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THE CONVERGENCE THEOREM OF N-INTEGRAL

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Abstract. This paper develops convergence theorems for the N-integral and extends the resulting framework to topological measure spaces. The N-integral is a Henstock-type integral whose class of integrable functions lies between the Riemann and Henstock integrable functions. Motivated by classical convergence results, this study formulates conditions under which the pointwise or uniform limit of a sequence of N-integrable functions remains N-integrable. A uniform form of the double Lusin condition, called the N-uniform double Lusin condition, is used to establish convergence results for sequences of N-integrable functions. The paper proves uniform convergence, pointwise convergence under N-UDL, monotone-type convergence, dominated-type convergence, and equiintegrable convergence theorems for the N-integral. It also presents examples and counterexamples showing the necessity of the imposed conditions. Furthermore, the study extends the N-integral to topological measure spaces by defining topological gauges, tagged divisions, and topological N-equiintegrability. Under this framework, topological versions of uniform, equiintegrable, monotone, and dominated convergence theorems are established. These results contribute to the theoretical development of integration theory by providing a convergence framework for the N-integral and by clarifying its relationship with the Riemann and Henstock integrals.

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1. Introduction

The study of convergence theorems is a central topic in integration theory. Such theorems provide conditions under which the limiting process may be interchanged with integration, or under which the limit of a sequence of integrable functions remains integrable. Classical results such as the monotone convergence theorem and the dominated convergence theorem play this role in Lebesgue integration, while related convergence results also appear in the Riemann and Henstock–Kurzweil settings [1,8].

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The Henstock–Kurzweil integral extends the Riemann integral by allowing the use of gauges rather than constant mesh sizes in partitions. This makes it possible to integrate functions that are not Riemann integrable and, in some cases, functions that are not Lebesgue integrable [1,2,5]. Within this broader gauge-type framework, Racca and Cabral introduced the N-integral, a Henstock-type integral for real-valued functions on a compact interval $[a, b]$ [7]. The class of N-integrable functions lies between the class of Riemann integrable functions and the class of Henstock integrable functions. Thus,

$$\text{Riemann integrable} \subset \text{N-integrable} \subset \text{Henstock integrable}.$$

The N-integral is defined by allowing the exclusion of tags belonging to a null set, called an avoided set [7]. This feature makes the N-integral suitable for handling functions with exceptional behavior on small sets. In particular, the N-integral can integrate all improper Riemann integrable functions, even when such functions are not Lebesgue integrable [7]. Hence, the N-integral provides an intermediate framework between classical and generalized integration.

Although the basic properties and characterizations of the N-integral have already been established, the convergence behavior of sequences of N-integrable functions requires further investigation. In particular, pointwise convergence alone does not guarantee that the limit function remains N-integrable. This motivates the introduction of additional uniform conditions on sequences of functions. One such condition is the N-uniform double Lusin condition, which is formulated by making the gauge-null set condition in the characterization of N-integrability uniform with respect to the sequence. The use of double Lusin-type conditions is motivated by earlier work on Kurzweil–Henstock type integrals [6].

The first purpose of this paper is to establish convergence theorems for the N-integral. We prove that uniform convergence preserves N-integrability. We also establish a pointwise convergence theorem under the N-uniform double Lusin condition, together with monotone-type, dominated-type, and equiintegrable convergence results. Examples and counterexamples are included to show the relevance and necessity of the stated assumptions.

The second purpose of this paper is to extend the convergence framework of the N-integral to a topological setting. To do this, we introduce a topological measure space framework and define topological gauges, δ-fine tagged divisions, and topological N-equiintegrability. The connection between convergence behavior and topological structures is consistent with earlier investigations on Henstock–Kurzweil type integration in topological settings [3,4]. Under these definitions, we obtain topological versions of uniform, equiintegrable, monotone, and dominated convergence theorems.

The paper is organized as follows. Section 2 presents the necessary preliminaries, including the basic concepts of gauge integration, the N-integral, avoided sets, and convergence of functions, and recalls the key characterization of N-integrable functions. Section 3 establishes the main convergence theorems for the N-integral. Section 4 presents the topological extension of the N-integral and proves

the corresponding convergence results. Finally, Section 5 gives the conclusion and possible directions for further research.

2. Preliminaries

Throughout this paper, let $[a, b]$ be a compact interval in $\mathbb{R}$. We recall the basic definitions and notations needed in the discussion of the N-integral and its convergence theorems. The basic concepts on gauge integration, Henstock integration, and real analysis may be found in [1,5,8,9]. The definition and basic properties of the N-integral are due to Racca and Cabral [7].

Definition 2.1 (Partition)**.** A partition of $[a, b]$ is a finite ordered set

$$P = \{x_0, x_1, \ldots, x_n\}$$

such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The subintervals determined by the partition are

$$[x_{i-1}, x_i], \qquad i = 1, 2, \ldots, n.$$

Definition 2.2 (Tagged Partition)**.** A tagged partition of $[a, b]$ is a finite collection

$$P = \{(\xi_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\},$$

where

$$a = x_0 < x_1 < \cdots < x_n = b$$

and each tag ξ_i satisfies

$$\xi_i \in [x_{i-1}, x_i].$$

Definition 2.3 (Partial Division [7])**.** A partial division

$$D = \{(\xi, [u, v])\}$$

of $[a, b]$ is a finite collection of point-interval pairs such that

$$\xi \in [u, v], \qquad [u, v] \subset [a, b],$$

and the subintervals $[u, v]$ are nonoverlapping. If the union of the subintervals in D is $[a, b]$, then D is called a division of $[a, b]$.

