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STUDY OF PARETO OPTIMALITY OF EXPONENTIAL PENALTY E-FUNCTION METHOD FOR E-DIFFERENTIABLE VECTOR OPTIMIZATION PROBLEMS

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Abstract. This article presents a study of the optimality conditions for multiobjective optimisation problems in so-called E-convex sets. This study focused on the application of the reductive approach of exponential penalisation at the level of E-convex sets and functions and the study of Karush-Kuhn-Tucker and Pareto optimality conditions. The theoretical results from this study demonstrate the efficiency of the exponential penalty approach in finding Pareto optimal solutions in the space of E-convex sets and functions.

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1. Introduction

Multi-objective programming is characterised by considerable development in methods and approaches that enable the identification of good compromise solutions known as Pareto optimal solutions. Upstream, several forms of the problems have been developed, modelling real phenomena. These problems fall into the category of linear, non-linear, fractional, linear and non-linear problems with fuzzy variables. Some are categories mentioned above and define convex or non-convex geometries [6]. Abdulaleem [2] studies E-convex single-objective optimisation problems and optimality properties using Karush-Kuhn-Tucker conditions through the use of the exact penalisation method L_1. Tadeusz Antczac and Najeeb Abdulaleem [7] focused on E-fractional multi-objective optimisation problems. They studied the optimality conditions for this category of optimisation problems. It should be noted that several studies have been conducted on optimisation problems in the E-convex domain [1, 3–5].However,

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an extension to the study of multi-objective optimisation problems on E-convex domains using the exponential penalty function has not been considered.

In this article, we define another approach to solving E-multi-objective optimisation problems using the exponential penalty function. Indeed, the exponential penalty function [9] has proven its worth through its use in transforming multi-objective optimisation problems into single-objective optimisation problems via the Alienor transformation [8]. Using this approach will enable us to study the optimality conditions for E-multi-objective optimisation problems, particularly at the various stages of transforming the multi-objective optimisation problem into a single-objective optimisation problem. These stages are punctuated by demonstrations of the equivalence between . These stages are marked by demonstrations of the equivalence between these different stages.

In the main body of our work, we will first begin with a preliminary section consisting of establishing definitions and properties that will enable us to develop the theoretical approach of our technique. Next, in a second section, we will have the E-convex multi-objective programming at which point we will proceed with the transformation into a problem of minimising the sum of the values of the objective functions. , we will have the E-convex multi-objective programming at which point we will proceed to transform it into a multi-objective optimisation problem using the one-to-one and onto operator. $E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. This procedure will lead us to multi-objective programming using the operator E via the exponential penalty function. We will then apply this to $E-$convex test problems. We will conclude our work with a summary.

2. Preliminaries

In this section, we will define some concepts and properties of E-convex sets and functions. Let $S \subset \mathbb{R}^n$ be a subset and $E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a one-to-one and onto operator defined on $\mathbb{R}^n$.

Definition 1. *A set $S \subset \mathbb{R}^n$ is said to be E-convex if and only if the following relation holds:*

$$\lambda E(x) + (1-\lambda)E(y) \in S, \ \forall x, y \in S \text{ and } \lambda \in [0,1].$$

We observe that if $S \subseteq \mathbb{R}^n$ is E-convex, then $E(S) \subseteq S$. If $E(S)$ is convex, then $E(S) \subseteq S$ and S is E-convex.

Definition 2. *Let S be a non-empty E-convex set of $\mathbb{R}^n$. Let $\mu : S \longrightarrow \mathbb{R}$, then f is said to be E-convex if and only if:*

$$\mu(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda\mu(E(x)) + (1-\lambda)\mu(E(y)),$$

where $x, y \in S, \ \lambda \in [0,1]$.

Strict E-convexity is defined by strict inequality. We define the notion of differentiability in E-convex space through the operator $E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Definition 3. *Let $E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an operator and $\mu : S \longrightarrow R$ a differentiable function at a point $\alpha \in S$. We say that μ is E-differentiable at the point $\alpha \in S$ if and only if $\mu \circ E$ is differentiable at $\alpha \in S$ and according to: $(\mu \circ E)(x) = (\mu \circ E)(\alpha) + \nabla(\mu \circ E)(\alpha)(x - \alpha) + o(\alpha, x - \alpha)\|x - \alpha\|$; where $o(\alpha, x - \alpha) \longrightarrow 0$ when $x \longrightarrow \alpha$.*

In the remainder of our work, we introduce the concept of an E-multiobjective function. Note that a multiobjective optimisation problem is differentiable when all objective functions μ_j and all constraints h_i are differentiable.

