Hence all seven statements are equivalent. □

Theorem 5.1 shows, in particular, that the three almost notions do not only collapse separately in the ordered 1- and ordered 2-settings; the resulting classes also coincide across the two operation structures.

The common class has natural closure properties.

Proposition 5.1. *The following statements hold.*

(1) *Let* $\{A_i\}_{i\in I} \subseteq \mathfrak{L}_b(\mathcal{Z})$. *If*

$$\bigcap_{i\in I} A_i \neq \emptyset,$$

then

$$\bigcap_{i\in I} A_i \in \mathfrak{L}_b(\mathcal{Z}).$$

(2) *Let* $\{A_i\}_{i\in I} \subseteq \mathfrak{L}_b(\mathcal{Z})$ *be a nonempty chain under inclusion. Then*

$$\bigcup_{i\in I} A_i \in \mathfrak{L}_b(\mathcal{Z}).$$

Proof. For (1), put

$$A = \bigcap_{i\in I} A_i.$$

By assumption, $A \neq \emptyset$. Since each A_i is a lower subset of $\mathcal{Z}$, their intersection is also lower.

Let $x, y \in A$. Then $x, y \in A_i$ for every $i \in I$. Because each A_i is closed under both operations,

$$x \uplus y \in A_i \quad \text{and} \quad x \oslash y \in A_i$$

for every $i \in I$. Hence

$$x \uplus y,\ x \oslash y \in A.$$

Therefore, $A \in \mathfrak{L}_b(\mathcal{Z})$.

For (2), let

$$U = \bigcup_{i\in I} A_i.$$

Clearly, $U \neq \emptyset$, and U is a lower subset because every A_i is lower. Let $x, y \in U$. Then there exist $i, j \in I$ such that

$$x \in A_i \quad \text{and} \quad y \in A_j.$$

Since the family is a chain, either

$$A_i \subseteq A_j \quad \text{or} \quad A_j \subseteq A_i.$$

Thus, x and y belong to one common member of the chain. Closure of that member under $\uplus$ and $\oslash$ gives

$$x \uplus y,\ x \oslash y \in U.$$

Hence U is a lower sub-b-semiring of $\mathcal{Z}$. □

Corollary 5.1. *Each of the equivalent classes in Theorem 5.1 is closed under nonempty intersections whenever the intersection is nonempty, and under unions of nonempty chains.*

Proof. This follows immediately from Theorem 5.1 and Proposition 5.1. □

The preceding result also provides a natural generated structure.

Definition 5.1. *For a nonempty subset X of $\mathcal{Z}$, define*

$$\mathrm{Hull}_{\mathcal{Z}}(X) = \bigcap \{A \in \mathfrak{L}_b(\mathcal{Z}) \mid X \subseteq A\}.$$

We call $\mathrm{Hull}_{\mathcal{Z}}(X)$ *the* lower sub-b-semiring hull *of* X.

The definition is meaningful because $\mathcal{Z} \in \mathfrak{L}_b(\mathcal{Z})$, so the family in Definition 5.1 is nonempty.

Proposition 5.2. *For every nonempty subset X of $\mathcal{Z}$, the following statements hold:*

(1)
$$X \subseteq \mathrm{Hull}_{\mathcal{Z}}(X) \in \mathfrak{L}_b(\mathcal{Z}).$$

(2) *If $X \subseteq Y$, then*
$$\mathrm{Hull}_{\mathcal{Z}}(X) \subseteq \mathrm{Hull}_{\mathcal{Z}}(Y).$$

(3)
$$\mathrm{Hull}_{\mathcal{Z}}\big(\mathrm{Hull}_{\mathcal{Z}}(X)\big) = \mathrm{Hull}_{\mathcal{Z}}(X).$$

(4) $\mathrm{Hull}_{\mathcal{Z}}(X)$ *is the smallest subset of $\mathcal{Z}$ containing X that is simultaneously an ordered* 1*-BAId,* 1*-QAId,* 1*-BQAId,* 2*-BAId,* 2*-QAId, and* 2*-BQAId.*

Proof. Every member of the family defining $\mathrm{Hull}_{\mathcal{Z}}(X)$ contains X. Hence their intersection contains X and is therefore nonempty. Part (1) now follows from Proposition 5.1.

