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PIVOTAL INFERENCE FOR THE WEIBULL DISTRIBUTION BASED ON PROGRESSIVE TYPE-II CENSORING SCHEME

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Abstract. This paper proposes a pivotal-based parameter estimation method for the Weibull distribution under progressive Type-II censoring. With calculation simplicity, the proposed estimators show favoring uniqueness and consistency properties. Furthermore, both numerical simulations and real-world applications illustrate that the proposed approach performs competitively against the maximum likelihood estimation method across different censoring schemes. Consequently, this method serves as a compelling alternative parameter estimation method for the Weibull distribution under progressive Type-II censoring.

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1. Introduction

The Weibull distribution, developed by Wallodi Weibull, is a widely used statistical distribution in diverse fields such as survival analysis, reliability engineering, and insurance. Its flexibility in capturing different failure rates makes it a popular choice for modeling lifetime data. The probability density function (pdf) and cumulative distribution function (cdf) of the Weibull distribution with scale parameter α ($\alpha > 0$) and shape parameter β ($\beta > 0$) are given by

$$f_X(x;\alpha,\beta) = \left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^{\beta}}, \qquad x \geq 0, \tag{1}$$

and

$$F_X(x;\alpha,\beta) = 1 - e^{-\left(\frac{x}{\alpha}\right)^{\beta}}, \qquad x \geq 0, \tag{2}$$

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respectively. The Weibull distribution is defined by two parameters: the shape parameter (β) and the scale parameter (α). The shape parameter affects the distribution's hazard function, allowing it to model various failure patterns, while the scale parameter affects the overall scale, influencing the location of failure times. When β equals 1, the Weibull distribution simplifies to the exponential distribution, commonly used in reliability and survival analysis. In survival analysis, the Weibull distribution is widely used to model lifetime data with different hazard rates. For example, Mudholkar *et al.* [7] used the Weibull distribution to model lifetime data with monotone hazard rates, showing its adaptability to different failure scenarios. Similarly, in reliability engineering, the Weibull distribution has been valuable for analyzing failure data. Calabria and Pulcini [3] used the Weibull distribution to study failure data obtained through randomly censored sampling methods and conducted Bayesian reliability analysis, highlighting its importance in assessing reliability and failure probabilities. The Weibull distribution also has practical applications in engineering beyond statistical analysis. Corzo and Bracho [4] used the Weibull distribution to explain the vacuum pulse osmotic dehydration of sardine sheets, demonstrating its significance in modeling various engineering processes. Additionally, the Weibull distribution has been used to model claim durations. Das and Nath [5] employed the Weibull distribution to fit insurance claim data, showing its ability to capture complex patterns in claim durations and assess the likelihood of eventual ruin for Weibull-distributed claim data.

Despite the versatility of the Weibull distribution, accurately estimating its parameters becomes more challenging with censored data. Censoring occurs when some observations are only partially known due to study termination or technical limitations. To address this issue, we focus on progressive Type-II censoring [2]. This is a common method in long-term reliability studies, where individuals are progressively censored as failures occur, making it suitable for capturing rare or long-lasting failure events. In this scheme, the pre-fixed failure number is $m \in \{1, \ldots, n\}$, and R_i specifies the number of units removed when the ith failure occurs. When the first failure happens, R_1 units are randomly removed from the remaining $n-1$ units. When the second failure occurs, R_2 units are removed from the $n-2-R_1$ surviving units, and so on. When the mth failure occurs, the experiment ends, and all remaining units are removed.

Censored data from the Weibull distribution have been widely studied in different contexts. Traditionally, the maximum likelihood estimation (MLE) method is used to estimate the parameters. However, MLE does not provide simple closed-form solutions for the parameter estimates. To overcome this limitation, several alternative approaches have been proposed in literature. For example, Pareek *et al.* [8] introduced approximate maximum likelihood estimators (AMLEs) in the context of competing risks. Beside the variations of the maximum likelihood method, the pivotal-based parameter estimation method has been shown to outperform for several underlying distributions and censoring schemes. For example, Kim et al. [6] proposed a parameter estimation method for the two-parameter

Pareto distribution using a pivotal quantity. Shang and Gui [9] proposed a pivotal inference method for the two-parameter Rayleigh distribution under progressive first-failure censoring.

