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OPTIMAL HARVESTING IN A PREDATOR-PREY MODEL WITH DUAL EFFORT

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Abstract. This work is part of the sustainable management of certain aquatic resources through the study of predator–prey models integrating harvesting. We propose an approach that structures the predator population into age classes (immature and mature) and introduces two independent fishing efforts applied respectively to the prey and the mature predators. Starting from a qualitative analysis of the model, we study the equilibria, their stability, and the impact of harvesting efforts on population dynamics. Then, by using Pontryagin's Maximum Principle, we propose a harvesting strategy that maximizes revenue while ensuring species sustainability. The results show that dissociating the efforts allows for a strategy adapted to the biological specificities of the species, thus contributing to better population sustainability while optimizing economic yield.

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1. Introduction

Prey–predator models are fundamental tools in the study of population dynamics and sustainable resource management, particularly in aquatic ecosystems. Numerous works have investigated such models under various biological and environmental assumptions [12–15]. In these studies, special attention has been paid to the impact of harvesting and environmental factors such as fluctuating water levels on fish populations.

In recent years, harvesting strategies in predator–prey systems have been increasingly studied using the optimal control approach [4,7,22]. Some studies have addressed variants of the predator–prey model with optimal harvesting and complex ecological factors, including the consideration of Allee

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effects [10, 17, 19], spatial approaches [11], multi-species optimization [3], and stochastic effects [16]. In a previous work, we proposed a model where only the prey population was exploited, with the fishing effort considered as a time-dependent variable [20]. Subsequently, we introduced a structure in the predator population by distinguishing between immature and mature individuals, and considered harvesting both the prey and the mature predators [21]. However, a harvesting strategy using a single effort E applied to both species was considered, which does not accurately reflect the diversity of practices observed in real-world fisheries [1, 5]. In practice, harvesting techniques and effort intensity often differ between prey and mature predators, making it necessary to consider two independent fishing efforts to better represent and optimize management strategies. In this article, we extend the analysis of a predator–prey model with harvesting by considering two distinct control efforts, denoted E_1 and E_2, applied respectively to the prey and the mature predators. This approach allows for targeted adjustment of fishing pressure on each population, promoting both species sustainability and economic yield. The introduction of two independent efforts modifies the formulation and the resolution of the optimal control problem. The main objective of this work is therefore to propose a new harvesting strategy that accounts for this dissociation of efforts, derive the optimality conditions using Pontryagin's Maximum Principle [18].

The diagram below illustrates the interactions between the different species, highlighting the dynamics between prey, predators, and harvesting.

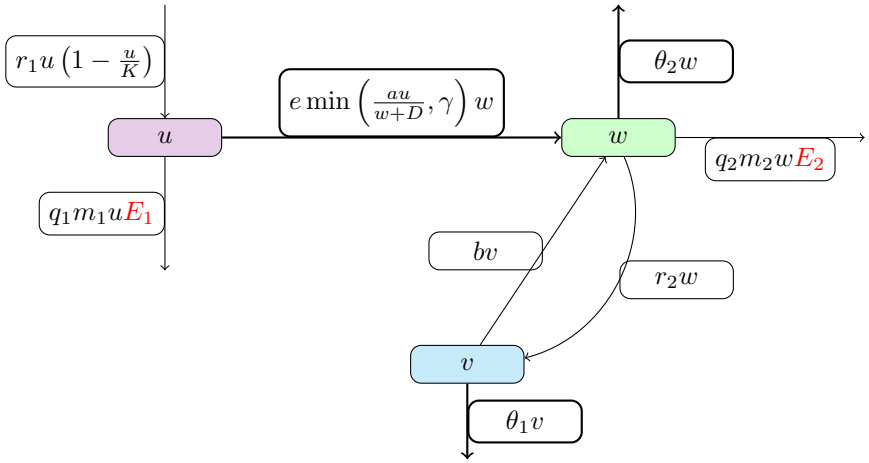


FIGURE 1.

• u, v and w represent prey, immature predators and mature predators densities, respectively, at time t;

• r_1, r_2, K, E_1, E_2, θ_1, θ_2, e, a, b, D, γ, m_1, m_2, q_1, q_2 are positive constants: r_1 is the growth rate of prey, r_2 is the growth rate of predators, K is the limiting capacity of the environment, θ_1 is the mortality rate

of immature predators, θ_2 is the mortality rate of mature predators, e is the conversion rate of prey to predators, a is the predation rate, D measures other causes of prey mortality outside of predation, γ is the maximum amount of food mature predator needs per unit time, b denotes the proportionality constant of transformation of immature to mature predators, E_1, E_2: constant fishing efforts applied to prey and mature predators, m_1 is the fraction of the prey stock available for harvesting and m_2 is the fraction of the mature predators stock available for harvesting with $m_1, m_2 \in]0,1[$, q_2 is the catchability coefficient of prey, q_2 is the catchability coefficient of mature predator;

- $\dfrac{au}{w+D}$ is the amount of food a mature predator has access to per unit time;
- $\min\left(\dfrac{au}{w+D}, \gamma\right)$ is the amount of food a mature predator receives per unit time.

Thus, the model under consideration is given by the following system:

$$\begin{cases} \dfrac{du}{dt} = r_1 u\left(1 - \dfrac{u}{K}\right) - \min\left(\dfrac{au}{w+D}, \gamma\right) w - m_1 q_1 E_1 u, \\ \dfrac{dv}{dt} = r_2 w - bv - \theta_1 v, \\ \dfrac{dw}{dt} = bv + e\min\left(\dfrac{au}{w+D}, \gamma\right) w - \theta_2 w - m_2 q_2 E_2 w, \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0, \quad w(0) = w_0 \geq 0. \end{cases} \tag{1.1}$$

The remainder of the paper is structured as follows: Section 2 the mathematical analysis of the system. In Section 3, we propose the optimal harvesting strategy for effective management. Finally, Section 4 offers a conclusion and perspectives.