3. Multiobjective E-convex programming

Let us begin by outlining a multi-objective optimisation problem. It is defined by the representation:

$$\begin{aligned} &\min \ \big(\mu_1(x), \mu_2(x), \cdots, \mu_j(x), \cdots, \mu_p(x)\big) \\ &s.t: \begin{cases} h(x) \leq 0 \\ x \in \mathbb{R}^n. \end{cases} \end{aligned} \tag{1}$$

Considering the operator $E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, we define the E-multiobjective problem as

$$\begin{aligned} &\min \ \big(\mu_1(E(x)), \mu_2(E(x)), \cdots, \mu_j(E(x)), \cdots, \mu_p(E(x))\big) \\ &s.t: \begin{cases} h(E(x)) \leq 0 \\ x \in \mathbb{R}^n, \end{cases} \end{aligned} \tag{2}$$

where $\mu_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, this implies the presence of m inequality constraints in problem (2). Multi-objective programming requires the characterisation of certain domains. This is the domain of admissible solutions. It is defined by the set

$$D = \{x \in \mathbb{R}^n,\ h_i(x) \leq 0, i \in \overline{1, m}\}.$$

A projection into E-convex space is also defined by

$$D_E = \{x \in \mathbb{R}^n,\ h_i(E(x)) \leq 0, i \in \overline{1, m}\},$$

resulting from problem (2) and E is one-to-one and onto operator. We have $E(D_E) = D$.

Let us now examine the optimality conditions for the E-multiobjective problem (2). We begin with the concept of E-Pareto optimality, which reflects the Pareto optimality of problem (2).

Definition 4. *Let $\mu : S \longrightarrow \mathbb{R}^p$ be an E-differentiable vector and $E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ an operator. Let $x^* \in S$ where S is an E-convex subset. x^* is said to be an E-Pareto optimal solution if and only if*

$$\nexists\, y \in S, \mu_j(E(y)) \leq \mu_j(E(x^*)), \forall j \in \overline{1, p}, \text{ and for some } k \in \{1, \cdots, p\},\ \mu_k(E(y)) < \mu_k(E(x^*))$$

.

Theorem 5. *Let $E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a one-to-one operator. Let $x^* \in D$ be a Pareto optimal solution to problem(1), then there exists $z^* \in D_E$ such that $x^* = E(z^*)$ and z^* is a Pareto optimal solution to problem(2), and vice versa.*

Proof. Suppose that $x^* \in D$ is a Pareto optimal solution of problem(1), then there exists $z^* \in D_E$ such that $x^* = E(z^*)$. Suppose that z^* is not a Pareto optimal solution of (2). So there exist $x \in D_E$ such that $\mu_j(E(x)) \leq \mu_j(E(z^*)), \forall j \in \overline{1,p}$ and for one $k \in \{1, \cdots, p\}$, $\mu_k(E(x)) < \mu_k(E(z^*))$. Since $E(z^*) = x^*$, we have $\mu_j(E(x)) \leq \mu_j(x^*), \forall j \in \{1, \cdots, p\}$ and for one $k \in \{1, \cdots, p\}$, $\mu_k(E(x)) < \mu_k(x^*)$. Setting $E(x) = \tilde{x}$, we have $\mu_j(\tilde{x}) \leq \mu_j(x^*), \forall j \in \overline{1,p}$ and for one $k \in \{1, \cdots, p\}$, $\mu_k(\tilde{x}) < \mu_k(x^*)$. This is absurd because x^* is a Pareto optimal solution to problem(1).

Conversely, suppose that $z^* \in D_E$ is a Pareto optimal solution to problem (2). There exists $\overline{x} \in D$ such that $\overline{x} = E(z^*)$. Suppose that $\overline{x}$ is not a Pareto optimal solution to problem (1), then there exists $y \in D$ such that $\mu_j(y) \leq \mu_j(\overline{x}), \forall j \in \overline{1,p}$ and for some $k \in \{1, \cdots, p\}$, $\mu_k(y) < \mu_k(\overline{x})$. Then there exists $\tilde{x} \in D_E$ such that $y = E(\tilde{x})$. Thus we have $\mu_j(E(\tilde{x})) \leq \mu_j(E(z^*)), \forall j \in \overline{1,p}$ and for one $k \in \{1, \cdots, p\}$, $\mu_k(E(\tilde{x})) < \mu_k(E(z^*))$. This is absurd, so $\overline{x} \in D$ is a Pareto optimal solution to problem (1). □

Proposition 6. *Given that $x^* \in D_E$ is a Pareto optimal solution to problem(2), then $E(x^*)$ is an E-Pareto optimal solution to problem(1) and vice versa.*

Proof. Suppose that $x^* \in D_E$ is a Pareto optimal solution to problem (2), then since $x^* \in D_E$, there exists $\tilde{x} \in D$ such that $\tilde{x} = E(x^*)$. Now suppose that $E(x^*)$ is not a Pareto optimal solution to problem (1), then there exists $y \in D$ such that $\mu_j(y) \leq \mu_j(E(x^*)), \forall j \in \overline{1,p}$ and for one $k \in \{1, \cdots, p\}$, $\mu_k(y) < \mu_k(E(x^*))$. Since $y \in D$, then there exists $\tilde{y} \in D_E$ such that $y = E(\tilde{y})$, which leads to $\mu_j(E(\tilde{y})) \leq \mu_j(E(x^*)), \forall j \in \overline{1,p}$ and for one $k \in \{1, \cdots, p\}$, $\mu_k(E(\tilde{y})) < \mu_k(E(x^*))$, which is absurd because x^* is a Pareto optimal solution to problem (2).