For a partial division

$$D = \{(\xi, [u, v])\}$$

of $[a, b]$, define

$$I(D) = \{[u, v] : (\xi, [u, v]) \in D\}.$$

Definition 2.4 (S-tagged Partial Division [7]). Let $S \subset [a,b]$. A partial division

$$D = \{(\xi, [u,v])\}$$

of $[a,b]$ is said to be S-tagged if

$$\xi \in S$$

for every $(\xi, [u,v]) \in D$.

Definition 2.5 (Gauge [1,5]). A gauge on $[a,b]$ is a function

$$\delta : [a,b] \to (0,\infty).$$

Definition 2.6 (δ-fine Partial Division [7]). Let δ be a gauge on $[a,b]$. A point-interval pair

$$(\xi, [u,v])$$

is said to be δ-fine if

$$[u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi)).$$

A partial division D is said to be δ-fine if every point-interval pair in D is δ-fine.

Definition 2.7 (Null Set [8]). A subset $S \subset [a,b]$ is called a null set if it has measure zero. Equivalently, for every $\varepsilon > 0$, there exists a countable collection of intervals covering S such that the sum of their lengths is less than ε.

Given a null set $S \subset [a,b]$ and a partial division D of $[a,b]$, define

$$\phi_S(D) = \{(\xi, [u,v]) \in D : \xi \notin S\}.$$

Also define

$$D_S = \{(\xi, [u,v]) \in D : \xi \in S\}.$$

These notations are used in the formulation of the N-integral [7].

Definition 2.8 (Compatible Division [7]). Let $D = \{(\xi, [u,v])\}$ be a partial division of $[a,b]$. A partial division

$$D^* = \{(\xi^*, [u,v])\}$$

is said to be D-compatible, written

$$D^* \sim D,$$

if

$$\xi^* \in [u,v]$$

and

$$I(D^*) = I(D).$$

That is, D^* has the same collection of subintervals as D, although the tags may be replaced.

Definition 2.9 (Henstock Integral [1, 5]). A function f on $[a, b]$ is said to be Henstock integrable if there exists a real number A such that for every $\varepsilon > 0$, there exists a gauge δ on $[a, b]$ such that for every δ-fine division

$$D = \{(\xi, [u, v])\}$$

of $[a, b]$,

$$\left| \sum_{(\xi,[u,v])\in D} f(\xi)(v - u) - A \right| < \varepsilon.$$

The number A is called the Henstock integral of f over $[a, b]$, and we write

$$(H) \int_a^b f = A.$$

Definition 2.10 (Primitive [7]). If f is Henstock integrable on $[a, b]$, then its primitive F is defined by

$$F(a) = 0$$

and

$$F(x) = (H) \int_a^x f$$

for $x \in (a, b]$.

Definition 2.11 (N-integral [7]). A function f on $[a, b]$ is said to be N-integrable if there exists a null set $S \subset [a, b]$ and a real number A such that for every $\varepsilon > 0$, there exists a gauge δ on $[a, b]$ such that for every δ-fine division

$$D = \{(\xi, [u, v])\}$$

of $[a, b]$ and every

$$D^* = \{(\xi^*, [u, v])\} \sim \phi_S(D),$$

we have

$$\left| \sum_{(\xi^*,[u,v])\in D^*} f(\xi^*)(v - u) - A \right| < \varepsilon.$$

The number A is called the N-integral of f over $[a, b]$, and we write

$$(N) \int_a^b f = A.$$

Definition 2.12 (Avoided Set [7]). A null set $S \subset [a, b]$ satisfying the N-integrability condition for a function f is called an avoided set for f on $[a, b]$.

For a function f on $[a, b]$ and $\varepsilon > 0$, define

$$\Gamma_\varepsilon = \{(\xi, [u, v]) : |f(v) - f(u)| \geq \varepsilon,\ \xi \in [u, v],\ [u, v] \subset [a, b]\}.$$

The set Γ_ε depends on the function f and the number ε. This notation is used in the characterization of N-integrable functions [7].

For a sequence of functions $\{f_n\}$ on $[a,b]$ and $\varepsilon > 0$, define

$$\Gamma_{\varepsilon,n} = \{(\xi,[u,v]) : |f_n(v) - f_n(u)| \geq \varepsilon,\ \xi \in [u,v],\ [u,v] \subset [a,b]\}.$$

Definition 2.13 (Pointwise Convergence [9]). A sequence of functions $\{f_n\}$ on $[a,b]$ is said to converge pointwise to a function f on $[a,b]$ if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for every $x \in [a,b]$.

Definition 2.14 (Uniform Convergence [9]). A sequence of functions $\{f_n\}$ on $[a,b]$ is said to converge uniformly to a function f on $[a,b]$ if for every $\varepsilon > 0$, there exists a positive integer N such that

$$|f_n(x) - f(x)| < \varepsilon$$

for all $n \geq N$ and for all $x \in [a,b]$.

