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QUASI-INTEGRAL TIMES IN SEQUENTIAL PROJECTIVE DYNAMICS OVER FUNCTION FIELDS

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Abstract. Let K be the function field of a smooth projective variety over an algebraically closed field of characteristic zero. We study sequential projective dynamics $P_{n+1} = \Phi_n(P_n)$, where the morphisms have fixed degree $r \geq 2$, relative to moving hypersurfaces $D_n = V(G_n)$ of fixed degree δ. Assume that $h(P_n) \to \infty$, that $h(\Phi_n) + h(G_{n+1}) = o(h(P_n))$, and that each pullback $G_{n+1} \circ \Phi_n$ admits q selected pairwise nonassociate K-defined factors in general position. Put $C_N = \lfloor (N+2)^2/4 \rfloor$. If $q > C_N$, then for every $0 \leq \theta < (q - C_N)/(\delta r)$ only finitely many n satisfy $P_{n+1} \notin D_{n+1}$ and

$$N_S(D_{n+1}, P_{n+1}) \leq \theta\big(\delta h(P_{n+1}) + h(G_{n+1})\big).$$

In particular, the set of strict integral times is finite. The proof uses a function-field Subspace Theorem for moving hypersurfaces without algebraic nondegeneracy, a minimal coefficient-field variety, and resultant estimates for pullback errors.

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1. Introduction

Integral and quasi-integral points in arithmetic orbits have been studied since Silverman's work on rational maps [1]. Over function fields, integral orbits and effective quasi-integrality were studied in [4,5]; higher-dimensional and uniform variants are treated in [2,3,6,7]. Here we study finiteness of quasi-integral times over function fields when both the morphism and the target divisor vary along the sequence.

We consider the non-autonomous system

$$P_{n+1} = \Phi_n(P_n), \qquad n \geq 0, \tag{1}$$

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where $\Phi_n : \mathbb{P}^N_K \to \mathbb{P}^N_K$ has fixed degree $r \geq 2$. Height theory for sequences of morphisms was developed in [8, 9]; semigroup quasi-integrality over number fields was considered in [10]. Our base field is $K = k(B)$, with B smooth projective and k algebraically closed of characteristic zero. Fix a finite set S of prime divisors and moving hypersurfaces $D_n = V(G_n)$ of fixed degree δ. For $P \notin D_n$, put

$$N_S(D_n, P) = \sum_{\mathfrak{p} \notin S} \lambda_{\mathfrak{p},G_n}(P),$$

where $\lambda_{\mathfrak{p},G_n}$ is the associated local Weil function. The product formula gives

$$\sum_{\mathfrak{p} \in M_K} \lambda_{\mathfrak{p},G_n}(P) = \delta h(P) + h(G_n).$$

Thus $N_S(D_n, P) = 0$ is strict S-integrality relative to the moving target, while an upper bound by a fixed proportion of the total height expresses quasi-integrality. Pulling D_{n+1} back through Φ_n produces moving hypersurface factors; the main Diophantine difficulty is that the points may be algebraically degenerate over their moving coefficient field.

This one-step formulation differs from iteration by a fixed morphism. The factor family and its coefficient field change with n, so a moving-target theorem requiring algebraic nondegeneracy cannot be applied directly. We therefore seek an argument that depends only on relative height growth and the distinct pullback factors, without requiring canonical heights or a finite collection of maps. These requirements motivate the two auxiliary estimates developed below.

The main Diophantine input is a Subspace Theorem for moving hypersurfaces over function fields. Fixed-target results include [12–15]; moving-target forms were developed in [16–21]. Over function fields, moving hyperplanes were treated in [24], while the moving-hypersurface theorem of [25] assumes algebraic nondegeneracy over the coefficient field. Degenerate fixed-target results are given in [22,23]. Over number fields, Ji, Yan, and Yu [28, Theorems 1.1 and 1.2] proved a moving-hypersurface estimate without algebraic nondegeneracy and applied it to sequences of semi-decomposable form inequalities.

We follow the minimal-variety strategy of [26, Theorem 8 and its proof]; see also Section 3 of that paper. Quang's theorem is stated over number fields and involves a distributive constant; it does not directly supply the function-field, general-position, cofinite estimate proved here. Persistent relations define a minimal projective variety over the moving coefficient field; Wang's theorem supplies the hyperplane estimate, divisorial valuations replace number-field norms, and an explicit colon-ideal argument avoids any Cohen–Macaulay assumption. Unlike [25], no algebraic nondegeneracy is imposed, although the resulting universal constant is larger than the coefficient $N + 1$ in the nondegenerate case. The proof uses one filtration at a time.

The minimal variety is defined over $\mathcal{R}_{A,\mathcal{Q}}$ and need not descend to a fixed K-variety, so replacement and filtration are carried out in its homogeneous coordinate ring. The argument has three components: a

cofinite moving-hypersurface estimate over function fields without nondegeneracy, a resultant-transfer estimate uniform in the varying morphism, and the finiteness criterion obtained by combining these tools with the product formula. The dynamical step uses only relative height growth and does not require a Northcott property.

For functions on an infinite index set, $a = o(g)$ means that $|a(\alpha)| \le \varepsilon g(\alpha)$ outside a finite subset for every $\varepsilon > 0$; the same convention applies after restriction.

Put

$$C_N = \left\lfloor \frac{(N+2)^2}{4} \right\rfloor. \tag{2}$$

Theorem 4.4 proves the required moving-hypersurface estimate with coefficient C_N and no algebraic nondegeneracy assumption. On a minimal coefficient-field variety of dimension ℓ, its proof first obtains the sharper coefficient $(N-\ell+1)(\ell+1)$ and then maximizes over $1 \le \ell \le N$.

The main result is the following.

Theorem 1.1 (Main theorem)**.** *Let (P_n) satisfy (1), where every Φ_n has degree $r \ge 2$, and let $D_n = V(G_n)$ have degree δ. Assume that*

$$h(P_n) \longrightarrow \infty \quad \textit{and} \quad h(\Phi_n) + h(G_{n+1}) = o(h(P_n)).$$

Suppose that, for every sufficiently large n, there is a factorization with the same fixed integer $q \ge 1$:

$$G_{n+1} \circ \Phi_n = c_n U_n \prod_{j=1}^{q} Q_{n,j}^{a_{n,j}},$$

where $c_n \in K^*$, U_n is a nonzero homogeneous polynomial, and the $Q_{n,j}$ are pairwise nonassociate nonconstant homogeneous polynomials over K, with $d_{n,j} := \deg Q_{n,j} \in \mathbb{Z}_{>0}$ and $a_{n,j} \in \mathbb{Z}_{\ge 1}$. Thus q counts selected distinct factors; neither their multiplicities $a_{n,j}$ nor any further factor occurring in U_n increases q. Assume that every $N+1$ of the hypersurfaces $V(Q_{n,j})$ have empty intersection in $\mathbb{P}^N(\overline{K})$. If $q > C_N$, then, for every real number

$$0 \le \theta < \frac{q - C_N}{\delta r},$$

there are only finitely many indices n for which $P_{n+1} \notin D_{n+1}$ and

$$N_S(D_{n+1}, P_{n+1}) \le \theta\big(\delta h(P_{n+1}) + h(G_{n+1})\big). \tag{3}$$

The condition $P_{n+1} \notin D_{n+1}$ ensures that the counting function appearing in the statement is defined.

The case $\theta = 0$ gives finiteness of strict integral times. The theorem does not require the maps or divisors to belong to a finite collection, and it imposes no algebraic nondegeneracy condition on the orbit.

The proof combines the following two estimates:

$$(3),\ \text{product formula} \Longrightarrow \sum_{\mathfrak{p} \in S} \sum_{j=1}^{q} \frac{1}{d_{n,j}} \lambda_{\mathfrak{p}, Q_{n,j}}(P_n) \ge (q - \theta\delta r) h(P_n) - o(h(P_n)),$$

$$(41) \Longrightarrow \sum_{\mathfrak{p}\in S}\sum_{j=1}^{q}\frac{1}{d_{n,j}}\lambda_{\mathfrak{p},Q_{n,j}}(P_n) \le (C_N + o(1))h(P_n).$$

The resultant estimates contribute only an $o(h(P_n))$ error. Hence an infinite set of quasi-integral times would imply $q - C_N \le \theta\delta r$.

Sections 2–4 develop the Diophantine input, Sections 5–6 prove the dynamical result, and Section 7 gives examples.

2. Function-Field Heights and Moving Targets

We fix the notation used throughout the paper. We use the normalization for moving targets over function fields from [24,25].

2.1. **Valuations and heights.** Let B be a smooth projective variety over an algebraically closed field k of characteristic zero, and let $K = k(B)$. Fix a projective embedding of B. We denote by M_K the set of prime divisors of B and by $\deg\mathfrak{p}$ the projective degree of $\mathfrak{p}$. We use the convention $\mathrm{ord}_{\mathfrak{p}}(0) = +\infty$ when taking minima of coefficient or coordinate orders. For $a \in K^*$, the product formula is

$$\sum_{\mathfrak{p}\in M_K}\mathrm{ord}_{\mathfrak{p}}(a)\deg\mathfrak{p} = 0. \tag{4}$$

For $a \in K$, set $h(a) = h([1 : a])$. Thus $h(0) = 0$, and for $a \neq 0$,

$$h(a) = -\sum_{\mathfrak{p}\in M_K}\min\{0, \mathrm{ord}_{\mathfrak{p}}(a)\}\deg\mathfrak{p}.$$

In particular, for $a \in K^*$ and each fixed $\mathfrak{p}$,

$$|\mathrm{ord}_{\mathfrak{p}}(a)|\deg\mathfrak{p} \le h(a). \tag{5}$$

Let $P = [x_0 : \cdots : x_N] \in \mathbb{P}^N(K)$. Set $e_{\mathfrak{p}}(\mathbf{x}) = \min_i \mathrm{ord}_{\mathfrak{p}}(x_i)$. The height of P is

$$h(P) = -\sum_{\mathfrak{p}\in M_K} e_{\mathfrak{p}}(\mathbf{x})\deg\mathfrak{p}. \tag{6}$$

By (4), this definition is independent of the chosen homogeneous coordinates.

Let

$$Q = \sum_{|I|=d} a_I X^I \in K[X_0, \ldots, X_N]$$

be a nonzero homogeneous polynomial of degree d. Set

$$e_{\mathfrak{p}}(Q) = \min_I \mathrm{ord}_{\mathfrak{p}}(a_I).$$

We define

$$h(Q) = -\sum_{\mathfrak{p}\in M_K} e_{\mathfrak{p}}(Q)\deg\mathfrak{p}. \tag{7}$$

Thus $h(Q)$ is the height of its coefficient point, and $h(cQ) = h(Q)$ for every $c \in K^*$.

If $\Phi = [F_0 : \cdots : F_N] \colon \mathbb{P}^N \to \mathbb{P}^N$ is a morphism of degree r, where the F_i have no common nontrivial zero, set

$$e_{\mathfrak{p}}(\Phi) = \min_{0 \le i \le N} e_{\mathfrak{p}}(F_i), \qquad h(\Phi) = -\sum_{\mathfrak{p} \in M_K} e_{\mathfrak{p}}(\Phi) \deg \mathfrak{p}.$$

This is again independent of a common scalar multiple of the F_i.

2.2. **Local Weil functions.** For $P \notin V(Q)$, define

$$\lambda_{\mathfrak{p},Q}(P) = \big(\operatorname{ord}_{\mathfrak{p}} Q(\mathbf{x}) - d e_{\mathfrak{p}}(\mathbf{x}) - e_{\mathfrak{p}}(Q)\big) \deg \mathfrak{p}. \tag{8}$$

The ultrametric inequality shows that $\lambda_{\mathfrak{p},Q}(P) \ge 0$. This quantity is unchanged if either $\mathbf{x}$ or Q is multiplied by a nonzero element of K.