In this study, we extend the pivotal method to estimate parameters of the Weibull distribution under progressive Type-II censoring. The performance of the proposed estimators is compared with that of the maximum likelihood estimators through a simulation study and real data applications. The organization of this paper is as follows. Section 2 briefly reviews the existing estimation method, namely the maximum likelihood estimation (MLE). Section 3 introduces the proposed estimation method based on pivotal quantities, and establishes the existence and uniqueness of the proposed estimators. Section 4 performs simulation study, followed by two real-data applications in Section 5. Finally, concluding remarks are given in Section 6.

2. Maximum Likelihood Estimation

Let $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$ be a progressively Type-II censored sample from a Weibull distribution with shape parameter β and scale parameter α. Under the progressive censoring scheme $\boldsymbol{R} = (R_1, R_2, \ldots, R_m)$, the likelihood function is given by

$$L(\alpha, \beta) = C \prod_{i=1}^{m} f(x_i; \alpha, \beta) \left[1 - F(x_i; \alpha, \beta)\right]^{R_i}, \tag{3}$$

where C is the progressive censoring constant. Substituting the Weibull density and distribution functions into (3) gives

$$L(\alpha, \beta) = C \left(\frac{\beta}{\alpha}\right)^m \prod_{i=1}^{m} \left(\frac{x_i}{\alpha}\right)^{\beta-1} \exp\left\{-\sum_{i=1}^{m}(1+R_i)\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}.$$

The corresponding log-likelihood function is

$$\begin{aligned}\ell(\alpha, \beta) = \log C + m \log \beta - m\beta \log \alpha + (\beta - 1) \sum_{i=1}^{m} \log x_i \\ - \alpha^{-\beta} \sum_{i=1}^{m} (1 + R_i) x_i^{\beta}.\end{aligned}$$

Differentiating the log-likelihood function with respect to α and β yields the likelihood equations

$$\begin{aligned}\frac{\partial \ell(\alpha, \beta)}{\partial \alpha} &= -\frac{m\beta}{\alpha} + \frac{\beta}{\alpha} \sum_{i=1}^{m} (1 + R_i) \left(\frac{x_i}{\alpha}\right)^{\beta} = 0, \\ \frac{\partial \ell(\alpha, \beta)}{\partial \beta} &= \frac{m}{\beta} + \sum_{i=1}^{m} \log\left(\frac{x_i}{\alpha}\right) - \sum_{i=1}^{m} (1 + R_i) \left(\frac{x_i}{\alpha}\right)^{\beta} \log\left(\frac{x_i}{\alpha}\right) = 0.\end{aligned}$$

Since these equations do not have closed-form solutions, the maximum likelihood estimates can be solved numerically using a standard nonlinear equation solver.

3. Point Estimations Based on Pivotal Quantities (PQs)

Suppose $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$ denoted as $X_1 < X_2 < \cdots < X_m$ is a progressive type-II censored sample from the weibull distribution with the progressive type-II censoring scheme described as the vector of the numbers of censored observations $(R_1, R_2, \ldots, R_m)$. To obtain a pivotal quantity, we transform the censored sample into a set of transformed variables Y_i, defined as $Y_i = -\log(1-F(X_i)) = (\frac{X_i}{\alpha})^\beta$. Therefore, the transformed variables $Y_1 < Y_2 < \cdots < Y_m$ form a type-II censored sample from a standard exponential distribution. We then further transform Y_i to T_i, where the transformation is given by

$$T_1 = 2nY_1,$$

and, for $i = 2, 3, \ldots, m$,

$$T_i = 2\Big[n - \sum_{j=1}^{i-1}(R_j + 1)\Big](Y_i - Y_{i-1}) = 2\Big[m - i + 1 + \sum_{j=1}^{m}(R_j + 1)\Big](Y_i - Y_{i-1}).$$

Applying a multivariate distribution transformation, we can show that the sequence of T_i $(i = 1, 2, \ldots, m)$ is independent and identically distributed with the $\chi^2(2)$ distribution. Define $U(\alpha, \beta)$ and $S(\alpha, \beta)$ as

$$U(\alpha, \beta) = \sum_{i=1}^{m} T_i = 2\sum_{i=1}^{m}(1 + R_i)\Big(\frac{X_i}{\alpha}\Big)^\beta, \tag{4}$$

and

$$S(\alpha, \beta) = \sum_{i=2}^{m} T_i = U(\alpha, \beta) - T_1 = \frac{2}{\alpha^\beta}\Big[\sum_{i=1}^{m}(1 + R_i)X_i^\beta - nX_1^\beta\Big].$$