Definition 2.15 (Monotone Increasing Sequence [8]). A sequence of functions $\{f_n\}$ on $[a,b]$ is said to be monotone increasing if

$$f_1(x) \leq f_2(x) \leq \cdots$$

for every $x \in [a,b]$.

Definition 2.16 (Bounded Function [9]). A function $f : [a,b] \to \mathbb{R}$ is said to be bounded if there exists a real number $M > 0$ such that

$$|f(x)| \leq M$$

for every $x \in [a,b]$.

Definition 2.17 (Almost Everywhere [8]). A property is said to hold almost everywhere on $[a,b]$ if the set of points where the property fails is a null set.

Definition 2.18 (Continuous Almost Everywhere [8]). A function $f : [a,b] \to \mathbb{R}$ is said to be continuous almost everywhere on $[a,b]$ if the set of points at which f is discontinuous is a null set.

The following characterization of N-integrable functions will be used in the development of the convergence theorems.

Theorem 2.19 (Characterization of N-integrability [7]). *Let f be a function on $[a,b]$. Then the following statements are equivalent:*

(1) *f is N-integrable;*

(2) *f is Henstock integrable on $[a,b]$ and there exists a null set $S \subset [a,b]$ such that for every $\varepsilon > 0$, there exists a gauge δ on $[a,b]$ such that for every δ-fine partial division $D = \{(\xi,[u,v])\}$ of $[a,b]$,*

$$\sum_{(\xi,[u,v])\in\phi_S(D)\cap\Gamma_\varepsilon} |f(v)-f(u)|(v-u) < \varepsilon;$$

(3) *f is Henstock integrable on $[a,b]$ and continuous almost everywhere on $[a,b]$.*

3. Convergence Theorems for the N-Integral

In this section, we establish convergence theorems for the N-integral. The main idea is to make the condition in the characterization of N-integrability uniform with respect to a sequence of functions.

Theorem 3.1 (Uniform Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of N-integrable functions on $[a,b]$, and let f be a function on $[a,b]$. If*

$$f_n \to f$$

uniformly on $[a,b]$, then f is N-integrable.

Proof. Let S_n be an avoided set corresponding to f_n. Put

$$S = \bigcup_{n=1}^{\infty} S_n.$$

Then S is a null set. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly on $[a,b]$, there exists a positive integer N such that for all $x \in [a,b]$,

$$|f(x) - f_N(x)| < \varepsilon_0,$$

where

$$\varepsilon_0 = \min\left\{\frac{\varepsilon}{4}, \frac{\varepsilon}{4(b-a)}\right\}.$$

Since f_N is N-integrable, there exists a gauge δ on $[a,b]$ such that for every δ-fine partial division

$$D = \{(\xi,[u,v])\}$$

of $[a,b]$,

$$\sum_{(\xi,[u,v])\in\phi_S(D)\cap\Gamma_{\varepsilon/2,N}} |f_N(v)-f_N(u)|(v-u) < \frac{\varepsilon}{2},$$

where

$$\Gamma_{\varepsilon/2,N} = \left\{(\xi,[u,v]) : |f_N(v)-f_N(u)| \geq \frac{\varepsilon}{2}\right\}.$$

Let $D = \{(\xi,[u,v])\}$ be a δ-fine partial division of $[a,b]$. Since

$$|f_N(x) - f(x)| < \frac{\varepsilon}{4}$$

for all $x \in [a,b]$, it follows that

$$D \cap \Gamma_\varepsilon \subset D \cap \Gamma_{\varepsilon/2,N}.$$

Therefore,

$$\begin{aligned}
&\sum_{(\xi,[u,v])\in\phi_S(D)\cap\Gamma_\varepsilon} |f(v)-f(u)|(v-u)\\
&\le \sum_{(\xi,[u,v])\in\phi_S(D)\cap\Gamma_\varepsilon} |(f(v)-f(u))-(f_N(v)-f_N(u))|(v-u)\\
&\quad+\sum_{(\xi,[u,v])\in\phi_S(D)\cap\Gamma_\varepsilon} |f_N(v)-f_N(u)|(v-u)\\
&< \frac{\varepsilon}{4(b-a)}(b-a)+\frac{\varepsilon}{4(b-a)}(b-a)+\frac{\varepsilon}{2}\\
&= \varepsilon.
\end{aligned}$$

Moreover, by the uniform convergence theorem for the Henstock integral, f is Henstock integrable. Hence, by the characterization of N-integrability, f is N-integrable. □

Definition 3.2 (N-Uniform Double Lusin Condition)**.** A sequence $\{f_n\}$ of functions on $[a,b]$ is said to satisfy the *N-uniform double Lusin condition*, or simply the *N-UDL condition*, if there exists a null set $S\subset[a,b]$ such that for every $\varepsilon>0$, there exists a gauge δ on $[a,b]$, independent of n, such that for every δ-fine partial division

$$D=\{(\xi,[u,v])\}$$

of $[a,b]$,

$$\sum_{(\xi,[u,v])\in\phi_S(D)\cap\Gamma_{\varepsilon,n}} |f_n(v)-f_n(u)|(v-u)<\varepsilon$$

for every n, where

$$\Gamma_{\varepsilon,n}=\{(\xi,[u,v]):|f_n(v)-f_n(u)|\ge\varepsilon\}.$$

Lemma 3.3. *Let $\{f_n\}$ be a sequence of functions on $[a,b]$ satisfying the N-UDL condition. If $f_n\to f$ pointwise on $[a,b]$, then there exists a null set $S\subset[a,b]$ such that for every $\varepsilon>0$, there exists a gauge δ on $[a,b]$ such that for every δ-fine partial division*

$$D=\{(\xi,[u,v])\}$$

of $[a,b]$,

$$\sum_{(\xi,[u,v])\in\phi_S(D)\cap\Gamma_\varepsilon} |f(v)-f(u)|(v-u)<\varepsilon.$$

