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ON SOLVING NEUTROSOPHIC CONSTRAINED OPTIMIZATION PROBLEMS BY AUGMENTED LAGRANGIAN METHOD

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Abstract. In this article, we investigate the solution of constrained optimization problems using the Augmented Lagrangian method with single-valued neutrosophic triangular number coefficients. Our approach introduces a new neutrosophic arithmetic operations applied to the parametric representations of neutrosophic numbers, along with the neutrosophic ranking of the parametric forms of Triangular neutrosophic numbers. The primary objective of this study is to solve constrained Single-Valued Neutrosophic Nonlinear Programming Problems using the Augmented Lagrangian method, actively managing uncertainty and imprecision in optimization. We present and prove an important theorem for the Augmented Lagrangian condition under neutrosophic environments, contributing to the theoretical foundation of this method. Furthermore, a detailed numerical example illustrates the practical application of the proposed approach. The results are compared with those of existing methods, demonstrating the activeness and advantages of the neutrosophic-based solution.

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1. Introduction

Nonlinear Programming Problems (NLPPs) are widely used to model optimization problems in engineering, economics, management, and decision sciences where the objective function or constraints are nonlinear in nature. Traditional nonlinear programming assumes that all parameters are precisely known; however, real-world problems often involve uncertainty, vagueness, and indeterminacy. To address these limitations, neutrosophic set theory, introduced by Florentin Smarandache, provides a powerful framework by representing information through truth, indeterminacy, and falsity membership functions. A Neutrosophic Nonlinear Programming Problem (NNLPP) incorporates neutrosophic

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numbers into the objective function and constraints, enabling a more realistic representation of uncertain and inconsistent data.

The integration of neutrosophic theory with nonlinear optimization has gained significant attention because it offers effective tools for handling complex decision-making situations under uncertainty. Consequently, NNLPPs have become an important research area, with applications in resource allocation, production planning, engineering design, and multi-criteria decision-making problems where imprecise information cannot be ignored. A new fuzzy arithmetic designed for fuzzy calculus and to solve fuzzy linear equations was developed by Ming Ma et al. [2]. Chakraborty et al. [1] extended the concept of neutrosophic numbers, suggesting that nonlinear neutrosophic numbers are more suitable for decision-makers than their linear counterparts. Sudha and Hepzibah [14] explored the substitution of fuzzy coefficients in the objective functions and constraints with single-valued triangular neutrosophic numbers. They reformulated a quadratic programming problem with triangular neutrosophic number coefficients into a Fuzzy Neutrosophic Linear Complementarity Problem (FNLCP) and developed an algorithm to solve the resulting model effectively. Purnima Raj and Ranjana [11] presented a method for solving FNLPPs with inequality constraints often encountered in practical cases and involves transforming the FNLPP into sharp equality constraints by applying the Karush–Kuhn–Tucker conditions.

Reig-Mullor and Salas-Molina [12] explored an alternative definition of nonlinear neutrosophic numbers (NLNN) to address certain limitations highlighted in existing research. They also examined the fundamental properties of NLNN, including α, β, γ-cuts, variance, standard deviation and possibilistic mean. Irene and Sudha [3] examined a new method to address Neutrosophic Fuzzy Quadratic Programming Problems (FNQPP) by employing a Taylor Series expansion. A comparison of the convergence performance of widely used solvers for both constrained and unconstrained cases was conducted by Lavezzi et al. [6] provided recommendations for choosing the solvers for nonlinear programming problems.

Khalifa et al. [4] addressed a challenging nonlinear programming problem in which the coefficients of the objective function are expressed as neutrosophic numbers, and the constraints are governed by fuzzy inequalities. Lachhwani [5] conducted extensive research on nonlinear neutrosophic numbers (NLNNs) and nonlinear neutrosophic linear programming problems (NLN-LPPs). Maissam Jdid [7,8] illustrated how binary integers can be utilized to transform specific nonlinear models into linear forms, ensuring integer solutions that align with the inherent characteristics of the problems under consideration. Also, their work explored significant methods for solving nonlinear models focusing on the Lagrangian multiplier approach for models with equality constraints, which were subsequently reformulated using neutrosophic science principles.

A fuzzy mathematical model incorporating Beale's condition to address such nonlinear programming problems (NLPPs) was discussed by Palanivel and Muralikrishna [9]. This model illustrated how quadratic programming problems could be solved using membership functions (MFs). Vanaja and Ganesan [15,16] proposed an innovative interior fuzzy penalty function approach for addressing Fuzzy Nonlinear Programming Problems (FNLPPs). Their methodology incorporated advanced fuzzy arithmetic and ranking techniques based on the parametric representation of triangular fuzzy numbers. Additionally, they developed an exterior penalty method utilizing fuzzy-valued functions, offering an effective framework for solving these fuzzy nonlinear programming problems. Our approach introduces new neutrosophic arithmetic operations applied to the parametric representations of neutrosophic numbers, along with the neutrosophic ranking of the parametric forms of Triangular Neutrosophic Numbers. The primary objective of this study is to develop a robust framework for solving constrained Single-Valued Neutrosophic Nonlinear Programming Problems using the Karush–Kuhn–Tucker condition, effectively managing uncertainty and imprecision in optimization.

2. Preliminaries

We explored Neutrosophic definitions, which capture the core of Neutrosophy by presenting subtle viewpoints on concepts frequently experienced in situations of uncertainty. By employing these definitions, our objective is to shed light on the complex nature of ambiguity and offer valuable perspectives on its impact on both theoretical frameworks and practical applications.

Definition 2.1. [13] *A single valued neutrosophic set (SVNS) $\tilde{M}^{\mathcal{N}}$ over X is characterized by a truth-membership function $T_{\tilde{M}^{\mathcal{N}}}$, indeterminacy-membership function $I_{\tilde{M}^{\mathcal{N}}}$ and falsity-membership function $F_{\tilde{M}^{\mathcal{N}}}$, where $T_{\tilde{M}^{\mathcal{N}}}; I_{\tilde{M}^{\mathcal{N}}}; F_{\tilde{M}^{\mathcal{N}}} : X \to [0,1]$. The single valued neutrosophic set (SVNS) is defined as $\tilde{M}^{\mathcal{N}} = \{< x, (T_{\tilde{M}^{\mathcal{N}}}(x), I_{\tilde{M}^{\mathcal{N}}}(x), F_{\tilde{M}^{\mathcal{N}}}(x) > / x \in X\}$, where $0 \leq T_{\tilde{M}^{\mathcal{N}}}(x) + I_{\tilde{M}^{\mathcal{N}}}(x) + F_{\tilde{M}^{\mathcal{N}}}(x) \leq 3$ for all $x \in X$.*

Definition 2.2. *The (α, β, γ)-cut of neutrosophic set is denoted by $F(\alpha, \beta, \gamma)$, where $(\alpha, \beta, \gamma) \in [0,1]$ and are fixed numbers, such that $\alpha + \beta + \gamma \leq 3$ and is defined as $F(\alpha, \beta, \gamma) = (\mu_T(x), \mu_I(x), \mu_F(x))$, where $x \in X, \mu_T(x) \geq \alpha, \mu_I(x) \leq \beta, \mu_F(x) \leq \gamma$.*

Definition 2.3. [13] *A single valued neutrosophic set (SVNS)*
$\tilde{M}^{\mathcal{N}} = \{< x, (T_{\tilde{M}^{\mathcal{N}}}(x), I_{\tilde{M}^{\mathcal{N}}}(x), F_{\tilde{M}^{\mathcal{N}}}(x) > / x \in X\}$ is called neutrosophic normal, i.e. there exists at least three points $x_0, x_1, x_2 \in X$ such that $T_{\tilde{M}^{\mathcal{N}}}(x_0) = 1; I_{\tilde{M}^{\mathcal{N}}}(x_1) = 1; F_{\tilde{M}^{\mathcal{N}}}(x_2) = 1$.

Definition 2.4. [13] *A single valued neutrosophic set (SVNS)*
$\tilde{M}^{\mathcal{N}} = \{< x, (T_{\tilde{M}^{\mathcal{N}}}(x), I_{\tilde{M}^{\mathcal{N}}}(x), F_{\tilde{M}^{\mathcal{N}}}(x) > / x \in X\}$ is called neutrosophic convex, if for all $x_1, x_2 \in X$ and $\lambda \in [0,1]$ the following conditions are satisfied.