The Gauss property of the divisorial valuation gives $e_{\mathfrak{p}}(QR) = e_{\mathfrak{p}}(Q) + e_{\mathfrak{p}}(R)$. Hence, whenever the values involved are defined,

$$\lambda_{\mathfrak{p},QR}(P) = \lambda_{\mathfrak{p},Q}(P) + \lambda_{\mathfrak{p},R}(P), \qquad h(QR) = h(Q) + h(R). \tag{9}$$

Both identities in (9) are exact. Scalar invariance follows from the product formula. Applying (4) to $Q(\mathbf{x}) \in K^*$ yields the global identity

$$\sum_{\mathfrak{p} \in M_K} \lambda_{\mathfrak{p},Q}(P) = d h(P) + h(Q). \tag{10}$$

2.3. **Moving hypersurfaces and coefficient fields.** Let Λ be an infinite index set. A moving hypersurface of positive degree $d \in \mathbb{Z}_{>0}$ is a family

$$Q(\alpha) = \sum_{|I| = d} a_I(\alpha) X^I, \qquad \alpha \in \Lambda,$$

where $a_I(\alpha) \in K$ and the coefficients are not all zero. We consider a finite family $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$, with $\deg Q_j = d_j$.

Let $A \subset \Lambda$ be infinite. Denote by $\mathcal{R}_A^0$ the ring of equivalence classes of maps from cofinite subsets of A to K, where two maps are identified if they agree outside a finite subset. Constant maps identify K with a subfield of $\mathcal{R}_A^0$.

Write $\mathbf{a}_j(\alpha)$ for the coefficient vector of $Q_j(\alpha)$. The set A is coherent with respect to $\mathcal{Q}$ if, for every polynomial F over K that is homogeneous in each block of variables corresponding to $\mathbf{a}_1, \ldots, \mathbf{a}_q$, the function

$$\alpha \longmapsto F\big(\mathbf{a}_1(\alpha), \ldots, \mathbf{a}_q(\alpha)\big)$$

either vanishes for all $\alpha \in A$ or vanishes at only finitely many indices. Every infinite index set contains an infinite coherent subset; see [24, Lemma 4] for the hyperplane coefficient-block precedent. The same Noetherian argument applied to the enlarged finite family of hypersurface coefficient blocks and one additional point block gives the extension used here. Moreover, every infinite subset of a coherent set

remains coherent, and the corresponding normalized coefficient fields are canonically identified by restriction.

Choose homogeneous coordinates $\mathbf{x}(\alpha) = (x_0(\alpha), \dots, x_N(\alpha))$ for a moving point $x(\alpha)$. We say that A is coherent with respect to $\mathcal{Q}$ and x if the same zero alternative holds for every polynomial over K that is homogeneous in each coefficient block $\mathbf{a}_j$ and homogeneous in the additional block $\mathbf{x}$. This property is independent of rescaling the individual blocks after restricting to a cofinite subset. The usual Noetherian coherence argument applies to the enlarged finite collection of blocks, exactly as in the joint-coherence construction of [26, Section 3]. Thus every infinite index set has an infinite subset coherent with respect to $\mathcal{Q}$ and x.

For each j, choose a coefficient a_{j,I_j} that is nonzero on a cofinite subset of A. The normalized coefficient field $\mathcal{R}_{A,\mathcal{Q}}$ is the fraction field of the integral domain over K generated in $\mathcal{R}_A^0$ by

$$\frac{a_{j,I}}{a_{j,I_j}}, \qquad 1 \le j \le q, \quad |I| = d_j.$$

Coherence implies that this definition is independent of the discarded finite subsets. A different choice of the normalizing coefficients gives the same field.

If A is jointly coherent and $F \in \mathcal{R}_{A,\mathcal{Q}}[X_0, \dots, X_N]$ is homogeneous, then

$$F(x(\alpha)) = 0 \text{ for infinitely many } \alpha \quad \Longrightarrow \quad F(x(\alpha)) = 0 \text{ outside a finite subset of } A.$$

This follows by clearing denominators and multihomogenizing the coefficient and point blocks.

Lemma 2.1 (Coefficient–point domain and persistent kernel)**.** *Assume that A is jointly coherent with respect to a finite moving family $\mathcal{Q}$ and a point family x. Choose a coordinate x_{i_0} which is nonzero outside a finite subset of A, and put*

$$z_i = x_i/x_{i_0} \quad (0 \le i \le N), \qquad z_{i_0} = 1.$$

Let $T_{A,x}$ be the subring of $\mathcal{R}_A^0$ generated by $\mathcal{R}_{A,\mathcal{Q}}$ and the germs z_i. Then $T_{A,x}$ is an integral domain. If $E = \mathrm{Frac}(T_{A,x})$, the homomorphism

$$\mathrm{ev}_x : \mathcal{R}_{A,\mathcal{Q}}[X_0, \dots, X_N] \longrightarrow E[T], \qquad X_i \longmapsto z_i T,$$

has a homogeneous prime kernel. For each $d \ge 0$, put

$$I_{x,d} = \big\{ F \in \mathcal{R}_{A,\mathcal{Q}}[X_0, \dots, X_N]_d : F(\alpha)(x(\alpha)) = 0 \text{ outside a finite subset of } A \big\},$$

where the subscript d denotes the homogeneous degree-d component. Then

$$\ker(\mathrm{ev}_x) = I_x := \bigoplus_{d \ge 0} I_{x,d}.$$

Proof. Take an element of $T_{A,x}$ and write it as a polynomial in finitely many coefficient-field elements and the z_i. Replace each coefficient-field element by a quotient of polynomials in the normalized coefficient ratios, clear all denominators, and restrict to the common cofinite subset on which these denominators and x_{i_0} are nonzero. If the resulting homogeneous polynomial were the zero germ, the original element would be zero in $\mathcal{R}_A^0$. Otherwise joint coherence says that its value vanishes at only finitely many indices. Thus every nonzero element of $T_{A,x}$ is nonzero outside a finite subset. The product of two such elements has the same property, so $T_{A,x}$ has no zero divisors.

The evaluation map is therefore well-defined, and its kernel is prime because $E[T]$ is a domain. If $F = \sum_d F_d$ is the homogeneous decomposition, then

$$\mathrm{ev}_x(F) = \sum_d F_d(z_0, \ldots, z_N) T^d.$$

This is zero exactly when every coefficient $F_d(z_0, \ldots, z_N)$ is zero in E. Since F_d is homogeneous and x_{i_0} is nonzero outside a finite subset, this is equivalent to $F_d \in I_{x,d}$ for every d. Hence the kernel is homogeneous and equals $\bigoplus_{d \geq 0} I_{x,d}$. □

Remark 2.2 (Descent obstruction). For $K = k(t)$ and $n \in \mathbb{Z}_{\geq 0}$, let

$$Q_n = X_1 - t^n X_0, \qquad x(n) = [1 : t^n : t^{n^2}], \qquad a(n) = t^n.$$

Then $X_1 - aX_0 = 0$ is persistent, whereas no fixed K-hyperplane contains a cofinite subfamily, since

$$\det(1, t^{n_i}, t^{n_i^2})_{i=1}^3 \neq 0 \qquad (n_1 < n_2 < n_3).$$

Indeed, in the expansion of this determinant the monomial $t^{n_2+n_3^2}$ has strictly larger exponent than every other monomial and occurs with coefficient 1. Here $Q_n(x(n)) = 0$, so the avoidance hypothesis fails.

Lemma 2.3 (Cofinite extraction principle)**.** *Let $\mathcal{P}(\alpha)$ be a property indexed by an infinite set Λ. Assume that every infinite subset $T \subset \Lambda$ contains an infinite subset $A \subset T$ such that $\mathcal{P}(\alpha)$ holds for all but finitely many $\alpha \in A$. Then $\mathcal{P}(\alpha)$ holds outside a finite subset of Λ.*

Proof. If the failure set were infinite, take it for T. The hypothesis would produce an infinite subset A of failures on which the property holds after deleting finitely many indices, a contradiction. □

When A is jointly coherent, every infinite subset of A is jointly coherent as well. In particular, restriction induces an injective map on the normalized coefficient field: a nonzero class cannot become zero after restriction, since that would give infinitely many zeros on A. Thus a persistent relation cannot be created by any of the further infinite refinements used below.

The family $\mathcal{Q}$ is in general position if there is a cofinite subset $\Lambda_0 \subset \Lambda$ such that, for every $\alpha \in \Lambda_0$ and every set of distinct indices $j_0, \dots, j_N$, the equations

$$Q_{j_0}(\alpha) = \cdots = Q_{j_N}(\alpha) = 0$$

have no common point in $\mathbb{P}^N(\overline{K})$. Equivalently, every $N+1$ members have nonzero resultant on a cofinite subset.

Let $x\colon \Lambda \to \mathbb{P}^N(K)$ be a family of points with $h(x(\alpha)) \to \infty$. We say that Q_j moves slowly with respect to x if

$$h(Q_j(\alpha)) = o(h(x(\alpha))). \tag{11}$$

If every member of $\mathcal{Q}$ moves slowly, then each $f \in \mathcal{R}_{A,\mathcal{Q}}$ satisfies

$$h(f(\alpha)) = o(h(x(\alpha))) \tag{12}$$

on a cofinite subset on which f is defined. This follows from the standard height inequalities for sums, products, and quotients in K.

The family x is linearly nondegenerate with respect to $\mathcal{Q}$ if, for every infinite coherent subset $A \subset \Lambda$, its coordinate functions are linearly independent over $\mathcal{R}_{A,\mathcal{Q}}$. It is algebraically nondegenerate with respect to $\mathcal{Q}$ if, for every such A, no nonzero homogeneous polynomial in $\mathcal{R}_{A,\mathcal{Q}}[X_0, \dots, X_N]$ vanishes at $x(\alpha)$ outside a finite subset of A. These notions will be used only inside the proofs of the moving-target theorems. No algebraic nondegeneracy assumption is imposed on the dynamical sequence in the main theorem.

2.4. **Moving counting functions.** Let $D(\alpha) = V(G(\alpha))$ be a moving hypersurface of degree δ and let $P(\alpha) \notin D(\alpha)$. For a finite subset $S \subset M_K$, set

$$N_S(D(\alpha), P(\alpha)) = \sum_{\mathfrak{p} \notin S} \lambda_{\mathfrak{p}, G(\alpha)}(P(\alpha)). \tag{13}$$

By (10), the total height relative to $D(\alpha)$ is

$$T_{D(\alpha)}(P(\alpha)) = \delta h(P(\alpha)) + h(G(\alpha)). \tag{14}$$

We call $P(\alpha)$ strictly S-integral with respect to $D(\alpha)$ if $N_S(D(\alpha), P(\alpha)) = 0$. Given $\theta \geq 0$, it is θ-quasi-integral if

$$N_S(D(\alpha), P(\alpha)) \leq \theta T_{D(\alpha)}(P(\alpha)). \tag{15}$$

All these definitions are unchanged when $G(\alpha)$ is multiplied by an arbitrary element of K^*.

3. A Maximum Theorem for Moving Hyperplanes

Let $L_1, \dots, L_t$ be fixed K-linear forms on K^{m+1}, let $\mathcal{I}_L$ be the collection of subsets on which they are linearly independent over K.

