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APPROXIMATE ANALYTICAL SOLUTIONS OF MULTIDIMENSIONAL FRACTIONAL BRUSSELATOR MODELS USING THE GENERAL RESIDUAL POWER SERIES METHOD

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Abstract. This study presents an approximate analytical approach for solving two- and three-dimensional fractional-order Brusselator reaction–diffusion models using a General Residual Power Series Method (General-RPSM). The method combines the Jafari integral transform with the residual power series technique to construct approximate solutions as fractional power series with undetermined coefficients. The coefficients are systematically determined by constructing the residual functions and imposing appropriate residual conditions. The proposed approach provides a straightforward computational procedure for obtaining truncated series solutions without requiring linearization or discretization of the governing equations. The effectiveness and accuracy of the method are demonstrated through three fractional Brusselator models in two and three spatial dimensions. The results obtained are in good agreement with the available exact solutions and previously published results. Furthermore, the graphical representations and absolute-error analyzes are also provided to demonstrate the method's accuracy and applicability.

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1. Introduction

The Brusselator model plays an important role in theoretical chemistry and nonlinear dynamics. It was originally introduced as a theoretical model of an autocatalytic chemical oscillator and has since become a prototypical model for investigating nonlinear chemical dynamics and oscillatory behavior [21]. When diffusion is incorporated, the Brusselator model takes the form of a nonlinear reaction–diffusion system and can show complex space-time behavior, including instabilities and pattern formation [19]. Consequently, the Brusselator system has been extensively studied in relation to chemical oscillations, bifurcations, limit cycles, and spatial pattern formation. The fractional Brusselator model extends the classical model by incorporating fractional-order derivatives. This remarkable model demonstrates the potential of fractional calculus, which has attracted considerable scholarly attention

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because fractional derivatives provide an effective mathematical framework for describing memory and non-local effects that are not directly captured by conventional integer-order differential operators [16]. As a result, fractional reaction–diffusion models have been increasingly used to describe complex dynamical phenomena arising in physical, chemical, and biological systems.

Unfortunately, problems involving fractional derivatives have not yet been addressed through a fully explicit approach. Researchers around the world have attempted to assemble and improve novel techniques to handle this challenging problem. Over the years, several analytical and semi-analytical approaches have been developed for fractional Brusselator reaction–diffusion systems. These include the Variational Iteration Method (VIM) [9], the Laplace–Adomian Decomposition Method (LADM) [12], Polynomial Least Squares [5], the Reduced Differential Transform Method (RDTM) [10], the q-Homotopy Analysis Transform Method (q-HATM) [22], Bernstein polynomials and operational matrix methods [13], the Residual Power Series Method (RPSM) [18], wavelet-based methods, and conformable fractional derivative approaches [17]. However, such methods all have limitations and specific conditions for their application.

The Residual Power Series Method (RPSM) and its modifications are effective strategies for solving scientific and engineering problems. These processes are particularly popular in the academic community due to their straightforwardness, the small number of steps involved, and their superior convenience and adaptability. One of the most notable and widely used recent improvements is the application of various integral transform techniques in conjunction with the residual power series method. Recently, the general integral transform (Jafari transform) has been combined with the residual power series approach to yield the general residual power series method (General-RPSM) for solving fractional gas dynamics and drainage equations [14]. This technique not only improves the efficiency of the traditional RPSM method but can also be applied to a wide range of Laplace-like transforms, while simultaneously simplifying the process of finding solutions [15]. Moreover, the recommended procedure is applied to a nonlinear fractional Zakharov–Kuznetsov equation (FZKE) and outperforms the well-known techniques on test models, with the error decreasing as more series terms are included [1]. This approach not only provides high accuracy with low-degree truncation but also generates an approximate series solution that tends to capture the behavior of the solution, making the proposed method effective for solving nonlinear fractional PDEs. A key finding is that the Yang residual power series transform method, which combines the Yang transform and the residual power series technique, has been utilized to obtain analytical approximations for fractional Brusselator reaction–diffusion models, including two-dimensional formulations [4]. Similarly, the Laplace residual power series technique has been employed to construct rapidly convergent series solutions for

nonlinear fractional Brusselator reaction–diffusion models [3]. These advances demonstrate the effectiveness of combining series-based methods with integral transforms for solving nonlinear fractional reaction–diffusion systems.

However, the application of generalized integral transformations combined with the residual power series method to identify solutions to multidimensional fractional Brusselator models has not yet been thoroughly investigated. This, together with the remarkable findings in [4] and [2], formed the initial motivation for this work. The present study develops a general residual power series method for obtaining analytical series solutions of two- and three-dimensional fractional Brusselator reaction–diffusion models and compares the results with existing benchmarks. The main objective is to apply the General-RPSM to the initial-value problem for the multidimensional fractional Brusselator model,

$$\begin{cases} D_t^{\alpha} u = \varepsilon_1 + u^2 v - (\varepsilon_2 + 1)u + \varepsilon_3 \nabla^2 u, \\ D_t^{\alpha} v = \varepsilon_2 u - u^2 v + \varepsilon_3 \nabla^2 v, \end{cases} \tag{1.1}$$

where $0 < \alpha \leqslant 1$, subjects to the initial condition $u(x, y, z, 0) = f(x, y, z),\ v(x, y, z, 0) = g(x, y, z)$. Here $\varepsilon_i, i = 1, 2, 3$ are positive constants and f, g is given function. D_t^{α} stands for the fractional differential operator with respect to t.

The rest of this article is organized as follows: Section 2 presents the basic definitions and theorems of fractional calculus, general integral transforms, and the residual power series method. The implementation of the General-RPSM for the fractional Brusselator model is described in Section 3. Moreover, the two- and three-dimensional fractional Brusselator models are examined in this section, and graphical comparisons of the results are presented. Section 4 discusses the study and concludes it.

2. General Integral Transform and the Residual Power Series Method

The definitions and vital properties of fractional calculus and the general integral transformation are reviewed. Moreover, the elementary details of the residual power series technique are mentioned in this section.

Definition 2.1. [20] *The fractional integral operator of order $\gamma(\gamma \geqslant 0)$ of $\phi(t)$ of Riemann-Liouville type is defined as*

$$I^{\gamma}\phi(t) = \begin{cases} \frac{1}{\Gamma(\gamma)} \int_0^t \frac{\phi(\tau)}{(t-\tau)^{1-\gamma}} d\tau,\ \gamma > 0,\ t > 0, \\ \phi(t),\ \gamma = 0. \end{cases}$$

Definition 2.2. *The Caputo fractional derivative operator D^{γ} of order γ, $(n-1 < \gamma \leqslant n, n \in \mathbb{N})$ is defined in the following form,*

$$D^{\gamma}\phi(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-\tau)^{-\gamma+n-1} \phi^{(n)}(\tau) d\tau, \tag{2.1}$$

$\gamma > 0$, $t > 0$, *where the function* $\phi(t)$ *has absolutely continuous derivatives up to order* $n-1$. *In particular, if* $0 < \gamma < 1$, *we have*

$$D^{\gamma}\phi(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} \phi'(\tau) d\tau.$$

Definition 2.3. [8] *Let* $f(t)$ *be an integrable function defined for* $t \geqslant 0$, $p(s)$ *and* $q(s)$ *are positive real functions. The general integral transform* $\mathcal{F}(s)$ *of* $f(t)$ *is defined by*

$$\mathbb{J}\{f(t)\} = \mathcal{F}(s) = p(s) \int_0^{\infty} f(t) e^{-q(s)t} dt$$

provide the improper integral exists for some $q(s)$.