In each part, our main theoretical results are validated through numerical simulations.

Remark 1.1. If the fishing efforts are equal, i.e., $E_1 = E_2 = E$, this model reduces to the one already studied in [21].

2. Mathematical Analysis

Proposition 2.1. *If the fishing effort E_1 satisfies $E_1 > \dfrac{r_1}{q_1 m_1}$, then the prey population $u(t)$ strictly decreases, i.e., $\dfrac{du(t)}{dt} < 0$. In the long run, this decline will lead to the extinction of both species.*

Proof. The first equation of system (1.1) gives us:

$$\frac{du}{dt} = u(r_1 - m_1 q_1 E_1) - \frac{r_1 u^2}{K} - \min\left(\frac{au}{w+D}, \gamma\right) w.$$

We observe that if $m_1 q_1 E_1 > r_1$, then the term $x(r_1 - m_1 q_1 E_1) < 0$ for $u > 0$. Consequently, we have $\dfrac{du(t)}{dt} < 0$, since the other two terms are also negative. □

Therefore, we make the following assumption:

$$(H_0): \quad E_1 < \frac{r_1}{q_1 m_1}.$$

Lemma 2.2. [20] *Let G be banach spaces, $f: G \to \mathbb{R}$ be a function and $\gamma \in \mathbb{R}$. Set $h: G \to \mathbb{R}$, $x \longmapsto \min(f(x), \gamma)$.*

If f is locally lipschitzian, then the function h is locally lipschitzian.

Lemma 2.3. [4] *Let a and b be two strictly positive real numbers.*

- *If $\frac{dx}{dt} \leq x(t)(a - bx(t))$ with $x(0) > 0$, then $\lim_{t\to+\infty} \sup x(t) \leq \frac{a}{b}$.*
- *If $\frac{dx}{dt} \geq x(t)(a - bx(t))$ with $x(0) > 0$, then $\lim_{t\to+\infty} \inf x(t) \geq \frac{a}{b}$.*

Theorem 2.4. *Let us consider the following assumption:*

$$E_2 > \frac{1}{q_2 m_2}(r_2 - \theta_2) \geq 0 \qquad (H_1)$$

Under this assumption, the system (1.1) admits a unique solution for any initial condition $(u(0), v(0), w(0)) \in \mathbb{R}^3_+$.

Moreover, if for all $t \geq 0$, $v(t)$ remains non-negative, then the solution is positive and belongs to the following set:

$$A = \{(u, v, w) \in \mathbb{R}^3_+ \mid 0 \leq u \leq K, \quad 0 \leq eu + v + w \leq Q\},$$

where $Q = \frac{er_1K}{4\tau}$, *with* $\tau = \min(q_1 m_1 E_1, \ \mu_1, \ q_2 m_2 E_2 + \theta_2 - r_2)$.

Proof. According to Lemma (2.2) and the Cauchy–Lipschitz theorem, for any initial conditions $u(0) = u_0$, $v(0) = v_0$, and $w(0) = w_0$, there exists a real number $T_{\max}$ such that the system (1.1) admits a unique solution $(u(t), v(t), w(t))$ for all $t \in [0, T_{\max})$.

Let us show that the solution belongs to the set A.

From the first equation of system (1.1), we can write:

$$\frac{du(t)}{dt} \geq \left\{ r_1 \left(1 - \frac{u}{K}\right) - \frac{aw}{z(t) + D} - q_1 m_1 E_1 \right\} u,$$

which implies:

$$\frac{du(t)}{u(t)} \geq \varphi(u, v, w)\, dt, \tag{2.1}$$

where $\varphi(u, v, w) = r_1 \left(1 - \frac{u}{K}\right) - \frac{aw}{w + D} - q_1 m_1 E_1$.

Integrating equation (2.1) over the interval $[0, t]$, we obtain:

$$u(t) \geq u(0) \exp\left(\int \varphi(u, v, w)\, dt \right) \geq 0, \quad \forall t \geq 0.$$

By assumption, $v(t) \geq 0$ for all $t \geq 0$.

We can also show that $\frac{dv(t)}{dt} \geq 0$ for all $t \geq 0$. Suppose the contrary: there exists $t_1 > 0$ such that

$$t_1 = \inf\{t > 0 : v(t) = 0\}, \quad \text{and } \dot{v}(t_1)\big|_{v(t_1)=0} = r_2 w(t_1) < 0,$$

which implies $w(t_1) < 0$. However, since $v(t) > 0$ for $t \in [0, t_1)$ and $w(0) \geq 0$, the function $w(t)$ must vanish at some earlier time and then become negative. That is, there exists $t_2 \in [0, t_1)$ defined by:

$$t_2 := \inf \{t \in [0, t_1) \mid w(t) = 0 \text{ and there exists } \delta > 0 \text{ such that } w(s) < 0 \text{ for all } s \in (t, t + \delta)\}.$$

By definition of t_2, we then have:

$$\dot{w}_+(t_2) = \lim_{h \to 0^+} \frac{w(t_2 + h) - w(t_2)}{h} = \lim_{h \to 0^+} \frac{w(t_2 + h)}{h} < 0.$$

But according to the last equation of the system, $\dot{w}(t_2) = bv(t_2) > 0$, which is a contradiction.

Hence, $\dot{v}(t) \geq 0$ for all $t \geq 0$.