Proof. Since $\{f_n\}$ satisfies the N-UDL condition, there exists a null set $S\subset[a,b]$ such that for every $\varepsilon>0$, there exists a gauge δ, independent of n, such that for every δ-fine partial division D of $[a,b]$,

$$\sum_{(\xi,[u,v])\in\phi_S(D)\cap\Gamma_{\varepsilon/2,n}} |f_n(v)-f_n(u)|(v-u)<\frac{\varepsilon}{2}.$$

Let $\varepsilon>0$ be given and choose such a gauge δ. Let $D=\{(\xi,[u,v])\}$ be any δ-fine partial division of $[a,b]$.

Since $f_n \to f$ pointwise and D is finite, there exists a positive integer N such that for every $(\xi, [u, v]) \in D$,

$$|(f(v) - f(u)) - (f_N(v) - f_N(u))| < \min\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2(b-a)}\right\}.$$

It follows that

$$|f(v) - f(u)| - \frac{\varepsilon}{2} < |f_N(v) - f_N(u)|.$$

Hence,

$$D \cap \Gamma_\varepsilon \subset D \cap \Gamma_{\varepsilon/2,N}.$$

Therefore,

$$\begin{aligned}
&\sum_{(\xi,[u,v])\in\phi_S(D)\cap\Gamma_\varepsilon} |f(v) - f(u)|(v-u) \\
&\leq \sum_{(\xi,[u,v])\in\phi_S(D)\cap\Gamma_\varepsilon} |(f(v) - f(u)) - (f_N(v) - f_N(u))|(v-u) \\
&\quad + \sum_{(\xi,[u,v])\in\phi_S(D)\cap\Gamma_\varepsilon} |f_N(v) - f_N(u)|(v-u) \\
&< \sum_{(\xi,[u,v])\in\phi_S(D)\cap\Gamma_\varepsilon} \frac{\varepsilon}{2(b-a)}(v-u) + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

This proves the assertion. □

Remark. The proof of the uniform convergence theorem can be revised using the preceding lemma.

The conditions in the preceding lemma are not enough to conclude that the limit function is N-integrable. The following example shows this limitation.

Example 3.4. Let f be a function on $[0, 1]$ defined by

$$f(x) = \begin{cases} x^{-1}, & x \in (0, 1], \\ 0, & x = 0. \end{cases}$$

For each positive integer n, define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} x^{-n/(n+1)}, & x \in (0, 1], \\ 0, & x = 0. \end{cases}$$

Then $f_n(x) \to f(x)$ for every $x \in [0, 1]$. Moreover, for any $u, v \in [0, 1]$ and every n,

$$|f_n(v) - f_n(u)| < |f(v) - f(u)|.$$

Set $S = \{0\}$. Since f is continuous on $(0, 1]$, for every $\varepsilon > 0$, there exists a gauge δ on $[0, 1]$ such that for any δ-fine partial division D,

$$D \cap \Gamma_\varepsilon = \varnothing.$$

Thus, there exists a sequence of N-integrable functions satisfying the N-UDL condition, but the limit function f is not N-integrable.

Theorem 3.5. *Let $\{f_n\}$ be a sequence of N-integrable functions on $[a,b]$, and let f be a Henstock integrable function on $[a,b]$. If*

$$f_n(x) \to f(x)$$

for every $x \in [a,b]$, and if $\{f_n\}$ satisfies the N-UDL condition, then f is N-integrable.

Proof. By the preceding lemma, the limit function f satisfies the gauge-null set condition appearing in the characterization of N-integrability. Since f is Henstock integrable by assumption, the characterization of N-integrability implies that f is N-integrable. □

Remark. In the preceding theorem, the Henstock integrability of f may be replaced by conditions on $\{f_n\}$ that imply the Henstock integrability of f. Thus, the following results are obtained.

Theorem 3.6 (Monotone Type Convergence)**.** *Let $\{f_n\}$ be a sequence of N-integrable functions on $[a,b]$ such that*

$$\lim_{n\to\infty} (N)\int_a^b f_n$$

exists. Suppose that for every $x \in [a,b]$,

$$f_1(x) \le f_2(x) \le \cdots$$

and

$$f_n(x) \to f(x).$$

If $\{f_n\}$ satisfies the N-UDL condition, then f is N-integrable and

$$\lim_{n\to\infty} (N)\int_a^b f_n = (N)\int_a^b f.$$

Theorem 3.7 (Dominated Type Convergence)**.** *Let $\{f_n\}$ be a sequence of N-integrable functions on $[a,b]$, and let f be a function on $[a,b]$. Suppose that*

$$g(x) \le f_n(x) \le h(x)$$

for every $x \in [a,b]$ and every n, where g and h are N-integrable functions on $[a,b]$. Suppose also that

$$f_n(x) \to f(x)$$

for every $x \in [a,b]$. If $\{f_n\}$ satisfies the N-UDL condition, then f is N-integrable and

$$\lim_{n\to\infty} (N)\int_a^b f_n = (N)\int_a^b f.$$

We now justify the need for the N-UDL condition in the preceding convergence theorems.