Theorem 2.1. [8] *Let* $f(t)$ *be a differentiable and* $p(s)$ *and* $q(s)$ *are positive real functions, then*

$$\mathbb{J}\{f^{(n)}(t)\} = q^n(s)\mathbb{J}\{f(t)\} - p(s) \sum_{k=0}^{n-1} q^{n-1-k}(s) f^{(k)}(0). \tag{2.2}$$

Theorem 2.2. [11] *If* $n \in \mathbb{Z}^+$ *where* $n-1 < \alpha \leqslant n$ *and* $\mathcal{F}(s)$ *is the general integral transform of the function* $f(t)$, *then the general integral transform of the Caputo fractional derivative of order* $\alpha > 0$ *is*

$$\mathbb{J}\{D^{\alpha} f(t)\} = q^{\alpha}(s)\mathcal{F}(s) - p(s) \sum_{k=0}^{n-1} q^{\alpha-1-k}(s) f^{(k)}(0). \tag{2.3}$$

For additional information, the general integral transforms and crucial properties of some fundamental functions are provided in [8].

Definition 2.4. [6,7] *If* $n \in \mathbb{N}$ *where* $n-1 < \alpha \leqslant n$. *A power series expansion of the form*

$$\sum_{n=0}^{\infty} a_n (t-t_0)^{n\alpha} = a_0 + a_1(t-t_0)^{\alpha} + a_2(t-t_0)^{2\alpha} + \ldots, \; t \geqslant t_0$$

is called fractional power series about t_0.

Theorem 2.3. [6,7] *Suppose that* $u(t)$ *has a fractional power series representation at* $t = t_0$ *of the form*

$$u(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^{n\alpha}, \quad 0 \leqslant m-1 < \gamma \leqslant m, \; t_0 \leqslant t < t_0 + R.$$

If $u(t) \in C[t_0, t_0+R)$ *and* $D^{n\alpha}u(t) \in C(t_0, t_0+R)$, $n = 0, 1, 2, \ldots$, *then* $a_n = \frac{D^{n\alpha}u(t)}{\Gamma(n\alpha+1)}$.

Definition 2.5. [6,7] *A power series of the form*

$$\sum_{n=0}^{\infty} f_n(x)(t-t_0)^{n\alpha} = f_0(x) + f_1(x)(t-t_0)^{\alpha} + f_2(x)(t-t_0)^{2\alpha} + \ldots$$

is called multiple fractional power series about $t = t_0$, *where* t *is a variable and* f_m*'s are functions of* x *called the coefficients of the series.*

Theorem 2.4. [6,7] *Suppose that $u(x,t)$ has a multiple fractional power series representation at $t=t_0$ of the form*

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=\sum_{n=0}^{\infty}f_n(x)(t-t_0)^{n\alpha},$$

$0\leqslant m-1<\alpha\leqslant m$, $x\in I$, $t_0\leqslant t<t_0+R$. *If $D_t^{n\alpha}u(x,t)$, $n=0,1,2,\ldots$ are continuous on $I\times(t_0,t_0+R)$, then*

$$f_n(x)=\frac{D_t^{n\alpha}u(x,t)}{\Gamma(n\alpha+1)},\ n=0,1,2,\ldots$$

Here $D_t^{n\alpha}(\cdot)=\frac{\partial^{n\alpha}}{\partial t^{n\alpha}}(\cdot)=\frac{\partial^{\alpha}}{\partial t^{\alpha}}(\frac{\partial^{\alpha}}{\partial t^{\alpha}}(\ldots(\frac{\partial^{\alpha}}{\partial t^{\alpha}}(\cdot))))$ $(n-times)$, and $R=\min_{c\in I}R_c$, in which R_c is the radius of convergence of the fractional power series $\sum_{k=0}^{\infty}f_n(c)(t-t_0)^{n\alpha}$.

The convergence of the traditional residual power series approach assures that there is a real number $0<\lambda<1$, such that $||u_{n+1}(x,t)||\leqslant\lambda||u_n(x,t)||$, $t\in(t_0,t_0+R)$.

3. Main Results

3.1. **Application of General-RPSM to Fractional Brusselator Model.** This section demonstrates how to apply the suggested technique to the fractional Brusselator model. Consider the fractional-order Brusselator system of equations,

$$D_t^{\alpha}u=\varepsilon_1+u^2v-(\varepsilon_2+1)u+\varepsilon_3\nabla^2u,$$
$$D_t^{\alpha}v=\varepsilon_2u-u^2v+\varepsilon_3\nabla^2v,$$

subject to the initial condition $u(x,y,z,0)=f(x,y,z)$, $v(x,y,z,0)=g(x,y,z)$.

Applying the General transformation with the initial conditions yields

$$q^{\alpha}(s)\mathbb{J}\{u\}-p(s)q^{\alpha-1}(s)f(x,y,z)=\varepsilon_1\mathbb{J}\{1\}+\mathbb{J}\{u^2v\}-(\varepsilon_2+1)\mathbb{J}\{u\}+\varepsilon_3\mathbb{J}\{\nabla^2u\}$$
$$q^{\alpha}(s)\mathbb{J}\{v\}-p(s)q^{\alpha-1}(s)g(x,y,z)=\varepsilon_2\mathbb{J}\{u\}-\mathbb{J}\{u^2v\}+\varepsilon_3\mathbb{J}\{\nabla^2v\}.$$

Rearranging the equations and using the inverse General transformation provides

$$\begin{aligned}u=&\mathbb{J}^{-1}\left\{\frac{p(s)}{q^{\alpha}(s)}q^{\alpha-1}(s)f(x,y,z)\right\}+\mathbb{J}^{-1}\left\{\frac{\varepsilon_1}{q^{\alpha}(s)}\mathbb{J}\{1\}\right\}+\mathbb{J}^{-1}\left\{\frac{1}{q^{\alpha}(s)}\mathbb{J}\{u^2v\}\right\}\\&-\mathbb{J}^{-1}\left\{\frac{\varepsilon_2+1}{q^{\alpha}(s)}\mathbb{J}\{u\}\right\}+\mathbb{J}^{-1}\left\{\frac{\varepsilon_3}{q^{\alpha}(s)}\mathbb{J}\{\nabla^2u\}\right\}\\v=&\mathbb{J}^{-1}\left\{\frac{p(s)}{q^{\alpha}(s)}q^{\alpha-1}(s)g(x,y,z)\right\}+\mathbb{J}^{-1}\left\{\frac{\varepsilon_2}{q^{\alpha}(s)}\mathbb{J}\{u\}\right\}-\mathbb{J}^{-1}\left\{\frac{1}{q^{\alpha}(s)}\mathbb{J}\{u^2v\}\right\}\\&+\mathbb{J}^{-1}\left\{\frac{\varepsilon_3}{q^{\alpha}(s)}\mathbb{J}\{\nabla^2v\}\right\}.\end{aligned}$$

Simplifying the equations indicates that

$$\begin{aligned} u &= f(x,y,z) + \frac{\varepsilon_1 t^\alpha}{\Gamma(\alpha+1)} + \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u^2v\}\right\} - (\varepsilon_2+1)\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u\}\right\} \\ &\quad + \varepsilon_3\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 u\}\right\} \\ v &= g(x,y,z) + \varepsilon_2\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u\}\right\} - \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u^2v\}\right\} + \varepsilon_3\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 v\}\right\}. \end{aligned}$$

The General-RPSM supposes that the solution of the problem is in the form of a power series with unknown coefficients

$$u(x,y,z,t) = \sum_{n=0}^{\infty} a_n(x,y,z)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \text{ and } v(x,y,z,t) = \sum_{n=0}^{\infty} b_n(x,y,z)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{3.1}$$

and the residual functions are defined by

$$\begin{aligned} Res_{1,\infty} &= u - f(x,y,z) - \frac{\varepsilon_1 t^\alpha}{\Gamma(\alpha+1)} - \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u^2v\}\right\} + (\varepsilon_2+1)\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u\}\right\} \\ &\quad - \varepsilon_3\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 u\}\right\} \\ Res_{2,\infty} &= v - g(x,y,z) - \varepsilon_2\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u\}\right\} + \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u^2v\}\right\} - \varepsilon_3\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 v\}\right\}. \end{aligned}$$