Since $\frac{dv(t)}{dt} \geq 0$, by the second equation of the system (1.1), we obtain:

$$w(t) \geq \frac{b + \theta_1}{r_2} v(t) \geq 0, \quad \forall t \geq 0.$$

Thus, the solution $(u(t), v(t), w(t)) \in \mathbb{R}^3_+$.

Now, using the first equation again, we get:

$$\frac{du(t)}{dt} \leq r_1 u \left(1 - \frac{u}{K}\right).$$

From Lemma (2.2), we deduce:

$$\limsup_{t \to +\infty} u(t) \leq K.$$

Therefore, for any $\varepsilon_1 > 0$, there exists $T_1 > 0$ such that

$$u(t) \leq K + \varepsilon_1, \quad \forall t > T_1.$$

Now define:

$$X(t) = eu(t) + v(t) + w(t).$$

Then,

$$\begin{aligned} \frac{dX(t)}{dt} &= e\frac{du(t)}{dt} + \frac{dv(t)}{dt} + \frac{dw(t)}{dt}, \\ &\leq er_1 u \left(1 - \frac{u}{K}\right) - \tau(eu + v + w), \end{aligned}$$

with $\tau = \min(q_1 m_1 E_1,\ \theta_1,\ q_2 m_2 E_2 + \theta_2 - r_2)$, under hypothesis (H_1).

This implies:

$$\frac{dX(t)}{dt} \leq \frac{er_1 K}{4} - \tau X(t).$$

Therefore:

$$\limsup_{t \to +\infty} X(t) \leq \frac{er_1 K}{4\tau} := Q.$$

So for any $\varepsilon_2 > 0$, there exists $T_2 > 0$ such that $X(t) \leq Q + \varepsilon_2$ for all $t > T_2$. This completes the proof. □

Proposition 2.5. *Assume that:*

$$a < min\left(\frac{\gamma_(w_0+D)}{u_0}, \quad \frac{4r_1\gamma D(\theta_2+q_2m_2E_2)}{K(r_1+q_2m_2E_2-q_1m_1E_1+\theta_2)^2}\right) \qquad (H_2),$$

then for all $t \geq 0$, *we have:*

$$au(t) < \gamma\left(w(t)+D\right).$$

Proof. Let $k(t) = au(t) - \gamma\left(w(t)+D\right).$

Let's show that if the hypothesis (H_2) is verified, then $u(t)$ is strictly negative for all $t \geq 0$.

For $t = 0$, we have: $k(0) = au(0) - \gamma(w(0)+D) = u_0\left(a - \frac{\gamma(w_0+D)}{u_0}\right)$, with $u_0 > 0$. Now according to the hypothesis (H_2), $a < \frac{\gamma(w_0+D)}{u_0}$. Thus $k(0) < 0$.

Suppose there is a positive t_0 such that $k(t_0) = 0$ and $\frac{dk(t_0)}{dt} \geq 0$.

Using the system (1.1), we obtain:

$$\begin{aligned}\frac{dk(t_0)}{dt} &= -\frac{ar_1}{K}(u(t_0))^2 + a(r_1+\theta_2+q_2m_2E_2-q_1m_1E_1)u(t_0)\\ &\quad - a(a+e\gamma)\frac{w(t_0)}{w(t_0)+D}u(t_0) - \gamma bv(t_0) - \gamma D(\theta_2+q_2m_2E_2),\\ &\leq -\frac{ar_1}{K}(u(t_0))^2 + a(r_1+\theta_2+q_2m_2E_2-q_1m_1E_1)u(t_0) - \gamma D(\theta_2+q_2m_2E_2).\end{aligned}$$

So under the hypothesis (H_2), $\frac{dk(t_0)}{dt} < 0$, and this constitutes a contradiction.

Thus, for all $t > 0$, we have $k(t) \neq 0$ or $\frac{dk(t)}{dt} < 0$.

This means that for all $t > 0$, $k(t) < 0$. □

Remark 2.6. Under the hypothesis of the previous proposition, the system (1.1) can be rewritten in a simplified form as follows:

$$\begin{cases}\frac{du}{dt} = r_1u\left(1-\frac{u}{K}\right) - \frac{auw}{w+D} - q_1m_1E_1u,\\ \frac{dv}{dt} = r_2w - bv - \theta_1v,\\ \frac{dw}{dt} = bv + \frac{eauw}{w+D} - \theta_2z - q_2m_2E_2w.\end{cases} \qquad (2.2)$$

Theorem 2.7. *In addition to assumption* (H_0),(H_1) *and* (H_2), *let's assume that the following assumption is true:*

$$\frac{ea\beta_1}{\mu_2+q_2m_2E_2} > D \qquad (H_3), \quad witht \quad \beta_1 = \frac{K}{r_1}(r_1-a-q_1m_1E_1),$$

then system (1.1) *is uniformly persistence.*

Proof. Note that according to thoeorem (2.4), we have:

$$\lim_{t\to+\infty}\sup u(t) \leq K, \quad \lim_{t\to+\infty}\sup v(t) \leq Q, \quad \lim_{t\to+\infty}\sup w(t) \leq Q.$$

with $Q = \dfrac{er_1K}{4\tau}$.

Posing $M_2 = \max(K, Q)$ and $U(t) = (u(t), v(t), w(t))^T$.

We have:

$$\lim_{t\to+\infty} \sup U(t) \leq M_2.$$

Using the first equation of system (2.2), we have:

$$\begin{aligned}\frac{du(t)}{dt} &\geq r_1u - \frac{r_1}{K}u^2 - \frac{auQ}{Q+D} - q_1m_1E_1u,\\ &\geq (r_1 - a - q_1m_1E_1)\, u - \frac{r_1}{K}u^2.\end{aligned}$$

According to the lemma (2.3), we get:

$$\lim_{t\to+\infty} \inf u(t) \geq \frac{K}{r_1}(r_1 - a - q_1m_1E_1) := \beta_1.$$

Under hypothesis (H_3), $\beta_1 > 0$.