Conversely, let $E(x^*)$, $x \in D_E$ be a Pareto optimal solution to problem(1). Suppose that x^* is not a Pareto optimal solution to problem(2), then there exists $\overline{y} \in D_E$ such that $\mu_j(E(\overline{y})) \leq \mu_j(E(x^*)), \forall j \in \overline{1,p}$ and for one $k \in \{1, \cdots, p\}$, $\mu_k(E(\overline{y})) < \mu_k(E(x^*))$. There exists $y \in D$ such that $y = E(\overline{y}) \Rightarrow \mu_j(y) \leq \mu_j(E(x^*)), \forall j \in \overline{1,p}$ and for a certain $k \in \{1, \cdots, p\}$, $\mu_k(y) < \mu_k(E(x^*))$. This is absurd because $E(x^*)$ is a Pareto optimal solution to problem(1). □

Let us now examine Pareto optimality using the Karush-Kuhn-Tucker theorem defined by the theorem below.

Definition 7. *Let μ_j and h_i, $\forall j \in J$ and $i \in I$ be differentiable functions. Let E be a univocal operator and $x^* \in D_E$. $E(x^*)$ is an E-Pareto optimal solution if $E(x^*)$ satisfies: $\exists \alpha_j$ and $\gamma_i, \forall j \in J,\ i \in I$ such that*

$$\sum_{j=1}^{p} \alpha_j \nabla(\mu_j \circ E)(x^*) + \sum_{i=1}^{m} \gamma_i \nabla(h_i \circ E)(x^*) = 0, \tag{3}$$

$$\gamma_i \nabla(h_i \circ E)(x^*) = 0; \quad \forall i \in I, \tag{4}$$

$$\gamma_i \in \mathbb{R}^+, \quad \forall i \in I. \tag{5}$$

Definition 8. *It is said that $(E(x^*), \alpha, \gamma) \in D \times \mathbb{R}^p \times \mathbb{R}^m$ is called an E-Karush-Kuhn-Tucker point or an E-KKT point for problem (1) if the necessary optimality conditions (3)-(4) are satisfied at $E(x^*)$ under the Lagrange multipliers α, γ.*

4. Main resuslts

In this part of our work, we establish the results from the study on the theoretical convergence of the application of exponential penalisation to E-multiobjective problems. This is done by developing the theoretical foundations that underpin this theoretical convergence. This involves studying Pareto optimality in the E-convex domain using the exponential penalty technique.

4.1. **E-Convex multiobjective programming with exponential penalty E-function.** Most multi-objective optimisation problems are commonly solved by programming, which involves transforming the multi-objective problem into a single-objective optimisation problem. This is known as scalarisation. One of the techniques also used in multi-objective programming is penalisation. This involves converting the multi-objective problem into a single-objective problem without constraints. Exponential penalisation is one such technique used in programming and solving multi-objective optimisation problems. The technique we refer to as the "exponential penalty E-multiobjective function" is defined by its general form:

$$P(E(x), \rho) = \frac{1}{\rho} \sum_{i \in I} e^{\rho h_i(E(x))}. \tag{6}$$

For our programming, the function f in relation (7) will represent our scalarisation function, which will be the weighted Chebyshev distance. This function is renowned for its use in various studies concerning multi-objective programming and, above all, for its ability to solve convex optimisation problems. It is defined by the relation:

$$f(x, \lambda) = \max_{j \in J} \{\lambda_j |\mu_j - z_j^*|\}, J = \{1, 2, \cdots, p\}. \tag{7}$$

Using this function transforms problem (1) into this optimisation problem:

$$\begin{aligned} &\min \ f(x, \lambda) = \max_{j \in J} \{\lambda_j |\mu_j - z_j^*|\} \\ &s.t: \begin{cases} h(x) \leq 0 \\ x \in \mathbb{R}^n. \end{cases} \end{aligned} \tag{8}$$

We define, with regard to problem (8), the E-scalar problem defined by the following problem:

$$\begin{aligned} &\min\ f(E(x),\lambda) = \max_{j\in J}\{\lambda_j|\mu_j(E(x)) - \tau_j^*|\} \\ &s.t: \begin{cases} h(E(x)) \leq 0 \\ x \in D_E. \end{cases} \end{aligned} \tag{9}$$