In practice, the k-order approximate solution is expressed as

$$u_k(x,y,z,t) = \sum_{n=0}^{k} a_n(x,y,z)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \text{ and } v_k(x,y,z,t) = \sum_{n=0}^{k} b_n(x,y,z)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{3.2}$$

Moreover, the k-order residual functions are determined by

$$\begin{aligned} Res_{1,k} &= u_k - f(x,y,z) - \frac{\varepsilon_1 t^\alpha}{\Gamma(\alpha+1)} - \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_k^2 v_k\}\right\} + (\varepsilon_2+1)\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_k\}\right\} \\ &\quad - \varepsilon_3\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 u_k\}\right\}, \\ Res_{2,k} &= v_k - g(x,y,z) - \varepsilon_2\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_k\}\right\} + \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_k^2 v_k\}\right\} - \varepsilon_3\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 v_k\}\right\}. \end{aligned}$$

Due to the initial conditions, the initial approximate solution is $a_0(x,y,z) = f(x,y,z)$, $b_0(x,y,z) = g(x,y,z)$. By constructing the recursive relation

$$\begin{cases} u_0(x,y,z) = f(x,y,z), \quad v_0(x,y,z) = g(x,y,z), \\ Res_{1,k} = u_k - f(x,y,z) - \dfrac{\varepsilon_1 t^\alpha}{\Gamma(\alpha+1)} - \mathbb{J}^{-1}\left\{\dfrac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}^2 v_{k-1}\}\right\} \\ \qquad + (\varepsilon_2+1)\mathbb{J}^{-1}\left\{\dfrac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}\}\right\} - \varepsilon_3\mathbb{J}^{-1}\left\{\dfrac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 u_{k-1}\}\right\}, k \geqslant 1, \\ Res_{2,k} = v_k - g(x,y,z) - \varepsilon_2\mathbb{J}^{-1}\left\{\dfrac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}\}\right\} + \mathbb{J}^{-1}\left\{\dfrac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}^2 v_{k-1}\}\right\} \\ \qquad - \varepsilon_3\mathbb{J}^{-1}\left\{\dfrac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 v_{k-1}\}\right\}, \quad k \geqslant 1, \end{cases} \tag{3.3}$$

and employing the conditions

$$t^{-k\alpha}Res_{1,k}(x,y,z,t)\Big|_{t=0}=0 \quad \text{and} \quad t^{-k\alpha}Res_{2,k}(x,y,z,t)\Big|_{t=0}=0,$$

the coefficients a_k, b_k are easily obtained as follows. For $k=1$. The 1st-order approximate solution is assumed

$$u_1(x,y,z,t)=f(x,y,z)+a_1(x,y,z)\frac{t^\alpha}{\Gamma(\alpha+1)} \text{ and } v_1(x,y,z,t)=g(x,y,z)+b_1(x,y,z)\frac{t^\alpha}{\Gamma(\alpha+1)}.$$

Substituting into (3.3), the first-order residual functions are given as

$$\begin{aligned}
Res_{1,1}(x,y,z,t)&=a_1(x,y,z)\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{\varepsilon_1 t^\alpha}{\Gamma(\alpha+1)}-\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_0^2v_0\}\right\}\\
&\quad+(\varepsilon_2+1)\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_0\}\right\}-\varepsilon_3\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2u_0\}\right\},\\
&=\big(a_1-\varepsilon_1-f^2g+(\varepsilon_2+1)f-\varepsilon_3\nabla^2f\big)\frac{t^\alpha}{\Gamma(\alpha+1)}\\
Res_{2,1}(x,y,z,t)&=b_1(x,y,z)\frac{t^\alpha}{\Gamma(\alpha+1)}-\varepsilon_2\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_0\}\right\}+\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_0^2v_0\}\right\}\\
&\quad-\varepsilon_3\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2v_0\}\right\},\\
&=\big(b_1-\varepsilon_2f+f^2g-\varepsilon_3\nabla^2g\big)\frac{t^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

By imposing the condition $t^{-\alpha}Res_{1,1}(x,y,z,t)\Big|_{t=0}=0$ and $t^{-\alpha}Res_{2,1}(x,y,z,t)\Big|_{t=0}=0$, the coefficients $a_1(x,y,z), b_1(x,y,z)$ are assigned as

$$a_1(x,y,z)=\varepsilon_1+f^2g-(\varepsilon_2+1)f+\varepsilon_3\nabla^2f \quad \text{and} \quad b_1(x,y,z)=\varepsilon_2f-f^2g+\varepsilon_3\nabla^2g.$$

Hence, the 1st-order approximate solution is discovered

$$\begin{aligned}
u_1(x,y,z,t)&=f+\big(\varepsilon_1+f^2g-(\varepsilon_2+1)f+\varepsilon_3\nabla^2f\big)\frac{t^\alpha}{\Gamma(\alpha+1)},\\
v_1(x,y,z,t)&=g+\big(\varepsilon_2f-f^2g+\varepsilon_3\nabla^2g\big)\frac{t^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

For $k=2$. Assume the 2nd-order approximate solution is

$$\begin{aligned}
u_2(x,y,z,t)&=f+\big(\varepsilon_1+f^2g-(\varepsilon_2+1)f+\varepsilon_3\nabla^2f\big)\frac{t^\alpha}{\Gamma(\alpha+1)}+a_2(x,y,z)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
v_2(x,y,z,t)&=g+\big(\varepsilon_2f-f^2g+\varepsilon_3\nabla^2g\big)\frac{t^\alpha}{\Gamma(\alpha+1)}+b_2(x,y,z)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned}$$

After substituting the previous solution into (3.3), the 2-order residual functions are constructed

$$\begin{aligned}
Res_{1,2}(x,y,z,t) &= \big(\varepsilon_1 + f^2 g - (\varepsilon_2+1)f + \varepsilon_3\nabla^2 f\big)\frac{t^\alpha}{\Gamma(\alpha+1)} + a_2(x,y,z)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\varepsilon_1 t^\alpha}{\Gamma(\alpha+1)} \\
&\quad - \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_1^2 v_1\}\right\} + (\varepsilon_2+1)\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_1\}\right\} + \varepsilon_3\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 u_1\}\right\}, \\
&= \big(a_2 - f^2 b_1 - 2fga_1 + (\varepsilon_2+1)a_1 - \varepsilon_3\nabla^2 a_1\big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\
Res_{2,2}(x,y,z,t) &= \big(\varepsilon_2 f - f^2 g + \varepsilon_3\nabla^2 g\big)\frac{t^\alpha}{\Gamma(\alpha+1)} b_2(x,y,z)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \varepsilon_2\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_1\}\right\} \\
&\quad + \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_1^2 v_1\}\right\} - \varepsilon_3\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 v_1\}\right\}, \\
&= \big(b_2 - \varepsilon_2 a_1 + f^2 b_1 + 2fga_1 - \varepsilon_3\nabla^2 b_1\big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned}$$

Imposing the conditions $t^{-2\alpha}Res_{1,2}(x,y,z,t)\Big|_{t=0} = 0$ and $t^{-2\alpha}Res_{2,2}(x,y,z,t)\Big|_{t=0} = 0$, the coefficients $a_2(x,y,z)$ and $b_2(x,y,z)$ are found:

$$a_2(x,y,z) = f^2 b_1 + 2fga_1 - (\varepsilon_2+1)a_1 + \varepsilon_3\nabla^2 a_1 \text{ and } b_2(x,y,z) = \varepsilon_2 a_1 - f^2 b_1 - 2fga_1 + \varepsilon_3\nabla^2 b_1.$$

Therefore, the 2nd-order approximate solution of this problem is obtained:

$$\begin{aligned}
u_2(x,y,z,t) &= f + \big(\varepsilon_1 + f^2 g - (\varepsilon_2+1)f + \varepsilon_3\nabla^2 f\big)\frac{t^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \big(f^2 b_1 + 2fga_1 - (\varepsilon_2+1)a_1 + \varepsilon_3\nabla^2 a_1\big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\
v_2(x,y,z,t) &= g + \big(\varepsilon_2 f - f^2 g + \varepsilon_3\nabla^2 g\big)\frac{t^\alpha}{\Gamma(\alpha+1)} + \big(\varepsilon_2 a_1 - f^2 b_1 - 2fga_1 + \varepsilon_3\nabla^2 b_1\big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned}$$

For $k \geqslant 3$. By repeating the previously discussed procedure, we establish a k-order approximation solution $u_k(x,y,z,t), v_k(x,y,z,t)$, in which

$$u(x,y,z,t) = \lim_{k\to\infty} u_k(x,y,z,t) \quad \text{and} \quad v(x,y,z,t) = \lim_{k\to\infty} v_k(x,y,z,t).$$

3.2. **Illustrative Examples.** This section illustrates the implementation of the suggested method for the fractional Brusselator model. The problem of knowing and not knowing the exact solutions is investigated.