Proposition 3.1 (Fixed-target maximum form of Wang)**.** *Let $S \subset M_K$ be finite and let $L_1, \dots, L_t$ be nonzero fixed K-linear forms on K^{m+1}. There is a finite union Z of proper K-linear subspaces of $\mathbb{P}^m$, depending only on the forms. For every $\varepsilon > 0$ there are constants $c_0(\varepsilon), c_1(\varepsilon)$, depending on the displayed data and ε, such that every $z \in \mathbb{P}^m(K)$ satisfying*

$$z \notin Z \cup \bigcup_{j=1}^{t} V(L_j), \qquad h(z) > c_0(\varepsilon),$$

also satisfies

$$\sum_{\mathfrak{p} \in S} \max_{J \in \mathcal{I}_L} \sum_{j \in J} \lambda_{\mathfrak{p}, L_j}(z) \le (m + 1 + \varepsilon) h(z) + c_1(\varepsilon). \tag{16}$$

This is Wang's Main Theorem [11, Main Theorem] with the local normalization $-e_{\mathfrak{p}}(L_j) \deg \mathfrak{p}$ inserted. For fixed forms and finite S, the resulting difference is $O(1)$. The exceptional set Z is independent of ε.

Let $\mathcal{H} = \{H_1, \dots, H_q\}$ be moving hyperplanes in $\mathbb{P}^M(K)$ indexed by an infinite set Λ. Write

$$L_j(\alpha)(X) = \sum_{i=0}^{M} a_{j,i}(\alpha) X_i$$

for a defining linear form. For each α, let $\mathcal{I}(\alpha)$ be the collection of subsets $J \subset \{1, \dots, q\}$ for which the forms $L_j(\alpha)$, $j \in J$, are linearly independent over K.

Proposition 3.2 (Maximum moving-hyperplane theorem)**.** *Let $S \subset M_K$ be finite and let $y \colon \Lambda \to \mathbb{P}^M(K)$ satisfy $h(y(\alpha)) \to \infty$. Assume that:*

1. *y is linearly nondegenerate with respect to $\mathcal{H}$;*
2. *$h(H_j(\alpha)) = o(h(y(\alpha)))$ for $1 \le j \le q$;*
3. *$y(\alpha) \notin \bigcup_{j=1}^{q} H_j(\alpha)$ outside a finite subset.*

Then, for every $\varepsilon > 0$, inequality

$$\sum_{\mathfrak{p} \in S} \max_{J \in \mathcal{I}(\alpha)} \sum_{j \in J} \lambda_{\mathfrak{p}, H_j(\alpha)}(y(\alpha)) \le (M + 1 + \varepsilon) h(y(\alpha)) \tag{17}$$

holds outside a finite subset of Λ.

Proof. If $S = \varnothing$, the left-hand side is zero and the proposition is immediate. Assume from now on that $S \neq \varnothing$.

Fix $\varepsilon > 0$. By Lemma 2.3, it suffices to work on an infinite coherent subset A. Normalize each L_j by a coefficient that is nonzero outside a finite subset of A:

$$L_j(\alpha)(X) = \sum_{i=0}^{M} \xi_{j,i}(\alpha) X_i,$$

where one coefficient is 1 and every $\xi_{j,i}$ belongs to $\mathcal{R}_{A,\mathcal{H}}$. By the slow-movement assumption and (12),

$$h(\xi_{j,i}(\alpha)) = o(h(y(\alpha))). \tag{18}$$

Each rank condition is expressed by finitely many coefficient minors, so coherence makes the collection $\mathcal{I}(\alpha)$ constant outside a finite subset of A. Since S and this collection are finite, we may pass to a further infinite subset and choose, for every $\mathfrak{p} \in S$, a fixed set $J_{\mathfrak{p}}$ at which the maximum in (17) is attained. Over $\mathcal{R}_{A,\mathcal{H}}$, complete the forms with indices in $J_{\mathfrak{p}}$ to a basis

$$L_{\mathfrak{p},0}, \dots, L_{\mathfrak{p},M}$$

of the dual vector space, choosing the additional forms among the coordinate forms. The determinant of this basis is a nonzero element of $\mathcal{R}_{A,\mathcal{H}}$; by coherence, after deleting finitely many indices its specialization is a K-basis for every $\mathfrak{p} \in S$. Since local Weil functions are nonnegative, it is enough to prove

$$\sum_{\mathfrak{p}\in S} \sum_{l=0}^{M} \lambda_{\mathfrak{p},L_{\mathfrak{p},l}(\alpha)}(y(\alpha)) \le (M+1+\varepsilon)h(y(\alpha)). \tag{19}$$

Let $\mathcal{C}$ be the finite set of normalized coefficients of all the forms $L_{\mathfrak{p},l}$. For $s \ge 0$, let $V(s)$ be the K-vector subspace of $\mathcal{R}_{A,\mathcal{H}}$ generated by the monomials of total degree at most s in the elements of $\mathcal{C}$. Put $u_s = \dim_K V(s)$. Then $V(s) \subset V(s+1)$, and the standard Hilbert-function argument gives

$$\liminf_{s\to\infty} \frac{u_{s+1}}{u_s} = 1. \tag{20}$$

This is the coefficient-space estimate used in Wang's proof of the function-field subspace theorem [11]; see also [24].

Choose $\eta > 0$ with $(M+1)\eta < \varepsilon/3$, and then choose s such that $u_{s+1} \le (1+\eta)u_s$. Take a K-basis $b_1, \dots, b_{u_{s+1}}$ of $V(s+1)$ such that $b_1 = 1$ and $b_1, \dots, b_{u_s}$ is a basis of $V(s)$. Define

$$Y(\alpha) = [b_\mu(\alpha)y_i(\alpha)]_{1\le\mu\le u_{s+1},\, 0\le i\le M} \in \mathbb{P}^{(M+1)u_{s+1}-1}(K). \tag{21}$$

The functions b_μ are small with respect to y. From the definition of height,

$$h(Y(\alpha)) = h(y(\alpha)) + o(h(y(\alpha))). \tag{22}$$

Write

$$L_{\mathfrak{p},l} = \sum_{i=0}^{M} a_{\mathfrak{p},l,i}X_i.$$

For $1 \le \nu \le u_s$, multiplication by each coefficient maps b_ν into $V(s+1)$, so there are unique $c_{\mathfrak{p},l,\nu,i,\mu} \in K$ such that

$$a_{\mathfrak{p},l,i}b_\nu = \sum_{\mu=1}^{u_{s+1}} c_{\mathfrak{p},l,\nu,i,\mu}b_\mu.$$

Thus the fixed K-linear form

$$\widetilde{L}_{\mathfrak{p},l,\nu}(Y)=\sum_{i=0}^{M}\sum_{\mu=1}^{u_{s+1}}c_{\mathfrak{p},l,\nu,i,\mu}Y_{\mu,i}$$

satisfies $\widetilde{L}_{\mathfrak{p},l,\nu}(Y)=b_\nu L_{\mathfrak{p},l}(y)$. Denote its hyperplane in $\mathbb{P}^{(M+1)u_{s+1}-1}(K)$ by $\widetilde{H}_{\mathfrak{p},l,\nu}$. For each fixed $\mathfrak{p}\in S$, the hyperplanes

$$\widetilde{H}_{\mathfrak{p},l,\nu},\qquad 0\le l\le M,\quad 1\le \nu\le u_s,$$

are linearly independent over K. Indeed, a linear relation among their defining forms gives

$$\sum_{l=0}^{M}c_l L_{\mathfrak{p},l}=0,\qquad c_l\in V(s).$$

The forms $L_{\mathfrak{p},l}$ are independent over $\mathcal{R}_{A,\mathcal{H}}$, so every c_l is zero. The independence of $b_1,\dots,b_{u_s}$ then gives the assertion. By the injectivity of coherent restriction noted in Section 2, this K-linear independence persists after every coherent refinement used below.

After a finite deletion, $b_\nu L_{\mathfrak{p},l}(y)\neq 0$. This follows from hypothesis (3) for the original forms, from linear nondegeneracy for the added coordinate forms, and from $b_\nu\neq 0$ in the coefficient field.

Apply Proposition 3.1 in dimension $(M+1)u_{s+1}-1$. If Y lay in a fixed exceptional subspace on an infinite subset, then, after a coherent refinement, one of the nonzero fixed linear equations defining that subspace would give

$$\sum_{i=0}^{M}\sum_{\mu=1}^{u_{s+1}}c_{\mu,i}b_\mu(\alpha)y_i(\alpha)=0,\qquad c_{\mu,i}\in K,$$

outside a finite subset. Put $a_i=\sum_{\mu=1}^{u_{s+1}}c_{\mu,i}b_\mu$. Then $a_i\in\mathcal{R}_{A,\mathcal{H}}$. Not all the a_i are zero: otherwise the K-linear independence of $b_1,\dots,b_{u_{s+1}}$ would force every $c_{\mu,i}$ to vanish. We would therefore obtain the nontrivial relation

$$\sum_{i=0}^{M}a_i y_i=0$$

over $\mathcal{R}_{A,\mathcal{H}}$, contrary to linear nondegeneracy. Thus the fixed exceptional set is avoided outside a finite subset. The bounded-height alternative is excluded by (22). Proposition 3.1 therefore yields, for any $\varepsilon_0>0$, the following estimate. For each $\mathfrak{p}$, the displayed subfamily is linearly independent and is therefore bounded by the local maximum in Wang's theorem:

$$\sum_{\mathfrak{p}\in S}\sum_{l=0}^{M}\sum_{\nu=1}^{u_s}\lambda_{\mathfrak{p},\widetilde{H}_{\mathfrak{p},l,\nu}}(Y(\alpha))\le\big((M+1)u_{s+1}+\varepsilon_0\big)h(Y(\alpha))+O(1).\tag{23}$$

By construction,

$$\widetilde{L}_{\mathfrak{p},l,\nu}(Y(\alpha))=b_\nu(\alpha)L_{\mathfrak{p},l}(\alpha)(y(\alpha)).$$

Moreover,

$$e_{\mathfrak{p}}(Y(\alpha))=e_{\mathfrak{p}}(y(\alpha))+\min_{1\le\mu\le u_{s+1}}\operatorname{ord}_{\mathfrak{p}}b_\mu(\alpha).$$

Since S is finite and all the b_μ and the coefficients of the moving forms are small, (5) gives

$$\lambda_{\mathfrak{p},\widetilde{H}_{\mathfrak{p},l,\nu}}(Y(\alpha)) = \lambda_{\mathfrak{p},L_{\mathfrak{p},l}(\alpha)}(y(\alpha)) + o(h(y(\alpha))) + O(1), \tag{24}$$

uniformly for the finitely many indices involved.

Summing (24), using (22) in (23), and dividing by u_s, we obtain

$$\sum_{\mathfrak{p}\in S}\sum_{l=0}^{M} \lambda_{\mathfrak{p},L_{\mathfrak{p},l}(\alpha)}(y(\alpha)) \le \left((M+1)\frac{u_{s+1}}{u_s} + \frac{\varepsilon_0}{u_s} + o(1)\right) h(y(\alpha)).$$

Choose $\varepsilon_0 > 0$ sufficiently small and restrict A so that the $o(1)$ term is smaller than the remaining part of ε. The choice of s then gives (19), and hence (17). Lemma 2.3 now shows that the failure set of (17) is finite. □

4. Slowly Moving Hypersurfaces without Nondegeneracy

We next pass from hyperplanes to hypersurfaces. The point family may be algebraically degenerate over the moving coefficient field. We replace the ambient projective space by the smallest variety defined over that field on which the family is algebraically nondegenerate.

We use the following replacement statement.