Where $\tau = \{\tau_1, \cdots, \tau_p\}$ is the ideal point defined by the above problem and $\lambda = \{\lambda_1, \cdots, \lambda_p\}$ such that $\sum_{j=1}^{p} \lambda_j = 1$. Applying the exact L_1 penalty E-multiobjective penalty technique gives us the following relationship:

$$\varGamma_E(x, \rho, \lambda) = f(E(x), \lambda) + P(E(x), \rho). \tag{10}$$

We define the properties defined by the following theorem:

Lemma 9. *Considering the set D_E compact, we have:*

- *if $x \in D_E$,then $\lim_{\rho\to+\infty} \frac{1}{\rho}\sum_{i\in I} e^{\rho h_i(E(x))} = 0$,*
- *if $x \notin D_E$, then $\lim_{\rho\to+\infty} \frac{1}{\rho}\sum_{i\in I} e^{\rho h_i(E(x))} = +\infty$.*

Thus, the problem shifts from a multi-objective optimisation problem to a single objective function problem according to:

$$\begin{aligned} &\min \varGamma_E(x, \rho, \lambda) \\ &s.t: \{\ x \in D_E. \end{aligned} \tag{11}$$

Theorem 10. *Let E be a one-to-one and onto operator and $\overline{x} \in D_E$. Then there exists $x \in D$ such that $x = E(\overline{x})$ is an optimal solution to problem (9). Then $x = E(\overline{x})$ is an optimal solution to problem (8) and vice versa.*

Proof. Let $x \in D$ be the optimal solution to problem(8) such that $x = E(\overline{x})$ where $\overline{x} \in D_E$. Suppose that $E(\overline{x})$ is not the optimal solution to problem(9), then there exists $\overline{z} \in D_E$ such that $f(E(\overline{z}), \lambda) < f(E(\overline{x}), \lambda) \Rightarrow f(z, \lambda) < f(x, \lambda)$, with $z = E(\overline{z})$, which is absurd because $\overline{x}$ is the optimal solution.

Conversely, suppose that $x \in D$ is not the optimal solution to problem(8), then there exists $x^* \in D_E$ such that $x = E(x^*)$ is the optimal solution to problem(9). Then $\exists \overline{x} \in D,\ f(\overline{x}, \lambda) < f(x, \lambda)$. Since $E(D_E) = D$, there exists $\tilde{x} \in D_E$ such that $\overline{x} = E(\tilde{x})$, which leads to à $f(E(\tilde{x}), \lambda) < f(E(x^*), \lambda)$, which is absurd because $E(x^*)$ is the optimal solution to problem(9). □

Theorem 11. *Let $x \in \mathbb{R}^n$ be the optimal solution to problem(8). Then there exists $x^* \in D_E$ such that $x = E(x^*)$ and x^* is the optimal solution to problem(11), and vice versa.*

Proof. Suppose that $x \in \mathbb{R}^n$ is the optimal solution to problem (8). Suppose that $x^* \in D_E$ is not the optimal solution to problem (11), then there exists $z \in D_E$ and $Z \in D$ such that $Z = E(z)$ and $\varGamma(Z, \lambda) < \varGamma(x, \lambda) \Leftrightarrow f(E(z), \lambda) + P(E(z), \rho) < f(E(x^*), \lambda) + P(E(x^*), \rho)$.

Since $z, x^* \in D_E$, then $\lim_{\rho\to+\infty} \frac{1}{\rho}\sum_{j=1}^{p} e^{\rho(h_j\circ E)(z)} = 0$ and $\lim_{\rho\to+\infty} \frac{1}{\rho}\sum_{j=1}^{p} e^{\rho(h_j\circ E)(x^*)} = 0$.

Then $\exists\rho_0 \geq 0,\ \rho \geq \rho_0$, we have the inequality $f(E(z),\lambda) < f(E(x^*),\lambda)$ because $x^* \in D_E$. This implies $f(Z,\lambda) < f(x,\lambda)$, which is absurd because x is the optimal solution to problem(8).

Conversely, suppose that $x^* \in D_E$ is the optimal solution to problem(11). Suppose that $x \in \mathbb{R}^n$ is not the optimal solution to problem(8), then

$$\exists y \in D;\ f(y,\lambda) < f(x,\lambda). \tag{12}$$

Since $D = E(D_E)$, then there exist $y^* \in D_E$ such that $y = E(y^*)$ and we have

$$\lim_{\rho\to+\infty} \frac{1}{\rho}\sum_{j=1}^{p} e^{\rho(h_j\circ E)(y^*)} = 0 \text{ and } \lim_{\rho\to+\infty} \frac{1}{\rho}\sum_{j=1}^{p} e^{\rho(h_j\circ E)(x^*)} = 0.$$

Then $\exists\eta_0 \geq 0$ such that $\forall\rho \geq \eta_0$, we have

$$\frac{1}{\rho}\sum_{j=1}^{p} e^{\rho(h_j\circ E)(y^*)} < \frac{1}{\rho}\sum_{j=1}^{p} e^{\rho(h_j\circ E)(x^*)}. \tag{13}$$