Example 3.1. [2] *Consider the fractional Brusselator system as follows,*

$$\begin{aligned}
D_t^\alpha u &= -2u + u^2 v + \frac{1}{6}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\
D_t^\alpha v &= u - u^2 v + \frac{1}{6}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)
\end{aligned}$$

where $0 < \alpha \leqslant 1$, *subjects to the initial conditions*

$$u(x,y,z,0) = e^{-(x+y+z)} \text{ and } v(x,y,z,0) = e^{x+y+z}.$$

The exact solutions of the problem when $\alpha = 1$ *is*

$$u(x,y,z,t) = e^{-(\frac{t}{2}+x+y+z)}, \;\; v(x,y,z,t) = e^{\frac{t}{2}+x+y+z}.$$

Here, $\varepsilon_1 = 0, \varepsilon_2 = 1, \; \varepsilon_3 = \frac{1}{6}, f(x,y,z) = e^{-(x+y+z)}$ and $g(x,y,z) = e^{x+y+z}$. The iterative relation for this problem is defined by

$$\begin{aligned}
&u_0(x,y,z) = e^{-(x+y+z)}, \quad v_0(x,y,z) = e^{x+y+z},\\
&Res_{1,k} = u_k - e^{-(x+y+z)} - \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}^2 v_{k-1}\}\right\} + 2\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}\}\right\}\\
&\qquad - \frac{1}{6}\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 u_{k-1}\}\right\},\\
&Res_{2,k} = v_k - e^{x+y+z} - \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}\}\right\} + \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}^2 v_{k-1}\}\right\}\\
&\qquad - \frac{1}{6}\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 v_{k-1}\}\right\},
\end{aligned}$$

$k = 1, 2, 3, \ldots$. Assume a solution is written as (3.2), and upon repeating the preceding recursive relation a few times, it is found that

$$\begin{aligned}
&a_0(x,y,z) = e^{-(x+y+z)}, \; a_1(x,y,z) = -\frac{1}{2}e^{-(x+y+z)}, \; a_2(x,y,z) = \frac{1}{4}e^{-(x+y+z)},\\
&a_3(x,y,z) = \Big(\frac{3}{8} - \frac{\Gamma(2\alpha+1)}{4\Gamma(\alpha+1)^2}\Big)e^{-(x+y+z)}\\
&a_4(x,y,z) = \Big(\frac{5}{16} - \frac{\Gamma(2\alpha+1)}{8\Gamma(\alpha+1)^2} + \frac{\Gamma(3\alpha+1)}{8\Gamma(\alpha+1)^3} - \frac{\Gamma(3\alpha+1)}{4\Gamma(2\alpha+1)\Gamma(\alpha+1)}\Big)e^{-(x+y+z)}\\
&\qquad - \Big(\frac{1}{2} + \frac{\Gamma(2\alpha+1)}{4\Gamma(\alpha+1)^2}\Big)e^{-3(x+y+z)}\\
&b_0(x,y,z) = e^{x+y+z}, \; b_1(x,y,z) = \frac{1}{2}e^{x+y+z}, \; b_2(x,y,z) = \frac{1}{4}e^{x+y+z},\\
&b_3(x,y,z) = \Big(\frac{\Gamma(2\alpha+1)}{4\Gamma(\alpha+1)} - \frac{1}{2}\Big)e^{-(x+y+z)} + \frac{1}{8}e^{x+y+z}\\
&b_4(x,y,z) = \Big(\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)} + \frac{3\Gamma(2\alpha+1)}{8\Gamma(\alpha+1)^2} - \frac{\Gamma(3\alpha+1)}{8\Gamma(\alpha+1)^3} - \frac{3}{4}\Big)e^{-(x+y+z)} + \frac{1}{16}e^{x+y+z}
\end{aligned}$$

Hence, the 4-order approximate solution is

$$\begin{aligned}
u_4(x,y,z,t) &= e^{-(x+y+z)}\Big[1 - \frac{t^\alpha}{2\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{4\Gamma(2\alpha+1)} - \Big(\frac{3}{8} - \frac{\Gamma(2\alpha+1)}{4\Gamma(\alpha+1)^2}\Big)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \Big(\frac{5}{16}\\
&- \frac{\Gamma(2\alpha+1)}{8\Gamma(\alpha+1)^2} + \frac{\Gamma(3\alpha+1)}{8\Gamma(\alpha+1)^3} - \frac{\Gamma(3\alpha+1)}{4\Gamma(2\alpha+1)\Gamma(\alpha+1)} - (\frac{1}{2} + \frac{\Gamma(2\alpha+1)}{4\Gamma(\alpha+1)^2})e^{-2(x+y+z)}\Big)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\Big]\\
v_4(x,y,z,t) &= e^{x+y+z}\Big[1 + \frac{t^\alpha}{2\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{4\Gamma(2\alpha+1)} + \Big((\frac{\Gamma(2\alpha+1)}{4\Gamma(\alpha+1)} - \frac{1}{2})e^{-2(x+y+z)} + \frac{1}{8}\Big)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\
&+ \Big((\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)} + \frac{3\Gamma(2\alpha+1)}{8\Gamma(\alpha+1)^2} - \frac{\Gamma(3\alpha+1)}{8\Gamma(\alpha+1)^3} - \frac{3}{4})e^{-2(x+y+z)} + \frac{1}{16}\Big)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\Big].
\end{aligned}$$

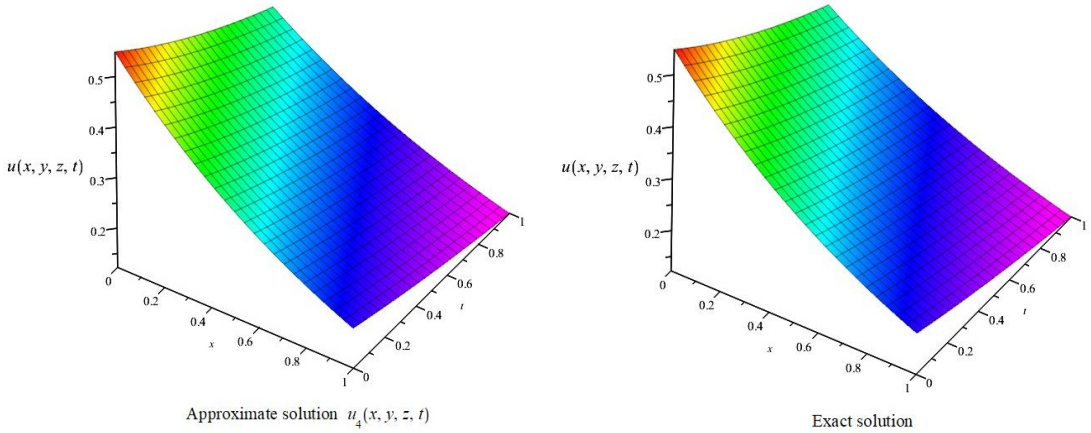


FIGURE 1. Show the comparison of the surface of an approximate solution $u_4(x, y, z, t)$ and the exact solution of Example 3.1 when $y = z = 0.3$ and $\alpha = 1$.