Now from the third equation of the system (2.2), we have:

$$\begin{aligned}\frac{dw}{dt} &\geq \frac{eauw}{w+D} - (\theta_2 + q_2m_2E_2)w,\\ &\geq \frac{1}{Q+D}w\left[ea\beta_1 - (\theta_2 + q_2m_2E_2)w - D(\theta_2 + q_2m_2E_2)\right].\end{aligned}$$

So, according to the lemma (2.3), we have:

$$\lim_{t\to+\infty} \inf w(t) \geq \frac{ea\beta_1}{\theta_2 + q_2m_2E_2} - D := \beta_2.$$

Under hypothesis (H_3), $\beta_2 > 0$.

Similarly, the second equation of the system (2.2) yields:

$$\frac{dv}{dt} \geq r_2\beta_2 - (b + \theta_1)v.$$

As a resutl, we obtain:

$$\lim_{t\to+\infty} \inf v(t) \geq \frac{r_2\beta_2}{b+\theta_1} := \beta_3.$$

Taking $M_1 = \min(\beta_1, \beta_2, \beta_3)$, we have $\lim_{t\to+\infty} \inf U(t) \geq M_1$. This completes the proof. □

Proposition 2.8. *The system (2.2) admits three equilibrium points:*

(i) The point $P_0(0,0,0)$,

(ii) The point $P_1(\bar{u},0,0)$ exists under hypothesis (H_0),

(iii) A positive equilibrium point $P^*(u^*,v^*,w^*)$ exists under hypothesis (H_0), and the following additional hypotheses:

$$E_1 > \frac{1}{q_1m_1}(r_1 - a) \geq 0 \qquad (H_3^c),$$

$$E_2 > \frac{1}{q_2 m_2}\left[\frac{eaDK}{(\beta_2 + D)^2} - \theta_2\right] \qquad (H_4),$$

$$w_b < w_a \qquad (H_5),$$

where

$$w_a = \frac{D(r_1 - q_1 m_1 E_1)}{a + q_1 m_1 E_1 - r_1} \quad \textit{and} \quad w_b = \frac{b}{\theta_2 + q_2 m_2 E_2} v, \quad \forall v > 0.$$

Proof. • The case $i)$ is obtained by substituting the variables (u, v, w) with $(0, 0, 0)$ in system (2.2).

• The abscissa of point P_1 is the solution to the equation:

$$r_1\left(1 - \frac{u}{K}\right) - q_1 m_1 E_1 = 0.$$

• To determine the existence of the equilibrium point P^*, we use the method of isoclines.

Let us show that P^* is the solution of the following equations:

$$r_1(1 - \frac{u}{K}) - \frac{aw}{w + D} - q_1 m_1 E_1 = 0, \tag{2.3}$$

$$r_2 w - bv - \theta_1 v = 0, \tag{2.4}$$

$$bv + \frac{eauw}{w + D} - (\theta_2 + q_2 m_2 E_2) w = 0. \tag{2.5}$$

Using equations (2.3) and (2.5), we check for the existence of u^* and w^*. The existence of v^* follows directly from equation (2.4), which can be written as:

$$v^* = \frac{r_2 w^*}{b + \theta_1}.$$

(i) Study of the isocline (2.3)

- **If** $u = 0$, then:

$$w = \frac{D(r_1 - q_1 m_1 E_1)}{a + q_1 m_1 E_1 - r_1} := w_a.$$

Under hypothesis $(H_0) : r_1 > q_1 m_1 E_1$, the numerator is strictly positive. Therefore, the value w_a is strictly positive under hypotheses (H_0) and (H_3^c), where:

$$(H_3^c) : E_1 > \frac{1}{q_1 m_1}(r_1 - a) \geq 0.$$

- **If** $w = 0$:

$$r_1\left(1 - \frac{u}{K}\right) - q_1 m_1 E_1 = 0 \quad \Rightarrow \quad u = \frac{K}{r_1}(r_1 - q_1 m_1 E_1) := u_a.$$

Under (H_0), we have $u_a > 0$.

Slope of the isocline

Let:

$$f(u, w) = r_1\left(1 - \frac{u}{K}\right) - \frac{aw}{w + D} - q_1 m_1 E_1 = 0.$$

Then:

$$\frac{dw}{du} = -\frac{\dfrac{\partial f}{\partial u}}{\dfrac{\partial f}{\partial w}} = -\frac{r_1}{KaD}(w+D)^2 < 0.$$

This analysis shows that the curve defined by isocline (2.3) is **strictly decreasing**. Under hypotheses (H_0) and (H_3^c), it passes through the points $(u_a, 0)$ and $(0, w_a)$ in the positive quadrant.

(ii) Study of the isocline (2.5)

- **If** $u = 0$:

$$bv - (\theta_2 + q_2 m_2 E_2)w = 0 \quad \Rightarrow \quad w = \frac{b}{\theta_2 + q_2 m_2 E_2} v := w_b,$$

  which is strictly positive for all $v > 0$.

- **If** $w = 0$:

$$bv + 0 = 0 \quad \Rightarrow \quad v = 0.$$

  In this case, no specific value of u can be deduced, hence $u = u_b$ can take any strictly positive value, i.e., $u_b \in \mathbb{R}_+^*$.