Example 3.8. Let $\{r_k\}$ be an enumeration of the rational numbers in $[0,1]$. For each positive integer n, define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1, & x \in \{r_1, r_2, \ldots, r_n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \text{ is rational,} \\ 0, & x \text{ is irrational.} \end{cases}$$

Then $f_n(x) \to f(x)$ pointwise on $[0,1]$. For each n, the function f_n is N-integrable. However, the limit function f is not N-integrable. Thus, the sequence satisfies the conditions in the monotone type and dominated type convergence theorems except for the N-UDL condition.

Definition 3.9 (N-Equiintegrability)**.** A sequence $\{f_n\}$ of functions on $[a,b]$ is said to be *N-equiintegrable* if there exists a null set $S \subset [a,b]$ such that for every $\varepsilon > 0$, there exists a gauge δ on $[a,b]$ such that for every δ-fine division

$$D = \{(\xi, [u,v])\}$$

of $[a,b]$, every

$$D^* = \{(\xi^*, [u,v])\} \sim \phi_S(D),$$

and every n, we have

$$\left| \sum_{(\xi^*,[u,v]) \in D^*} f_n(\xi^*)(v-u) - (N)\int_a^b f_n \right| < \varepsilon.$$

Theorem 3.10 (N-Equiintegrable Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of N-integrable functions on $[a,b]$, and let f be a function on $[a,b]$. Suppose that $\{f_n\}$ is N-equiintegrable and*

$$f_n(x) \to f(x)$$

for every $x \in [a,b]$. Then f is N-integrable and

$$\lim_{n\to\infty} (N)\int_a^b f_n = (N)\int_a^b f.$$

Proof. Since $\{f_n\}$ is N-equiintegrable, there exists a null set $S \subset [a,b]$ such that for every $\varepsilon > 0$, there exists a gauge δ on $[a,b]$ such that for every δ-fine division $D = \{(\xi, [u,v])\}$, every $D^* \sim \phi_S(D)$, and every n,

$$\left| \sum_{(\xi^*,[u,v]) \in D^*} f_n(\xi^*)(v-u) - (N)\int_a^b f_n \right| < \varepsilon.$$

We first show that the sequence

$$\left\{ (N)\int_a^b f_n \right\}$$

is Cauchy. Fix a δ-fine division $D_0 = \{(\xi, [u, v])\}$ of $[a, b]$. Since D_0 is finite and $f_n(x) \to f(x)$ for each x, there exists a positive integer N_{D_0} such that for all $n, m \geq N_{D_0}$,

$$\left| \sum_{\phi_S(D_0)} f_n(\xi^*)(v-u) - \sum_{\phi_S(D_0)} f_m(\xi^*)(v-u) \right| < \varepsilon.$$

Therefore, for $n, m \geq N_{D_0}$,

$$\begin{aligned} \left| (N)\int_a^b f_n - (N)\int_a^b f_m \right| &\leq \left| (N)\int_a^b f_n - \sum_{\phi_S(D_0)} f_n(\xi^*)(v-u) \right| \\ &\quad + \left| \sum_{\phi_S(D_0)} f_n(\xi^*)(v-u) - \sum_{\phi_S(D_0)} f_m(\xi^*)(v-u) \right| \\ &\quad + \left| \sum_{\phi_S(D_0)} f_m(\xi^*)(v-u) - (N)\int_a^b f_m \right| \\ &< 3\varepsilon. \end{aligned}$$

Thus,

$$\left\{ (N)\int_a^b f_n \right\}$$

is Cauchy. Since $\mathbb{R}$ is complete, there exists $A \in \mathbb{R}$ such that

$$A = \lim_{n\to\infty} (N)\int_a^b f_n.$$

We now show that $A = (N)\int_a^b f$. Let $\varepsilon > 0$. Choose M such that for all $n \geq M$,

$$\left| (N)\int_a^b f_n - A \right| < \varepsilon.$$

Choose the gauge δ from N-equiintegrability. Let $D = \{(\xi, [u, v])\}$ be any δ-fine division of $[a, b]$ and let $D^* = \{(\xi^*, [u, v])\} \sim \phi_S(D)$. Since D^* is finite and $f_n(\xi^*) \to f(\xi^*)$, there exists $N(D)$ such that for all $n \geq N(D)$,

$$\left| \sum_{D^*} f(\xi^*)(v-u) - \sum_{D^*} f_n(\xi^*)(v-u) \right| < \varepsilon.$$