Adding (12) and (13), we have $\Gamma(y,\lambda) < \Gamma(x,\lambda)$, which is absurd. □

Theorem 12. *Let $x^* \in D_E$ be a KKT point of problem(2) satisfying conditions (3) and (4) of Theorem 7 at point x^* with Lagrange multipliers $\alpha \in \mathbb{R}^p$ and $\beta \in \mathbb{R}^m$. Suppose the following hypotheses hold:*

a) *if $\mu_j, \forall j \in \{1,2,\cdots,p\}$ is E-differentiable and E-convex at point x^*,*

b) *each constraint h_i, $\forall i \in \{1,2,\cdots,m\}$ is E-differentiable and E-convex at point x^*,*

c) *If the penalty parameter $\rho \leq \frac{1}{\max\{\alpha_i\}}, i \in I$,*

then x^* is an optimal solution to the penalised problem(11).

Proof. Let $x^* \in D_E$ be a KKT point of problem (2). Then there exist Lagrange multipliers $\alpha \in \mathbb{R}^p$ and $\mathbb{R}^m$ satisfying conditions (3) and (4) at point x^*. Suppose that there exists $\rho \leq \frac{1}{\max\{\alpha_i\}}, i \in I$. Then we have $\Gamma(x,\rho) = f(E(x),\lambda) + \frac{1}{\rho}\sum_{i=1}^{p} e^{\rho(h_i\circ E)(x)}$. Since f and h_i, $\forall i \in I$ are convex, then we have:

$$f(E(x),\lambda) - f(E(x^*),\lambda) \geq \nabla f(E(x^*),\lambda)(x - x^*), \tag{14}$$

$h_i(E(x)) - h_i(E(x^*)) \geq \nabla h_i(E(x^*))(x - x^*)$, $\forall i \in I$. We have:

$$\sum_{i=1}^{m} \alpha_i h_i(E(x)) - \sum_{i=1}^{m} \alpha_i h_i(E(x^*)) \geq \sum_{i=1}^{m} \alpha_i \nabla h_i(E(x^*))(x - x^*), \tag{15}$$

Adding (14) and (15), we obtain:

$$f(E(x),\lambda) - f(E(x^*),\lambda) + \sum_{i=1}^{m} \alpha_i h_i(E(x)) - \sum_{i=1}^{m} \alpha_i h_i(E(x^*)) \geq$$
$$\nabla h_i(E(x^*))(x - x^*) + \sum_{i=1}^{m} \alpha_i \nabla h_i(E(x^*))(x - x^*),$$

$$\Rightarrow f(E(x),\lambda) - f(E(x^*),\lambda) + \sum_{i=1}^{m} \alpha_i h_i(E(x)) - \sum_{i=1}^{m} \alpha_i h_i(E(x^*)) \geq$$
$$\left[\nabla h_i(E(x^*)) + \sum_{i=1}^{m} \alpha_i \nabla h_i(E(x^*))\right](x - x^*),$$

According to condition(3) of KKT's theorem, we have

$f(E(x),\lambda) - f(E(x^*),\lambda) + \sum_{i=1}^{m} \alpha_i h_i(E(x)) - \sum_{i=1}^{m} \alpha_i h_i(E(x^*)) \geq 0$, then

$$f(E(x),\lambda) + \sum_{i=1}^{m} \alpha_i h_i(E(x)) \geq f(E(x^*),\lambda) + \sum_{i=1}^{m} \alpha_i h_i(E(x^*)). \tag{16}$$

Since $\rho \leq \dfrac{1}{\max\{\alpha_i\}}, i \in I$, then the inequality (16) becomes

$f(E(x),\lambda) + \dfrac{1}{\rho}\sum_{i=1}^{m} e^{\rho h_i(E(x))} \geq f(E(x^*),\lambda) + \dfrac{1}{\rho}\sum_{i=1}^{m} e^{\rho h_i(E(x^*))}$. Then x^* is the optimal solution to problem(11). □

Definition 13. *Suppose that $E(x^*)$ is a Pareto optimal solution to problem(1). Suppose also that the assumptions of theorem(12) are satisfied and $E(x^*)$ is a KKT point. Then $E(x^*)$ is an E-Pareto optimal solution to problem(2).*

Theorem 14. *Let $\overline{x} \in D$ be an optimal solution to problem(8), then there exists $\overline{z} \in D_E$ such that $\overline{x} = E(\overline{z})$ and $\overline{z}$ is a solution to problem(9) and vice versa.*