Lemma 4.1 (Uniform replacement)**.** *Let F be a field containing the infinite field k, let $W \subset \mathbb{P}^N_F$ be an irreducible projective variety of dimension $\ell \ge 1$, and let $Q_1, \dots, Q_{N+1} \in F[X_0, \dots, X_N]$ be homogeneous polynomials of the same positive degree. Assume that the Q_i have no common zero in $\mathbb{P}^N(\overline{F})$ and that W is not contained in any $V(Q_i)$. Then there are constants $c_{t,j} \in k$ and polynomials*

$$P_0 = Q_1, \qquad P_t = \sum_{j=1}^{N-\ell+t+1} c_{t,j} Q_j, \quad 1 \le t \le \ell,$$

such that $W \not\subset V(P_t)$ for $0 \le t \le \ell$ and

$$\dim\left(W \cap \bigcap_{i=0}^{t} V(P_i)\right) \le \ell - t - 1, \qquad 0 \le t \le \ell. \tag{25}$$

In particular, $W \cap \bigcap_{i=0}^{\ell} V(P_i) = \varnothing$.

Proof. Since $W \not\subset V(Q_1)$, the assertion holds for $t = 0$. Suppose that $P_0, \dots, P_{t-1}$ have been chosen. Every component of dimension $\ell - t$ in

$$W \cap \bigcap_{i=0}^{t-1} V(P_i)$$

fails to be contained in at least one of $V(Q_1), \dots, V(Q_{N-\ell+t+1})$. Otherwise that component would be contained in an intersection of dimension at most $\ell - t - 1$. To verify this bound, put $m = N - \ell + t + 1$. Since the full collection has no common projective zero, the homogeneous ideal $(Q_1, \dots, Q_{N+1})$ has

height $N+1$. The generalized principal ideal theorem, applied successively in the catenary polynomial ring $F[X_0,\dots,X_N]$, says that deleting one generator can lower the height by at most one. Hence

$$\operatorname{ht}(Q_1,\dots,Q_m) \geq (N+1)-(N+1-m) = m.$$

The reverse inequality follows from Krull's height theorem. The same height calculation holds after extension to $\overline{F}$, so the corresponding projective dimension bound is valid geometrically. Therefore $\operatorname{ht}(Q_1,\dots,Q_m) = m$, and

$$\dim V(Q_1,\dots,Q_m) \leq N-m = \ell-t-1.$$

For each of the finitely many components, the coefficient vectors producing a linear combination that vanishes identically on that component form a proper F-linear subspace of $F^{N-\ell+t+1}$. The coefficient vectors for which the same combination vanishes identically on W form another proper F-linear subspace, because none of the Q_j contains W. The intersection of each of these subspaces with $k^{N-\ell+t+1}$ is a proper k-linear subspace, since the standard basis over k spans the ambient space over F. Because k is infinite, finitely many such subspaces cannot cover $k^{N-\ell+t+1}$. We may therefore choose the $c_{t,j}$ in k so that P_t contains neither W nor any component of maximal dimension. This proves (25) by induction. □

The following algebraic computation is used in the filtration argument and is valid without assuming that the homogeneous coordinate ring is Cohen–Macaulay.

Lemma 4.2 (Hilbert-weight asymptotics)**.** *Let R be a field and fix an algebraic closure $\overline{R}$ of R. Let*

$$\mathcal{A} = R[X_0,\dots,X_N]/I$$

be a standard graded domain such that $W = \operatorname{Proj}(\mathcal{A})$ has dimension $\ell \geq 1$ and degree Δ. Let $D_1,\dots,D_\ell \in \mathcal{A}$ be homogeneous elements of the same positive degree d. Assume that the geometric intersection $W_{\overline{R}} \cap V(D_1,\dots,D_\ell)$ has dimension at most zero, where $W_{\overline{R}} = W \times_R \overline{R}$. For L divisible by d, put

$$\tau_L = \{\mathbf{i} \in \mathbb{Z}^\ell_{\geq 0} : d|\mathbf{i}| \leq L\}$$

and order τ_L lexicographically, with the convention that at the first coordinate where two multi-indices differ, the one with the larger coordinate comes later. Filter $\mathcal{A}_L$ by

$$\mathcal{F}_{\mathbf{i}}(L) = \sum_{\substack{\mathbf{e}\in\tau_L \\ \mathbf{e}\geq_{\text{lex}}\mathbf{i}}} D_1^{e_1}\cdots D_\ell^{e_\ell}\mathcal{A}_{L-d|\mathbf{e}|}.$$

If $\mathbf{i}^+$ is the successor of $\mathbf{i}$ in τ_L, set

$$\Delta_{\mathbf{i}}(L) = \dim_R\big(\mathcal{F}_{\mathbf{i}}(L)/\mathcal{F}_{\mathbf{i}^+}(L)\big),$$

and set $\mathcal{F}_{\mathbf{i}^+}(L) = 0$ *when* $\mathbf{i}$ *is maximal in* τ_L*. Then*

$$\sum_{\mathbf{i}\in\tau_L} \Delta_{\mathbf{i}}(L) i_t = \frac{\Delta L^{\ell+1}}{(\ell+1)!d} + O(L^{\ell}), \qquad 1 \le t \le \ell. \tag{26}$$

The implied constant may depend on W *and the* D_i*, but not on* L*.*

Proof. Put $\mathcal{J} = (D_1, \dots, D_\ell)$. The hypothesis on the common zero set and faithful flatness of $\overline{R}/R$ give $\dim(\mathcal{A}/\mathcal{J}) \le 1$. For every minimal prime $\mathfrak{q}$ over $\mathcal{J}$, Krull's height theorem and the dimension formula give

$$\dim(\mathcal{A}/\mathcal{J}) \ge \dim(\mathcal{A}/\mathfrak{q}) = \ell + 1 - \operatorname{ht}\mathfrak{q} \ge 1.$$

Thus $\dim(\mathcal{A}/\mathcal{J}) = 1$ and $\operatorname{ht}\mathcal{J} = \ell$. For the Corvaja–Zannier filtration [27], we use the colon-space argument of [29, Section 3]; no Cohen–Macaulay hypothesis is imposed.

For $\mathbf{i} \in \mathbb{Z}^{\ell}_{\ge 0}$ and $n \ge 0$, put

$$\mathcal{J}_{\mathbf{i},n} = \left\{ \gamma \in \mathcal{A}_n : D^{\mathbf{i}}\gamma \in \sum_{\substack{\mathbf{e}\in\mathbb{Z}^{\ell}_{\ge 0} \\ \mathbf{e}>_{\mathrm{lex}}\mathbf{i}}} D^{\mathbf{e}}\mathcal{A}_{n+d|\mathbf{i}|-d|\mathbf{e}|} \right\},$$

where $D^{\mathbf{i}} = D_1^{i_1}\cdots D_\ell^{i_\ell}$ and a graded component of negative degree is understood to be zero. Then $\mathcal{J}_{\mathbf{i}} = \bigoplus_{n\ge 0}\mathcal{J}_{\mathbf{i},n}$ is a homogeneous ideal: multiplication of a defining relation by $a \in \mathcal{A}_s$ puts $a\gamma$ in $\mathcal{J}_{\mathbf{i},n+s}$. Multiplication by $D^{\mathbf{i}}$ induces an isomorphism

$$\mathcal{F}_{\mathbf{i}}(L)/\mathcal{F}_{\mathbf{i}^+}(L) \simeq \mathcal{A}_{L-d|\mathbf{i}|}/\mathcal{J}_{\mathbf{i},L-d|\mathbf{i}|}. \tag{27}$$

The map $\gamma \mapsto D^{\mathbf{i}}\gamma$ is surjective and has kernel $\mathcal{J}_{\mathbf{i},L-d|\mathbf{i}|}$.

If $\mathbf{j} - \mathbf{i} \in \mathbb{Z}^{\ell}_{\ge 0}$, multiplying a defining relation for $\mathcal{J}_{\mathbf{i}}$ by $D^{\mathbf{j}-\mathbf{i}}$ gives

$$\mathcal{J}_{\mathbf{i}} \subseteq \mathcal{J}_{\mathbf{j}}. \tag{28}$$

The set of ideals $\{\mathcal{J}_{\mathbf{i}} : \mathbf{i} \in \mathbb{Z}^{\ell}_{\ge 0}\}$ is finite. Indeed, if infinitely many distinct ideals occurred, choose one multi-index for each of them. Dickson's lemma gives an infinite componentwise nondecreasing subsequence

$$\mathbf{i}_1 \le \mathbf{i}_2 \le \cdots.$$

After deleting repetitions of multi-indices, the corresponding ideals remain pairwise distinct. Equation (28) would then give the strictly increasing chain

$$\mathcal{J}_{\mathbf{i}_1} \subsetneq \mathcal{J}_{\mathbf{i}_2} \subsetneq \cdots,$$

contrary to the ascending-chain condition in the Noetherian ring $\mathcal{A}$.

For every $\mathbf{i}$, $\mathcal{J} \subseteq \mathcal{J}_{\mathbf{i}}$, since $D^{\mathbf{i}} D_t \gamma = D^{\mathbf{i}+\mathbf{e}_t}\gamma$ occurs later in the lexicographic order. Since $\mathrm{Proj}(\mathcal{A}/\mathcal{J})$ is zero-dimensional,

$$c_{\mathbf{i}}(n) = \dim_R\big(\mathcal{A}_n/\mathcal{J}_{\mathbf{i},n}\big)$$

is uniformly bounded and eventually constant. Since only finitely many ideals $\mathcal{J}_{\mathbf{i}}$ occur, the maximum of their stabilization indices gives an integer n_0 valid for every $\mathbf{i}$. The same finiteness, together with $\mathcal{J} \subseteq \mathcal{J}_{\mathbf{i}}$, gives a single bound for all the values $c_{\mathbf{i}}(n)$, including the finitely many degrees $n \le n_0$.

Let $c_{\mathbf{i}}$ denote this stable value, choose $\mathbf{i}_0$ for which $c_{\mathbf{i}_0}$ is minimal, and set $c_0 = c_{\mathbf{i}_0}$. If $\mathbf{i} - \mathbf{i}_0 \in \mathbb{Z}^{\ell}_{\ge 0}$, then (28) gives $c_{\mathbf{i}} \le c_{\mathbf{i}_0}$; minimality therefore gives $c_{\mathbf{i}} = c_0$. For $m = L/d$, define the interior set

$$\tau_L^0 = \left\{ \mathbf{i} \in \tau_L : L - d|\mathbf{i}| > n_0,\ \mathbf{i} - \mathbf{i}_0 \in \mathbb{Z}^{\ell}_{\ge 0} \right\}.$$

Its complement has $O(L^{\ell-1})$ elements: a fixed-width boundary shell and the strips $i_t < (\mathbf{i}_0)_t$. By (27),

$$\Delta_{\mathbf{i}}(L) = c_0 \quad (\mathbf{i} \in \tau_L^0), \qquad \Delta_{\mathbf{i}}(L) = O(1) \quad (\mathbf{i} \in \tau_L), \tag{29}$$

uniformly in $\mathbf{i}$ and L.

The successive quotients exhaust $\mathcal{A}_L$. Hence the Hilbert polynomial of W and (29) give

$$\frac{\Delta L^{\ell}}{\ell!} + O(L^{\ell-1}) = \dim_R \mathcal{A}_L = \sum_{\mathbf{i} \in \tau_L} \Delta_{\mathbf{i}}(L) = c_0 \frac{L^{\ell}}{d^{\ell}\ell!} + O(L^{\ell-1}).$$

Thus

$$c_0 = \Delta d^{\ell}. \tag{30}$$

$i_t = O(L)$ on τ_L. The boundary estimate in (29), together with (30), therefore gives

$$\sum_{\mathbf{i} \in \tau_L} \Delta_{\mathbf{i}}(L) i_t = \Delta d^{\ell} \sum_{\mathbf{i} \in \tau_L} i_t + O(L^{\ell}).$$

The lattice identity

$$\sum_{\substack{\mathbf{i} \in \mathbb{Z}^{\ell}_{\ge 0} \\ |\mathbf{i}| \le m}} i_t = \binom{m+\ell}{\ell+1} = \frac{m^{\ell+1}}{(\ell+1)!} + O(m^{\ell})$$

now gives

$$\sum_{\mathbf{i} \in \tau_L} \Delta_{\mathbf{i}}(L) i_t = \Delta d^{\ell} \binom{L/d+\ell}{\ell+1} + O(L^{\ell}) = \frac{\Delta L^{\ell+1}}{(\ell+1)!d} + O(L^{\ell}).$$

This proves (26). □

The next lemma applies this computation to a different system of parameters at each place in S.