If $X \subseteq Y$, then every member of $\mathfrak{L}_b(\mathcal{Z})$ containing Y also contains X. Therefore,

$$\mathrm{Hull}_{\mathcal{Z}}(X) \subseteq \mathrm{Hull}_{\mathcal{Z}}(Y),$$

which proves (2).

Part (3) follows because $\mathrm{Hull}_{\mathcal{Z}}(X)$ already belongs to $\mathfrak{L}_b(\mathcal{Z})$. Finally, (4) follows from the minimality in Definition 5.1 together with Theorem 5.1. □

5.2. **Regularity and fixed-point characterizations.** We next examine how the product expressions occurring in the ideal conditions behave under regularity. Motivated by the standard regularity condition for ordered semirings [9], we introduce operation-specific versions adapted to the two symmetric operations of an ordered b-semiring.

Definition 5.2. *An ordered b-semiring $\mathcal{Z}$ is called* ordered 1-regular *if, for every $a \in \mathcal{Z}$, there exists $x \in \mathcal{Z}$ such that*

$$a \preccurlyeq a \oslash x \oslash a.$$

It is called ordered 2-regular *if, for every $a \in \mathcal{Z}$, there exists $x \in \mathcal{Z}$ such that*

$$a \preccurlyeq a \uplus x \uplus a.$$

Clearly, $\mathcal{Z}$ is ordered 2-regular if and only if $\mathcal{Z}^{\partial}$ is ordered 1-regular.

The next theorem gives fixed-point descriptions of the ordinary ordered 1-ideal classes in an ordered 1-regular b-semiring.

Theorem 5.2. *Let $\mathcal{Z}$ be an ordered 1-regular b-semiring. Then:*

(1) *for every ordered 1-right ideal R,*

$$R = (R \oslash \mathcal{Z}];$$

(2) *for every ordered 1-left ideal L,*

$$L = (\mathcal{Z} \oslash L];$$

(3) *for every ordered 1-BId B,*

$$B = (B \oslash \mathcal{Z} \oslash B];$$

(4) *for every ordered 1-QId Q,*

$$Q = (Q \oslash \mathcal{Z}] \cap (\mathcal{Z} \oslash Q].$$

Proof. Let R be an ordered 1-right ideal. By definition,

$$(R \oslash \mathcal{Z}] \subseteq R.$$

Let $r \in R$. Since $\mathcal{Z}$ is ordered 1-regular, there exists $x \in \mathcal{Z}$ such that

$$r \preccurlyeq r \oslash x \oslash r.$$

Since $x \oslash r \in \mathcal{Z}$,

$$r \oslash x \oslash r = r \oslash (x \oslash r) \in R \oslash \mathcal{Z}.$$

Hence,

$$r \in (R \oslash \mathcal{Z}],$$

and therefore

$$R \subseteq (R \oslash \mathcal{Z}].$$

This proves (1).

The proof of (2) is analogous. Indeed,

$$r \oslash x \oslash r = (r \oslash x) \oslash r \in \mathcal{Z} \oslash L$$

for $r \in L$.

For (3), let $b \in B$. Ordered 1-regularity gives $x \in \mathcal{Z}$ such that

$$b \preccurlyeq b \oslash x \oslash b.$$

Thus,

$$b \in (B \oslash \mathcal{Z} \oslash B].$$

Consequently,

$$B \subseteq (B \oslash \mathcal{Z} \oslash B].$$

The reverse inclusion follows from the ordered 1-BId condition. Hence,

$$B = (B \oslash \mathcal{Z} \oslash B].$$

Finally, let $q \in Q$. Choose $x \in \mathcal{Z}$ such that

$$q \preccurlyeq q \oslash x \oslash q.$$

Since

$$q \oslash x \oslash q = q \oslash (x \oslash q) \in Q \oslash \mathcal{Z}$$

and

$$q \oslash x \oslash q = (q \oslash x) \oslash q \in \mathcal{Z} \oslash Q,$$

we obtain

$$q \in (Q \oslash \mathcal{Z}] \cap (\mathcal{Z} \oslash Q].$$

Therefore,

$$Q \subseteq (Q \oslash \mathcal{Z}] \cap (\mathcal{Z} \oslash Q].$$

The reverse inclusion is precisely the absorption condition for an ordered 1-QId. This proves (4). □

A familiar intersection–product relation follows.