(1) $T_{\tilde{M}^{\mathcal{N}}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{T_{\tilde{M}^{\mathcal{N}}}(x_1), T_{\tilde{M}^{\mathcal{N}}}(x_2)\}$

(2) $I_{\tilde{M}^{\mathcal{N}}}(\lambda x_1 + (1-\lambda)x_2) \leq \max\{I_{\tilde{M}^{\mathcal{N}}}(x_1), I_{\tilde{M}^{\mathcal{N}}}(x_2)\}$

(3) $F_{\tilde{M}^{\mathcal{N}}}(\lambda x_1 + (1-\lambda)x_2) \leq \max\{F_{\tilde{M}^{\mathcal{N}}}(x_1), F_{\tilde{M}^{\mathcal{N}}}(x_2)\}$

i.e.,$\tilde{M}^{\mathcal{N}}$ is neutrosophic convex, if its truth membership function is fuzzy convex, indeterminacy membership function and falsity membership function is fuzzy concave.

Definition 2.5. [13] *A Single Valued Neutrosophic set (SVNN)*
$\tilde{M}^{\mathcal{N}} = \{< x, (T_{\tilde{M}^{\mathcal{N}}}(x), I_{\tilde{M}^{\mathcal{N}}}(x), F_{\tilde{M}^{\mathcal{N}}}(x)) > | x \in X\}$, *subset of a real line, is called a neutrosophic number if*

(1) $\tilde{M}^{\mathcal{N}}$ *is neutrosophic normal, i.e. there exists atleast three points* $x_0, x_1, x_2 \in X$ *such that* $T_{\tilde{M}^{\mathcal{N}}}(x_0) = 1; I_{\tilde{M}^{\mathcal{N}}}(x_1) = 1; F_{\tilde{M}^{\mathcal{N}}}(x_2) = 1.$

(2) $\tilde{M}^{\mathcal{N}}$ *is neutrosophic convex.*

(3) $T_{\tilde{M}^{\mathcal{N}}}(x)$ *is upper semi-continuous,* $I_{\tilde{M}^{\mathcal{N}}}(x)$ *is lower semi continuous and* $F_{\tilde{M}^{\mathcal{N}}}(x)$ *is lower semi continuous.*

(4) $\tilde{M}^{\mathcal{N}}$ *is support, i.e.* $S(\tilde{M}^{\mathcal{N}}) = x \in x : T_{\tilde{M}^{\mathcal{N}}} > 0, I_{\tilde{M}^{\mathcal{N}}} < 1, F_{\tilde{M}^{\mathcal{N}}} < 1$ *is bounded.*

Definition 2.6. [13] *A single valued neutrosophic number* $\tilde{M}^{\mathcal{N}}$ *is a Triangular Neutrosophic Number (TNN) and is denoted by*
$\tilde{M}^{\mathcal{N}} = (T_{\tilde{M}^{\mathcal{N}}}; I_{\tilde{M}^{\mathcal{N}}}; F_{\tilde{M}^{\mathcal{N}}}) = (\langle m_{T_1}, m_{T_2}, m_{T_3}\rangle; \langle m_{I_1}, m_{I_2}, m_{I_3}\rangle; \langle m_{F_1}, m_{F_2}, m_{F_3}\rangle)$ *having the membership function, indeterminacy function and non-membership function as follows.M*

$$\mu_T(x) = \begin{cases} \dfrac{x - m_{T_1}}{m_{T_2} - m_{T_1}}, & m_{T_1} \leq x \leq m_{T_2} \\ \dfrac{m_{T_3} - x}{m_{T_3} - m_{T_2}}, & m_{T_2} \leq x \leq m_{T_3} \\ 0, & \textit{elsewhere} \end{cases}$$

$$\mu_I(x) = \begin{cases} \dfrac{x - m_{I_1}}{m_{I_2} - m_{I_1}}, & m_{I_1} \leq x \leq m_{I_2} \\ \dfrac{m_{I_3} - x}{m_{I_3} - m_{I_2}}, & m_{I_2} \leq x \leq m_{I_3} \\ 0, & \textit{elsewhere} \end{cases}$$

$$\mu_F(x) = \begin{cases} \dfrac{x - m_{F_1}}{m_{F_2} - m_{F_1}}, & m_{F_1} \leq x \leq m_{F_2} \\ \dfrac{m_{F_3} - x}{m_{F_3} - m_{F_2}}, & m_{F_2} \leq x \leq m_{F_3} \\ 0, & \textit{elsewhere} \end{cases}$$

We use $F(R)$ to represent the set of all triangular neutrosophic numbers defined on R.

Definition 2.7. [13] *A triangular neutrosophic number* $\tilde{M}^{\mathcal{N}}$ *can also be represented as a pair* $\tilde{M}_T^{\mathcal{N}} = (\underline{m}_T; \overline{m}_T), \tilde{M}_I^{\mathcal{N}} = (\underline{m}_I; \overline{m}_I), \tilde{M}_F^{\mathcal{N}} = (\underline{m}_F; \overline{m}_F)$ *of functions*
$\underline{m}_T(\beta), \overline{m}_T(\beta), \underline{m}_I(\beta), \overline{m}_I(\beta), \underline{m}_F(\beta), \overline{m}_F(\beta), 0 \leq r \leq 1$ *which satisfy the following requirements:*

- $\underline{m}_T(\beta)$ *is a bounded monotonic increasing left continuous function for membership function.*

- $\overline{m}_T(\beta)$ *is a bounded monotonic decreasing left continuous function for membership function.*
- $\underline{m}_I(\beta)$ *is a bounded monotonic increasing left continuous function for indeterminacy function.*
- $\overline{m}_I(\beta)$ *is a bounded monotonic decreasing left continuous function for indeterminacy function.*
- $\underline{m}_F(\beta)$ *is a bounded monotonic increasing left continuous function for non-membership function.*
- $\overline{m}_F(\beta)$ *is a bounded monotonic decreasing left continuous function for non-membership function.*
- $\underline{m}_T(\beta) \leq \overline{m}_T(\beta), \underline{m}_I(\beta) \leq \overline{m}_I(\beta), \underline{m}_F(\beta) \leq \overline{m}_F(\beta), 0 \leq \beta \leq 1.$

Definition 2.8. (**Parametric Form**) [13]
Let $\tilde{M}^{\mathcal{N}} = (m_T, m_I, m_F)$ *be a triangular neutrosophic number and*

$$\overline{m}_T(\beta) = m_{T_3} - (m_{T_3} - m_{T_2})\beta,\ \underline{m}_T(\beta) = m_{T_1} + (m_{T_2} - m_{T_1})\beta$$

$$\overline{m}_I(\beta) = m_{I_3} - (m_{I_3} - m_{I_2})\beta,\ \underline{m}_I(\beta) = m_{I_1} + (m_{I_2} - m_{I_1})\beta$$

$$\overline{m}_F(\beta) = m_{F_3} - (m_{F_3} - m_{F_2})\beta,\ \underline{m}_F(\beta) = m_{F_1} + (m_{F_2} - m_{F_1})\beta,\ \beta \in [0,1]$$

The parametric form of the TNN is defined as

$$\tilde{M}^{\mathcal{N}} = (T_{\tilde{M}^{\mathcal{N}}}; I_{\tilde{M}^{\mathcal{N}}}; F_{\tilde{M}^{\mathcal{N}}}) = (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle)$$

where $m_{T_*} = m_{T_0} - \underline{m}_T$ *and* $m_{T^*} = \overline{m}_T - m_{T_0}$, $m_{I_*} = m_{I_0} - \underline{m}_I$ *and* $m_{I^*} = \overline{m}_I - m_{I_0}$, $m_{F_*} = m_{F_0} - \underline{m}_F$ *and* $m_{F^*} = \overline{m}_F - m_{F_0}$ *are the left and right fuzziness index functions respectively.*
The number $m_{T_0} = \left(\frac{\underline{m}_T(1) + \overline{m}_T(1)}{2}\right), m_{I_0} = \left(\frac{\underline{m}_I(1) + \overline{m}_I(1)}{2}\right)$,
$m_{F_0} = \left(\frac{\underline{m}_F(1) + \overline{m}_F(1)}{2}\right)$ *is called the location index number. When* $\beta = 1$, *we get* $m_{T_0} = m_{T_2}, m_{I_0} = m_{I_2}, m_{F_0} = m_{F_2}$.