Lemma 4.3 (Filtration estimate)**.** *Let $S \subset M_K$ be finite, let $\mathcal{Q}$ be a finite family of moving hypersurfaces on an infinite index set A, and let $x\colon A \to \mathbb{P}^N(K)$ satisfy $h(x(\alpha)) \to \infty$. Assume that A is jointly coherent with $\mathcal{Q}$*

and x, put $R = \mathcal{R}_{A,\mathcal{Q}}$, and assume that every element of R is small with respect to x. Let $I_x \subset R[X_0, \dots, X_N]$ be the full homogeneous ideal of relations of x, and fix an algebraic closure $\overline{R}$ of R. Put

$$W = \mathrm{Proj}(R[X_0, \dots, X_N]/I_x).$$

Assume that W is irreducible of dimension $\ell \geq 1$. For every $\mathfrak{p} \in S$, let $P_{\mathfrak{p},0}, \dots, P_{\mathfrak{p},\ell} \in R[X_0, \dots, X_N]$ be homogeneous polynomials of the same positive degree d, with coefficients small relative to x, and assume that their geometric common zero set on $W_{\overline{R}} = W \times_R \overline{R}$ is empty. Assume also that $P_{\mathfrak{p},t}(\alpha)(x(\alpha)) \neq 0$ outside a finite subset, for all $\mathfrak{p} \in S$ and $0 \leq t \leq \ell$. Then, for every $\varepsilon > 0$, after passing to an infinite subset of A,

$$\frac{1}{d} \sum_{\mathfrak{p} \in S} \sum_{t=0}^{\ell} \lambda_{\mathfrak{p},P_{\mathfrak{p},t}(\alpha)}(x(\alpha)) \leq (\ell + 1 + \varepsilon) h(x(\alpha)). \tag{31}$$

Proof. If $S = \varnothing$, the left-hand side of (31) is zero, so the assertion is immediate. Assume from now on that $S \neq \varnothing$.

The Nullstellensatz gives a common $m \geq d$ and homogeneous $C_{\mathfrak{p},k,t}$ of degree $m - d$ such that

$$X_k^m - \sum_{t=0}^{\ell} C_{\mathfrak{p},k,t} P_{\mathfrak{p},t} \in I_x.$$

Their coefficients are small. Fix $\mathfrak{p}$ and choose k with $\mathrm{ord}_{\mathfrak{p}}\, x_k = e_{\mathfrak{p}}(\mathbf{x})$. In the identity above, at least one nonzero summand on the right has order at most $m e_{\mathfrak{p}}(\mathbf{x})$. For such a summand, the definition of the local Weil function (and $\deg C_{\mathfrak{p},k,t} = m - d$) gives

$$\mathrm{ord}_{\mathfrak{p}}\big(C_{\mathfrak{p},k,t}(x(\alpha)) P_{\mathfrak{p},t}(x(\alpha))\big) \geq m e_{\mathfrak{p}}(\mathbf{x}) + e_{\mathfrak{p}}(C_{\mathfrak{p},k,t}(\alpha)) + e_{\mathfrak{p}}(P_{\mathfrak{p},t}(\alpha)) + \frac{\lambda_{\mathfrak{p},P_{\mathfrak{p},t}(\alpha)}(x(\alpha))}{\deg \mathfrak{p}},$$

and hence

$$\frac{\lambda_{\mathfrak{p},P_{\mathfrak{p},t}(\alpha)}(x(\alpha))}{\deg \mathfrak{p}} \leq -e_{\mathfrak{p}}(C_{\mathfrak{p},k,t}(\alpha)) - e_{\mathfrak{p}}(P_{\mathfrak{p},t}(\alpha)).$$

Taking the minimum over t and using (5) (coefficientwise for the finitely many $C_{\mathfrak{p},k,t}$ and $P_{\mathfrak{p},t}$), together with the nonnegativity of local Weil functions, yields

$$\min_{0 \leq t \leq \ell} \lambda_{\mathfrak{p},P_{\mathfrak{p},t}(\alpha)}(x(\alpha)) = o(h(x(\alpha))). \tag{32}$$

After restricting A, fix for each $\mathfrak{p}$ the ℓ largest local Weil functions. Let Z be a positive-dimensional component of their common zero set over $\overline{R}$. If Z were contained in the omitted hypersurface, it would lie in the common zero set of all $\ell + 1$ polynomials. Otherwise the projective variety Z would meet that positive-degree hypersurface. Both alternatives contradict the assumed empty intersection. Hence the selected ℓ polynomials have geometric common zero set of dimension at most zero. Faithful flatness descends this dimension statement to R. Since the homogeneous coordinate ring of W has dimension $\ell + 1$, Krull's height theorem now shows that the ideal generated by these ℓ polynomials has height exactly ℓ.

Enumerate the selected auxiliary parameter polynomials as $D_{\mathfrak{p},1}, \ldots, D_{\mathfrak{p},\ell}$ (these are distinct from the target divisors D_n). For a multi-index $\mathbf{i} = (i_1, \ldots, i_\ell)$, let $|\mathbf{i}| = i_1 + \cdots + i_\ell$ and put $I_{x,L} = I_x \cap R[X_0, \ldots, X_N]_L$. Set

$$\mathcal{F}_{\mathfrak{p},\mathbf{i}}(L) = \sum_{\substack{\mathbf{e}\in\tau_L \\ \mathbf{e}\geq_{\mathrm{lex}}\mathbf{i}}} D_{\mathfrak{p},1}^{e_1} \cdots D_{\mathfrak{p},\ell}^{e_\ell} R[X_0, \ldots, X_N]_{L-d|\mathbf{e}|} \pmod{I_{x,L}}.$$

These spaces form a finite filtration, and Lemma 4.2 applies.

Let $\Delta = \deg W$. For a sufficiently large integer L divisible by d, write

$$M_L = \dim_R(R[X_0, \ldots, X_N]_L / I_{x,L}).$$

The Hilbert polynomial gives

$$M_L = \frac{\Delta L^\ell}{\ell!} + O(L^{\ell-1}). \tag{33}$$

If $\Delta_{\mathfrak{p},\mathbf{i}}(L)$ denotes the dimension of the successive quotient at $\mathbf{i}$, Lemma 4.2 gives

$$\Gamma_{L,\mathfrak{p},t} = \sum_{\mathbf{i}} \Delta_{\mathfrak{p},\mathbf{i}}(L) i_t = \frac{\Delta L^{\ell+1}}{(\ell+1)!d} + O(L^\ell), \qquad 1 \leq t \leq \ell. \tag{34}$$

The error is uniform for $\mathfrak{p} \in S$, since S is finite. Define

$$\Gamma_L = \min_{\mathfrak{p}\in S,\, 1\leq t\leq \ell} \Gamma_{L,\mathfrak{p},t}. \tag{35}$$

Then Γ_L has the same asymptotic expansion as the right-hand side of (34). We use the minimum because the lower-order terms need not be identical for different parameters.

Fix an R-basis $\phi_1, \ldots, \phi_{M_L}$ of the degree-L component and define

$$F_L(x(\alpha)) = [\phi_1(\alpha)(x(\alpha)) : \cdots : \phi_{M_L}(\alpha)(x(\alpha))]. \tag{36}$$

Joint coherence and $\phi_j \notin I_x$ imply that, after a finite deletion, $\phi_j(\alpha)(x(\alpha)) \neq 0$ for every j. Hence $F_L(x(\alpha))$ is defined. The coefficients of the ϕ_i are small. Comparing this basis with the degree-L monomials gives

$$h(F_L(x(\alpha))) = Lh(x(\alpha)) + o(h(x(\alpha))). \tag{37}$$

The corresponding local comparison is as follows. Since $\{\phi_j\}_{j=1}^{M_L}$ is an R-basis of the degree-L component, choose finite transition coefficients $c_{j,I}, b_{I,j} \in R$ such that

$$\phi_j \equiv \sum_{|I|=L} c_{j,I} X^I, \qquad X^I \equiv \sum_{j=1}^{M_L} b_{I,j}\phi_j \pmod{I_x}.$$

For a fixed $\mathfrak{p} \in S$, let u_C and u_B be the minima of the $\mathfrak{p}$-orders of the nonzero transition coefficients $c_{j,I}$ and $b_{I,j}$, respectively. Their smallness gives $|u_C| + |u_B| = o(h(x(\alpha)))/\deg \mathfrak{p}$. If $e = e_{\mathfrak{p}}(\mathbf{x}(\alpha))$, then

$$e_{\mathfrak{p}}(F_L(x(\alpha))) \geq Le + u_C.$$

Choosing a nonzero coordinate x_k with $\operatorname{ord}_{\mathfrak{p}} x_k = e$ and evaluating the second transition for X_k^L gives

$$Le \geq e_{\mathfrak{p}}(F_L(x(\alpha))) + u_B.$$

Consequently,

$$e_{\mathfrak{p}}(F_L(x(\alpha))) = Le_{\mathfrak{p}}(\mathbf{x}(\alpha)) + o(h(x(\alpha)))/\deg \mathfrak{p}. \tag{38}$$

For each $\mathfrak{p}$, choose a basis adapted to the filtration. A basis element belonging to the quotient indexed by $\mathbf{i}$ may be represented in the form

$$D_{\mathfrak{p},1}^{i_1} \cdots D_{\mathfrak{p},\ell}^{i_\ell} \gamma,$$

where γ is homogeneous of degree $L - d|\mathbf{i}|$. These elements are linear combinations of the ϕ_i and hence define M_L moving hyperplanes in $\mathbb{P}^{M_L-1}$. They are linearly independent over R. The determinant of each adapted basis is therefore a nonzero element of R; joint coherence lets us delete finitely many indices so that its specialization is nonzero for every $\mathfrak{p} \in S$. Their coefficients belong to R, so their normalized coefficient field is contained in R and their heights are $o(h(F_L(x(\alpha))))$ by (12) and (37).

After a finite deletion, $F_L(x)$ avoids the adapted hyperplanes. Indeed, each adapted basis element represents a nonzero class in the degree-L component, and hence does not belong to I_x; joint coherence then excludes infinitely many zeros at $x(\alpha)$.

Their normalized coefficient field R_H embeds in R. Suppose that, on an infinite coherent subset, the coordinates of $F_L(x)$ satisfied a nontrivial relation

$$\sum_{j=1}^{M_L} a_j \phi_j(x) = 0, \qquad a_j \in R_H.$$

By the injectivity of coherent restriction, the coefficients may be viewed as elements of R, not all zero. Since $\phi_1, \ldots, \phi_{M_L}$ form an R-basis of the degree-L component, the polynomial

$$\psi = \sum_{j=1}^{M_L} a_j \phi_j$$

represents a nonzero class and therefore $\psi \notin I_x$. On the other hand, the displayed relation and joint coherence imply that $\psi(x(\alpha)) = 0$ outside a finite subset, so the definition of I_x gives $\psi \in I_x$, a contradiction. Hence $F_L(x)$ is linearly nondegenerate with respect to these hyperplanes.