Slope of the isocline

Let:

$$g(u, w) = bv + \frac{eauw}{w+D} - (\theta_2 + q_2 m_2 E_2)w = 0,$$

Then:

$$\frac{dw}{du} = -\frac{\dfrac{\partial g}{\partial u}}{\dfrac{\partial g}{\partial w}} = \frac{eau(w+D)}{(\theta_2 + q_2 m_2 E_2)(w+D)^2 - eaDu}$$

This slope is **positive** if:

$$\frac{eaDu}{(w+D)^2} < \theta_2 + q_2 m_2 E_2.$$

Since:

$$\lim_{t\to+\infty} \sup u(t) \le K \quad \text{and} \quad \lim_{t\to+\infty} \inf w(t) \ge \beta_2,$$

for sufficiently large t, we have:

$$\frac{eaDu(t)}{(w(t)+D)^2} \le \frac{eaDK}{(\beta_2+D)^2}.$$

Thus, $\dfrac{dw}{du} > 0$ if the following inequality holds:

$$\frac{eaDK}{(\beta_2+D)^2} < \theta_2 + q_2 m_2 E_2 \Rightarrow E_2 > \frac{1}{q_2 m_2}\left[\frac{eaDK}{(\beta_2+D)^2} - \theta_2\right] \ge 0 \quad (H_4)$$

Under hypothesis (H_4), the curve defined by isocline (2.5) is **strictly increasing**. It passes through the points $(0, w_b)$ and $(w_b, 0)$ in the positive quadrant.

In conclusion, the two isoclines (2.3) and (2.5) intersect at a unique point (u^*, w^*) in the positive quadrant if the following condition holds:

$$w_b < w_a. \qquad (H_5)$$

□

Proposition 2.9.
- *The point P_0 is stable under hypothesis (H_1), and unstable under the hypothesis* $(H_1^c) := E_2 < \dfrac{1}{q_2 m_2}(r_2 - \theta_2)$.
- *The equilibrium point P_1 is unstable under hypothesis (H_3), and stable under hypothesis (H_3^c) and the condition $S_2 < 0$, where*
$(H_3^c) := \dfrac{ea\beta_1}{\theta_2 + q_2 m_2 E_2} < D$, *and* $S_2 = -\left(b + \theta_1 + \theta_2 + q_2 m_2 E_2 - \dfrac{ea\bar{u}}{D}\right)$
- *The point P^* is locally asymptotically stable under hypothesis (H_4).*

Proof. The Jacobian matrix J of system (2.2) is given by:

$$J(u,v,w) = \begin{pmatrix} r_1 - \dfrac{2r_1 u}{K} - \dfrac{aw}{w+D} - q_1 m_1 E_1 & 0 & \dfrac{-aDu}{(w+D)^2} \\ 0 & -(b+\theta_1) & r_2 \\ \dfrac{eaw}{w+D} & b & \dfrac{eaDu}{(w+D)^2} - (\theta_2 + q_2 m_2 E_2) \end{pmatrix}.$$

- The characteristic polynomial P of the matrix $J(P_0)$ is:

$$P(\lambda) = (r_1 - q_1 m_1 E_1 - \lambda)\left[\lambda^2 + (b + \theta_1 + \theta_2 + q_2 m_2 E_2)\lambda + (b+\theta_1)(\theta_2 + q_2 m_2 E_2) - br_2\right].$$

Hence, the point P_0 is unstable under hypothesis (H_0).

- The characteristic polynomial P of the matrix $J(P_1)$ is:

$$P(\lambda) = \left(\frac{-r_1\bar{u}}{K} - \lambda\right) \times \left[\lambda^2 + \left(b + \theta_1 + \theta_2 + q_2 m_2 E_2 - \frac{ea\bar{u}}{D}\right)\lambda - (b+\theta_1)\left(\frac{ea\bar{u}}{D} - \theta_2 - q_2 m_2 E_2\right) - br_2\right].$$

Therefore, the equilibrium point P_1 is unstable under hypothesis (H_3), and stable under (H_3^c) and if $S_2 < 0$.

- Consider the matrix $J(P^*)$ defined by:

$$J(P^*) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The characteristic polynomial of $J(P^*)$ is of the form:

$$P(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3,$$

where:

$$\begin{cases} A_1 = -(a_{11} + a_{22} + a_{33}), \\ A_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{13}a_{31} - a_{23}a_{32}, \\ A_3 = a_{11}a_{23}a_{32} + a_{13}a_{22}a_{31} - a_{11}a_{22}a_{33}. \end{cases}$$

Under hypothesis (H_4), we have $A_1 > 0$, $A_3 > 0$, and $A_1A_2 - A_3 > 0$.
According to the Routh–Hurwitz criterion [2], all eigenvalues of the matrix $J(P^*)$ have negative real parts. Therefore, the point P^* is locally asymptotically stable.

□

For the figures below, the values of the constants are as: $r_1 = 12$,$r_2 = 9$, $K = 40$, $a = 11$, $D = 5$, $q_1 = 6$, $q_2 = 5$, $m_1 = m_2 = 0.75$, $e = 1.20$, $E_1 = 2.4$, $E_2 = 1.9$, $\theta_1 = 0.45$, $\theta_2 = 0,4$.

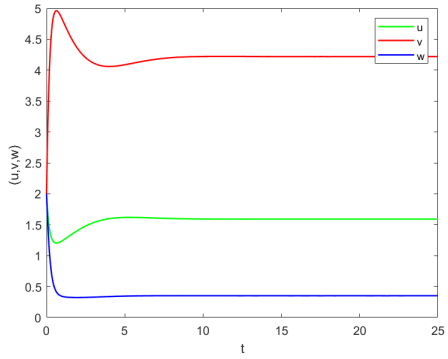


FIGURE 2. Convergence of the solution towards the equilibrium point P^*

Theorem 2.10. *The equilibrium point P^* is globally asymptotically stable if the assumption (H_5) and the following hypothesis*

$$E_2 > \frac{1}{q_2 m_2}\left[\frac{r_2(r_2+1)}{4(b+\theta_1)} - \theta_2\right], \qquad (H_6)$$

are satisfied.