2.1. **Arithmetic Operations on Neutrosophic Numbers.** We develop a new fuzzy arithmetic using the parametric form of triangular neutrosophic numbers. This involves expressing the triangular neutrosophic numbers in terms of location index and fuzziness index functions for membership, indeterminacy, and non-membership functions. The lattice rule, which involves the least upper bound and greatest lower bound in the lattice L, defines the fuzziness index functions and the location index number corresponds to standard arithmetic. That is for $m, n \in L$, $m \vee n = \max\{m, n\}$ and $m \wedge n = \min\{m, n\}$.

For any two neutrosophic numbers

$$\tilde{M}^{\mathcal{N}} = (T_{\tilde{M}^{\mathcal{N}}}; I_{\tilde{M}^{\mathcal{N}}}; F_{\tilde{M}^{\mathcal{N}}}) = (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle)$$

$$\tilde{N}^{\mathcal{N}} = (T_{\tilde{N}^{\mathcal{N}}}; I_{\tilde{N}^{\mathcal{N}}}; F_{\tilde{N}^{\mathcal{N}}}) = (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle)$$

The arithmetic operations are defined as

$$\begin{aligned}\tilde{M}^{\mathcal{N}} * \tilde{N}^{\mathcal{N}} =&(\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle)\\ &* (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle)\\ =&(\langle m_{T_0} * n_{T_0}, m_{T_*} \vee n_{T_*}, m_{T^*} \vee n_{T^*}\rangle, \langle m_{I_0} * n_{I_0}, m_{I_*} \vee n_{I_*}, m_{I^*} \vee n_{I^*}\rangle,\\ &\langle m_{F_0} * n_{F_0}, m_{F_*} \vee n_{F_*}, m_{F^*} \vee n_{F^*}\rangle)\end{aligned}$$

In particular for any two $\tilde{M}^{\mathcal{N}}, \tilde{N}^{\mathcal{N}} \in F(R)$, we have

Addition:

$$\begin{aligned}\tilde{M}^{\mathcal{N}} + \tilde{N}^{\mathcal{N}} =&(\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle)\\ &+ (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle)\\ =&(\langle m_{T_0} + n_{T_0}, m_{T_*} \vee n_{T_*}, m_{T^*} \vee n_{T^*}\rangle, \langle m_{I_0} + n_{I_0}, m_{I_*} \vee n_{I_*}, m_{I^*} \vee n_{I^*}\rangle,\\ &\langle m_{F_0} + n_{F_0}, m_{F_*} \vee n_{F_*}, m_{F^*} \vee n_{F^*}\rangle)\end{aligned}$$

Subtraction:

$$\begin{aligned}\tilde{M}^{\mathcal{N}} - \tilde{N}^{\mathcal{N}} =&(\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle)\\ &- (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle)\\ =&(\langle m_{T_0} - n_{T_0}, m_{T_*} \vee n_{T_*}, m_{T^*} \vee n_{T^*}\rangle, \langle m_{I_0} - n_{I_0}, m_{I_*} \vee n_{I_*}, m_{I^*} \vee n_{I^*}\rangle,\\ &\langle m_{F_0} - n_{F_0}, m_{F_*} \vee n_{F_*}, m_{F^*} \vee n_{F^*}\rangle)\end{aligned}$$

Multiplication:

$$\begin{aligned}\tilde{M}^{\mathcal{N}} * \tilde{N}^{\mathcal{N}} =&(\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle)\\ &\times (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle)\\ =&(\langle m_{T_0} \times n_{T_0}, m_{T_*} \vee n_{T_*}, m_{T^*} \vee n_{T^*}\rangle, \langle m_{I_0} \times n_{I_0}, m_{I_*} \vee n_{I_*}, m_{I^*} \vee n_{I^*}\rangle,\\ &\langle m_{F_0} \times n_{F_0}, m_{F_*} \vee n_{F_*}, m_{F^*} \vee n_{F^*}\rangle)\end{aligned}$$

Division:

$$\begin{aligned}\tilde{M}^{\mathcal{N}} \div \tilde{N}^{\mathcal{N}} =&(\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle)\\ &\div (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle)\\ =&(\langle m_{T_0} \div n_{T_0}, m_{T_*} \vee n_{T_*}, m_{T^*} \vee n_{T^*}\rangle, \langle m_{I_0} \div n_{I_0}, m_{I_*} \vee n_{I_*}, m_{I^*} \vee n_{I^*}\rangle,\\ &\langle m_{F_0} \div n_{F_0}, m_{F_*} \vee n_{F_*}, m_{F^*} \vee n_{F^*}\rangle),\end{aligned}$$

provided $n_{T_0}, n_{I_0}, n_{F_0} \neq 0$.

2.2. **Ranking of Neutrosophic Numbers.** [16] The ranking of neutrosophic numbers plays an essential role in the decision-making process within a fuzzy environment. Various authors in the literature have proposed different types of ranking methods. This article uses a highly effective ranking technique based on the graded mean.

For $\tilde{M}^{\mathcal{N}} = (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle) \in F(R)$, define $\mathcal{R} : F(R) \to R$ by

$$\mathcal{R}(T_{\tilde{M}^{\mathcal{N}}}) = \left(\frac{\langle m_{T_*} + 4m_{T_0} + m_{T^*}\rangle}{6}\right).$$

$$\mathcal{R}(I_{\tilde{M}^{\mathcal{N}}}) = \left(\frac{\langle m_{I_*} + 4m_{I_0} + m_{I^*}\rangle}{6}\right).$$

$$\mathcal{R}(F_{\tilde{M}^{\mathcal{N}}}) = \left(\frac{\langle m_{F_*} + 4m_{F_0} + m_{F^*}\rangle}{6}\right).$$

For any two triangular neutrosophic numbers

$$\tilde{M}^{\mathcal{N}} = (T_{\tilde{M}^{\mathcal{N}}}; I_{\tilde{M}^{\mathcal{N}}}; F_{\tilde{M}^{\mathcal{N}}}) = (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle) and$$
$$\tilde{N}^{\mathcal{N}} = (T_{\tilde{N}^{\mathcal{N}}}; I_{\tilde{N}^{\mathcal{N}}}; F_{\tilde{N}^{\mathcal{N}}}) = (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle) \in F(R),$$

we have the following comparison:

- If $\mathcal{R}(T_{\tilde{M}^{\mathcal{N}}}) < \mathcal{R}(T_{\tilde{N}^{\mathcal{N}}}); \mathcal{R}(I_{\tilde{M}^{\mathcal{N}}}) < \mathcal{R}(I_{\tilde{N}^{\mathcal{N}}}), \mathcal{R}(F_{\tilde{M}^{\mathcal{N}}}) < \mathcal{R}(F_{\tilde{N}^{\mathcal{N}}})$, then $\tilde{M}^{\mathcal{N}} \prec \tilde{N}^{\mathcal{N}}$
- If $\mathcal{R}(T_{\tilde{M}^{\mathcal{N}}}) > \mathcal{R}(T_{\tilde{N}^{\mathcal{N}}}); \mathcal{R}(I_{\tilde{M}^{\mathcal{N}}}) > \mathcal{R}(I_{\tilde{N}^{\mathcal{N}}}), \mathcal{R}(F_{\tilde{M}^{\mathcal{N}}}) > \mathcal{R}(F_{\tilde{N}^{\mathcal{N}}})$, then $\tilde{M}^{\mathcal{N}} \succ \tilde{N}^{\mathcal{N}}$
- If $\mathcal{R}(T_{\tilde{M}^{\mathcal{N}}}) = \mathcal{R}(T_{\tilde{N}^{\mathcal{N}}}); \mathcal{R}(I_{\tilde{M}^{\mathcal{N}}}) = \mathcal{R}(I_{\tilde{N}^{\mathcal{N}}}), \mathcal{R}(F_{\tilde{M}^{\mathcal{N}}}) = \mathcal{R}(F_{\tilde{N}^{\mathcal{N}}})$, then $\tilde{M}^{\mathcal{N}} \approx \tilde{N}^{\mathcal{N}}$.