Then, $U(\alpha, \beta)$ and $S(\alpha, \beta)$ follow $\chi^2(2m)$ and $\chi^2(2m-2)$ distributions, respectively. Therefore, by the properties of a chi-square distribution, we obtain

$$E(U(\alpha, \beta)) = 2m, \quad \text{and} \quad \text{Var}(U(\alpha, \beta)) = 4m, \tag{5}$$

and

$$E(S(\alpha, \beta)) = 2(m-1), \quad \text{and} \quad \text{Var}(S(\alpha, \beta)) = 4(m-1).$$

Since $S(\alpha, \beta)$ is independent of T_1, the statistic $Q(\alpha, \beta)$ defined as

$$Q(\alpha, \beta) = \frac{T_1/2}{S/[2m-2]} = \frac{nX_1^\beta(m-1)}{\sum_{i=1}^{m}(1 + R_i)X_i^\beta - nX_1^\beta}, \tag{6}$$

is a pivotal quantity following $F(2, 2(m-1))$ distribution with

$$E(Q(\alpha, \beta)) = \frac{2(m-1)}{2(m-1)-2} = \frac{m-1}{m-2}, \quad \text{and} \quad \text{Var}(Q(\alpha, \beta)) = \frac{(m-1)^2}{(m-2)^2(m-3)}. \tag{7}$$

From (5), the pivotal quantity $\frac{U(\alpha,\beta)}{2m}$ converges to one in probability as m converges to infinity. By the same technique, (7) suggests that $\frac{Q(\alpha,\beta)}{m-1} - \frac{1}{m-2}$ converges to zero in probability as m converges to infinity. Following Kirimi (2014), we apply a modified method of moments to (4) - (5)

$$U(\hat{\alpha},\hat{\beta}) = 2\sum_{i=1}^{m}(1+R_i)\Big(\frac{X_i}{\hat{\alpha}}\Big)^{\hat{\beta}} = 2m. \tag{8}$$

Similarly, we obtain from (6) - (7) that

$$Q(\hat{\alpha},\hat{\beta}) = \frac{nX_1^{\hat{\beta}}(m-1)}{\sum_{i=1}^{m}(1+R_i)X_i^{\hat{\beta}} - nX_1^{\hat{\beta}}} = \frac{m-1}{m-2},$$

which is equivalent to

$$\sum_{i=1}^{m}(1+R_i)\Big(\frac{X_i}{X_1}\Big)^{\hat{\beta}} = n(m-1). \tag{9}$$

The estimator $\hat{\beta}$ is obtained by solving (9). To facilitate this analysis, we first establish that the estimating equation (9) has the unique positive solution.

Proposition 1. *Suppose that*

$$0 < X_1 < X_2 < \cdots < X_m$$

and $m > 2$. Then the estimating equation

$$\sum_{i=1}^{m}(1+R_i)\left(\frac{X_i}{X_1}\right)^{\beta} = n(m-1) \tag{10}$$

has the unique positive solution.

Proof. For $\beta \geq 0$, define

$$h(\beta) = \sum_{i=1}^{m}(1+R_i)\left(\frac{X_i}{X_1}\right)^{\beta}.$$

Then, $h(\beta)$ is a continuous function on $[0,\infty)$ such that

$$h(0) = \sum_{i=1}^{m}(1+R_i) = m + \sum_{i=1}^{m}R_i = n.$$

Moreover, using the fact that $\frac{X_i}{X_1} > 1$ for all $i = 2,\ldots,m$, we can show that

$$\lim_{\beta\to\infty} h(\beta) = \infty,$$

and

$$h'(\beta) = \sum_{i=2}^{m}(1+R_i)\left(\frac{X_i}{X_1}\right)^{\beta}\log\left(\frac{X_i}{X_1}\right) > 0.$$

Therefore, there exists the unique positive solution of (10). Once $\widehat{\beta}$ is obtained, the estimator of the scale parameter α is uniquely obtained as

$$\widehat{\alpha} = \left[\frac{1}{m}\sum_{i=1}^{m}(1+R_i)X_i^{\widehat{\beta}}\right]^{1/\widehat{\beta}},$$

which is always positive since each observed failure time is positive. □

In the remainder of this section, we establish the consistency of the derived estimators. As a preliminary step, let (α_0, β_0) be the true parameter values. For $i = 1, \ldots, m$, define

$$Z_i = \left(\frac{X_i}{\alpha_0}\right)^{\beta_0}.$$

Under the Weibull model, the Z_i's are progressively Type-II censored observations from a standard exponential distribution.