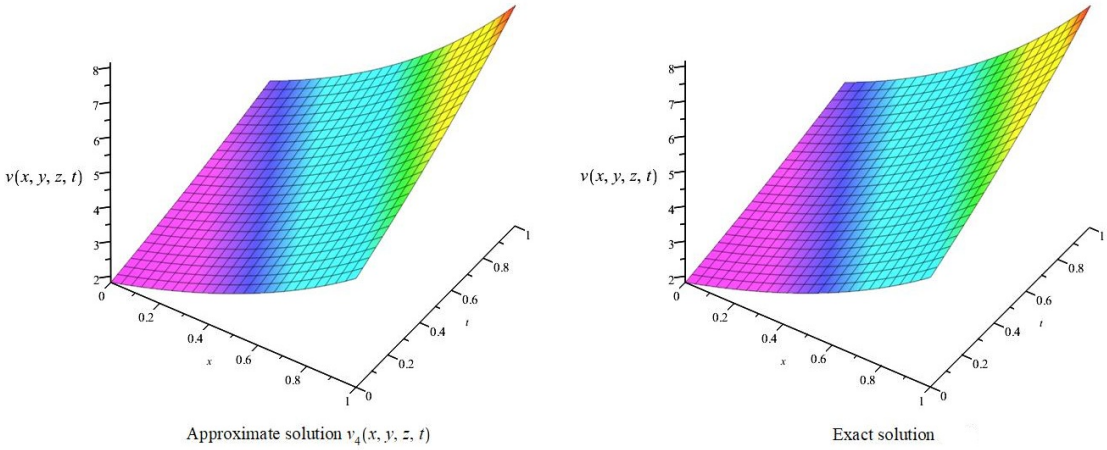


FIGURE 2. Show the comparison of the surface of an approximate solution $v_4(x, y, z, t)$ and the exact solution of Example 3.1 when $y = z = 0.3$ and $\alpha = 1$.

The remaining terms can be obtained by performing the previously mentioned procedure.

$$\begin{aligned} u(x,y,z,t) &= e^{-(x+y+z)}\Big[1 - \frac{t^{\alpha}}{2\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{4\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{8\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{16\Gamma(4\alpha+1)} + \dots\Big] \\ &= e^{-(x+y+z)}E(-\frac{t^{\alpha}}{2}) \\ v(x,y,z,t) &= e^{x+y+z}\Big[1 + \frac{t^{\alpha}}{2\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{4\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{8\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{16\Gamma(4\alpha+1)} + \dots\Big] \\ &= e^{(x+y+z)}E(\frac{t^{\alpha}}{2}). \end{aligned}$$

When $\alpha = 1$, the solution is reduced to

$$\begin{aligned} u(x,y,z,t) &= e^{-(x+y+z)}E(-\frac{t}{2}) = e^{-(\frac{t}{2}+x+y+z)} \\ v(x,y,z,t) &= e^{(x+y+z)}E(\frac{t}{2}) = e^{(\frac{t}{2}+x+y+z)}, \end{aligned}$$

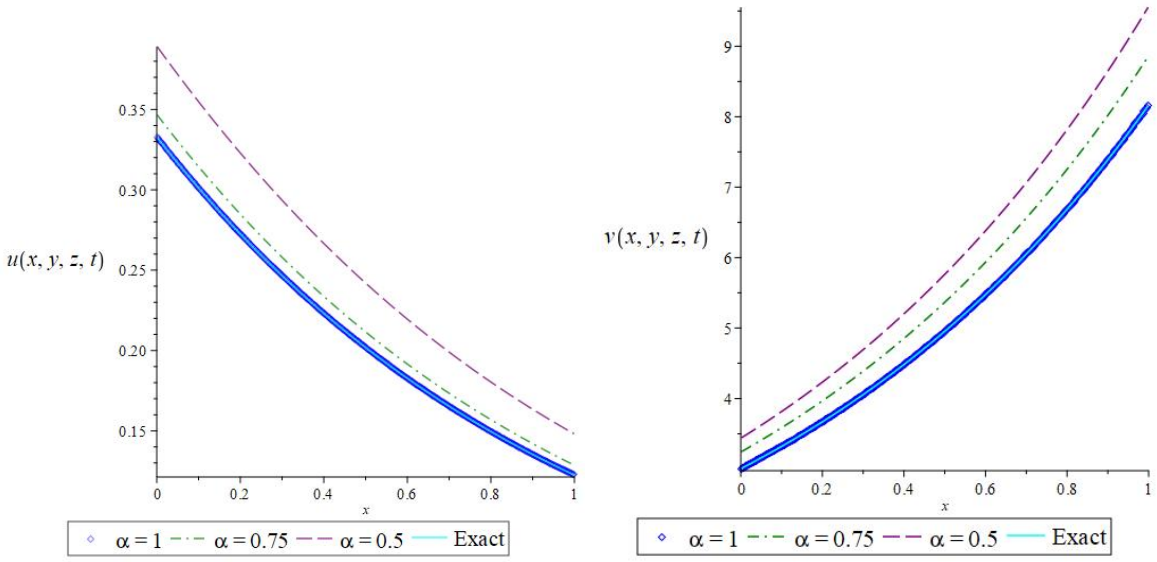


FIGURE 3. Curve comparison of the exact solution and approximate solution u_4, v_4 of Example 3.1 when $y = z = 0.3$ at different α values.

is the exact solution to the classic Brusselator model.

Example 3.2. [2] *Consider the Brusselator system with Caputo fractional derivative,*

$$D_t^\alpha u = 1 - 3u + u^2 v + \frac{1}{500}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$
$$D_t^\alpha v = 2u - u^2 v + \frac{1}{500}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

with respect to the initial conditions,

$$u(x, y, z, 0) = 1 + \sin(\lambda(x + y + z)) \quad \textit{and} \quad v(x, y, z, 0) = 3.$$

Note that $\varepsilon_1 = 1, \varepsilon_2 = 2,\ \varepsilon_3 = \frac{1}{500}, f(x, y, z) = 1 + \sin(\lambda(x + y + z))$ and $g(x, y, z) = 3$. Hence, the recurrence relation is proposed:

$$u_0(x, y, z) = 1 + \sin(\lambda(x + y + z)), \quad v_0(x, y, z) = 3,$$
$$Res_{1,k} = u_k - 1 - \sin(\lambda(x + y + z)) - \frac{t^\alpha}{\Gamma(\alpha + 1)} - \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}^2 v_{k-1}\}\right\}$$
$$+ 3\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}\}\right\} - \frac{1}{500}\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 u_{k-1}\}\right\},$$
$$Res_{2,k} = v_k - 3 - 2\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}\}\right\} + \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}^2 v_{k-1}\}\right\}$$
$$- \frac{1}{500}\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 v_{k-1}\}\right\},$$

For $k = 1$. The 1-order approximate solution is

$$u_1(x,y,z,t) = 1 + \sin(\lambda(x+y+z)) + a_1(x,y,z)\frac{t^\alpha}{\Gamma(\alpha+1)},$$
$$v_1(x,y,z,t) = 3 + b_1(x,y,z)\frac{t^\alpha}{\Gamma(\alpha+1)}.$$