Proof. Suppose that $\overline{x} \in D$ is an optimal E-solution to problem(8). Since $E(D_E) = D$, there exists $\overline{z} \in D_E$ such that $\overline{x} = E(\overline{z})$. Let us show that $\overline{z}$ is an optimal solution to problem(9). Suppose that $\overline{z}$ is not, then there exists $z^* \in D_E$ such that $f(E(z^*,\lambda)) \leq f(E(\overline{z},\lambda)) \Leftrightarrow \max_{j\in J}\{\lambda_j|\mu_j(E(z^*)) - \tau_j^*|\} \leq \max_{j\in J}\{\lambda_j|\mu_j(E(\overline{z})) - \tau_j^*|\}$. Since z^* and $\overline{z} \in D_E$, there exist $\overline{x}$ and $x^* \in D$ such that $\overline{x} = E(\overline{\overline{z}})$ and $x^* = E(z^*)$.Which gives $\Leftrightarrow \max_{j\in J}\{\lambda_j|\mu_j(x^*) - \tau_j^*|\} \leq \max_{j\in J}\{\lambda_j|\mu_j(\overline{x}) - \tau_j^*|\} \Leftrightarrow f(x^*,\lambda) \leq f(\overline{x},\lambda)$,which is absurd because $\overline{x}$ is the optimal solution to problem(8).

Conversely, suppose that $\overline{x}$ is an optimal solution to problem (8). Let us show that $\overline{x}$ is the optimal solution to problem(9), then there exists $y \in D$ such that $f(y,\lambda) \leq f(\overline{x},\lambda)$. Since $D = E(D_E)$, there exist α and $\overline{\alpha}$ such that $y = E(\alpha)$ and $\overline{x} = E(\overline{\alpha})$, so we have: $f(E(\alpha),\lambda) \leq f(E(\overline{x}),\lambda)$. Then $E(\alpha)$ is the optimal solution to problem (8), but $E(\alpha) = y$. This is absurd, hence the result. □

Theorem 15. *Let $\overline{x} \in D$ be a Pareto optimal solution to problem (1). Then there exists $\overline{y} \in D_E$ such that $\overline{x} = E(\overline{y})$ and $\overline{y}$ is a Pareto optimal solution to problem(2).*

Proof. Let $\overline{x} \in D$ be a Pareto optimal solution to problem(1), such that $E(\overline{y}) = \overline{x}$. Suppose that $\overline{y}$ is not a Pareto optimal solution, then there exists $y \in D_E$ such that $\forall j = \{1,\cdots,p\},\ quad\mu_j(E(y)) \leq \mu_j(E(\overline{y}))$ and for at least one $k \in \{1,2,\cdots,p\}$ $\mu_k(E(y)) < \mu_k(E(\overline{y}))$. Since $E(D_E) = D$, there exists $x \in D$ such that $E(y) = x$ and $E(\overline{y}) = \overline{x}$ and we have: $\mu_j(x) \leq \mu_j(\overline{x}),\quad \forall j = \{1,\cdots,p\}$ and $\mu_k(x) <$

$\mu_k(\overline{x})$, and for at least $k = \{1, \cdots, p\}$. This makes $x \in D$ an optimal solution to problem(1), which is absurd, so $\overline{x}$ is a Pareto optimal solution to problem (2).

Conversely, suppose that $\overline{y} \in D_E$ is a Pareto optimal or E-Pareto optimal solution to problem (2). Then there exists $\tilde{x} \in D$ such that $\tilde{x} = E(\overline{y})$. Suppose that $E(\overline{y})$ is not a Pareto optimal solution to problem (1), then there exists $x \in D$ such that $\forall j \in \{1, 2, \cdots, p\}, \mu_j(x) \leq \mu_j(E(\overline{y}))$ and for some $k \in \{1, 2, \cdots, p\}$ such that $\mu_k(x) < \mu_k(E(\overline{y}))$. Since $\overline{y} \in D_E,$ and $\overline{x} = E(\overline{y})$. Then $\forall j \in \{1, 2, \cdots, p\}$, $\mu_j(x) \leq \mu_j(\overline{x})$ and for some k, $\mu_k(x) < \mu_k(\overline{x})$. Then x is a Pareto optimal solution, which is absurd, hence equivalence. □

Theorem 16. *Let x^* be an optimal solution to problem(9), then $E(x^*)$ is a Pareto optimal solution to problem(1).*

Proof. Let $x^* \in D_E$ be an optimal solution to problem (9) and suppose that $E(x^*)$ is not a Pareto optimal solution to problem (1). Then there exists $y \in D$ such that $\forall j \in \{1, 2, \cdots, p\}$ $\mu_j(y) \leq \mu_j(E(x^*))$ and for some $k \in \{1, 2, \cdots, p\}$, $\mu_k(y) < \mu_k(E(x^*))$. Since $y^* \in \mathbb{R}^p$ is the ideal point and $\lambda \in \mathbb{R}^p$ is the weighted point, so we have $\max_j\{\lambda_j|\mu_j(y) - z_j^*|\} \leq \max_j\{\lambda_j|\mu_j(E(x^*)) - z_j^*|\}$ and for some $k \in \{1, 2, \cdots, p\}$ $\max_k\{\lambda_k|\mu_k(y) - z_j^*|\} < \max_k\{\lambda_k|\mu_k(E(x^*)) - z_k^*|\} \Rightarrow f(y, \lambda) \leq f(E(x^*), \lambda)$. Since $x^* \in D_E$, then there exist $\tilde{x} \in D$ such that $\tilde{x} = E(x^*) \Rightarrow f(y, \lambda) \leq f(\tilde{x}, \lambda)$ that is absurd because x^* is the optimal solution of the mono objective problem(9). □