2.3. **Differential Calculus of Neutrosophic Single Valued Function.**

Definition 2.9. [10] *Let $\tilde{f}^{\mathcal{N}} : \mathbb{R}^n \to F(\mathbb{R})$ be a neutrosophic-valued function. We say that $\tilde{f}^{\mathcal{N}}$ is continuous at $c \in \mathbb{R}^n$ if and only if for every $\epsilon > 0$, there exists a $\delta = \delta(c, \epsilon) > 0$ such that $d(\tilde{f}^{\mathcal{N}}(x), \tilde{f}^{\mathcal{N}}(c)) < \epsilon$, for all $x \in \mathbb{R}^n$ with $\|x - c\| < \delta$. That is,*

$$\lim_{x \to c} \tilde{f}^{\mathcal{N}}(x) = \tilde{f}^{\mathcal{N}}(c).$$

Definition 2.10. [10] *Let X be a subset of $\mathbb{R}$. A neutrosophic single valued function $\tilde{f}^{\mathcal{N}} : X \to F(\mathbb{R})$ is said to be differentiable at $x^0 \in X$ if and only if there exists a neutrosophic number $D\tilde{f}^{\mathcal{N}}(x^0)$ such that the limits (with respect to metric d) $D\tilde{f}(x^0) = \lim_{h \to 0^+} \frac{1}{h} \odot \left[\tilde{f}^{\mathcal{N}}(x^0 + h) - \tilde{f}^{\mathcal{N}}(x^0)\right]$ exists. In this case, $D\tilde{f}^{\mathcal{N}}(x^0)$ is called the derivative of $\tilde{f}^{\mathcal{N}}$ at x^0. If $\tilde{f}^{\mathcal{N}}$ is differentiable at every $x \in X$, we call $\tilde{f}^{\mathcal{N}}$ differentiable over X.*

Definition 2.11. [10] *Let $\tilde{f}^{\mathcal{N}}$ be a neutrosophic single valued function defined on an open subset X of $\mathbb{R}^n$ and let $x^0 = (x_1^0, \ldots, x_n^0) \in X$ be fixed. We say that $\tilde{f}^{\mathcal{N}}$ has the ith partial derivative $D_i\tilde{f}^{\mathcal{N}}(x^0)$ at x^0 if and only if the neutrosophic-valued function $\tilde{g}^{\mathcal{N}}(x_i) = \tilde{f}^{\mathcal{N}}(x_1^0, \ldots, x_{i-1}^0, x_i, x_{i+1}^0, \ldots, x_n^0)$ is differentiable at x_i^0 with derivative $D_i\tilde{f}^{\mathcal{N}}(x^0)$. We also write $D_i\tilde{f}^{\mathcal{N}}(x^0)$ as $\left(\frac{\partial \tilde{f}^{\mathcal{N}}}{\partial x_i}\right)(x^0)$.*

Definition 2.12. [10] *We say that $\tilde{f}^{\mathcal{N}}$ is differentiable at x^0 if and only if one of the partial derivatives $\partial\tilde{f}^{\mathcal{N}}/\partial x_1, \ldots, \partial\tilde{f}^{\mathcal{N}}/\partial x_n$ exists at x^0 and the remaining $(n-1)$ partial derivatives exist on some neighborhoods of x^0 and are continuous at x^0 (in the sense of neutrosophic-valued function). The gradient of $\tilde{f}^{\mathcal{N}}$ at x^0 is denoted by*

$$\nabla\tilde{f}^{\mathcal{N}}(x^0) = \left(D_1\tilde{f}^{\mathcal{N}}(x^0), \ldots, D_n\tilde{f}^{\mathcal{N}}(x^0)\right),$$

and it defines a neutrosophic single valued function from X to $F^n(\mathbb{R}) = F(\mathbb{R}) \times \cdots \times F(\mathbb{R})$ $(n$ times$)$, where each $D_i\tilde{f}^{\mathcal{N}}(x^0)$ is a neutrosophic number for $i = 1, \ldots, n$. We say that $\tilde{f}^{\mathcal{N}}$ is differentiable on X if it is differentiable at every $x^0 \in X$.

Definition 2.13. [10] *We say that $\tilde{f}^{\mathcal{N}}$ is continuously differentiable at x^0 if and only if all of the partial derivatives $\partial\tilde{f}^{\mathcal{N}}/\partial x_i$, $i = 1, \ldots, n$, exist on some neighborhoods of x^0 and are continuous at x^0 (in the sense of neutrosophic single valued function). We say that $\tilde{f}^{\mathcal{N}}$ is continuously differentiable on X if it is continuously differentiable at every $x^0 \in X$.*

3. Neutrosophic Non Linear Programming Problems (NNLPPs)

Consider a general neutrosophic nonlinear programming problem:

$$\begin{aligned} &\min \tilde{f}^{\mathcal{N}}(\tilde{x}) \\ &\text{subject to } \tilde{g}_j^{\mathcal{N}}(\tilde{x}) \preceq (or) \succeq \tilde{0} \text{ for } j = 1, 2, \cdots, m \\ &\tilde{h}_j^{\mathcal{N}}(\tilde{x}) \approx \tilde{0} \text{ for } j = 1, 2, \cdots, m, \\ &\tilde{\mathbf{x}} \succeq \tilde{0}, \end{aligned} \tag{3.1}$$

where $\tilde{f}^{\mathcal{N}}, \tilde{g}_1^{\mathcal{N}}, \cdots, \tilde{g}_m^{\mathcal{N}}, \tilde{h}_1^{\mathcal{N}}, \cdots, \tilde{h}_l^{\mathcal{N}}$ are continuous neutrosophic valued functions defined on R^n.

Definition 3.1. *A vector $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \cdots, \tilde{x}_n)$ is said to be a feasible solution to the NNLPP (3.1) if it meets all the constraints and adheres to the non-negativity condition of the NNLPP (3.1).*

The collection of all such feasible solutions is known as the feasible region, which is defined by these criteria.

$$F = \{\tilde{\mathbf{x}} \in F(R)^n / \tilde{h}_i^{\mathcal{N}}(x) \approx \tilde{0}, \text{ for } i = 1, 2, 3, \cdots, l, \ \tilde{g}_j^{\mathcal{N}}(x) \preceq \tilde{0}, \text{ for } j = 1, 2, 3, \cdots, m\}.$$

Definition 3.2. *A feasible solution $\tilde{\mathbf{x}}^0$ to the NNLPP (3.1) is said to be an optimal solution to the NNLPP (3.1) if $\tilde{f}^{\mathcal{N}}(\tilde{\mathbf{x}}^0) \preceq \tilde{f}^{\mathcal{N}}(\tilde{\mathbf{x}})$, for all $\tilde{\mathbf{x}} \in F$.*

Theorem 3.1. *Let $\tilde{\mathbf{x}}^*$ be a local solution at which the linear independence constraint qualification is satisfied (that is, the gradients $\nabla c_i(\tilde{\mathbf{x}}^*),\ i \in \mathcal{E}$, are linearly independent vectors), and the second-order sufficient conditions are satisfied for $\lambda = \lambda^*$. Then there is a threshold value $\bar{\mu}$ such that for all $\mu \geq \bar{\mu}$, $\tilde{\mathbf{x}}^*$ is a strict local minimizer of $\mathcal{L}_A(\tilde{\mathbf{x}}, \lambda^*; \mu)$.*

Proof. We prove the result by showing that $\tilde{\mathbf{x}}^*$ satisfies the second-order sufficient conditions to be a strict local minimizer of $\mathcal{L}_A(\tilde{\mathbf{x}}, \lambda^*; \mu)$ for all μ sufficiently large; that is,

$$\nabla_{\tilde{\mathbf{x}}} \mathcal{L}_A(\tilde{\mathbf{x}}^*, \lambda^*; \mu) \approx 0, \qquad \nabla^2_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}} \mathcal{L}_A(\tilde{\mathbf{x}}^*, \lambda^*; \mu) \succ 0. \tag{3.2}$$

Because $\tilde{\mathbf{x}}^*$ is a local solution for which linear independence constraint qualification is satisfied, we can apply theorem to deduce that

$$\nabla_{\tilde{\mathbf{x}}} \mathcal{L}_A(\tilde{\mathbf{x}}^*, \boldsymbol{\lambda}^*) \approx 0 \quad \text{and} \quad \nabla c_i(\tilde{\mathbf{x}}^*) \approx 0, \text{ for all } i \in \mathcal{E} \tag{3.3}$$

so that

$$\begin{aligned} \nabla_{\tilde{\mathbf{x}}} \mathcal{L}_A\left(\tilde{\mathbf{x}}^*, \boldsymbol{\lambda}^*; \mu\right) &\approx \tilde{f}^{\mathscr{N}}(\tilde{\mathbf{x}}^*) - \sum_{i \in \mathcal{E}} [\lambda_i^{(*)} - \mu c_i(\tilde{\mathbf{x}}^*)] \nabla c_i(\tilde{\mathbf{x}}^*) \\ &\approx \tilde{f}^{\mathscr{N}}(\tilde{\mathbf{x}}^*) - \sum_{i \in \mathcal{E}} \lambda_i^{(*)} \nabla c_i(\tilde{\mathbf{x}}^*) \approx \nabla_{\tilde{\mathbf{x}}} \mathcal{L}_A(\tilde{\mathbf{x}}^*, \boldsymbol{\lambda}^*) \approx 0 \end{aligned} \tag{3.4}$$

verifying the first part of (3.2), independently of μ.