Corollary 5.2. *Let $\mathcal{Z}$ be ordered 1-regular. If R is an ordered 1-right ideal and L is an ordered 1-left ideal of $\mathcal{Z}$, then*

$$R \cap L = (R \oslash L].$$

Proof. Since

$$R \oslash L \subseteq R \oslash \mathcal{Z},$$

we have

$$(R \oslash L] \subseteq R.$$

Similarly,

$$(R \oslash L] \subseteq L.$$

Hence,

$$(R \oslash L] \subseteq R \cap L.$$

Conversely, let $a \in R \cap L$. Choose $x \in \mathcal{Z}$ such that

$$a \preccurlyeq a \oslash x \oslash a.$$

Since $a \in R$ and R is an ordered 1-right ideal,

$$a \oslash x \in R.$$

Since $a \in L$,

$$(a \oslash x) \oslash a \in R \oslash L.$$

Therefore,

$$a \in (R \oslash L].$$

Thus,

$$R \cap L \subseteq (R \oslash L],$$

and equality follows. □

By operation duality, the corresponding ordered 2-results require no separate proofs.

Corollary 5.3. *Let $\mathcal{Z}$ be an ordered 2-regular b-semiring. Then:*

(1) *every ordered 2-right ideal R satisfies*

$$R = (R \uplus \mathcal{Z}];$$

(2) *every ordered 2-left ideal L satisfies*

$$L = (\mathcal{Z} \uplus L];$$

(3) *every ordered 2-BId B satisfies*

$$B = (B \uplus \mathcal{Z} \uplus B];$$

(4) *every ordered 2-QId Q satisfies*

$$Q = (Q \uplus \mathcal{Z}] \cap (\mathcal{Z} \uplus Q];$$

(5) *if R is an ordered 2-right ideal and L is an ordered 2-left ideal, then*

$$R \cap L = (R \uplus L].$$

Proof. Apply Theorem 5.2 and Corollary 5.2 to $\mathcal{Z}^{\partial}$, and use Proposition 4.1. □

Remark 5.1. *Theorem 5.2 clarifies the difference between ordinary ordered ideals and the almost notions introduced in this paper. The almost conditions require only that a lower subset have a nonempty intersection with the relevant lower product, whereas ordinary ideals satisfy absorption inclusions. Under regularity, the latter inclusions become fixed-point equalities. Thus, regularity places the ordinary ideal classes at a structurally stronger level than their corresponding almost counterparts.*

5.3. **Homomorphisms and preservation properties.** We conclude this section by studying the behavior of the newly introduced structures under mappings.

Definition 5.3. *Let*

$$\mathcal{Z} = (\mathcal{Z}, \uplus, \oslash, \preccurlyeq) \quad \textit{and} \quad \mathcal{Y} = (\mathcal{Y}, \uplus, \oslash, \preccurlyeq)$$

be ordered b-semirings. A map

$$f : \mathcal{Z} \longrightarrow \mathcal{Y}$$

is called an order-preserving b-semiring homomorphism *if for all $x, y \in \mathcal{Z}$,*

$$f(x \uplus y) = f(x) \uplus f(y), \qquad f(x \oslash y) = f(x) \oslash f(y),$$

and

$$x \preccurlyeq y \quad \Longrightarrow \quad f(x) \preccurlyeq f(y).$$

The lower closure of a homomorphic image is particularly natural in the ordered setting.