Proof. Let V be a function defined by:

$$V(u,v,w) = \alpha\left[(u-u^*) - u^*\ln(\frac{u}{u^*})\right] + \frac{1}{2}(v-v^*)^2 + \frac{1}{2}(w-w^*)^2,$$

where α is a positive constant to be determined.

- We have: $V(u^*,v^*,w^*) = 0 \quad$ and $\quad V(u,v,w) > 0, \quad \forall \quad (u,v,w) \neq (u^*,v^*,w^*)$.
- Let's calculate $\dfrac{dV}{dt}$.

$$\frac{dV}{dt} = \alpha\frac{(u-u^*)}{u} \times \frac{du}{dt} + (v-v^*) \times \frac{dv}{dt} + (w-w^*) \times \frac{dw}{dt}.$$

This implies that:

$$\frac{dV}{dt} = \frac{-\alpha r_1}{K}(u-u^*)^2 - (b+\theta_1)(v-v^*)^2 - \left[\theta_2 + q_2m_2E_2 - \frac{eaDu^*}{(w^*+D)(w+D)}\right](w-w^*)^2$$

$$+\left[\frac{eaw}{w+D}-\frac{\alpha aD}{(w^*+D)(w+D)}\right](u-u^*)(w-w^*)+(r_2+b)(v-v^*)(w-w^*).$$

By choosing $\alpha=\dfrac{ew(w^*+D)}{D}$, we have:

$$\frac{dV}{dt}=\frac{-\beta r_1}{K}(u-u^*)^2-(b+\theta_1)(v-v^*)^2-\left[\theta_2+q_2m_2E_2-\frac{eaDu^*}{(w^*+D)(w+D)}\right](w-w^*)^2$$
$$+(r_2+b)(v-v^*)(w-w^*).$$

The right hand side is a quadratic form in the variables $(u-u^*)$,$(v-v^*)$ and $(w-w^*)$ which can be written as

$$X^TAX,$$

where $X^T=[(u-u^*),(v-v^*),(w-w^*)]$ and

$$A=\begin{pmatrix}\dfrac{-\beta r_1u^*}{K} & 0 & 0\\ 0 & -(b+\theta_1) & \dfrac{1}{2}(r_2+b)\\ 0 & \dfrac{1}{2}(r_2+b) & \dfrac{eaDu^*}{(w^*+D)(w+D)}-(\theta_2+q_2m_2E_2)\end{pmatrix}.$$

$\dfrac{dV}{dt}<0$, if the matrix A is is negative definite.

Therefore, $\dfrac{dV}{dt}<0$ under the assumptions (H_4) and (H_6).

So according to Lyapunov's theorem [6] the positive equilibrium P^* is globally asymptotically stable.

□

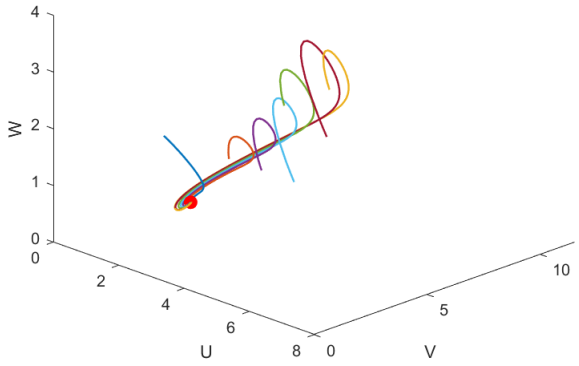
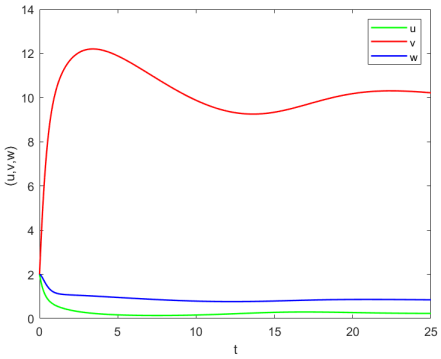
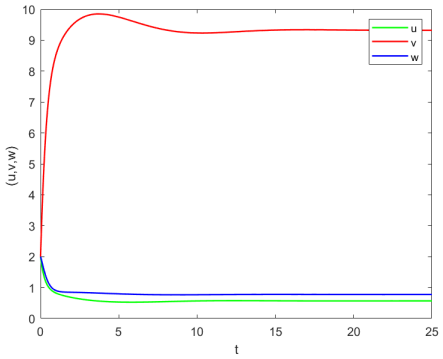


FIGURE 3. Phase portrait

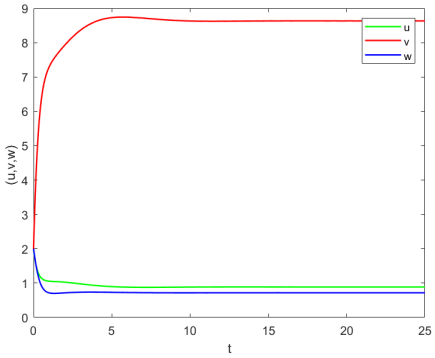
Remark 2.11. The distinction between fishing efforts allows maintaining a controlled pressure on the prey (E_1) while modulating the effort on mature predators (E_2), in order to regulate predation and support the prey biomass. For instance, for $E_1 = 2.3$ and $E_2 = 1$ (Figure 4)(a), the prey biomass is lower than that of mature predators and close to extinction, which does not favor either fishing activity or species sustainability. Therefore, fishers can maintain the same pressure (E_1) on the prey when it is economically advantageous, and adjust (E_2) to regulate predation and support the prey biomass (Figure 4)(b,c,d).