The residual functions are

$$\begin{aligned}
Res_{1,1} &= a_1\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^\alpha}{\Gamma(\alpha+1)} - \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_0^2v_0\}\right\} + 3\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_0\}\right\} \\
&\quad - \frac{1}{500}\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 u_0\}\right\}, \\
&= a_1\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^\alpha}{\Gamma(\alpha+1)} + u_0^2v_0\frac{t^\alpha}{\Gamma(\alpha+1)} + 3u_0\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{1}{500}\nabla^2 u_0\frac{t^\alpha}{\Gamma(\alpha+1)}, \\
&= \big(a_1 - 1 + u_0^2v_0 + 3u_0 - \frac{1}{500}\nabla^2 u_0\big)\frac{t^\alpha}{\Gamma(\alpha+1)}, \\
Res_{2,1} &= b_1\frac{t^\alpha}{\Gamma(\alpha+1)} - 2\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_0\}\right\} + \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_0^2v_0\}\right\} - \frac{1}{500}\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 v_0\}\right\}, \\
&= b_1\frac{t^\alpha}{\Gamma(\alpha+1)} - 2u_0\frac{t^\alpha}{\Gamma(\alpha+1)} + u_0^2v_0\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{1}{500}\nabla^2 v_0\frac{t^\alpha}{\Gamma(\alpha+1)}, \\
&= \big(b_1 - 2u_0 + u_0^2v_0 - \frac{1}{500}\nabla^2 v_0\big)\frac{t^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Apply the conditions $t^{-\alpha}Res_{1,1}(x,y,z,t)\big|_{t=0} = 0$ and $t^{-\alpha}Res_{2,1}(x,y,z,t)\big|_{t=0} = 0$, the coefficients are found

$$a_1(x,y,z) = 1 - f^2g - 3f + \frac{1}{500}\nabla^2 f, \quad b_1(x,y,z) = 2f - f^2g + \frac{1}{500}\nabla^2 g.$$

Hence, the 1-order approximate solution is delivered

$$u_1(x,y,z,t) = 1 + \sin(\lambda(x+y+z)) + \Big(1 - f^2g - 3f + \frac{1}{500}\nabla^2 f\Big)\frac{t^\alpha}{\Gamma(\alpha+1)},$$
$$v_1(x,y,z,t) = 3 + \Big(2f - f^2g + \frac{1}{500}\nabla^2 g\Big)\frac{t^\alpha}{\Gamma(\alpha+1)}.$$

After proceeding with a few previous iterations, the coefficients are reached

$$\begin{aligned}
a_2(x,y,z) &= f^2b_1 + 2fga_1 - 3a_1 + \frac{1}{500}\nabla^2 a_1, \\
b_2(x,y,z) &= 2a_1 - f^2b_1 - 2fga_1 + \frac{1}{500}\nabla^2 b_1, \\
a_3(x,y,z) &= -3a_2 + f^2b_2 + 2fga_2 + 2fa_1b_1\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + ga_1^2\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{1}{500}\nabla^2 a_2, \\
b_3(x,y,z) &= 2a_2 - f^2b_2 - 2fga_2 - 2fa_1b_1\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - a_1^2g\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{1}{500}\nabla^2 b_2.
\end{aligned}$$

Therefore, the 3-order approximate solution is obtained

$$u_3(x,y,z,t) = 1 + \sin(\lambda(x+y+z)) + \Big(1 - f^2 g - 3f + \frac{1}{500}\nabla^2 f\Big)\frac{t^\alpha}{\Gamma(\alpha+1)}$$
$$+ \Big(f^2 b_1 + 2fga_1 - 3a_1 + \frac{1}{500}\nabla^2 a_1\Big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+ \Big(- 3a_2 + f^2 b_2 + 2fga_2 + 2fa_1 b_1 \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + ga_1^2 \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{1}{500}\nabla^2 a_2\Big)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$v_3(x,y,z,t) = 3 + \Big(2f - f^2 g + \frac{1}{500}\nabla^2 g\Big)\frac{t^\alpha}{\Gamma(\alpha+1)}$$
$$+ \Big(2a_1 - u_0^2 b_1 - 2u_0 v_0 a_1 + \frac{1}{500}\nabla^2 b_1\Big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+ \Big(2a_2 - f^2 b_2 - 2fga_2 - 2fa_1 b_1 \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - a_1^2 g \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{1}{500}\nabla^2 b_2\Big)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.$$

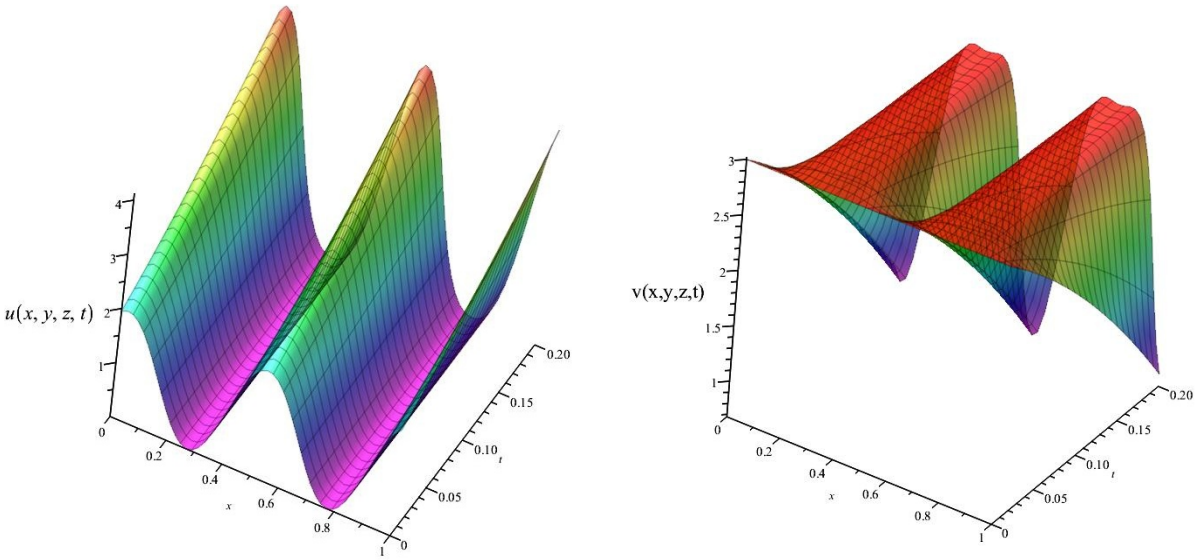


FIGURE 4. Show the surface of General-RPSM solution u_3, v_3 of Example 3.2 when $y = z = 0.3$ and $\lambda = 4\pi$ for $\alpha = 1$.

Example 3.3. [4] *Let us consider the fractional Brusselator system of equations*

$$D_t^\alpha u = 1 + u^2 v - \left(\frac{17}{5} + 1\right) u + \frac{1}{500}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$
$$D_t^\alpha v = -u^2 v + \frac{17}{5} u + \frac{1}{500}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

subject to the initial conditions,

$$u(x,y,0) = 2 + \frac{y}{4} \quad \textit{and} \quad v(x,y,0) = 1 + \frac{4x}{5}.$$

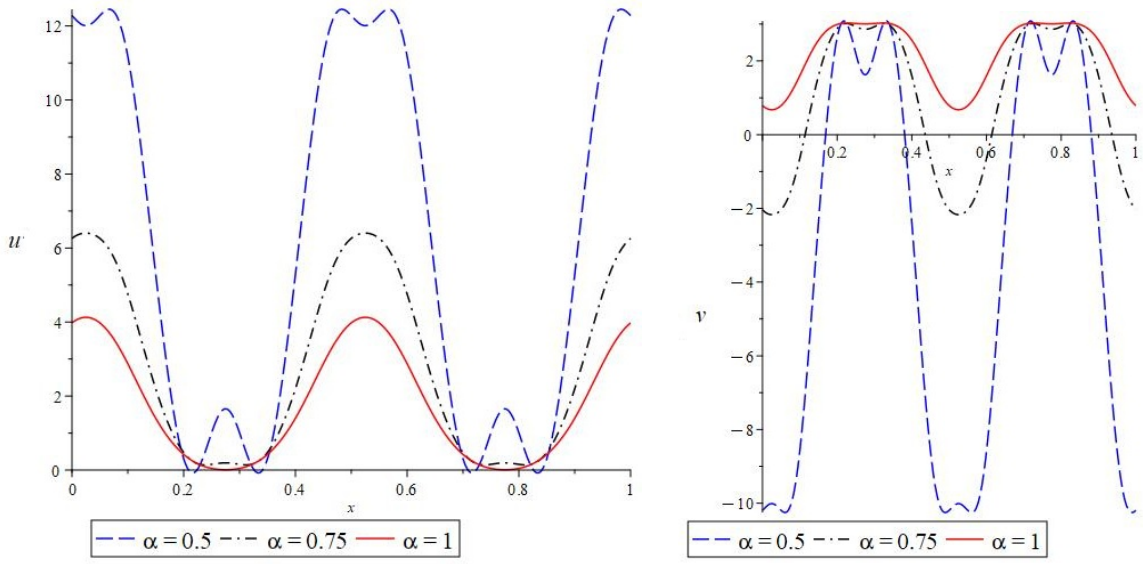


FIGURE 5. Show the curve of General-RPSM solution u_3, v_3 of Example 3.2 when $y = z = 0.3, t = 0.2$ and $\lambda = 4\pi$ for different values of $\alpha = 0.5, 0.75, 1$.