Lemma 5.1. *Let $f : \mathcal{Z} \to \mathcal{Y}$ be an order-preserving b-semiring homomorphism and let A be a nonempty subset of $\mathcal{Z}$. Then:*

(1) *$(f(A)]$ is a nonempty lower subset of $\mathcal{Y}$;*

(2) *if A is a sub-b-semiring of $\mathcal{Z}$, then $(f(A)]$ is a sub-b-semiring of $\mathcal{Y}$.*

Proof. Part (1) follows directly from the definition of lower closure.

For (2), let

$$x, y \in (f(A)].$$

Then there exist $a, b \in A$ such that

$$x \preccurlyeq f(a) \quad \text{and} \quad y \preccurlyeq f(b).$$

By compatibility of the order with both operations,

$$x \uplus y \preccurlyeq f(a) \uplus f(b) = f(a \uplus b).$$

Since A is closed under $\uplus$,

$$a \uplus b \in A,$$

and therefore

$$x \uplus y \in (f(A)].$$

Similarly,

$$x \oslash y \preccurlyeq f(a) \oslash f(b) = f(a \oslash b),$$

and $a \oslash b \in A$. Hence

$$x \oslash y \in (f(A)].$$

Thus, $(f(A)]$ is a sub-b-semiring of $\mathcal{Y}$. □

Theorem 5.3. *Let $f : \mathcal{Z} \to \mathcal{Y}$ be an order-preserving b-semiring homomorphism and let A be a nonempty subset of $\mathcal{Z}$.*

(1) *If A is an ordered 1-RAId, 1-LAId, or 1-AId of $\mathcal{Z}$, then $(f(A)]$ is of the same type in $\mathcal{Y}$.*
(2) *If A is an ordered 2-RAId, 2-LAId, or 2-AId of $\mathcal{Z}$, then $(f(A)]$ is of the same type in $\mathcal{Y}$.*
(3) *If A is an ordered i-BAId, i-QAId, or i-BQAId for some $i \in \{1, 2\}$, then $(f(A)]$ is simultaneously an ordered 1-BAId, 1-QAId, 1-BQAId, 2-BAId, 2-QAId, and 2-BQAId of $\mathcal{Y}$.*

Proof. Suppose first that A is an ordered 1-RAId. Then there exists $a \in A$ such that

$$a \in (A \oslash \mathcal{Z}].$$

Hence there exist $b \in A$ and $z \in \mathcal{Z}$ such that

$$a \preccurlyeq b \oslash z.$$

Applying f, we obtain

$$f(a) \preccurlyeq f(b) \oslash f(z).$$

Since

$$f(a), f(b) \in (f(A)] \quad \text{and} \quad f(z) \in \mathcal{Y},$$

it follows that

$$f(a) \in \big((f(A)] \oslash \mathcal{Y}\big].$$

Consequently,

$$\big((f(A)] \oslash \mathcal{Y}\big] \cap (f(A)] \neq \emptyset.$$

Since $(f(A)]$ is lower, it is an ordered 1-RAId of $\mathcal{Y}$.

The ordered 1-LAId case is proved similarly. Hence the ordered 1-AId case follows as well. This proves (1).

Part (2) follows by the same argument with $\uplus$ in place of $\oslash$, or equivalently by applying Proposition 4.1.

For (3), any ordered i-BAId, i-QAId, or i-BQAId is a nonempty lower sub-b-semiring of $\mathcal{Z}$. By Lemma 5.1, $(f(A)]$ is a nonempty lower sub-b-semiring of $\mathcal{Y}$. The conclusion now follows from Theorem 5.1. □

The generated hull introduced above also behaves naturally under homomorphisms.