(a) $(E_1, E_2) = (2.3, 1)$ **(b)** $(E_1, E_2) = (2.3, 1.2)$





(c) $(E_1, E_2) = (2.3, 1.4)$ **(d)** $(E_1, E_2) = (2.3, 1.8)$



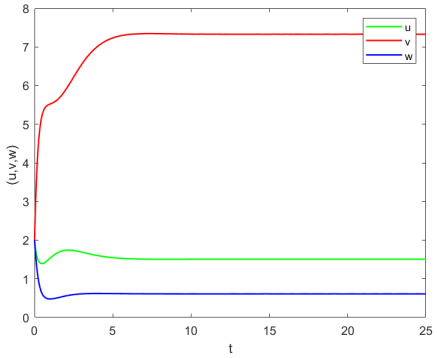


FIGURE 4.

3. Optimum Harvesting Strategy

Determining an optimal fishing strategy remains a key challenge for policymakers. The objective is to strike a balance between present and future harvests, while ensuring both economic viability and political feasibility. This section focuses on the profit-oriented aspect of fisheries management, as highlighted in [5]. In this context, we apply Pontryagin's Maximum Principle to derive an optimal harvesting strategy by considering a function that represents the continuous-time flow of income.

The **net economic revenue** is defined by:

$$\pi(u, v, w, E_1, E_2, t) = (p_1 q_1 m_1 E_1 u - c_1 E_1) + (p_2 q_2 m_2 E_2 w - c_2 E_2)\,, \tag{3.1}$$

where:

- c_1 and c_2 denote the harvesting costs per unit effort for the prey and mature predator species, respectively,
- p_1 is the price per unit biomass of the prey population u,
- p_2 is the price per unit biomass of the mature predator population w.

We assume that p_1, p_2, c_1, and c_2 are **positive constants**.

Let U_{ad} denote the admissible control set, defined by:

$$U_{ad} = \{E_i : [0, +\infty[\to [0, E_i^{\max}],\ i = 1, 2,\ \text{Lebesgue measurable}\}\,. \tag{3.2}$$

Consider J a cost function defined by:

$$\begin{aligned} J(E_1, E_2) &= \int_0^{+\infty} e^{-\delta t} \pi(u, v, w, E_1, E_2, t) dt, \\ &= \int_0^{+\infty} e^{-\delta t} \left[(p_1 q_1 m_1 u - c_1) E_1 + (p_2 q_2 m_2 w - c_2) E_2\right] dt. \end{aligned}$$

where δ is the instantaneous annual discount rate, $E_1, E_2 \in U_{ad}$ is the control function.

Our objective is to maximize the functional J. Since J is convex and U_{ad} is compact, then we can show that an optimal control exists [8].

The the Hamiltonian function is defined by:

$$\begin{aligned} H \quad &= e^{-\delta t} \left[(p_1 q_1 m_1 u - c_1) E_1 + (p_2 q_2 m_2 w - c_2) E_2\right] + \lambda_1 \left[r_1 u - \frac{r_1}{K} u^2 - \frac{auw}{w + D} - q_1 m_1 E_1 u\right] \\ &\quad + \lambda_2 \left[r_2 w - bv - \theta_1 v\right] + \lambda_3 \left[bv + \frac{eauw}{w + D} - \theta_2 z - q_2 m_2 E_2 w\right], \end{aligned} \tag{3.3}$$

with λ_1,λ_2 and λ_3 as adjoint variables.

Assuming that the control constraints are note binding. We must have singular control given by:

$$\frac{\partial H}{\partial E_1} = e^{-\delta t}(p_1 q_1 m_1 u - c_1) - \lambda_1 q_1 m_1 u = 0, \tag{3.4}$$

$$\frac{\partial H}{\partial E_2} = e^{-\delta t}(p_2 q_2 m_2 w - c_2) - \lambda_2 q_2 m_2 w = 0, \tag{3.5}$$

Using equations (3.4) and (3.5), respectively, we obtain:

$$\lambda_1 = e^{-\delta t} \left(p_1 - \frac{c_1}{q_1 m_1 u}\right), \tag{3.6}$$

$$\lambda_2 = e^{-\delta t} \left(p_2 - \frac{c_2}{q_2 m_2 w}\right). \tag{3.7}$$

According the Pontryagin maximum principle [9], we get:

$$\frac{d\lambda_1}{dt} = -e^{-\delta t}p_1q_1m_1E_1 - \lambda_1\left(r_1 - \frac{2r_1u}{K} - \frac{aw}{w+D} - q_1m_1E_1\right) - \lambda_3\frac{eaw}{w+D}, \tag{3.8}$$

$$\frac{d\lambda_2}{dt} = (b+\theta_1)\lambda_2 - b\lambda_3, \tag{3.9}$$

$$\frac{d\lambda_3}{dt} = -e^{-\delta t}p_2q_2m_2E_2 + \lambda_1\left[\frac{aDu}{(w+D)^2}\right] - r_2\lambda_2 - \lambda_3\left[\frac{eaDu}{(w+D)^2} - (\theta_2 + q_2m_2E_2)\right]. \tag{3.10}$$

According to equation (3.7), we have:

$$\dot{\lambda_2} = -\delta e^{-\delta t}\left(p_2 - \frac{c_2}{q_2m_2w}\right) \tag{3.11}$$

The use of equations (3.7) and (3.11) in equation (3.9) allows us to obtain:

$$\lambda_3 = \frac{1}{b}e^{-\delta t}\left(p_2 - \frac{c_2}{q_2m_2w}\right)(b+\theta_1+\delta) \tag{3.12}$$

Thus, the shadow price $e^{\delta t\lambda_i}$ $(i = 1, 2, 3)$ remain constant over time in optimal equilibrium when they satisfy the transversality condition at $t = \infty$, i.e they remain bounded as $t \to \infty$.