For the second part of (3.2), we define $\tilde{A}$ to be the constraint gradient matrix to be evaluated at $\tilde{\mathbf{x}}^*$, and write

$$\nabla^2_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}} \mathcal{L}_{\tilde{A}}(\tilde{\mathbf{x}}^*, \lambda^*; \mu) \approx \nabla^2_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}} \mathcal{L}(\tilde{\mathbf{x}}^*, \lambda^*) + \mu \tilde{A}^T \tilde{A}.$$

If the claim in (3.2) were not true, then for each integer $k \geq 1$, we could choose a vector $\tilde{\mathbf{w}}_k$ with $\|\tilde{\mathbf{w}}_k\| \approx 1$ such that

$$0 \succeq \tilde{\mathbf{w}}_k^T \nabla^2_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}} \mathcal{L}_{\tilde{A}}(\tilde{\mathbf{x}}^*, \lambda^*; k) \tilde{\mathbf{w}}_k \approx \tilde{\mathbf{w}}_k^T \nabla^2_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}} \mathcal{L}(\tilde{\mathbf{x}}^*, \lambda^*) \tilde{\mathbf{w}}_k + k \left\| \tilde{A} \tilde{\mathbf{w}}_k \right\|_2^2. \tag{3.5}$$

and therefore

$$\|\tilde{A} \tilde{\mathbf{w}}_k\|_2^2 \preceq -\frac{1}{k} \tilde{\mathbf{w}}_k^T \nabla^2_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}} \mathcal{L}(\tilde{\mathbf{x}}^*, \lambda^*) \tilde{\mathbf{w}}_k \to 0, \qquad \text{as } k \to \infty. \tag{3.6}$$

Since the vectors $\{\tilde{\mathbf{w}}_k\}$ lie in a compact set (the surface of the unit sphere), they have an accumulation point $\tilde{\mathbf{w}}$. The limit (3.6) implies that $\tilde{A}\tilde{\mathbf{w}} \approx 0$. Moreover, by rearranging (3.5), we have that

$$\tilde{\mathbf{w}}_k^T \nabla^2_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}} \mathcal{L}(\tilde{\mathbf{x}}^*, \lambda^*) \tilde{\mathbf{w}}_k \preceq -k \|\tilde{A} \tilde{\mathbf{w}}_k\|_2^2 \preceq 0,$$

so by taking limits we have

$$\tilde{\mathbf{w}}^T \nabla^2_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}} \mathcal{L}(\tilde{\mathbf{x}}^*, \lambda^*) \tilde{\mathbf{w}} \preceq 0.$$

However, this inequality contradicts the second-order conditions and hence we must have

$$\tilde{\mathbf{w}}^T \nabla^2_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}} \mathcal{L}(\tilde{\mathbf{x}}^*, \lambda^*) \tilde{\mathbf{w}} \succ 0$$

for all nonzero vectors $\tilde{\mathbf{w}}$ with $\tilde{A}\tilde{\mathbf{w}} \approx 0$. Hence, the second part of (3.2) holds for all μ sufficiently large.

□

4. Algorithm for Augmented Lagrangian Method

Step 1. Express the given NNLPP in its parametric form.

Step 2. Construct the augmented Lagrangian neutrosophic valued function:

$$\tilde{\mathcal{L}}_A(\tilde{x}, \lambda^{(\kappa)}, \mu^{(\kappa)}) = \tilde{f}^{\mathcal{N}}(\tilde{x}) + \sum_{i=1}^{m} \lambda_i^{(\kappa)} \tilde{h}_i(\tilde{x}) + \frac{\mu^{(\kappa)}}{2} \sum_{i=1}^{m} \tilde{h}_i(\tilde{x})^2$$

and convert the NNLPP as an unconstrained optimization problem.

Step 3. Apply the necessary condition for $\tilde{\mathcal{L}}_A(\tilde{x}, \lambda^{(\kappa)}, \mu^{(\kappa)})$ and find the extreme points.

Step 4. Initialize the Augmented Lagrange Multipliers(LM) $\lambda_i^{(\kappa)}$ for $\tilde{h}_i(\tilde{x})$ and choose an initial solution $\tilde{x}^{(\kappa)}$. Set the penalty parameter $\mu^{(\kappa)}$and iteration counter $k = 0$, $\lambda_i^{(0)} = 0, \mu^{(0)} = 1$. Update the penalty parameter if necessary: $\mu^{(\kappa+1)} = \gamma\mu^{(\kappa)}$, where $\gamma > 1$ is a constant.

Step 5. Update the LM: $\lambda_i^{(\kappa+1)} = \lambda_i^{(\kappa)} + \mu^{(\kappa)} \tilde{h}_i^{\kappa}(\tilde{x}^{(\kappa)})$.

Step 6. Check the convergence criteria: If $\|\tilde{g}^{\mathcal{N}}(\tilde{x}^{(\kappa+1)})\| \preceq \epsilon_1$ (for a small tolerance ϵ_1).

If the change in $(\tilde{x})$ is small enough, i.e., $(\|\tilde{x}^{(\kappa+1)} - \tilde{x}^{(\kappa)}\| \preceq \epsilon_2)$.

Step 7. If the convergence criteria are met, stop the algorithm. Otherwise, go to step 3.

5. Numerical Example

Consider a problem of constrained optimization with single-valued neutrosophic triangular coefficients discussed by Vanaja et.al. [16].

$$\begin{aligned}
\max \tilde{f}^{\mathcal{N}}(\tilde{x}_1, \tilde{x}_2) =& \langle (1,2,3); (0.6,0.4,0.1)\rangle \tilde{x}_1 + \langle (2,3,4); (0.2,0.3,0.5)\rangle \tilde{x}_2 \\
& - \langle (1,2,3); (0.6,0.4,0.1)\rangle \tilde{x}_1^2 \\
\text{subject to } \tilde{h}_1^{\mathcal{N}}(\tilde{x}) =& \langle (0,1,2); (0.2,0.3,0.5)\rangle \tilde{x}_1 + \langle (3,4,5); (0.4,0.3,0.2)\rangle \tilde{x}_2 \\
& \preceq \langle (3,4,5); (0.4,0.3,0.2)\rangle \\
\tilde{h}_2^{\mathcal{N}}(\tilde{\chi}) =& \langle (0,1,2); (0.2,0.3,0.5)\rangle \tilde{x}_1 + \langle (0,1,2); (0.2,0.3,0.5)\rangle \tilde{x}_2 \\
& \preceq \langle (1,2,3); (0.6,0.4,0.1)\rangle \\
& \tilde{x}_1, \tilde{x}_2 \succeq \tilde{0}.
\end{aligned} \tag{5.1}$$

Express the SVNNLPP (5.1) in its parametric form as

$$\begin{aligned}
\max \ \tilde{f}^{\mathcal{N}}(\tilde{x}_1, \tilde{x}_2) \ &= \ \langle (2, 1-\beta, 1-\beta); (0.6,0.4,0.1)\rangle \tilde{x}_1 \\
& \quad + \langle (3, 1-\beta, 1-\beta); (0.2,0.3,0.5)\rangle \tilde{x}_2 \\
& \quad - \langle (2, 1-\beta, 1-\beta); (0.6,0.4,0.1)\rangle \tilde{x}_1^2 \\
\text{subject to} \quad \tilde{g}_1^{\mathcal{N}}(\tilde{x}) \ &= \ \langle (1, 1-\beta, 1-\beta); (0.2,0.3,0.5)\rangle \tilde{x}_1 \\
& \quad + \langle (4, 1-\beta, 1-\beta); (0.4,0.3,0.2)\rangle \tilde{x}_2
\end{aligned}$$

$$\begin{aligned} &\preceq \langle(4,1-\beta,1-\beta);(0.4,0.3,0.2)\rangle, \\ \tilde{g}_2^{\mathscr{N}}(\tilde{x}) &= \langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_1 \\ &\quad +\langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_2 \\ &\preceq \langle(2,1-\beta,1-\beta);(0.6,0.4,0.1)\rangle, \\ &\tilde{x}_1,\tilde{x}_2 \succeq \tilde{0}. \end{aligned} \tag{5.2}$$