Proposition 2. *The proposed estimator $\widehat{\beta}$ is consistent for β_0; that is,*

$$\widehat{\beta} \xrightarrow{p} \beta_0 \qquad \text{as } m \to \infty.$$

Proof. Write

$$q = \frac{\beta}{\beta_0},$$

and define

$$H_m(q) = \sum_{i=1}^{m}(1+R_i)\left(\frac{Z_i}{Z_1}\right)^q.$$

Since

$$\left(\frac{X_i}{X_1}\right)^{\beta} = \left(\frac{Z_i}{Z_1}\right)^q,$$

the estimating equation for $\widehat{\beta}$ can be written as

$$H_m(\widehat{q}) = n(m-1), \qquad \widehat{q} = \frac{\widehat{\beta}}{\beta_0}.$$

From the previous proposition, $H_m(q)$ is strictly increasing in q. We now show that its root approaches $q = 1$. Let $0 < \varepsilon < 1$ and first consider $q = 1 - \varepsilon$. Since z^q is concave for $0 < q < 1$ and

$$\sum_{i=1}^{m}(1+R_i) = n,$$

Jensen's inequality gives

$$\sum_{i=1}^{m}(1+R_i)Z_i^q \le n^{1-q}\left[\sum_{i=1}^{m}(1+R_i)Z_i\right]^q.$$

It follows that

$$\frac{H_m(q)}{n(m-1)} \le \frac{1}{m-1}\left[\frac{\sum_{i=1}^{m}(1+R_i)Z_i}{nZ_1}\right]^q.$$

From the exponential pivotal quantities,

$$2\sum_{i=1}^{m}(1+R_i)Z_i \sim \chi^2_{2m},$$

and hence

$$\frac{1}{m}\sum_{i=1}^{m}(1+R_i)Z_i \xrightarrow{p} 1.$$

Moreover,

$$2nZ_1 \sim \chi^2_2,$$

so that nZ_1 has a nondegenerate exponential distribution. Therefore, for $q = 1-\varepsilon$,

$$\frac{H_m(1-\varepsilon)}{n(m-1)} = O_p\left(m^{-\varepsilon}\right) \xrightarrow{p} 0.$$

Thus,

$$P\left\{H_m(1-\varepsilon) < n(m-1)\right\} \longrightarrow 1.$$

Next, consider $q = 1+\varepsilon$. In this case z^q is convex, and Jensen's inequality gives

$$\sum_{i=1}^{m}(1+R_i)Z_i^q \geq n^{1-q}\left[\sum_{i=1}^{m}(1+R_i)Z_i\right]^q.$$

Hence,

$$\frac{H_m(q)}{n(m-1)} \geq \frac{1}{m-1}\left[\frac{\sum_{i=1}^{m}(1+R_i)Z_i}{nZ_1}\right]^q.$$

For $q = 1+\varepsilon$, the right-hand side is of order m^{ε}, and therefore

$$\frac{H_m(1+\varepsilon)}{n(m-1)} \xrightarrow{p} \infty.$$

Thus,

$$P\left\{H_m(1+\varepsilon) > n(m-1)\right\} \longrightarrow 1.$$

We have therefore shown that, with probability tending to one,

$$H_m(1-\varepsilon) < n(m-1) < H_m(1+\varepsilon).$$

Since $H_m(q)$ is strictly increasing and $\widehat{q}$ is its unique root,

$$P\left\{1-\varepsilon < \widehat{q} < 1+\varepsilon\right\} \longrightarrow 1.$$

Since this holds for every $\varepsilon > 0$,

$$\widehat{q} \xrightarrow{p} 1.$$

Finally,

$$\widehat{q} = \frac{\widehat{\beta}}{\beta_0},$$

and therefore

$$\widehat{\beta} \xrightarrow{p} \beta_0.$$

□

Proposition 3. *The proposed estimator $\widehat{\alpha}$ is consistent for α_0; that is,*

$$\widehat{\alpha} \xrightarrow{p} \alpha_0 \qquad \text{as } m \to \infty.$$

Proof. Recall that

$$\widehat{\alpha} = \left[\frac{1}{m}\sum_{i=1}^{m}(1+R_i)X_i^{\widehat{\beta}}\right]^{1/\widehat{\beta}}.$$