Theorem 17. *Let $E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous map and let x^* be an optimal solution to the E-penalised problem (11). Further, let $E(\tilde{x})$ be an E-KKT point of the multi-objective problem (1) and satisfying the conditions (3) and (4) of theorem (7) at point $E(\tilde{x})$ with Lagrange multipliers $\alpha \in \mathbb{R}^p$ and $\mathbb{R}^m$. Let us assume that the following hypotheses are verified:*

a) *if $\mu_j, \forall j \in \{1, 2, \cdots, p\}$ is E-differentiable and E-convex at the point $\tilde{x}$ on $\mathbb{R}^n$,*
b) *each constraint h_i, $\forall i \in \{1, 2, \cdots, m\}$ is E-differentiable E-convex at point $\tilde{x}$,*
c) *Suppose that domain D is compact and that the penalty parameter $\rho \leq \dfrac{1}{\max\{\alpha_i\}}, i \in I$,*

then $E(x^*)$ is an E-Pareto optimal solution to problem (1).

Proof. Suppose that x^* is the optimal solution to problem (11), then $\forall x \in D_E$, we have $f(E(x^*), \lambda) + \frac{1}{\rho}\sum_{i=1}^{m} e^{\rho h_i(E(x^*))} < f(E(x), \lambda) + \frac{1}{\rho}\sum_{i=1}^{m} e^{\rho h_i(E(x))}$. Since $x^* \in D_E, \exists \rho_0 \geq 0, \forall \rho \geq \rho_0$, we have $f(E(x^*), \lambda) < f(E(x), \lambda) + \frac{1}{\rho}\sum_{i=1}^{m} e^{\rho h_i(E(x))}$ and $f(E(x^*), \lambda) < f(E(x), \lambda)$ because $x \in D_E$.

This means that $f(E(.), \lambda)$ is bounded on the set D_E. Since f is continuous, then according to Weierstrass's theorem, $f(E(.), \lambda)$ has a global minimum $\tilde{x}$ on D_E. Let us show that $E(x^*) \in D$. Suppose the contrary, i.e. $E(x^*)\, notin D$. Since f has a global minimum $\tilde{x} \in D_E$, then $E(\tilde{x})$ is the global minimum of problem (8), and any global minimum of problem (8) is a Pareto optimal solution to problem (2). Thus, according to the KKT conditions, there exists $\alpha_i, \forall i \in I$ such that $E(\tilde{x})$ satisfies the KKT conditions.

Therefore, we have

$$f_j(E(x^*)) - f_j(E(\tilde{x})) \geq \nabla f_j(E(\tilde{x}))(x^* - \tilde{x}), \tag{17}$$

$$h_i(E(x^*)) - h_i(E(\tilde{x})) \geq \nabla h_i(E(\tilde{x}))(x^* - \tilde{x}). \tag{18}$$

Multiplying (18) by the Lagrange multiplier $\tilde{\alpha}$, we obtain the summation

$$\sum_{i=1}^{m} \tilde{\alpha}_i h_i(E(x^*)) - \sum_{i=1}^{m} \tilde{\alpha}_i h_i(E(\tilde{x})) \geq \sum_{i=1}^{m} \tilde{\alpha}_i \nabla h_i(E(\tilde{x}))(x^* - \tilde{x}). \tag{19}$$

Adding (17) and (19) together, we obtain:

$$f_j(E(x^*)) - f_j(E(\tilde{x})) + \sum_{i=1}^{m} \tilde{\alpha}_i h_i(E(x^*)) - \sum_{i=1}^{m} \tilde{\alpha}_i h_i(E(\tilde{x})) \geq \nabla f_j(E(\tilde{x}))(x^* - \tilde{x}) + \sum_{i=1}^{m} \tilde{\alpha}_i \nabla h_i(E(\tilde{x}))(x^* - \tilde{x})$$

$$\Rightarrow f_j(E(x^*)) - f_j(E(\tilde{x})) + \sum_{i=1}^{m} \tilde{\alpha}_i h_i(E(x^*)) - \sum_{i=1}^{m} \tilde{\alpha}_i h_i(E(\tilde{x})) \geq \left[\nabla f_j(E(\tilde{x})) + \sum_{i=1}^{m} \tilde{\alpha}_i \nabla h_i(E(\tilde{x}))\right](x^* - \tilde{x}).$$