Construct the augmented Lagrangian neutrosophic valued function

$$\tilde{\mathcal{L}}_A(\tilde{x},\lambda,\mu) = \tilde{f}(\tilde{x}) + \sum_{i=1}^{m}\lambda_i\tilde{h}_i(\tilde{x}) + \frac{\mu}{2}\sum_{i=1}^{m}(\tilde{h}_i(\tilde{x}))^2$$

Then the corresponding unconstrained optimization problem is

$$\begin{aligned} \mathcal{L}_A(\tilde{x},\lambda,\mu) &= \langle(2,1-\beta,1-\beta);(0.6,0.4,0.1)\rangle\tilde{x}_1 + \langle(3,1-\beta,1-\beta);(0.2,0.3,0.3)\rangle\tilde{x}_2 \\ &\quad - \langle(2,1-\beta,1-\beta);(0.6,0.4,0.1)\rangle\tilde{x}_1^2 + \lambda_1\Big(\langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_1 \\ &\quad + \langle(4,1-\beta,1-\beta);(0.4,0.3,0.2)\rangle\tilde{x}_2 - \langle(4,1-\beta,1-\beta);(0.4,0.3,0.2)\rangle\Big) \\ &\quad + \lambda_2\Big(\langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_1 + \langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_2 \\ &\quad - \langle(2,1-\beta,1-\beta);(0.6,0.4,0.1)\rangle\Big) + \frac{\mu}{2}\Big(\langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_1 \\ &\quad + \langle(4,1-\beta,1-\beta);(0.4,0.3,0.2)\rangle\tilde{x}_2 - \langle(4,1-\beta,1-\beta);(0.4,0.3,0.2)\rangle\Big)^2 \\ &\quad + \frac{\mu}{2}\Big(\langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_1 + \langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_2 \\ &\quad - \langle(2,1-\beta,1-\beta);(0.6,0.4,0.1)\rangle\Big)^2. \end{aligned}$$

The necessary condition for $\mathcal{L}_A(\tilde{x},\lambda,\mu)$ to have an extreme point can be written as

$$\begin{aligned} \frac{\partial\mathcal{L}_A}{\partial\tilde{x}_1} &= \langle(2,1-\beta,1-\beta);(0.6,0.4,0.1)\rangle - 2\langle(2,1-\beta,1-\beta);(0.6,0.4,0.1)\rangle\tilde{x}_1 \\ &\quad + \lambda_1\langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle + \lambda_2\langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle \\ &\quad + \mu\Big(\langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_1 + \langle(4,1-\beta,1-\beta);(0.4,0.3,0.2)\rangle\tilde{x}_2 \\ &\quad - \langle(4,1-\beta,1-\beta);(0.4,0.3,0.2)\rangle + \langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_1 \\ &\quad + \langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_2 - \langle(2,1-\beta,1-\beta);(0.6,0.4,0.1)\rangle\Big) \approx \tilde{0}. \end{aligned}$$

and

$$\begin{aligned} \frac{\partial\mathcal{L}_A}{\partial\tilde{x}_2} = \quad & \langle(3,1-\beta,1-\beta);(0.2,0.3,0.3)\rangle + \lambda_1\langle(4,1-\beta,1-\beta);(0.4,0.3,0.2)\rangle \\ & +\lambda_2\langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle \\ & +\mu\langle(4,1-\beta,1-\beta);(0.4,0.3,0.2)\rangle\Big(\langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_1 \end{aligned}$$

$$+\langle(4,1-\beta,1-\beta);(0.4,0.3,0.2)\rangle\tilde{x}_2-\langle(4,1-\beta,1-\beta);(0.4,0.3,0.2)\rangle\Big)$$
$$+\mu\Big(\langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_1+\langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{x}_2$$
$$-\langle(2,1-\beta,1-\beta);(0.6,0.4,0.1)\rangle\Big)\approx\tilde{0}.$$

Then the values of $\tilde{x}_1$ and $\tilde{x}_2$ are given by

$$x_1^{(k)}=\frac{\left(4\mu_k+2-\lambda_1^{(k)}\right)-\left(4\mu_k+\frac{3}{4}-\lambda_1^{(k)}\right)}{4}andx_2^{(k)}\qquad=\frac{4-x_1}{4}+\frac{\frac{3}{4}-\lambda_1^{(k)}}{4\mu_k}.\tag{5.3}$$

Table 1. Iterations of the Augmented Lagrangian Method.

k	μ	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{f}^{\mathcal{N}}(x)$
0	1	$\frac{(8,1-\beta,1-\beta);(0.6,0.4,0.5)}{25}$	$\frac{(18,1-\beta,1-\beta);(0.6,0.4,0.5)}{16}$	$\frac{(975,1-\beta,1-\beta);(0.6,0.4,0.5)}{256}$
1	10	$\frac{(7,1-\beta,1-\beta);(0.6,0.4,0.5)}{20}$	$\frac{(15,1-\beta,1-\beta);(0.6,0.4,0.5)}{16}$	$\frac{(890,1-\beta,1-\beta);(0.6,0.4,0.5)}{256}$
2	100	$\frac{(13,1-\beta,1-\beta);(0.6,0.4,0.5)}{40}$	$\frac{(29,1-\beta,1-\beta);(0.6,0.4,0.5)}{32}$	$\frac{(835,1-\beta,1-\beta);(0.6,0.4,0.5)}{256}$
3	1000	$\frac{(41,1-\beta,1-\beta);(0.6,0.4,0.5)}{128}$	$\frac{(117,1-\beta,1-\beta);(0.6,0.4,0.5)}{128}$	$\frac{(822,1-\beta,1-\beta);(0.6,0.4,0.5)}{256}$
4	10000	$\frac{(5,1-\beta,1-\beta);(0.6,0.4,0.5)}{16}$	$\frac{(59,1-\beta,1-\beta);(0.6,0.4,0.5)}{64}$	$\frac{(819,1-\beta,1-\beta);(0.6,0.4,0.5)}{256}$
5	100000	$\frac{(5,1-\beta,1-\beta);(0.6,0.4,0.5)}{16}$	$\frac{(59,1-\beta,1-\beta);(0.6,0.4,0.5)}{64}$	$\frac{(818,1-\beta,1-\beta);(0.6,0.4,0.5)}{256}$

Hence the optimal solution of the given SVNNLPP (5.2) is
$\tilde{x}_1=\frac{\langle(5,1-\beta,1-\beta);(0.6,0.4,0.5)\rangle}{\langle(16,1-\beta,1-\beta);(0.6,0.4,0.5)\rangle},\quad\tilde{x}_2=\frac{\langle(59,1-\beta,1-\beta);(0.6,0.4,0.5)\rangle}{\langle(64,1-\beta,1-\beta);(0.6,0.4,0.5)\rangle}$
with $\max\tilde{f}^{\mathcal{N}}(\tilde{x}_1,\tilde{x}_2)=\frac{\langle(818,1-\beta,1-\beta);(0.6,0.4,0.5)\rangle}{\langle(256,1-\beta,1-\beta);(0.6,0.4,0.5)\rangle}$.

That is the optimal solution of the SVNNLPP (5.1)is

$$\tilde{x}_1=\left\langle\left(\frac{-11000000}{16}+\beta,\frac{5}{16},\frac{21}{16}-\beta\right);(0.6,0.4,0.5)\right\rangle,$$

$$\tilde{x}_2=\left\langle\left(\frac{-5}{64}+\beta,\frac{59}{64},\frac{123}{64}-\beta\right);(0.6,0.4,0.5)\right\rangle$$

$$\text{with }\max\tilde{f}^{\mathcal{N}}(\tilde{x}_1,\tilde{x}_2)=\left\langle\left(\frac{562}{256}+\beta,\frac{818}{256},\frac{1074}{256}-\beta\right);(0.6,0.4,0.5)\right\rangle.$$

The Neutrosophic optimal solution of the SVNNLPP (5.1) for different values of β is shown in Table 2 and in figure 1.