Let

$$Z_i = \left(\frac{X_i}{\alpha_0}\right)^{\beta_0}.$$

Then

$$X_i = \alpha_0 Z_i^{1/\beta_0},$$

and hence

$$\begin{aligned}\widehat{\alpha} &= \left[\frac{1}{m}\sum_{i=1}^{m}(1+R_i)\left(\alpha_0 Z_i^{1/\beta_0}\right)^{\widehat{\beta}}\right]^{1/\widehat{\beta}} \\ &= \alpha_0\left[\frac{1}{m}\sum_{i=1}^{m}(1+R_i)Z_i^{\widehat{\beta}/\beta_0}\right]^{1/\widehat{\beta}}. \end{aligned} \tag{11}$$

From the consistency of $\widehat{\beta}$,

$$\frac{\widehat{\beta}}{\beta_0} \xrightarrow{p} 1.$$

Thus, $Z_i^{\widehat{\beta}/\beta_0}$ approaches Z_i as m increases. Under the stated moment condition, this also holds for the weighted average, so that

$$\frac{1}{m}\sum_{i=1}^{m}(1+R_i)Z_i^{\widehat{\beta}/\beta_0} - \frac{1}{m}\sum_{i=1}^{m}(1+R_i)Z_i \xrightarrow{p} 0.$$

From the pivotal quantity,

$$2\sum_{i=1}^{m}(1+R_i)Z_i \sim \chi^2_{2m}.$$

Therefore,

$$\frac{1}{m}\sum_{i=1}^{m}(1+R_i)Z_i \xrightarrow{p} 1.$$

Combining the two results gives

$$\frac{1}{m}\sum_{i=1}^{m}(1+R_i)Z_i^{\widehat{\beta}/\beta_0} \xrightarrow{p} 1.$$

Returning to (11), and using $\widehat{\beta} \xrightarrow{p} \beta_0$, we obtain $\dfrac{\widehat{\alpha}}{\alpha_0} \xrightarrow{p} 1^{1/\beta_0} = 1$.

Hence,

$$\widehat{\alpha} \xrightarrow{p} \alpha_0.$$

□

4. Simulation Study

In this section, we perform a simulation study to investigate the performance of the proposed pivotal-based estimators, compared with the maximum likelihood estimator (MLE). The parameter setting considered in our study is the case where $\beta = 1, \alpha = 1$. To ensure consistency, we consider different values of the the sample sizes n and the number of observed failures m, as follows

$$n = 20, \quad m = 18 \quad \text{and} \quad n = 30, \quad m = 18.$$

To examine how well each method performs under different conditions, several progressive censoring schemes are considered. Four censoring schemes are designed to test various censoring patterns: Scheme 1: All remaining $n - m$ units are censored at the last stage (stage m). Scheme 2: All $n - m$ units are censored immediately after the first failure. Scheme 3: Half of the $n - m$ units are censored at the first failure, to study partial early censoring. Scheme 4: All $n - m$ units are censored at the midpoint. These can be mathematically described as:

$$\begin{aligned}
&\text{Scheme 1:} \quad R_m = n - m, \quad R_i = 0, \quad i \neq m, \\
&\text{Scheme 2:} \quad R_1 = n - m, \quad R_i = 0, \quad i \neq 1, \\
&\text{Scheme 3:} \quad R_1 = \frac{n-m}{2}, \quad R_i = 0, \quad i \neq 1, m, \\
&\text{Scheme 4:} \quad R_{m/2} = n - m, \quad R_i = 0, \quad i \neq m/2,
\end{aligned}$$

where R_i denotes the number of units censored at stage i. To quantify the precision of the derived estimators, we consider the Mean Bias (MB) and the Mean Squared Error (MSE), defined as

$$\text{MB}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta), \quad \text{MSE}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2,$$

where N is the total number of simulation runs and $\hat{\theta}_i$ represents the estimate of the i -th replication. The corresponding results calculated over 10,000 replications are presented in Table 1.