It can be observed that $\varepsilon_1 = 1, \varepsilon_2 = \frac{17}{5}$, $\varepsilon_3 = \frac{1}{500}$, $f(x,y) = 2 + \frac{y}{4}$ and $g(x,y) = 1 + \frac{4x}{5}$. Therefore, the iterative relation for the problem is defined by

$$
\begin{aligned}
&u_0(x,y) = 2 + \frac{y}{4}, \quad v_0(x,y) = 1 + \frac{4x}{5},\\
&Res_{1,k} = u_k - f(x,y,z) - \frac{t^\alpha}{\Gamma(\alpha+1)} - \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}^2 v_{k-1}\}\right\} + \frac{22}{5}\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}\}\right\}\\
&\qquad - \frac{1}{500}\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 u_{k-1}\}\right\}, k = 1,2,3,\ldots,\\
&Res_{2,k} = v_k - g(x,y,z) - \frac{17}{5}\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}\}\right\} + \mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{u_{k-1}^2 v_{k-1}\}\right\}\\
&\qquad - \frac{1}{500}\mathbb{J}^{-1}\left\{\frac{1}{q^\alpha(s)}\mathbb{J}\{\nabla^2 v_{k-1}\}\right\}, \quad k = 1,2,3,\ldots.
\end{aligned}
$$

By using the iteration, the following coefficients are given,

$$
\begin{aligned}
&a_1(x,y) = 1 + f^2 g - \frac{22}{5}f + \frac{1}{500}\nabla^2 f, \quad b_1(x,y) = \frac{17}{5}f - f^2 g + \frac{1}{500}\nabla^2 g,\\
&a_2(x,y) = f^2 b_1 + 2fga_1 - \frac{22}{5}a_1 + \frac{1}{500}\nabla^2 a_1, \quad b_2(x,y) = \frac{17}{5}a_1 - f^2 b_1 - 2fga_1 + \frac{1}{500}\nabla^2 b_1.
\end{aligned}
$$

TABLE 1. Show the numerical simulation $u_4(x,y,z,t), v_4(x,y,z,t)$ and the corresponding absolute error for Example 3.1

t	x	y	z	$u_4(x,y,z,t)$	u_{Exact}	$\lvert u_4 - u_{Exact}\rvert$	$v_4(x,y,z,t)$	v_{Exact}	$\lvert v_4 - v_{Exact}\rvert$
	0.1	0.3	0.3	0.4723665540	0.4723665527	1.3×10^{-9}	2.117000009	2.117000017	8.0×10^{-9}
		0.5	0.5	0.3166367702	0.3166367694	8.0×10^{-10}	3.158192900	3.158192910	1.0×10^{-8}
		0.7	0.7	0.2122479743	0.2122479738	5.0×10^{-10}	4.711470167	4.711470183	1.6×10^{-8}
	0.2	0.3	0.3	0.4274149331	0.4274149319	1.2×10^{-9}	2.339646844	2.339646852	8.0×10^{-9}
0.1		0.5	0.5	0.2865047976	0.2865047969	7.0×10^{-10}	3.490342947	3.490342957	1.0×10^{-8}
		0.7	0.7	0.1920499091	0.1920499086	5.0×10^{-10}	5.206979810	5.206979827	1.7×10^{-8}
	0.3	0.3	0.3	0.3867410244	0.3867410235	9.0×10^{-10}	2.585709651	2.585709659	8.0×10^{-9}
		0.5	0.5	0.2592402613	0.2592402606	7.0×10^{-10}	3.857425519	3.857425531	1.2×10^{-8}
		0.7	0.7	0.1737739440	0.1737739435	5.0×10^{-10}	5.754602658	5.754602676	1.8×10^{-8}
	0.1	0.3	0.3	0.4723665540	0.4723665527	1.3×10^{-9}	2.585692563	2.585709659	1.7096×10^{-5}
		0.5	0.5	0.3166367702	0.3166367694	8.0×10^{-10}	3.857400026	3.857425531	2.5505×10^{-5}
		0.7	0.7	0.2122479743	0.2122479738	5.0×10^{-10}	5.754564627	5.754602676	3.8049×10^{-5}
	0.2	0.3	0.3	0.4274149331	0.4274149319	1.2×10^{-9}	2.857632223	2.857651118	1.8895×10^{-5}
0.5		0.5	0.5	0.2865047976	0.2865047969	7.0×10^{-10}	4.263086329	4.263114515	2.8186×10^{-5}
		0.7	0.7	0.1920499091	0.1920499086	5.0×10^{-10}	6.359777472	6.359819523	4.0251×10^{-5}
	0.3	0.3	0.3	0.3867410244	0.3867410235	9.0×10^{-10}	3.158172028	3.158192910	2.0882×10^{-5}
		0.5	0.5	0.2592402613	0.2592402606	7.0×10^{-10}	4.711439032	4.711470183	3.1151×10^{-5}
		0.7	0.7	0.1737739440	0.1737739435	5.0×10^{-10}	7.028641109	7.028687581	4.6472×10^{-5}
	0.1	0.3	0.3	0.4723665540	0.4723665527	1.3×10^{-9}	3.319545478	3.320116923	5.71445×10^{-4}
		0.5	0.5	0.3166367702	0.3166367694	8.0×10^{-10}	4.952179930	4.953032424	8.52494×10^{-4}
		0.7	0.7	0.2122479743	0.2122479738	5.0×10^{-10}	7.387784326	7.389056099	1.271773×10^{-3}
	0.2	0.3	0.3	0.4274149331	0.4274149319	1.2×10^{-9}	3.668665124	3.669296668	6.31544×10^{-4}
1		0.5	0.5	0.2865047976	0.2865047969	7.0×10^{-10}	5.473005240	5.473947392	9.42152×10^{-4}
		0.7	0.7	0.1920499091	0.1920499086	5.0×10^{-10}	8.164764386	8.166169913	1.405527×10^{-3}
	0.3	0.3	0.3	0.3867410244	0.3867410235	9.0×10^{-10}	4.054502003	4.055199967	6.97964×10^{-4}
		0.5	0.5	0.2592402613	0.2592402606	7.0×10^{-10}	6.048606226	6.049647464	1.041238×10^{-3}
		0.7	0.7	0.1737739440	0.1737739435	5.0×10^{-10}	9.023460154	9.025013499	1.553345×10^{-3}

Therefore, the 2-order approximate solution for the problem is

$$\begin{aligned} u_2(x,y,t) &= 2 + \frac{y}{4} + \Big(1 + f^2 g - \frac{22}{5} f + \frac{1}{500}\nabla^2 f\Big)\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \Big(f^2 b_1 + 2fga_1 - \frac{22}{5}a_1 + \frac{1}{500}\nabla^2 a_1\Big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ v_2(x,y,t) &= 1 + \frac{4x}{5} + \Big(\frac{17}{5} f - f^2 g + \frac{1}{500}\nabla^2 g\Big)\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \Big(\frac{17}{5}a_1 - f^2 b_1 - 2fga_1 + \frac{1}{500}\nabla^2 b_1\Big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned}$$