Proposition 5.3. *Let $f : \mathcal{Z} \to \mathcal{Y}$ be an order-preserving b-semiring homomorphism. For every nonempty subset X of $\mathcal{Z}$,*

$$\mathrm{Hull}_{\mathcal{Y}}(f(X)) \subseteq \big(f(\mathrm{Hull}_{\mathcal{Z}}(X))\big].$$

Proof. By Proposition 5.2,

$$\mathrm{Hull}_{\mathcal{Z}}(X) \in \mathfrak{L}_b(\mathcal{Z}).$$

Therefore, Lemma 5.1 implies that

$$\big(f(\mathrm{Hull}_{\mathcal{Z}}(X))\big] \in \mathfrak{L}_b(\mathcal{Y}).$$

Moreover,

$$f(X) \subseteq f(\mathrm{Hull}_{\mathcal{Z}}(X)) \subseteq \big(f(\mathrm{Hull}_{\mathcal{Z}}(X))\big].$$

By the minimality of $\mathrm{Hull}_{\mathcal{Y}}(f(X))$, the required inclusion follows. □

Corollary 5.4. *Let $f : \mathcal{Z} \to \mathcal{Y}$ be an order isomorphism that is also a b-semiring isomorphism. For $i \in \{1, 2\}$, a nonempty subset A of $\mathcal{Z}$ is an ordered i-RAId, i-LAId, i-AId, i-BAId, i-QAId, or i-BQAId if and only if $f(A)$ has the corresponding property in $\mathcal{Y}$.*

Proof. Since f and f^{-1} are both order-preserving b-semiring homomorphisms, apply Theorem 5.3 in both directions. Moreover, the image of a lower subset under an order isomorphism is again lower, so no additional lower closure is required. □

6. Conclusion

In this paper, we introduced and investigated ordered 1- and 2-almost ideals in ordered b-semirings, together with their bi-almost, quasi-almost, and bi-quasi-almost forms. The ordered definitions were formulated by retaining the closure requirements of the corresponding nonordered notions and incorporating the lower-set condition through lower closures of the relevant products.

A main structural consequence of these definitions is the collapse of several apparently different almost ideal classes. For $i \in \{1, 2\}$, we obtained

$$\text{ordered } i\text{-BQAId} \iff \text{ordered } i\text{-BAId} \iff \text{ordered } i\text{-QAId} \implies \text{ordered } i\text{-AId},$$

where the first three notions are considered within the class of nonempty lower sub-b-semirings. The converse of the final implication does not hold in general. In particular, a finite example with a nontrivial partial order was constructed to show that the lower closure may play an essential role in obtaining the almost ideal property.

We further showed that the ordered 1- and 2-bi-almost, quasi-almost, and bi-quasi-almost ideals all determine the same class, namely the class of nonempty lower sub-b-semirings. This common class is closed under nonempty intersections whenever the intersection is nonempty and under unions of chains. Based on this structure, we introduced the lower sub-b-semiring hull of a nonempty subset and proved that it is the smallest member of the common class containing that subset.

To strengthen the connection with the structure of the ambient ordered b-semiring, we also considered operation-specific regularity. Under ordered 1-regularity, ordinary ordered right ideals, left ideals, bi-ideals, and quasi-ideals admit fixed-point descriptions involving lower products. In particular, if R is an ordered 1-right ideal and L is an ordered 1-left ideal, then

$$R \cap L = (R \oslash L].$$

The corresponding ordered 2-results follow from operation duality.

Preservation properties under mappings were also established. For an order-preserving b-semiring homomorphism, the lower closure of the image of an ordered right, left, or two-sided almost ideal preserves the corresponding almost ideal property. The common lower sub-b-semiring characterization yields a unified preservation result for bi-almost, quasi-almost, and bi-quasi-almost ideals. We also obtained a compatibility relation for the lower sub-b-semiring hull under homomorphisms and proved that the ordered almost ideal properties considered here are invariant under ordered b-semiring isomorphisms.

Finally, the symmetry of the two operations of a b-semiring was used to formulate an operation-dual ordered b-semiring. This duality allows the ordered 2-theory to be derived systematically from the ordered 1-theory and avoids unnecessary repetition of parallel arguments.

The results of this paper show that the interaction between closure conditions, lower sets, regularity, and homomorphisms provides a structural framework for ordered almost ideals in ordered b-semirings. Further investigation may consider converse regularity characterizations, generated ideal structures under additional assumptions, and fuzzy or other generalized versions of the ordered almost ideals introduced here.

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