Table 1 demonstrates that the PQ estimator produces a smaller absolute mean bias than the MLE in most cases for β, especially under Schemes 2–4. For example, when $n = 30$, $m = 18$, and Scheme 4 is used, the absolute mean bias of the PQ estimator is 0.0892, compared with 0.7182 for the MLE. For α, no single method consistently outperforms the other across all scenarios. The MLE usually has a smaller absolute mean bias under Scheme 1, while the results under the other schemes depend on n and m. For example, when $n = 30$ and $m = 18$ under Scheme 1, the absolute mean biases of the PQ and MLE estimators are 0.0954 and 0.0185, respectively. The results also show that the censoring scheme affects the performance of both methods. In particular, relatively large biases are observed for some cases with $n = 30$ and $m = 18$, especially under Schemes 2 and 4. In term of mean square error (MSE), the PQ estimator has a smaller MSE than MLE in several cases. For example, for $n = 20$

and $m = 18$, the PQ estimator produces smaller MSEs than MLE under Schemes 2 and 4 for the shape parameter β. Under Scheme 2, the MSEs were 0.1131 for the PQ estimator and 0.1450 for the MLE, while under Scheme 4, the corresponding MSEs were 0.1109 and 0.1686, respectively. For the scale parameter α, the PQ estimator also gave a slightly smaller MSE under Scheme 1 $(0.0564$ vs. $0.0571)$. Moreover, For $n = 30$ and $m = 18$ the PQ estimator gave smaller MSE under Schemes 3 and 4 for both β and α. Overall, the PQ estimator gives competitive results compared with the MLE, although the relative performance of the two methods varies across the censoring schemes.

TABLE 1. The mean bias and mean squared error (MSE) of the estimators of the shape parameter β and scale parameter α, given that $\beta = 1$ and $\alpha = 1$.

n	m	Scheme	β				α			
			PQ		MLE		PQ		MLE	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	18	1	-0.0677	0.0941	0.1049	0.0694	-0.0186	0.0564	0.0079	0.0571
		2	0.0120	0.1131	0.2571	0.1450	-0.2345	0.0946	-0.1843	0.0749
		3	-0.0317	0.1003	0.1674	0.0938	-0.1284	0.0636	-0.0903	0.0563
		4	-0.0106	0.1109	0.2921	0.1686	-0.1705	0.0716	-0.1318	0.0625
30	18	1	-0.0745	0.0999	0.1029	0.0729	0.0954	0.1183	-0.0185	0.0537
		2	0.2229	0.2389	0.5155	0.3906	-0.5719	0.3395	-0.5527	0.3183
		3	0.0324	0.1226	0.2168	0.1228	-0.2776	0.1061	-0.3018	0.1162
		4	0.0892	0.1772	0.7182	0.6756	-0.3325	0.1631	-0.4302	0.2056

5. REAL DATA

In this section, we illustrate the performance of the proposed method through two real-world applications, analyzing daily wind speed values and times to breakdown data. The first dataset is originated from the work of Abd-Elfattah [1], and the second is based on a progressive Type-II censoring study conducted by Viveros and Balakrishnan [10]. For each dataset, we obtain parameter estimates and conduct goodness-of-fit evaluations using several statistical measures. To assess the fit of the proposed model under progressive Type-II censoring, we apply three commonly used tests: the Anderson-Darling (AD), Cramér-von Mises (CM), and Kolmogorov-Smirnov (KS) tests, following the definitions provided by Zhu [11]. These tests help determine how well the estimated distribution matches the observed data. For each of the datasets, we compute parameter estimates corresponding with their bootstrap standard errors ($\text{SE}_{boot}(\hat{\theta})$) and coefficient of variation (CV). The SE_{boot} and CV of

any estimator $\hat{\theta}$ are calculated as

$$\mathrm{SE}_{boot}(\hat{\theta}) = \sqrt{\frac{1}{B-1}\sum_{b=1}^{B}(\hat{\theta}_b - \bar{\theta}_{\mathrm{boot}})^2} \quad \text{and} \quad \mathrm{CV} = \frac{\mathrm{SE}_{\mathrm{boot}}}{\bar{\theta}_{\mathrm{boot}}},$$

where

$$\bar{\theta}_{\mathrm{boot}} = \frac{1}{B}\sum_{b=1}^{B}\hat{\theta}_b,$$

with B as the size of bootstrap sample and $\hat{\theta}_b$ $(b = 1, 2, ..., B)$ is the estimate from the bth bootstrap sample.