The KKT conditions imply

$$f_j(E(x^*)) - f_j(E(\tilde{x})) + \sum_{i=1}^{m} \tilde{\alpha}_i h_i(E(x^*)) - \sum_{i=1}^{m} \tilde{\alpha}_i h_i(E(\tilde{x})) \geq 0.$$

$$\Rightarrow f_j(E(x^*)) + \sum_{i=1}^{m} \tilde{\alpha}_i h_i(E(x^*)) \geq f_j(E(\tilde{x})) + \sum_{i=1}^{m} \tilde{\alpha}_i h_i(E(\tilde{x})).$$

Since $\tilde{x} \in D_E$ et $x^* \in D_E$, $\exists \rho \leq \dfrac{1}{\max\{\tilde{\alpha}_i\}}$, $i \in I$ such that

$$f_j(E(x^*)) + \frac{1}{\rho} \sum_{i=1}^{m} e^{\rho h_i(E(x^*))} \geq f_j(E(\tilde{x})) + \frac{1}{\rho} \sum_{i=1}^{m} e^{\rho h_i(E(\tilde{x}))}.$$

Since $x^*, \tilde{x} \in D_E$ then $\exists \rho_0 \geq 0$, $\forall \rho \geq \rho_0$ we have $f_j(E(x^*)) \geq f_j(E(\tilde{x}))$. Since τ_j^* is the ideal point of problem(13), then it follows directly that.

$$\max_j \{\lambda_j |f_j(E(x^*)) - \tau_j^*|\} \geq \max_j \{\lambda_j |f_j(E(\tilde{x})) - \tau_j^*|\}.$$

Furthermore, given that $x^*, \tilde{x} \in D_E$

$$\exists \tilde{\rho} \geq 0, \forall \rho \geq \tilde{\rho}, \max_j \{\lambda_j |f_j(E(x^*)) - \tau_j^*|\} + \frac{1}{\rho} \sum_{i=1}^{m} e^{\rho h_i(E(x^*))} \geq \max_j \{\lambda_j |f_j(E(\tilde{x})) - \tau_j^*|\} + \frac{1}{\rho} \sum_{i=1}^{m} e^{\rho h_i(E(\tilde{x}))}.$$

This is absurd because x^* is the optimal solution to problem(11). Hence, $E(x^*)$ is the global minimum of problem(1). □

4.2. **Discussion of the new approach.**

The algorithm for finding Pareto optimal solutions is defined as follows:

Algorithm 1.

(1) *Begin*

(2) *Enter the value of* ρ*;*

(3) *Define the operator* $E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *and the E-multiobjective problem;*

(4) *Choose* λ_j *such as* $\sum_{j=1}^{p} \lambda_j = 1$*;*

(5) *for j from* 1 *to* p

  (a) $f(E(x), \lambda) \longleftarrow \max_{j \in J}\{\lambda_j|\mu_j(E(x)) - \tau_j^*|\}$;

  (b) $\mathit{\Gamma}_E(x, \rho, \lambda) \longleftarrow \left\{ f(E(x), \lambda) + \frac{1}{\rho} \sum_{i \in I} e^{\rho h_i(E(x))} \right\}$;

  (c) $x^* \longleftarrow Argmin(\mathit{\Gamma}_E(x, \rho, \lambda))$;

  end for

(6) *Display the solution* x *of the problem which is one of the best compromise corresponding to fixed* λ_k*;*

(7) *End*

Table 1. E-exponential penalty optimization algorithm

In this section, we propose a study of the complexity or efficiency of the above algorithm. We will conduct a complexity study for each step of the algorithm. The scalarisation step of the problem consists of a comparison in which we seek to find the maximum of the quantity $\max_{j \in J}\{\lambda_j|\mu_j(E(x)) - \tau_j^*|\}$. This operation has a complexity of (p^2). The penalty stage is characterised by a sum of $m + 1$ elements. This operation has a complexity of (m). For a given value λ_j, the complexity of finding the solution x^* depends on the algorithm that will search for the solution. We can estimate the complexity of this search algorithm as (η), $\eta \geq 0$. Thus, for $\imath$ pairs of weightings, the complexity of our work is defined by the table below:

Steps	Complexity
Scalarization	(p^2)
Penalisation	(m)
Search for a solutions	(η)
Final complexity	$\max\{(\imath p^2); (\imath m); (\imath\eta)\}$

Table 2. Table of complexity

The complexity of the method is in the class of polynomial complexity algorithms. We can therefore conclude that our approach is effective.

5. Conclusion

This study enabled us to lay the theoretical foundations for the impact of exponential penalisation in the search for Pareto optimal solutions for multi-objective optimisation problems in E-convex sets. Indeed, this study has established satisfactory results regarding the guarantee of obtaining Pareto solutions. However, the difficulty looming on the horizon is the availability of E-convex test problems for an in-depth study of Pareto front visualisation and performance analysis of methods on multi-objective optimisation problems in the E-convex domain. This could be a starting point for further research.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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