Proposition 3.2, in dimension $M_L - 1$ with error 1, gives

$$\sum_{\mathfrak{p} \in S} \sum_{i=1}^{M_L} \lambda_{\mathfrak{p}, H_{\mathfrak{p},i}(\alpha)}(F_L(x(\alpha))) \leq (M_L + 1) h(F_L(x(\alpha))) \tag{39}$$

outside a finite subset. For an adapted basis element indexed by $\mathbf{i}$, write $\psi = D_{\mathfrak{p},1}^{i_1} \cdots D_{\mathfrak{p},\ell}^{i_\ell} \gamma$ and write $\psi = \sum_j v_j \phi_j$ modulo I_x. Put $u_v = \min_{v_j \neq 0} \operatorname{ord}_{\mathfrak{p}}(v_j)$ and let H_ψ be the corresponding hyperplane. Then

$$\frac{\lambda_{\mathfrak{p}, H_\psi}(F_L(x(\alpha)))}{\deg \mathfrak{p}} = \operatorname{ord}_{\mathfrak{p}} \psi(x(\alpha)) - e_{\mathfrak{p}}(F_L(x(\alpha))) - u_v,$$

$$\sum_{t=1}^{\ell} i_t \frac{\lambda_{\mathfrak{p},D_{\mathfrak{p},t}(\alpha)}(x(\alpha))}{\deg \mathfrak{p}} = \operatorname{ord}_{\mathfrak{p}}(D_{\mathfrak{p},1}^{i_1}\cdots D_{\mathfrak{p},\ell}^{i_\ell})(x(\alpha)) - d|\mathbf{i}|e_{\mathfrak{p}}(\mathbf{x}(\alpha)) - \sum_{t=1}^{\ell} i_t e_{\mathfrak{p}}(D_{\mathfrak{p},t}).$$

Here $\operatorname{ord}_{\mathfrak{p}}\gamma(x(\alpha)) \geq (L-d|\mathbf{i}|)e_{\mathfrak{p}}(\mathbf{x}(\alpha)) + e_{\mathfrak{p}}(\gamma)$. Using (38), all coefficient-order terms are small (for fixed L). Subtracting the two displayed expressions and using $e_{\mathfrak{p}}(F_L) \leq Le_{\mathfrak{p}}(\mathbf{x}) - u_B$ gives

$$\begin{aligned}\frac{\lambda_{\mathfrak{p},H_\psi}(F_L(x(\alpha)))}{\deg \mathfrak{p}} - \sum_{t=1}^{\ell} i_t \frac{\lambda_{\mathfrak{p},D_{\mathfrak{p},t}(\alpha)}(x(\alpha))}{\deg \mathfrak{p}} &\geq u_B + e_{\mathfrak{p}}(\gamma) - u_v + \sum_{t=1}^{\ell} i_t e_{\mathfrak{p}}(D_{\mathfrak{p},t}) \\ &\geq -o(h(x(\alpha)))/\deg \mathfrak{p}.\end{aligned}$$

Hence

$$\lambda_{\mathfrak{p},H_{\mathfrak{p},i}(\alpha)}(F_L(x(\alpha))) \geq \sum_{t=1}^{\ell} i_t \lambda_{\mathfrak{p},D_{\mathfrak{p},t}(\alpha)}(x(\alpha)) - o(h(x(\alpha))).$$

Summing over an adapted basis and using nonnegativity with (34)–(35) gives

$$\Gamma_L \sum_{\mathfrak{p}\in S}\sum_{t=1}^{\ell} \lambda_{\mathfrak{p},D_{\mathfrak{p},t}(\alpha)}(x(\alpha)) \leq \sum_{\mathfrak{p}\in S}\sum_{i=1}^{M_L} \lambda_{\mathfrak{p},H_{\mathfrak{p},i}(\alpha)}(F_L(x(\alpha))) + o(h(x(\alpha))). \tag{40}$$

Here L is fixed before $\alpha \to \infty$, so all coefficient errors are $o(h(x(\alpha)))$.

Combining (33)– (40), we obtain

$$\frac{1}{d}\sum_{\mathfrak{p}\in S}\sum_{t=1}^{\ell} \lambda_{\mathfrak{p},D_{\mathfrak{p},t}(\alpha)}(x(\alpha)) \leq \left(\ell + 1 + O\left(\frac{1}{L}\right) + o(1)\right) h(x(\alpha)).$$

The local comparison errors are uniform over $\mathfrak{p} \in S$, since S is finite. The order of choices is as follows: first fix a sufficiently large multiple L of d so that the term $O(1/L)$ is smaller than $\varepsilon/2$; then restrict A by deleting finitely many indices so that the remaining $o(1)$ term is smaller than $\varepsilon/2$. Adding the omitted term via (32) proves (31). □

Theorem 4.4 (Moving hypersurfaces without nondegeneracy)**.** *Let $S \subset M_K$ be finite, let Λ be an infinite index set, and let $Q_1,\dots,Q_q$ be moving hypersurfaces in $\mathbb{P}^N(K)$ of fixed degrees $d_1,\dots,d_q \in \mathbb{Z}_{>0}$, where $q \geq N+1$. Let $x\colon \Lambda \to \mathbb{P}^N(K)$ satisfy $h(x(\alpha)) \to \infty$. Assume that:*

(1) *$Q_1,\dots,Q_q$ are in general position;*

(2) *$h(Q_j(\alpha)) = o(h(x(\alpha)))$ for every j;*

(3) *$x(\alpha) \notin \bigcup_{j=1}^{q} V(Q_j(\alpha))$ outside a finite subset.*

Then, for every $\varepsilon > 0$, inequality

$$\sum_{\mathfrak{p}\in S}\sum_{j=1}^{q} \frac{1}{d_j}\lambda_{\mathfrak{p},Q_j(\alpha)}(x(\alpha)) \leq (C_N + \varepsilon)h(x(\alpha)) \tag{41}$$

holds outside a finite subset of Λ, where C_N is given by (2).

Proof. If $S = \varnothing$, the left-hand side is zero and the theorem is immediate. Assume from now on that $S \neq \varnothing$.

By Lemma 2.3, begin with an arbitrary infinite subset of Λ. Let $d = \operatorname{lcm}(d_1, \dots, d_q)$. Replacing Q_j by Q_j^{d/d_j} gives

$$\frac{1}{d_j}\lambda_{\mathfrak{p},Q_j}(x) = \frac{1}{d}\lambda_{\mathfrak{p},Q_j^{d/d_j}}(x).$$

This preserves supports, general position, avoidance, and slow movement. After a joint coherent refinement A, assume $\deg Q_j = d$ and put $R = \mathcal{R}_{A,\mathcal{Q}}$.

For each j, choose a coefficient a_{j,I_j} that is nonzero outside a finite subset of A and replace Q_j by the normalized representative

$$\widetilde{Q}_j = \frac{Q_j}{a_{j,I_j}}. \tag{42}$$

From now on, write Q_j for $\widetilde{Q}_j$. Then $Q_j \in R[X_0, \dots, X_N]$, one of its coefficients is 1, and all of its coefficients are small with respect to x. Heights, local Weil functions, supports, and general position are unchanged; see [20, Remark 1.5(ii)].

The resultant of every selected $N+1$ normalized forms is nonzero in R. Hence these forms have no common zero in $\mathbb{P}^N(\overline{R})$.

Joint coherence gives a coordinate x_{i_0} that is nonzero outside a finite subset. Apply Lemma 2.1 to obtain the coefficient–point domain $T_{A,x}$ and let $E = \operatorname{Frac}(T_{A,x})$. The graded homomorphism

$$R[X_0, \dots, X_N] \longrightarrow E[T], \qquad X_i \longmapsto (x_i/x_{i_0})T,$$

has a homogeneous prime kernel I_x. By the lemma, this kernel is the full homogeneous ideal of persistent relations. Consequently,

$$W = \operatorname{Proj}(R[X_0, \dots, X_N]/I_x)$$

is irreducible. Write $\ell = \dim W$.

We have $\ell \geq 1$. Indeed, if $\ell = 0$, every coordinate ratio of x is algebraic over R. For each i, choose a relation of minimal positive degree

$$a_{i,0} + a_{i,1}Z + \cdots + a_{i,m_i}Z^{m_i} = 0, \qquad a_{i,m_i} \neq 0,$$

with coefficients in R, satisfied by $Z = x_i/x_{i_0}$. Divide by a_{i,m_i} and put $c_{i,\nu} = a_{i,\nu}/a_{i,m_i}$. If $\mathfrak{p}$ is a pole of Z and $w = -\operatorname{ord}_{\mathfrak{p}} Z > 0$, the leading term has order $-m_i w$. The ultrametric inequality forces some $\nu < m_i$ with

$$\operatorname{ord}_{\mathfrak{p}} c_{i,\nu} - \nu w \leq -m_i w,$$

and therefore

$$(m_i - \nu)w \leq -\operatorname{ord}_{\mathfrak{p}} c_{i,\nu}.$$

Summing over all poles gives the root-height estimate

$$h(x_i(\alpha)/x_{i_0}(\alpha)) \le \sum_{\nu=0}^{m_i-1} h(c_{i,\nu}(\alpha)).$$

Every term on the right is $o(h(x(\alpha)))$ by (12). On the other hand,

$$h(x(\alpha)) \le \sum_{i\neq i_0} h(x_i(\alpha)/x_{i_0}(\alpha)).$$

This gives $h(x(\alpha)) = o(h(x(\alpha)))$, a contradiction. Thus $1 \le \ell \le N$.

Fix $\mathfrak{p} \in S$. On a further infinite subset, order the local Weil functions as $\mu_1 \ge \cdots \ge \mu_q$, with corresponding indices $j_{\mathfrak{p},1}, \ldots, j_{\mathfrak{p},q}$. General position implies that, for some integer m and every $0 \le k \le N$, there are homogeneous polynomials $B_{k,i} \in R[X_0, \ldots, X_N]$ of degree $m-d$ such that

$$X_k^m = \sum_{i=1}^{N+1} B_{k,i} Q_{j_{\mathfrak{p},i}}. \tag{43}$$

All coefficients occurring here are fixed elements of R and are therefore small relative to x. Specialize (43) at α, evaluate it at $x(\alpha)$, and choose k with $\operatorname{ord}_{\mathfrak{p}} x_k(\alpha) = e_{\mathfrak{p}}(\mathbf{x}(\alpha))$. At least one nonzero summand $B_{k,i}(\alpha)(x(\alpha))Q_{j_{\mathfrak{p},i}}(\alpha)(x(\alpha))$ has order at most $me_{\mathfrak{p}}(\mathbf{x}(\alpha))$. Since $\deg B_{k,i} = m-d$, the definition of the local Weil function gives

$$\frac{\lambda_{\mathfrak{p},Q_{j_{\mathfrak{p},i}}(\alpha)}(x(\alpha))}{\deg \mathfrak{p}} \le -e_{\mathfrak{p}}(B_{k,i}(\alpha)) - e_{\mathfrak{p}}(Q_{j_{\mathfrak{p},i}}(\alpha)).$$

Coefficientwise application of (5) gives

$$\left(-e_{\mathfrak{p}}(B_{k,i}(\alpha)) - e_{\mathfrak{p}}(Q_{j_{\mathfrak{p},i}}(\alpha))\right) \deg \mathfrak{p} = o(h(x(\alpha))).$$

Only finitely many coefficient polynomials occur, so the bound is uniform in i and k. Therefore

$$\min_{1\le i\le N+1} \mu_i = \mu_{N+1} = o(h(x(\alpha))).$$

Since every omitted function is at most μ_{N+1} and q is fixed, we obtain

$$\sum_{j=1}^{q} \lambda_{\mathfrak{p},Q_j(\alpha)}(x(\alpha)) \le \sum_{i=1}^{N+1} \lambda_{\mathfrak{p},Q_{j_{\mathfrak{p},i}}(\alpha)}(x(\alpha)) + o(h(x(\alpha))). \tag{44}$$

Lemma 4.1 gives degree-d polynomials $P_{\mathfrak{p},0}, \ldots, P_{\mathfrak{p},\ell}$ with no common zero on W and $P_{\mathfrak{p},t} \notin I_x$. Joint coherence gives $P_{\mathfrak{p},t}(\alpha)(x(\alpha)) \neq 0$ outside a finite subset. The triangular form gives

$$\sum_{i=1}^{N+1} \mu_i \le (N-\ell+1)\mu_1 + \sum_{t=1}^{\ell} \mu_{N-\ell+t+1}.$$

The definition of $P_{\mathfrak{p},t}$ and slow movement imply $\mu_1 = \lambda_{\mathfrak{p},P_{\mathfrak{p},0}}$ and

$$\mu_{N-\ell+t+1} \le \lambda_{\mathfrak{p},P_{\mathfrak{p},t}} + o(h(x(\alpha))).$$

The inequality follows from the ultrametric inequality and the smallness of the normalized coefficients. Consequently,

$$\sum_{i=1}^{N+1} \lambda_{\mathfrak{p},Q_{j_{\mathfrak{p},i}(\alpha)}}(x(\alpha)) \le (N-\ell+1)\sum_{t=0}^{\ell} \lambda_{\mathfrak{p},P_{\mathfrak{p},t}(\alpha)}(x(\alpha)) + o(h(x(\alpha))). \tag{45}$$

Each $P_{\mathfrak{p},t}$ is a fixed k-linear combination of slowly moving hypersurfaces and is therefore slow relative to x.