Using the equilibrium conditions, equation (3.8) becomes:

$$\frac{d\lambda_1}{dt} = -e^{-\delta t}p_1q_1m_1E_1 + \lambda_1\left(\frac{r_1u}{K}\right) - \lambda_3\frac{eaw}{w+D} \tag{3.13}$$

By substituting λ_1 and λ_3 into equation (3.13), and then integrating, we obtain:

$$\lambda_1 = \frac{1}{\delta}e^{-\delta t}\left[p_1q_1m_1E_1 - \left(p_1 - \frac{c_1}{q_1m_1u}\right)\left(\frac{r_1u}{K}\right) + \frac{1}{b}\left(p_2 - \frac{c_2}{q_2m_2w}\right)(b+\theta_1+\delta)\right] \tag{3.14}$$

From equations (3.6) and (3.14), we have the following equality:

$$\begin{aligned} e^{-\delta t}\left(p_1 - \frac{c_1}{q_1m_1u}\right) = \frac{1}{\delta}e^{-\delta t}\Bigg[&p_1q_1m_1E_1 - \left(p_1 - \frac{c_1}{q_1m_1u}\right)\left(\frac{r_1u}{K}\right) \\ &+ \frac{1}{b}\left(p_2 - \frac{c_2}{q_2m_2w}\right)(b+\theta_1+\delta)\Bigg] \end{aligned} \tag{3.15}$$

From equation (3.15), the optimal control E_1 is given by:

$$E_1 = \frac{\delta\left(p_1 - \frac{c_1}{q_1m_1u}\right) + \left(p_1 - \frac{c_1}{q_1m_1u}\right)\left(\frac{r_1u}{K}\right) - \frac{1}{b}\left(p_2 - \frac{c_2}{q_2m_2w}\right)(b+\theta_1+\delta)}{p_1q_1m_1} \tag{3.16}$$

Similarly, using the equilibrium conditions, equation (3.10) becomes:

$$\frac{d\lambda_3}{dt} = -e^{-\delta t}p_2q_2m_2E_2 + \lambda_1\left[\frac{aDu}{(w+D)^2}\right] - r_2\lambda_2 + \lambda_3\left[\frac{1}{w}\left(b + \frac{eaDuw^2}{(w+D)^2}\right)\right] \tag{3.17}$$

By substituting λ_1 and λ_3 into equation (3.13), and then integrating, we obtain:

$$\begin{aligned} \lambda_3 = \frac{1}{\delta}e^{-\delta t}\Bigg[&p_2q_2m_2E_2 - \left(p_1 - \frac{c_1}{q_1m_1u}\right)\left(\frac{aDu}{(w+D)^2}\right) + r_2\left(p_2 - \frac{c_2}{q_2m_2w}\right) \\ &- \frac{1}{bw}\left(p_2 - \frac{c_2}{q_2m_2w}\right)(b+\theta_1+\delta)\left(b + \frac{eaDuw^2}{(w+D)^2}\right)\Bigg] \end{aligned} \tag{3.18}$$

From equations (3.12) and (3.18), we obtain the following equality:

$$\frac{1}{b}e^{-\delta t}\left(p_2-\frac{c_2}{q_2m_2w}\right)(b+\theta_1+\delta)=\frac{1}{\delta}e^{-\delta t}\Bigg[p_2q_2m_2E_2-\left(p_1-\frac{c_1}{q_1m_1u}\right)\left(\frac{aDu}{(w+D)^2}\right)+r_2\left(p_2-\frac{c_2}{q_2m_2w}\right)-\frac{1}{bw}\left(p_2-\frac{c_2}{q_2m_2w}\right)(b+\theta_1+\delta)\left(b+\frac{eaDuw^2}{(w+D)^2}\right)\Bigg] \tag{3.19}$$

From equation (3.19), the optimal control E_2 is given by:

$$E_2=\frac{1}{p_2q_2m_2}\Bigg[\frac{\delta}{b}\left(p_2-\frac{c_2}{q_2m_2w}\right)(b+\theta_1+\delta)+\left(p_1-\frac{c_1}{q_1m_1u}\right)\left(\frac{aDu}{(w+D)^2}\right)-r_2\left(p_2-\frac{c_2}{q_2m_2w}\right)+\frac{1}{bw}\left(p_2-\frac{c_2}{q_2m_2w}\right)(b+\theta_1+\delta)\left(b+\frac{eaDuw^2}{(w+D)^2}\right)\Bigg] \tag{3.20}$$

Thus, by solving the equilibrium equations using the relations (3.16) and (3.20), we obtain the optimal solution and the corresponding optimal efforts.

4. Conclusion

In this work, we studied a prey–predator model with harvesting, in which the predator population is structured into age classes (immature and mature). Two distinct fishing efforts were applied respectively to the prey and to the mature predators. Based on a qualitative analysis of the model, we examined the equilibria, their stability, and the impact of harvesting efforts on population dynamics. This approach makes it possible to maintain the prey biomass at a viable level, regulate predation, and optimize economic yield. The application of Pontryagin's Maximum Principle allowed us to propose a harvesting strategy aimed at maximizing revenue while ensuring the sustainability of the species.

Authors' Contributions. All authors (1, 2, 3, and 4) have read and approved the final version of the manuscript.

- Author 1: Model development and theoretical analysis.
- Author 2: Numerical simulations and interpretation of results.
- Author 3: Scientific discussions, contributions to certain aspects of the mathematical analysis, and input on the overall direction of the work.
- Author 4: Validation of calculations and manuscript review.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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