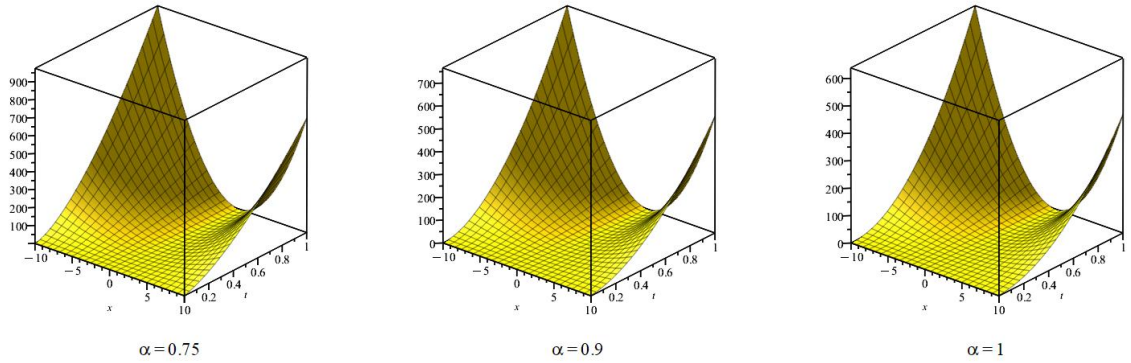


FIGURE 6. Show the surface of General-RPSM solution u_2 of Example 3.3 in different α

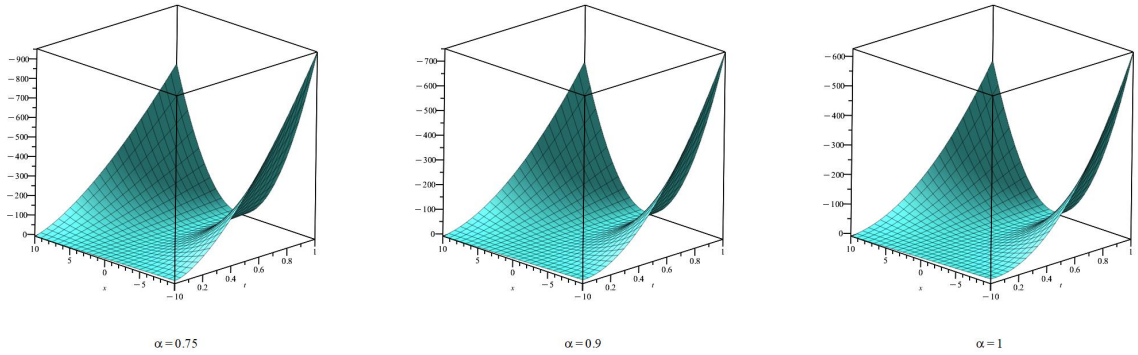


FIGURE 7. Show the surface of General-RPSM solution v_2 of Example 3.3 in different α

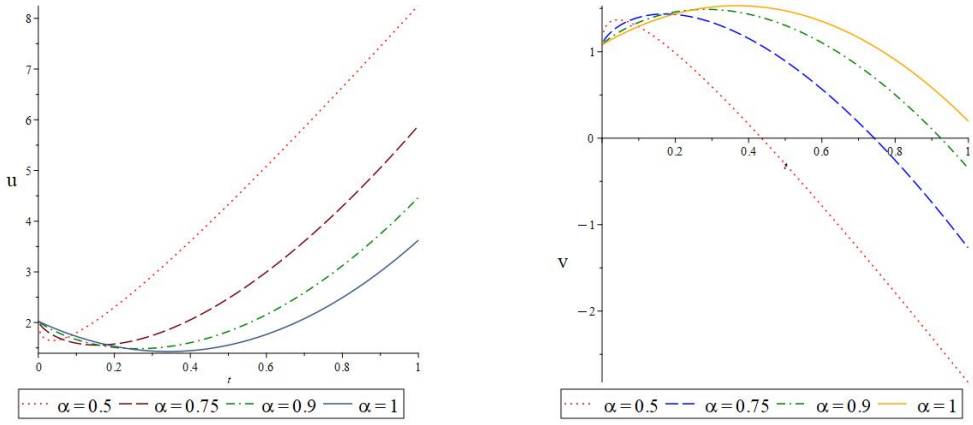


FIGURE 8. Show the curve of the General-RPSM solution u_2, v_2 of Example 3.3 at different α

TABLE 2. Approximate solution $u_2(x, y, t)$ of Example 3.3 at different values of α when $y = 1$.

t	x	Approx soln. at $\alpha = 0.5$	Approx soln. at $\alpha = 0.75$	Approx soln. at $\alpha = 1$
0.1	0.2	1.821346223	1.819226501	1.979831669
	0.4	1.607075830	1.856225843	2.035666744
	0.6	1.509445436	1.920971882	2.097333819
	0.8	1.528455043	2.013464619	2.164832894
	1.0	1.664104650	2.133704055	2.238163969
0.25	0.2	2.170999474	1.698094844	1.696760430
	0.4	1.369746786	1.536433443	1.741979649
	0.6	0.860094099	1.484450498	1.823648867
	0.8	0.642041411	1.542146008	1.941768086
	1.0	0.715588724	1.709519974	2.096337305

TABLE 3. Approximate solution $v_2(x, y, t)$ of Example 3.3 at different values of α when $y = 1$.

t	x	Approx soln. at $\alpha = 0.5$	Approx soln. at $\alpha = 0.75$	Approx soln. at $\alpha = 1$
0.1	0.2	1.445372748	1.420932139	1.320305831
	0.4	1.738643141	1.524664256	1.420420756
	0.6	1.915273535	1.600649677	1.514703681
	0.8	1.975263928	1.648888398	1.603154606
	1.0	1.918614321	1.669380422	1.685773531
0.25	0.2	1.290638546	1.515724370	1.495348945
	0.4	2.049391234	1.761220176	1.584817226
	0.6	2.516543921	1.897037527	1.637835508
	0.8	2.692096609	1.923176423	1.654403789
	1.0	2.576049296	1.839636864	1.634522070

4. Discussion and Conclusion

In this work, the generalized RPSM is successfully applied to resolve the multidimensional fractional Brusselator model. Three initial-value problems are investigated for both known and unknown exact solutions under different initial conditions.

In Example 3.1, the suggested procedure is used to solve the initial value problem for the three-dimensional fractional Brusselator model with a known exact solution. After four iterations, a five-term approximate power series solution is obtained, and the exact solution is derived by mathematical induction. To clarify the effect of the repeated iterations, Figures 1 and 2 compare the approximate and exact solutions for the case $\alpha = 1$. Figure 3 presents the approximate solution curves for $\alpha = 0.5$, 0.75, and 1, together with the exact solution; it also shows that the computed solution curve converges

to the exact solution as α approaches 1. Moreover, Table 1 presents the numerical results and the corresponding absolute errors for this problem.

In Example 3.2, the initial value problem for the fractional Brusselator system, which has no exact solution, is solved using the proposed method. The third-order truncated series solution is obtained after three iterations. The resulting solution surfaces are shown in Figure 4 and are consistent with the results reported in [2]. The curves of the approximate solutions u_3 and v_3, evaluated at $y = 0.3$, $z = 0.3$, $t = 0.2$, and $\lambda = 4\pi$ for $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 1$, are shown in Figure 5.

In Example 3.3, the General-RPSM is used to construct approximate power-series solutions u_2 and v_2 for an initial value problem with no exact solution. Solution surfaces for different values of α are shown in Figures 6 and 7, and the corresponding two-dimensional solution curves at $x = y = 0.1$ are compared in Figure 8. In addition, Tables 2 and 3 present numerical simulations of u_2 and v_2, respectively, for $\alpha = 0.5, 0.75$, and 1 at $y = 1$. These results show a trend similar to that reported in [4].

From the above discussion, it can be concluded that the General-RPSM is an efficient technique for solving the multidimensional fractional Brusselator model and provides a straightforward way to handle this nonlinear problem without linearization, perturbation techniques, or Adomian polynomials.

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