TABLE 2. Single-valued neutrosophic optimal solution for various values of $\beta \in [0,1]$.

β	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{f}^{\mathcal{N}}(\tilde{x})$
0	$\frac{(-5,59,123)\,;(0.6,0.4,0.5)}{64}$	$\frac{(-11,5,21)\,;(0.6,0.4,0.5)}{16}$	$\frac{(562,818,1074)\,;(0.6,0.4,0.5)}{256}$
0.25	$\frac{(11,59,107)\,;(0.6,0.4,0.5)}{64}$	$\frac{(-7,5,17)\,;(0.6,0.4,0.5)}{16}$	$\frac{(626,818,1010)\,;(0.6,0.4,0.5)}{256}$
0.50	$\frac{(27,59,91)\,;(0.6,0.4,0.5)}{64}$	$\frac{(-3,5,13)\,;(0.6,0.4,0.5)}{16}$	$\frac{(690,818,946)\,;(0.6,0.4,0.5)}{256}$
0.75	$\frac{(43,75,123)\,;(0.6,0.4,0.5)}{64}$	$\frac{(1,5,9)\,;(0.6,0.4,0.5)}{16}$	$\frac{(754,818,882)\,;(0.6,0.4,0.5)}{256}$
1	$\frac{(59,59,59)\,;(1,1,1)}{64} = \frac{59}{64}$	$\frac{(5,5,5)\,;(1,1,1)}{16} = \frac{5}{16}$	$\frac{(818,818,818)\,;(1,1,1)}{256} = \frac{818}{256}$

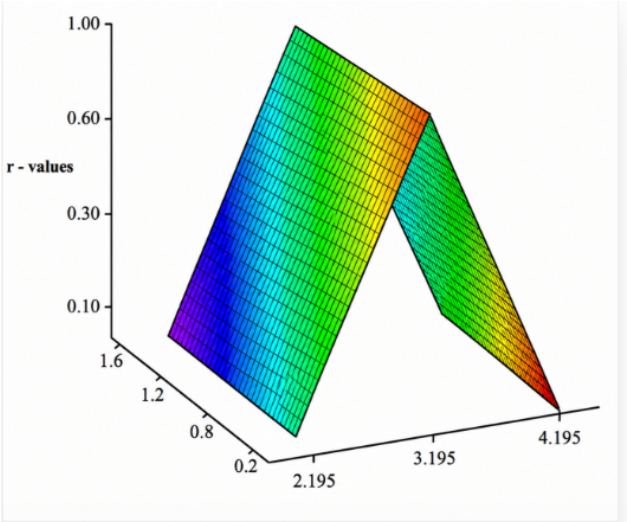


FIGURE 1. Single-valued neutrosophic optimal solution for various values of $\beta \in [0,1]$

6. An Application to Advertising Strategy

Himaja and Co. wishes to plan its advertising strategy for which there are two media under consideration, namely *Raja Cable* and *Srija Channel*. Both Raja Cable and Srija Channel have a reach of potential customers proportional to the square of the number of appearances. The cost per one-minute advertisement is approximately Rs. 6,000 in Raja Cable and Rs. 9,000 in Srija Channel, respectively. The total advertising budget available to Himaja and Co. is approximately Rs. 80,000 per month. Furthermore, the total reach among customers whose annual income is below Rs. 60,000 should not exceed approximately 3,000 potential customers. The reach in Raja Cable and Srija Channel for this income group is approximately 300 and 150 potential customers, respectively. The objective is to determine the optimal number of one-minute advertisements in Raja Cable and Srija Channel so as to maximize the total customer reach.

The Single-valued neutrosophic nonlinear programming model is formulated as follows:

$$\begin{aligned} & \max \ \tilde{f}(\tilde{\mathbf{x}}) \approx \tilde{x}_1^2 + \tilde{x}_2^2 \\ \text{subject to} \quad & \tilde{6}\tilde{x}_1 + \tilde{9}\tilde{x}_2 \preceq \tilde{80}, \\ & \tilde{2}\tilde{x}_1 + \tilde{x}_2 \preceq \tilde{20}, \\ & \tilde{x}_1,\ \tilde{x}_2 \succeq \tilde{0}. \end{aligned} \tag{6.1}$$

where

$$\tilde{x}_1 = \text{number of one-minute advertisements in Raja Cable,}$$

$$\tilde{x}_2 = \text{number of one-minute advertisements in Srija Channel.}$$

Suppose that the coefficients in the objective function and in the constraints are uncertain and are modeled as Single-valued triangular neutrosophic numbers, then the SVNNLPP (6.1) becomes

$$\begin{aligned} \min \tilde{f}^{\mathcal{N}}(\tilde{x}) \approx & \tilde{1}\tilde{x}_1^2 + \tilde{1}\tilde{x}_2^2 \\ \text{sub. to } \tilde{h}_1^{\mathcal{N}}(\tilde{x}) \approx & \quad \tilde{6}\tilde{x}_1 + \tilde{9}\tilde{x}_2 \preceq \tilde{80} \\ \tilde{h}_2^{\mathcal{N}}(\tilde{x}) \approx & \quad \tilde{2}\tilde{x}_1 + \tilde{1}\tilde{x}_2 \preceq \tilde{80} \\ & \tilde{x}_1, \tilde{x}_2 \succeq \tilde{0}. \end{aligned} \tag{6.2}$$

Assume, $\tilde{2} \approx (1,2,3);(0.6,0.3,0.2), \tilde{1} \approx (0,1,2);(0.9,0.7,0.5), \tilde{6} \approx (5,6,7);(0.8,0.6,0.4), \tilde{9} \approx (8,9,10);(0.6,0.3,0.6), \tilde{80} \approx (79,80,81);(0.9,0.7,0.6)$, and $\tilde{20} \approx (19,20,21);(0.8,0.6,0.4)$.
Now the SVNNLPP (6.2) becomes,

$$\begin{aligned} & \max \tilde{f}^{\mathcal{N}}(\tilde{x}) \approx (0,1,2);(0.9,0.7,0.5)\tilde{x}_1^2 + (0,1,2);(0.9,0.7,0.5)\tilde{x}_2^2 \\ & \text{subject to } (5,6,7);(0.8,0.6,0.4)\tilde{x}_1 + (8,9,10);(0.6,0.3,0.6)\tilde{x}_2 \preceq (79,80,81);(0.9,0.7,0.6) \\ & \qquad (1,2,3);(0.6,0.3,0.2)\tilde{x}_1 + (0,1,2);(0.9,0.7,0.5)\tilde{x}_2 \preceq (19,20,21);(0.8,0.6,0.4)) \\ & \qquad \tilde{x}_1, \tilde{x}_2 \succeq \tilde{0}. \end{aligned} \tag{6.3}$$

The parametric form of (6.3) is given by

$$\begin{aligned} & \max \tilde{f}^{\mathcal{N}}(\tilde{x}) \approx (1,1-r,1-r);(0.9,0.7,0.5)\tilde{x}_1^2 + (0,1,2);(0.9,0.7,0.5)\tilde{x}_2^2 \\ & \text{subject to } (5,6,7);(0.8,0.6,0.4)\tilde{x}_1 + (8,9,10);(0.6,0.3,0.6)\tilde{x}_2 \preceq (79,80,81);(0.9,0.7,0.6) \\ & \qquad (1,2,3);(0.6,0.3,0.2)\tilde{x}_1 + (0,1,2);(0.9,0.7,0.5)\tilde{x}_2 \preceq (19,20,21);(0.8,0.6,0.4)) \\ & \qquad \tilde{x}_1, \tilde{x}_2 \succeq \tilde{0}. \end{aligned} \tag{6.4}$$

Now the SVNNLPP (6.4) is converted into and equivalent minimization SVNNLPP as

$$\begin{aligned}\min \tilde{f}^{\mathcal{N}}(\tilde{x}) \approx & -(1,1-r,1-r);(0.9,0.7,0.5)\tilde{x}_1^2-(0,1,2);(0.9,0.7,0.5)\tilde{x}_2^2\\ \text{subject to } & (5,6,7);(0.8,0.6,0.4)\tilde{x}_1+(8,9,10);(0.6,0.3,0.6)\tilde{x}_2 \prec (79,80,81);(0.9,0.7,0.6)\\ & (1,2,3);(0.6,0.3,0.2)\tilde{x}_1+(0,1,2);(0.9,0.7,0.5)\tilde{x}_2 \prec (19,20,21);(0.8,0.6,0.4))\\ & \tilde{x}_1,\tilde{x}_2 \succeq \tilde{0}.\end{aligned} \tag{6.5}$$