Sum (44) and (45) over S, and apply Lemma 4.3. For every positive η, after restricting the index set,

$$\frac{1}{d}\sum_{\mathfrak{p}\in S}\sum_{j=1}^{q} \lambda_{\mathfrak{p},Q_j(\alpha)}(x(\alpha)) \le \big((N-\ell+1)(\ell+1)+\eta\big)h(x(\alpha)).$$

Finally,

$$(N-\ell+1)(\ell+1) \le \left\lfloor \frac{(N+2)^2}{4} \right\rfloor = C_N.$$

Choosing $\eta < \varepsilon$ proves (41). Lemma 2.3 now shows that the failure set of (41) is finite. □

Remark 4.5. The proof gives the coefficient $(N-\ell+1)(\ell+1)$ on the coherent subset selected in the argument. The dimension ℓ is the dimension of the minimal coefficient-field variety of that refinement and need not be uniform under a different refinement. We use the universal constant C_N in the dynamical theorem. If the point family is algebraically nondegenerate over the moving coefficient field, then $\ell = N$ and the coefficient reduces to $N+1$.

5. Resultants and Pullback Estimates

Let

$$\Phi = [F_0 : \cdots : F_N] \colon \mathbb{P}^N_K \longrightarrow \mathbb{P}^N_K$$

be a morphism of degree $r \ge 1$. Denote its Macaulay resultant by $\mathrm{Res}(\Phi)$. This resultant is homogeneous of degree r^N in the coefficients of each F_i. Its total coefficient degree is

$$D_{N,r} = (N+1)r^N. \tag{46}$$

For $\mathfrak{p} \in M_K$, define the resultant defect

$$\rho_{\mathfrak{p}}(\Phi) = \mathrm{ord}_{\mathfrak{p}}\,\mathrm{Res}(\Phi) - D_{N,r}e_{\mathfrak{p}}(\Phi). \tag{47}$$

We use the local resultant inequality, in additive valuation form, [30, Proposition 9]. Although that proposition is stated over a complete algebraically closed nonarchimedean field, for each $\mathfrak{p}$ it applies after passing to the completed algebraic closure of the completion of K at $\mathfrak{p}$; all quantities below are valuations of elements of K, so the resulting inequality descends to K. For $\mathbf{x} \in K^{N+1} \setminus \{\mathbf{0}\}$, after scaling the coefficients of Φ and $\mathbf{x}$ so that $e_{\mathfrak{p}}(\Phi) = e_{\mathfrak{p}}(\mathbf{x}) = 0$, one has

$$\min_{0\le i\le N} \mathrm{ord}_{\mathfrak{p}}\, F_i(\mathbf{x}) \le \mathrm{ord}_{\mathfrak{p}}\,\mathrm{Res}(\Phi). \tag{48}$$

The invariant form used below is $\tau_{\mathfrak{p}}(\Phi, P) \le \rho_{\mathfrak{p}}(\Phi)$, with the defects defined in the proposition immediately below. This form is unchanged when the coordinate vector of P or all coefficients of Φ are rescaled. After the local normalizations above, it is equivalent to (48), and it is the only resultant estimate needed below.

Proposition 5.1 (Resultant and composition estimates)**.** *Let $P = [x_0 : \cdots : x_N] \in \mathbb{P}^N(K)$ and set*

$$\mathbf{F}(\mathbf{x}) = \big(F_0(\mathbf{x}), \dots, F_N(\mathbf{x})\big).$$

For a chosen homogeneous coordinate vector $\mathbf{x}$ of P, set

$$\tau_{\mathfrak{p}}(\Phi, P) = e_{\mathfrak{p}}\big(\mathbf{F}(\mathbf{x})\big) - re_{\mathfrak{p}}(\mathbf{x}) - e_{\mathfrak{p}}(\Phi). \tag{49}$$

This quantity is unchanged when $\mathbf{x}$ is multiplied by an element of K^*. It is also unchanged when all the F_i are multiplied by a common element of K^*. Thus the notation $\tau_{\mathfrak{p}}(\Phi, P)$ is well defined. The same coefficient rescaling leaves $\rho_{\mathfrak{p}}(\Phi)$ unchanged. Then

$$0 \le \tau_{\mathfrak{p}}(\Phi, P) \le \rho_{\mathfrak{p}}(\Phi) \tag{50}$$

and

$$\sum_{\mathfrak{p} \in M_K} \rho_{\mathfrak{p}}(\Phi) \deg \mathfrak{p} = D_{N,r} h(\Phi). \tag{51}$$

Consequently,

$$rh(P) - (D_{N,r} - 1)h(\Phi) \le h(\Phi(P)) \le rh(P) + h(\Phi). \tag{52}$$

Let $G \in K[Y_0, \dots, Y_N]$ be homogeneous of degree δ and put $H = G \circ \Phi$. Here H denotes only this pullback polynomial; it is unrelated to the hyperplane family $\mathcal{H}$ in Section 3. Define

$$\sigma_{\mathfrak{p}}(G, \Phi) = e_{\mathfrak{p}}(H) - e_{\mathfrak{p}}(G) - \delta e_{\mathfrak{p}}(\Phi). \tag{53}$$

Then $\sigma_{\mathfrak{p}}(G, \Phi) \ge 0$ and, whenever $H(P) \ne 0$,

$$\lambda_{\mathfrak{p},H}(P) = \lambda_{\mathfrak{p},G}(\Phi(P)) + \big(\delta\tau_{\mathfrak{p}}(\Phi, P) - \sigma_{\mathfrak{p}}(G, \Phi)\big) \deg \mathfrak{p}. \tag{54}$$

In particular,

$$\lambda_{\mathfrak{p},H}(P) \le \lambda_{\mathfrak{p},G}(\Phi(P)) + \delta\rho_{\mathfrak{p}}(\Phi) \deg \mathfrak{p}, \tag{55}$$

and

$$h(H) \le h(G) + \delta h(\Phi). \tag{56}$$

Proof. The resultant is a polynomial with coefficients in the prime field, every monomial of which has total degree $D_{N,r}$. Hence $\rho_{\mathfrak{p}}(\Phi) \ge 0$. The first inequality in (50) follows directly from the ultrametric inequality applied to $F_i(\mathbf{x})$.

To prove the second inequality, scale the coefficients of Φ and the coordinates of P locally so that $e_{\mathfrak{p}}(\Phi) = e_{\mathfrak{p}}(\mathbf{x}) = 0$. The normalization gives (48) and hence

$$\min_{0\le i\le N} \operatorname{ord}_{\mathfrak{p}} F_i(\mathbf{x}) \le \operatorname{ord}_{\mathfrak{p}} \operatorname{Res}(\Phi).$$

Thus $\tau_{\mathfrak{p}}(\Phi, P) \le \rho_{\mathfrak{p}}(\Phi)$. Formula (51) follows from the product formula applied to $\operatorname{Res}(\Phi)$.

By definition,

$$e_{\mathfrak{p}}\big(\mathbf{F}(\mathbf{x})\big) = re_{\mathfrak{p}}(\mathbf{x}) + e_{\mathfrak{p}}(\Phi) + \tau_{\mathfrak{p}}(\Phi, P).$$

Summing over M_K gives the exact identity

$$h(\Phi(P)) = rh(P) + h(\Phi) - \sum_{\mathfrak{p}\in M_K} \tau_{\mathfrak{p}}(\Phi, P) \deg \mathfrak{p}. \tag{57}$$

Equations (50) and (51) now give (52).

Each coefficient of H is a sum of products consisting of one coefficient of G and δ coefficients of the F_i. Thus $\sigma_{\mathfrak{p}}(G, \Phi) \ge 0$. Directly from (8),

$$\begin{aligned}&\lambda_{\mathfrak{p},H}(P) - \lambda_{\mathfrak{p},G}(\Phi(P))\\ &\quad = \big(\delta e_{\mathfrak{p}}(\mathbf{F}(\mathbf{x})) + e_{\mathfrak{p}}(G) - \delta re_{\mathfrak{p}}(\mathbf{x}) - e_{\mathfrak{p}}(H)\big) \deg \mathfrak{p}.\end{aligned}$$

This is (54). The upper estimate follows from (50) and $\sigma_{\mathfrak{p}} \ge 0$. Finally,

$$h(H) = h(G) + \delta h(\Phi) - \sum_{\mathfrak{p}\in M_K} \sigma_{\mathfrak{p}}(G, \Phi) \deg \mathfrak{p},$$

which proves (56). □

Corollary 5.2 (Factor and transfer estimates)**.** *Retain the notation above. Put $D = V(G)$ and suppose that*

$$H = cU \prod_{j=1}^{q} Q_j^{a_j}, \tag{58}$$

where $c \in K^*$, the Q_j are pairwise nonassociate nonconstant homogeneous polynomials of degrees $d_j \in \mathbb{Z}_{>0}$, U is a nonzero homogeneous polynomial, and $a_j \in \mathbb{Z}_{\ge 1}$. Then

$$h(H) = h(U) + \sum_{j=1}^{q} a_j h(Q_j). \tag{59}$$

If $H(P) \ne 0$, then

$$\sum_{\mathfrak{p}\notin S} \sum_{j=1}^{q} \frac{1}{d_j} \lambda_{\mathfrak{p},Q_j}(P) \le N_S(D, \Phi(P)) + \delta D_{N,r} h(\Phi), \tag{60}$$

and

$$\sum_{\mathfrak{p}\in S} \sum_{j=1}^{q} \frac{1}{d_j} \lambda_{\mathfrak{p},Q_j}(P) \ge qh(P) - N_S(D, \Phi(P)) - \delta D_{N,r} h(\Phi). \tag{61}$$

Moreover,

$$h(Q_j) \le h(G) + \delta h(\Phi), \qquad 1 \le j \le q. \tag{62}$$

Proof. Equation (59) follows from (9). In particular,

$$h(Q_j) \le h(H) \le h(G) + \delta h(\Phi),$$

which proves (62).