Take $n \geq \max\{M, N(D)\}$. Then

$$\begin{aligned} \left| \sum_{D^*} f(\xi^*)(v-u) - A \right| &\leq \left| \sum_{D^*} f(\xi^*)(v-u) - \sum_{D^*} f_n(\xi^*)(v-u) \right| \\ &\quad + \left| \sum_{D^*} f_n(\xi^*)(v-u) - (N)\int_a^b f_n \right| \\ &\quad + \left| (N)\int_a^b f_n - A \right| \\ &< 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, f is N-integrable and

$$(N)\int_a^b f = A.$$

Hence,

$$\lim_{n\to\infty}(N)\int_a^b f_n = (N)\int_a^b f.$$

□

Lemma 3.11. *Let $\{f_n\}$ be a sequence of N-equiintegrable functions on $[a,b]$. Then there exists a null set $S \subset [a,b]$ such that for every $\varepsilon > 0$, there exists a gauge δ on $[a,b]$ such that for every δ-fine division $D = \{(\xi,[u,v])\}$, every $D^* \sim \phi_S(D)$, and every n,*

$$\sum_{D^*}\left|f_n(\xi^*)(v-u) - (N)\int_u^v f_n\right| < \varepsilon.$$

Theorem 3.12. *Let $\{f_n\}$ be a sequence of N-equiintegrable functions on $[a,b]$. Then $\{f_n\}$ satisfies the N-UDL condition.*

Proof. Let $\{f_n\}$ be a sequence of N-equiintegrable functions on $[a,b]$. Choose a corresponding null set S, and let $\varepsilon > 0$. By the preceding lemma, there exists a gauge δ on $[a,b]$ such that for every δ-fine division $D = \{(\xi,[u,v])\}$, every $D^* \sim \phi_S(D)$, and every n,

$$\sum_{D^*}\left|f_n(\xi^*)(v-u) - (N)\int_u^v f_n\right| < \frac{\varepsilon}{2}.$$

For a partial division D, define

$$D_u = \{(u,[u,v]) : (\xi,[u,v]) \in \phi_S(D)\}$$

and

$$D_v = \{(v,[u,v]) : (\xi,[u,v]) \in \phi_S(D)\}.$$

Then $D_u \sim \phi_S(D)$ and $D_v \sim \phi_S(D)$. Hence, for every δ-fine partial division D and every n,

$$\begin{aligned}
&\sum_{(\xi,[u,v])\in\phi_S(D)\cap\Gamma_{\varepsilon,n}} |f_n(v) - f_n(u)|(v-u)\\
&\le \sum_{\phi_S(D)\cap\Gamma_{\varepsilon,n}}\left|f_n(u)(v-u) - (N)\int_u^v f_n\right|\\
&\quad + \sum_{\phi_S(D)\cap\Gamma_{\varepsilon,n}}\left|(N)\int_u^v f_n - f_n(v)(v-u)\right|\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, $\{f_n\}$ satisfies the N-UDL condition. □

4. Topological Extension of the N-Integral

In this section, we extend the convergence framework of the N-integral to a topological measure space. Throughout this section, let $(X, \mathcal{T}, \mu)$ be a topological measure space satisfying the following conditions:

(1) X is a nonempty set;
(2) $\mathcal{T}$ is a topology on X;
(3) μ is a positive finite measure defined on a σ-algebra containing $\mathcal{T}$;
(4) every gauge on X admits a finite δ-fine tagged division of X.

Definition 4.1 (Topological Gauge)**.** A *topological gauge* on X is a function

$$\delta : X \to \mathcal{T}$$

such that

$$x \in \delta(x)$$

for every $x \in X$.

Definition 4.2 (δ-fine Tagged Division)**.** A finite tagged family

$$\mathcal{D} = \{(x_i, U_i)\}_{i=1}^{m}$$

is called a δ-fine tagged division of X if the following conditions hold:

(1) $x_i \in U_i \subset \delta(x_i)$;
(2) the sets U_i are pairwise disjoint and measurable;
(3) $\bigcup_{i=1}^{m} U_i = X$;
(4) $\mu(U_i) < \infty$ for each $i = 1, 2, \ldots, m$.

Definition 4.3. Let $S \subset X$ be a null set and let

$$\mathcal{D} = \{(x_i, U_i)\}_{i=1}^{m}$$

be a tagged division of X. Define

$$\phi_S(\mathcal{D}) = \{(x_i, U_i) \in \mathcal{D} : x_i \notin S\}.$$

Definition 4.4 (Compatible Tagged Division)**.** Let $\mathcal{D}$ be a tagged division of X. A tagged family

$$\mathcal{D}^* = \{(x_i^*, U_i)\}$$

is said to be compatible with $\mathcal{D}$, written

$$\mathcal{D}^* \sim \mathcal{D},$$

if $x_i^* \in U_i$ for every i and $\mathcal{D}^*$ has the same measurable sets as $\mathcal{D}$.

Definition 4.5 (Topological N-integral)**.** A function $f : X \to \mathbb{R}$ is said to be *topologically N-integrable* on X if there exists a null set $S \subset X$ and a real number A such that for every $\varepsilon > 0$, there exists a gauge δ on X such that for every δ-fine tagged division $\mathcal{D}$ of X and every

$$\mathcal{D}^* \sim \phi_S(\mathcal{D}),$$

we have

$$\left| \sum_{(x_i^*, U_i) \in \mathcal{D}^*} f(x_i^*)\mu(U_i) - A \right| < \varepsilon.$$

In this case, we write

$$(N) \int_X f = A.$$

Theorem 4.6 (Topological Uniform Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of topologically N-integrable functions on X. Suppose that $f_n \to f$ uniformly on X. Then f is topologically N-integrable on X, and*

$$\lim_{n\to\infty} (N) \int_X f_n = (N) \int_X f.$$

Proof. For each n, let

$$A_n = (N) \int_X f_n.$$

We first show that $\{A_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly on X, the sequence $\{f_n\}$ is uniformly Cauchy. Thus, there exists a positive integer N_1 such that for all $n, m \geq N_1$ and all $x \in X$,