Construct the augmented Lagrangian neutrosophic valued function

$$\tilde{\mathcal{L}}_A(\tilde{x},\lambda,\mu)=\tilde{f}(\tilde{x})+\sum_{i=1}^{m}\lambda_i\tilde{h}_i(\tilde{x})+\frac{\mu}{2}\sum_{i=1}^{m}(\tilde{h}_i(\tilde{x}))^2 \tag{6.6}$$

Then the corresponding unconstrained optimization problem is

$$\begin{aligned}\mathcal{L}_A(\tilde{x},\lambda,\mu) = & -\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle\tilde{x}_1^2-\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle\tilde{x}_2^2\\ & +\lambda_1\big(\langle(6,1-\beta,1-\beta);(0.8,0.6,0.4)\rangle\tilde{x}_1+\langle(9,1-\beta,1-\beta);(0.6,0.3,0.6)\rangle\tilde{x}_2\\ & -\langle(80,1-\beta,1-\beta);(0.9,0.7,0.6)\rangle\big)+\lambda_2\big(\langle(2,1-\beta,1-\beta);(0.6,0.3,0.2)\rangle\tilde{x}_1\\ & +\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle\tilde{x}_2-\langle(20,1-\beta,1-\beta);(0.8,0.6,0.4)\rangle\big)\\ & +\frac{\mu}{2}\big(\langle(6,1-\beta,1-\beta);(0.8,0.6,0.4)\rangle\tilde{x}_1+\langle(9,1-\beta,1-\beta);(0.6,0.3,0.6)\rangle\tilde{x}_2\\ & -\langle(80,1-\beta,1-\beta);(0.9,0.7,0.6)\rangle\big)^2+\frac{\mu}{2}\big(\langle(2,1-\beta,1-\beta);(0.6,0.3,0.2)\rangle\tilde{x}_1\\ & +\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle\tilde{x}_2-\langle(20,1-\beta,1-\beta);(0.8,0.6,0.4)\rangle\big)^2.\end{aligned}$$

By applying the the Augmented Lagrangian Method, the optimal solution of the SVNNLPP (6.2) is

$$\tilde{x}_1=(8.3333,\ 10-10\beta,\ 10-10\beta)\,, \tag{6.7}$$

$$\tilde{x}_2=(3.3333,\ 10-10\beta,\ 10-10\beta)\,, \tag{6.8}$$

with the maximum objective function value

$$\max\tilde{f}(\tilde{x})=(80.2948,\ 10-10\beta,\ 10-10\beta)\,. \tag{6.9}$$

Therefore, the optimal solution of the SVNNLPP (6.1) is

$$\tilde{x}_1=(-1.6667+10\beta,\ 8.3333,\ 18.3333-10\beta)\,, \tag{6.10}$$

$$\tilde{x}_2=(-6.6667+10\beta,\ 3.3333,\ 13.3333-10\beta)\,, \tag{6.11}$$

with the corresponding optimal objective function value

$$\tilde{f}(\tilde{x})=(70.2948+10\beta,\ 80.2948,\ 90.2948-10\beta)\,. \tag{6.12}$$

TABLE 3. Optimal solution of the SVNNLPP (6.1) for different values of β.

β	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{f}(\tilde{x})$
0	$(-1.6667,\ 8.3333,\ 18.3333)$	$(-6.6667,\ 3.3333,\ 13.3333)$	$(70.2948,\ 80.2948,\ 90.2948)$
0.25	$(0.8333,\ 8.3333,\ 15.8333)$	$(-4.1667,\ 3.3333,\ 10.8333)$	$(72.7948,\ 80.2948,\ 87.7948)$
0.5	$(3.3333,\ 8.3333,\ 13.3333)$	$(-1.6667,\ 3.3333,\ 8.3333)$	$(75.2948,\ 80.2948,\ 85.2948)$
0.75	$(5.8333,\ 8.3333,\ 10.8333)$	$(0.8333,\ 3.3333,\ 5.8333)$	$(77.7948,\ 80.2948,\ 82.7948)$
1	$(8.3333,\ 8.3333,\ 8.3333)$	$(3.3333,\ 3.3333,\ 3.3333)$	$(80.2948,\ 80.2948,\ 80.2948)$
	$= 8.3333$	$= 3.3333$	$= 80.2948$

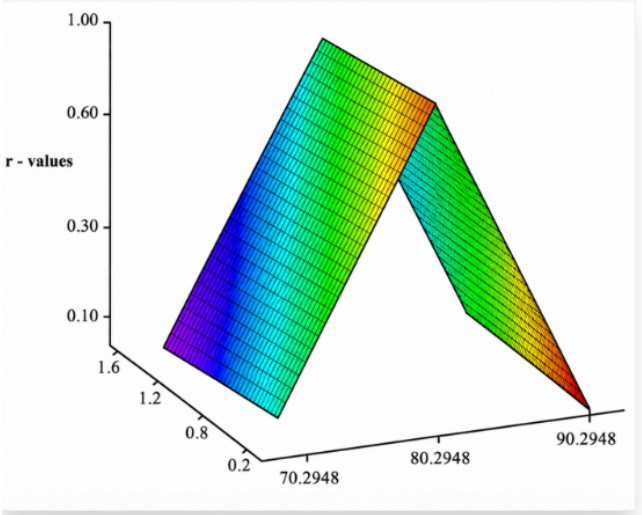


FIGURE 2. Single-valued neutrosophic optimal solution for various values of $\beta \in [0, 1]$

Hence, Himaja and Co. should plan for

$$\tilde{x}_1 = (-1.6667 + 10\beta,\ 8.3333,\ 18.3333 - 10\beta)$$

and

$$\tilde{x}_2 = (-6.6667 + 10\beta,\ 3.3333,\ 13.3333 - 10\beta)$$

appearances of one-minute advertisements in Raja Cable and Srija Channel, respectively, to achieve the maximum total reach

$$\tilde{f}(\tilde{x}) = (70.2948 + 10\beta,\ 80.2948,\ 90.2948 - 10\beta)\,.$$

7. Results and Discussions

From Table 4, we see that the single-valued neutrosophic optimal solution obtained by the proposed Augmented Lagrangian Method coincides with the single-valued neutrosophic optimal solution obtained by Vanaja et al. [16] using the Karush–Kuhn–Tucker method. But the optimal solution is obtained with fewer iterations by applying the proposed method.

Table 4. Comparison of the Single-Valued Neutrosophic Optimal Solution

Value	**Vanaja *et al.***	**Proposed method**
$\tilde{x}_1$	$\frac{(-11,\ 5,\ 21);(0.6,0.4,0.5)}{16}$	$\frac{(-11,\ 5,\ 21);(0.6,0.4,0.5)}{16}$
$\tilde{x}_2$	$\frac{(-5,\ 59,\ 123);(0.6,0.4,0.5)}{64}$	$\frac{(-5,\ 59,\ 123);(0.6,0.4,0.5)}{64}$
$\tilde{f}^{\mathcal{N}}(\tilde{x})$	$\frac{(562,\ 818,\ 1074);(0.6,0.4,0.5)}{256}$	$\frac{(562,\ 818,\ 1074);(0.6,0.4,0.5)}{256}$

8. Conclusion

We discussed a solution concept for SVNNLPP involving neutrosophic triangular numbers. First, the given SVNNLPP is expressed in terms of its location index number, left and right fuzziness index functions. In the parametric form of neutrosophic numbers, a new type of neutrosophic arithmetic and neutrosophic ranking are introduced and utilized. The neutrosophic versions of the Augmented Lagrangian (ALM) method is used, and the neutrosophic optimal solution of the SVNNLPP is obtained without having to convert the given problem into its equivalent crisp form. The neutrosophic optimal solution of the given SVNNLPP is tabulated for different values of $\beta \in [0, 1]$. It is important to note that by utilizing the suggested procedure and selecting an appropriate value for $\beta \in [0, 1]$, the decision maker has the flexibility to select his or her preferred optimal solution based on the situation. The numerical solutions have been presented and discussed.

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