Since $d_j \ge 1$ and $a_j \ge 1$, multiplicativity of local Weil functions gives

$$\sum_{j=1}^{q} \frac{1}{d_j} \lambda_{\mathfrak{p},Q_j}(P) \le \sum_{j=1}^{q} a_j \lambda_{\mathfrak{p},Q_j}(P) \le \lambda_{\mathfrak{p},H}(P).$$

Sum this inequality over $\mathfrak{p} \notin S$ and use (55) and (51). This proves (60).

For each j, (10) gives

$$\sum_{\mathfrak{p}\in M_K} \frac{1}{d_j} \lambda_{\mathfrak{p},Q_j}(P) = h(P) + \frac{h(Q_j)}{d_j} \ge h(P).$$

Summing over j and subtracting (60) gives (61). □

In particular, if $h(\Phi) + h(G) = o(h(P))$, then every selected factor Q_j moves slowly with respect to P, and the resultant term in (60) is $o(h(P))$.

6. Sequential Dynamics and Quasi-Integral Times

Proof of Theorem 1.1. Fix a real number θ in the stated range. Suppose, contrary to the conclusion, that there is an infinite set Λ of indices n such that $P_{n+1} \notin D_{n+1}$ and (3) holds.

For every sufficiently large $n \in \Lambda$, use the factorization supplied by the hypothesis:

$$G_{n+1} \circ \Phi_n = c_n U_n \prod_{j=1}^{q} Q_{n,j}^{a_{n,j}}. \tag{63}$$

The degrees of its selected factors belong to the finite set $\{1, \dots, \delta r\}$. After passing to an infinite subset and relabeling the factors, we may assume that $\deg Q_{n,j} = d_j$ is independent of n for every j.

Since $P_{n+1} = \Phi_n(P_n)$ and $P_{n+1} \notin D_{n+1}$, we have

$$(G_{n+1} \circ \Phi_n)(P_n) = G_{n+1}(P_{n+1}) \ne 0.$$

In particular, $Q_{n,j}(P_n) \ne 0$ for all j. Moreover, (62) gives

$$h(Q_{n,j}) \le h(G_{n+1}) + \delta h(\Phi_n) = o(h(P_n)). \tag{64}$$

Because $q > C_N \ge N + 1$, all assumptions of Theorem 4.4 are satisfied by the moving points P_n and the moving hypersurfaces $Q_{n,1}, \dots, Q_{n,q}$. Moreover,

$$q \le \sum_{j=1}^{q} d_j \le \delta r.$$

Thus the upper endpoint $(q - C_N)/(\delta r)$ is always strictly smaller than 1. Consequently, for every $\varepsilon > 0$, outside a finite subset of Λ we have

$$\sum_{\mathfrak{p}\in S}\sum_{j=1}^{q}\frac{1}{d_j}\lambda_{\mathfrak{p},Q_{n,j}}(P_n) \le (C_N + \varepsilon)h(P_n). \tag{65}$$

On the other hand, Corollary 5.2, applied to (63), yields

$$\sum_{\mathfrak{p}\in S}\sum_{j=1}^{q}\frac{1}{d_j}\lambda_{\mathfrak{p},Q_{n,j}}(P_n) \ge qh(P_n) - N_S(D_{n+1}, P_{n+1}) - \delta D_{N,r}h(\Phi_n). \tag{66}$$

The quasi-integrality assumption and the upper estimate in (52) imply

$$\begin{aligned} N_S(D_{n+1}, P_{n+1}) &\le \theta\big(\delta h(P_{n+1}) + h(G_{n+1})\big) \\ &\le \theta\delta r h(P_n) + \theta\delta h(\Phi_n) + \theta h(G_{n+1}). \end{aligned} \tag{67}$$

Combining (65)– (67), we obtain

$$\big(q - C_N - \theta\delta r - \varepsilon\big)h(P_n) \le \big(\theta\delta + \delta D_{N,r}\big)h(\Phi_n) + \theta h(G_{n+1}). \tag{68}$$

Put $\gamma = q - C_N - \theta\delta r$. By the choice of θ, we have $\gamma > 0$. Take $\varepsilon = \gamma/2$ in (68) and divide by $h(P_n)$. The left-hand side is $\gamma/2$, whereas the right-hand side tends to zero by the slow-movement hypothesis. This is a contradiction. Hence Λ is finite. □

Corollary 6.1 (Strict integral times). *In the setting of Theorem 1.1, if $q > C_N$, then there are only finitely many indices n such that $P_{n+1} \notin D_{n+1}$ and*

$$N_S(D_{n+1}, P_{n+1}) = 0.$$

Proof. Apply Theorem 1.1 with $\theta = 0$. □

Corollary 6.2 (Nondegenerate refinement). *Retain the setting of Theorem 1.1 and all its hypotheses except the numerical condition $q > C_N$. Assume additionally that, for every infinite set Λ of sufficiently large indices, there is an infinite subset $A \subset \Lambda$ on which the q factors can be relabeled as $Q_{n,1}, \dots, Q_{n,q}$ so that $\deg Q_{n,j}$ is independent of n and the point family $n \mapsto P_n$ is algebraically nondegenerate with respect to this moving family in the sense of Section 2. The factors remain in general position by the hypotheses of Theorem 1.1. If $q > N + 1$, then, for every*

$$0 \le \theta < \frac{q - N - 1}{\delta r},$$

there are only finitely many indices n for which $P_{n+1} \notin D_{n+1}$ and

$$N_S(D_{n+1}, P_{n+1}) \le \theta\big(\delta h(P_{n+1}) + h(G_{n+1})\big).$$

Proof. If the conclusion failed, apply the additional hypothesis to the resulting infinite failure set. Write $d_j = \deg Q_{n,j}$ on the resulting subset, put $d = \operatorname{lcm}(d_1, \dots, d_q)$, and set

$$\widehat{Q}_{n,j} = Q_{n,j}^{d/d_j}.$$

Take a further infinite subset jointly coherent with the original factors, the powered factors, and the point family. Let R and R' be the normalized coefficient fields of the original and powered families, respectively. Every normalized coefficient of $\widehat{Q}_{n,j}$ is a rational expression in the normalized coefficients of $Q_{n,j}$; hence $R' \subseteq R$. Algebraic nondegeneracy over R therefore implies algebraic nondegeneracy over R'. Thus the ideal I_x used in the proof of Theorem 4.4 is zero, so $W = \mathbb{P}^N_{R'}$ and $\ell = N$. By Remark 4.5, the moving-hypersurface upper bound therefore has coefficient $N+1$. Repeating the proof of Theorem 1.1 with C_N replaced by $N+1$ gives the stated range. □

7. Examples

7.1. **A varying constant-coefficient family.** Let $K = k(t)$, let $B = \mathbb{P}^1_k$, and let S consist of the places $t = 0$ and $t = \infty$. Fix an integer $r \geq 3$. Choose arbitrary sequences $\alpha_n, \beta_n, \eta_n \in k^*$ and define

$$\Phi_n([X : Y]) = [\alpha_n X^r : \beta_n Y^r], \qquad G_n = X - \eta_n Y, \qquad P_0 = [t : 1]. \tag{69}$$

Thus $D_n = V(G_n)$ has degree $\delta = 1$. Since all coefficients belong to the constant field,

$$h(\Phi_n) = h(G_n) = 0.$$

Define $A_0 = B_0 = 1$ and

$$A_{n+1} = \alpha_n A_n^r, \qquad B_{n+1} = \beta_n B_n^r.$$

Then the sequential orbit is

$$P_n = [A_n t^{r^n} : B_n], \qquad h(P_n) = r^n. \tag{70}$$

Hence $h(P_n) \to \infty$, and the slow-movement hypothesis holds.

For each n, choose $\xi_n \in k^*$ such that $\xi_n^r = \eta_{n+1}\beta_n/\alpha_n$. Since k is algebraically closed, such a choice is always possible. The relevant pullback has the factorization

$$G_{n+1} \circ \Phi_n = \alpha_n X^r - \eta_{n+1}\beta_n Y^r = \alpha_n \prod_{\zeta \in \mu_r} (X - \zeta \xi_n Y), \tag{71}$$

where $\mu_r \subset k^*$ is the group of rth roots of unity. The factors in (71) are distinct because $\operatorname{char} k = 0$. Any two of them have no common zero in $\mathbb{P}^1$, so they are in general position. Hence we may take $q = r$. The restriction $r \geq 3$ is specific to this construction and is used only to ensure $q = r > C_1 = 2$; the main theorem itself allows every degree $r \geq 2$. Since $C_1 = 2$, Theorem 1.1 applies for every

$$0 \leq \theta < \frac{r-2}{r}. \tag{72}$$

The counting function can also be computed directly. From (70),

$$G_n(P_n) = A_n t^{r^n} - \eta_n B_n.$$

This polynomial has exactly r^n simple zeros in k^*. All corresponding places lie outside S, and the local Weil function equals one at each of them. Therefore

$$N_S(D_n, P_n) = r^n = h(P_n) = h(P_n) + h(G_n). \tag{73}$$

Thus $N_S(D_n, P_n) = h(P_n)$, so these points are not θ-quasi-integral for $0 \le \theta < 1$. The ratios α_n/β_n and the constants η_n may be chosen to take infinitely many values.

7.2. **Positive map height.** Choose $c_n, \eta_n \in k^*$ and set

$$\Phi_n([X : Y]) = [(c_n t)^r X^r : Y^r], \qquad D_n = V(X - \eta_n Y), \qquad P_0 = [t : 1].$$

Then $h(\Phi_n) = r$ and $h(G_n) = 0$. Writing $P_n = [\kappa_n t^{E_n} : 1]$ gives

$$\kappa_0 = 1, \qquad \kappa_{n+1} = c_n^r \kappa_n^r, \qquad E_0 = 1, \qquad E_{n+1} = r(E_n + 1), \qquad h(P_n) = E_n \longrightarrow \infty,$$

so $h(\Phi_n) = o(h(P_n))$. Since k is algebraically closed, choose $\omega_n \in k^*$ with $\omega_n^r = \eta_{n+1}$. The pullback

$$(c_n t)^r X^r - \eta_{n+1} Y^r = \prod_{\zeta \in \mu_r} (c_n t X - \zeta \omega_n Y)$$

again has r distinct K-defined factors in general position. Moreover, $\kappa_n t^{E_n} - \eta_n$ has E_n simple zeros in k^*, and hence

$$N_S(D_n, P_n) = E_n = h(P_n).$$

7.3. **Prescribed finite exceptional sets.** Let $F \subset \mathbb{Z}_{\ge 1}$ be finite and, for each $m \in F$, replace only the target polynomial G_m by

$$G_m^F = X - \big(\kappa_m t^{E_m} + t^{a_m}\big) Y, \qquad a_m \ge 1.$$

Then $G_m^F(P_m) = -t^{a_m} \neq 0$. For every $\mathfrak{p} \notin S$, the element t and all nonzero constants are $\mathfrak{p}$-units. Hence

$$e_{\mathfrak{p}}(P_m) = e_{\mathfrak{p}}(G_m^F) = 0, \qquad \operatorname{ord}_{\mathfrak{p}} G_m^F(P_m) = 0.$$

Thus every local Weil function outside S vanishes, and

$$N_S\big(V(G_m^F), P_m\big) = 0, \qquad m \in F.$$

Since F is finite, the asymptotic hypotheses are unchanged. The strict integral times in this construction are precisely the indices in F.

8. Conclusion

Theorem 1.1 combines the moving-hypersurface estimate, the resultant transfer inequality, and the product formula. It depends on the number of distinct pullback factors; their multiplicities do not enter the coefficient q. Under the algebraic nondegeneracy hypothesis in Corollary 6.2, the coefficient C_N is replaced by $N+1$. The theorem is stated for morphisms of fixed degree; rational maps with indeterminacy, unbounded degree sequences, and targets in subgeneral position are not considered.

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