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{3\mu(X)}.$$

Since f_n and f_m are topologically N-integrable, there exist gauges corresponding to f_n and f_m such that the corresponding N-integral sums approximate A_n and A_m, respectively, within $\varepsilon/3$. Let δ be a common refinement of these gauges. Then for every δ-fine tagged division

$$\mathcal{D} = \{(x_i, U_i)\}_{i=1}^r$$

and compatible tagged division $\mathcal{D}^*$, we have

$$\begin{aligned} |A_n - A_m| \leq & \left| A_n - \sum f_n(x_i^*)\mu(U_i) \right| \\ & + \left| \sum \big(f_n(x_i^*) - f_m(x_i^*)\big)\mu(U_i) \right| \\ & + \left| \sum f_m(x_i^*)\mu(U_i) - A_m \right|. \end{aligned}$$

Hence,

$$|A_n - A_m| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3\mu(X)} \sum_i \mu(U_i) + \frac{\varepsilon}{3}.$$

Since $\mathcal{D}$ is a tagged division of X,

$$\sum_i \mu(U_i) = \mu(X).$$

Therefore,

$$|A_n - A_m| < \varepsilon.$$

Thus, $\{A_n\}$ is Cauchy. Since $\mathbb{R}$ is complete, there exists $A \in \mathbb{R}$ such that

$$A_n \to A.$$

We now show that f is topologically N-integrable with integral A. Let $\varepsilon > 0$. Choose N such that

$$|A_N - A| < \frac{\varepsilon}{3}$$

and

$$|f(x) - f_N(x)| < \frac{\varepsilon}{3\mu(X)}$$

for all $x \in X$.

Since f_N is topologically N-integrable, there exists a gauge δ on X such that for every δ-fine tagged division $\mathcal{D}$ and every compatible tagged division $\mathcal{D}^*$,

$$\left|\sum f_N(x_i^*)\mu(U_i) - A_N\right| < \frac{\varepsilon}{3}.$$

Then

$$\begin{aligned}\left|\sum f(x_i^*)\mu(U_i) - A\right| &\le \left|\sum (f(x_i^*) - f_N(x_i^*))\mu(U_i)\right| \\ &\quad + \left|\sum f_N(x_i^*)\mu(U_i) - A_N\right| \\ &\quad + |A_N - A|.\end{aligned}$$

Therefore,

$$\left|\sum f(x_i^*)\mu(U_i) - A\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence, f is topologically N-integrable and

$$(N)\int_X f = A.$$

Consequently,

$$\lim_{n\to\infty} (N)\int_X f_n = (N)\int_X f.$$

□

Definition 4.7 (Topological N-equiintegrability)**.** A sequence $\{f_n\}$ of topologically N-integrable functions on X is said to be *topologically N-equiintegrable* if there exists a null set $S \subset X$ such that for every $\varepsilon > 0$, there exists a gauge δ on X, independent of n, such that for every δ-fine tagged division $\mathcal{D}$ and every

$$\mathcal{D}^* \sim \phi_S(\mathcal{D}),$$

we have

$$\left|\sum_{(x_i^*,U_i)\in\mathcal{D}^*} f_n(x_i^*)\mu(U_i) - (N)\int_X f_n\right| < \varepsilon$$

for all n.

Theorem 4.8 (Topological Equiintegrable Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of topologically N-integrable functions on X. Suppose that*

(1) *$\{f_n\}$ is topologically N-equiintegrable;*

(2) *$f_n(x) \to f(x)$ for all $x \in X$.*

Then f is topologically N-integrable and

$$\lim_{n\to\infty} (N)\int_X f_n = (N)\int_X f.$$

Proof. Since $\{f_n\}$ is topologically N-equiintegrable, there exists a null set $S \subset X$ such that for every $\varepsilon > 0$, there exists a gauge δ satisfying

$$\left|\sum f_n(x_i^*)\mu(U_i) - (N)\int_X f_n\right| < \varepsilon$$

for every n, every δ-fine tagged division $\mathcal{D}$, and every compatible tagged division $\mathcal{D}^* \sim \phi_S(\mathcal{D})$.

We first show that

$$\left\{(N)\int_X f_n\right\}$$

is a Cauchy sequence. Fix a δ-fine tagged division $\mathcal{D}$. Since $\mathcal{D}^*$ is finite and $f_n(x) \to f(x)$ for every $x \in X$, there exists a positive integer N_0 such that for all $n, m \geq N_0$,

$$\left|\sum f_n(x_i^*)\mu(U_i) - \sum f_m(x_i^*)\mu(U_i)\right| < \varepsilon.$$

Thus, for $n, m \geq N_0$,

$$\begin{aligned}\left|(N)\int_X f_n - (N)\int_X f_m\right| \leq{}& \left|(N)\int_X f_n - \sum f_n(x_i^*)\mu(U_i)\right| \\ &+ \left|\sum f_n(x_i^*)\mu(U_i) - \sum f_m(x_i^*)\mu(U_i)\right| \\ &+ \left|\sum f_m(x_i^*)\mu(U_i) - (N)\int_X f_m\right|.\end{aligned}$$