5.1. **Real Data 1: Daily wind speed values.** The first dataset consists of 30 monthly average wind speed observations, calculated using daily wind speed data obtained from the Swedish Meteorological and Hydrological Institute, covering the years 2006 to 2008. A progressive Type-II censored sample based on this dataset was constructed by Shang and Gui [9] and is presented in Table 2. Table 3 summarizes the results for the progressively Type-II censored monthly average wind speed data. The two methods produce similar parameter estimates. The MLE has smaller bootstrap standard errors and coefficients of variation for both parameters, while the PQ estimator gives comparable results. The goodness-of-fit statistics are also similar for both methods, however the bootstrap p-values corresponding to the PQ is higher than those of the MLE. The results indicate that the Weibull model provides an adequate fit to the data. Overall, the performance of the PQ estimator is close to that of the MLE for this dataset.

TABLE 2. A progressively type-II censored dataset was obtained from 30 months of average wind speed observations, as reported by Zhang and Gui [9]).

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$x_{i:18:13}$	2	2.7	2.8	2.9	3.2	3.3	3.3	3.4	3.5	3.9	4	4.7	5.3	5.4	5.5	5.7	6.7	6.9
R_i	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1

TABLE 3. Parameter estimates, bootstrap standard errors ($\mathrm{SE}_{\mathrm{boot}}$), coefficients of variation (CV), and goodness-of-fit statistics for the progressively Type-II censored monthly average wind speed data.

	β			α			AD	CM	KS
Estimator	$\hat{\beta}$	$\mathrm{SE}_{\mathrm{boot}}$	CV	$\hat{\alpha}$	$\mathrm{SE}_{\mathrm{boot}}$	CV			
PQ	3.3355	1.0364	0.3341	5.4503	0.4260	0.0772	7.1290 (0.386)	1.4889 (0.377)	2.2069 (0.528)
MLE	3.2693	0.7964	0.2253	5.4518	0.3139	0.0581	7.3051 (0.333)	1.5239 (0.335)	2.2031 (0.567)

5.2. **Real Data 2: Times to breakdown data.** The second dataset is based on a progressive Type-II censoring study carried out by Viveros and Balakrishnan [10]. It includes 8 observed values related to times to breakdown, as listed in Table 4. Applying the same analysis as in dataset 1, we present numerical results in Table 5. From the table, we can see that the two parameter estimation methods provide adequate fits to the data, as indicated by the goodness-of-fit statistics and the corresponding bootstrap p-values. The PQ and MLE estimators have comparable variability, although their parameter estimates are not identical. Both methods give similar goodness-of-fit results and lead to the same conclusion regarding the suitability of the Weibull model.

TABLE 4. A progressively type-II censored sample obtained from the time-to-breakdown data presented by Viveros and Balakrishnan [10].

i	1	2	3	4	5	6	7	8
$x_{i:n}$	0.19	0.78	0.96	1.31	2.78	4.85	6.50	7.35
R_i	0	0	3	0	3	0	0	5

TABLE 5. Parameter estimates, bootstrap standard errors ($\mathrm{SE}_{\mathrm{boot}}$), coefficients of variation (CV), and goodness-of-fit statistics for the progressively Type-II censored time-to-breakdown dataset.

	β			α			AD	CM	KS
Estimator	$\hat{\beta}$	$\mathrm{SE}_{\mathrm{boot}}$	CV	$\hat{\alpha}$	$\mathrm{SE}_{\mathrm{boot}}$	CV			
PQ	1.0867	0.3101	0.2648	4.1215	2.2073	0.5219	2.4583 (0.946)	0.4624 (0.955)	1.4710 (0.828)
MLE	0.9743	0.2393	0.2006	9.2254	10.9027	1.2070	2.6495 (0.805)	0.5182 (0.834)	1.4347 (0.806)

6. CONCLUSION

This paper proposed a pivotal-based parameter estimation method for the Weibull distribution under progressive Type-II censoring. The proposed estimators are straightforward to compute and exhibit desirable asymptotic properties. The simulation results and applications indicate that the proposed estimators exhibit competitive finite-sample properties relative to the maximum likelihood estimators across different censoring schemes. Consequently, owing to its computational simplicity and favorable asymptotic properties, the pivotal-based parameter estimation method serves as a viable alternative method for the Weibull data with progressive censoring.

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