Therefore,

$$\left|(N)\int_X f_n - (N)\int_X f_m\right| < 3\varepsilon.$$

Hence, $\{(N)\int_X f_n\}$ is Cauchy. Let

$$A = \lim_{n\to\infty} (N)\int_X f_n.$$

We now show that $A = (N)\int_X f$. Let $\varepsilon > 0$. Choose a positive integer M such that for all $n \geq M$,

$$\left|(N)\int_X f_n - A\right| < \varepsilon.$$

Choose the gauge δ from topological N-equiintegrability. Let $\mathcal{D}$ be any δ-fine tagged division and let $\mathcal{D}^* \sim \phi_S(\mathcal{D})$. Since $\mathcal{D}^*$ is finite and $f_n(x_i^*) \to f(x_i^*)$, there exists $N(\mathcal{D}^*)$ such that for all $n \geq N(\mathcal{D}^*)$,

$$\left|\sum f(x_i^*)\mu(U_i) - \sum f_n(x_i^*)\mu(U_i)\right| < \varepsilon.$$

Take

$$n \geq \max\{M, N(\mathcal{D}^*)\}.$$

Then

$$\begin{aligned}\left|\sum f(x_i^*)\mu(U_i) - A\right| \leq &\left|\sum f(x_i^*)\mu(U_i) - \sum f_n(x_i^*)\mu(U_i)\right| \\ &+ \left|\sum f_n(x_i^*)\mu(U_i) - (N)\int_X f_n\right| \\ &+ \left|(N)\int_X f_n - A\right|.\end{aligned}$$

Hence,

$$\left|\sum f(x_i^*)\mu(U_i) - A\right| < 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, f is topologically N-integrable and

$$(N)\int_X f = A.$$

Therefore,

$$\lim_{n\to\infty}(N)\int_X f_n = (N)\int_X f.$$

□

Theorem 4.9 (Topological Monotone Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of topologically N-integrable functions on X. Suppose that*

(1) $f_1(x) \leq f_2(x) \leq \cdots$ *for all* $x \in X$*;*

(2) $f_n(x) \to f(x)$ *for all* $x \in X$*;*

(3) $\{f_n\}$ *is topologically N-equiintegrable.*

Then f is topologically N-integrable and

$$\lim_{n\to\infty}(N)\int_X f_n = (N)\int_X f.$$

Proof. Since $\{f_n\}$ is topologically N-equiintegrable and $f_n(x) \to f(x)$ for all $x \in X$, the result follows directly from the Topological Equiintegrable Convergence Theorem. Hence, f is topologically N-integrable and

$$\lim_{n\to\infty}(N)\int_X f_n = (N)\int_X f.$$

□

Theorem 4.10 (Topological Dominated Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of topologically N-integrable functions on X. Suppose that*

(1) $f_n(x) \to f(x)$ *for all* $x \in X$*;*

(2) *there exist topologically N-integrable functions* $g, h : X \to \mathbb{R}$ *such that*

$$g(x) \leq f_n(x) \leq h(x)$$

for all $x \in X$ *and all* n*;*

(3) $\{f_n\}$ *is topologically N-equiintegrable.*

Then f *is topologically N-integrable and*

$$\lim_{n\to\infty} (N) \int_X f_n = (N) \int_X f.$$

Proof. Since $\{f_n\}$ is topologically N-equiintegrable and $f_n(x) \to f(x)$ for all $x \in X$, the result follows from the Topological Equiintegrable Convergence Theorem. Therefore, f is topologically N-integrable and

$$\lim_{n\to\infty} (N) \int_X f_n = (N) \int_X f.$$

□

5. Conclusion

In this paper, convergence theorems for the N-integral were established. The results show that the class of N-integrable functions is stable under uniform convergence. More precisely, if a sequence of N-integrable functions converges uniformly to a function f, then f is N-integrable.

A uniform condition for sequences of functions, called the N-uniform double Lusin condition, was introduced. This condition was used to obtain a pointwise convergence theorem for the N-integral. It was shown that if $\{f_n\}$ is a sequence of N-integrable functions satisfying the N-UDL condition, $f_n \to f$ pointwise on $[a, b]$, and f is Henstock integrable, then f is N-integrable.

Monotone type and dominated type convergence theorems for the N-integral were also obtained. These results provide conditions under which the limit of a sequence of N-integrable functions is again N-integrable and the limit may be interchanged with the N-integral. The examples and counterexamples presented in the paper show that the N-UDL condition is necessary in these convergence results.

The concept of N-equiintegrability was also considered. It was shown that N-equiintegrability implies the N-UDL condition. Furthermore, if a sequence of N-integrable functions is N-equiintegrable and converges pointwise to f, then f is N-integrable and

$$\lim_{n\to\infty} (N) \int_a^b f_n = (N) \int_a^b f.$$

Finally, the convergence framework of the N-integral was extended to topological measure spaces. By introducing topological gauges, δ-fine tagged divisions, topological N-integrability, and topological N-equiintegrability, topological versions of the uniform, equiintegrable, monotone, and dominated convergence theorems were obtained.

These results contribute to the further development of the N-integral by providing convergence theorems analogous to classical convergence principles in integration theory. They also suggest possible directions for future research, particularly in the study of N-integrability on more general topological and measure-theoretic structures.

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