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GENERALIZED INTUITIONISTIC FUZZY ACZEL–ALSINA SHAPLEY–CHOQUET OPERATORS FOR MACHINE-LEARNING MODEL SELECTION

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Abstract. Machine-learning model selection is usually shaped by evidence that is incomplete, graded, and mutually dependent rather than fully precise or independent. A classifier may perform well on validation data, yet it can still suffer from poor calibration, limited interpretability, excessive computational burden, or sensitivity to distributional changes. Reducing this type of assessment to a single acceptance degree may remove information that is relevant for practical ranking. Intuitionistic fuzzy (IF) information is appropriate in this setting because it keeps the membership, non-membership, and hesitation components attached to each criterion. In this paper, an IF Aczel–Alsina generalized Shapley–Choquet integral framework is constructed for multi-criteria decision-making. Average-type and geometric-type generalized Shapley–Choquet integral operators are introduced in the IF environment, and their closure, idempotency, boundedness, monotonicity, and main special cases are proved. The numerical part studies machine-learning model selection by comparing four learning alternatives under prediction quality, calibration reliability, explainability, and deployment economy. Decision tables, Shapley-based importance information, sensitivity results, comparative evaluations, graphical displays, and a methodological flowchart are

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arranged within one decision procedure. The final ranking shows that the proposed family can combine expert uncertainty and criterion interaction in a stable and interpretable way.

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1. Introduction

Many decision situations under uncertainty cannot be reduced to exact observations that act independently of one another. In model selection, medical diagnosis, financial screening, energy planning, and intelligent information systems, the information supplied to decision makers is commonly partial, vague, and sometimes conflicting [1]. One alternative may receive a certain level of support, another level of opposition, and a remaining part of undecided judgement [2]. Crisp decision models often eliminate this hesitation too early, and this may conceal information about reliability and risk [3], [4]. Fuzzy set theory was originally introduced by Zadeh (1965) [5] as a mathematical framework for handling uncertainty and imprecision through membership functions. Subsequently, Zadeh (1968) [6] extended this framework by introducing probability measures of fuzzy events. Zadeh (1971) [7] further developed similarity relations and fuzzy orderings, providing important mathematical foundations for fuzzy relations. Later, Zadeh (1975) [8] introduced linguistic variables and approximate reasoning to represent and process imprecise information. Furthermore, Zadeh (1978) [9] established the relationship between fuzzy sets and possibility theory, significantly extending the theoretical foundations of uncertainty modeling.

To address uncertainty and hesitation more explicitly, Atanassov introduced intuitionistic fuzzy sets (IFSs) as an extension of classical fuzzy sets [10]. An IFS assigns separate membership and non-membership degrees while allowing a remaining degree of hesitation. Atanassov and Gargov extended this framework to interval-valued intuitionistic fuzzy sets [11]. Atanassov subsequently established additional properties and operators for IFSs [12]. Further algebraic operations were developed to strengthen the mathematical structure of IFS theory [13]. Szmidt and Kacprzyk proposed distance measures that account for the three components of intuitionistic fuzzy information [14]. They also developed an entropy measure for evaluating intuitionistic fuzzy uncertainty [15]. De et al. demonstrated the application of IFSs to medical diagnosis [16].

The development of aggregation operators further expanded the role of IFSs in decision-making. Xu and Yager introduced geometric aggregation operators for combining intuitionistic fuzzy evaluations [17]. Xu developed weighted, ordered weighted, and hybrid averaging operators for intuitionistic fuzzy information [18]. An additional approach based on intuitionistic preference relations was established for group decision-making [19]. Boran et al. integrated intuitionistic fuzzy information with TOPSIS

for supplier selection [20]. Xu and Zhao reviewed information-fusion techniques for intuitionistic fuzzy decision-making [21].

The theory has continued to develop through generalized representations and aggregation methods. Atanassov introduced circular intuitionistic fuzzy sets to extend the representation of uncertain information [22]. Senapati et al. developed intuitionistic fuzzy geometric aggregation operators using Aczel–Alsina operations [23]. Wasim et al. proposed generalized intuitionistic fuzzy aggregation operators for decision-making applications in the technology sector [24]. Fahmi et al. investigated Hamacher aggregation operators within a circular intuitionistic fuzzy environment [25]. Kumar et al. developed Yager-based intuitionistic fuzzy aggregation operators for green supply-chain decisions [26]. Pratama et al. proposed improved ordering functions and aggregation operators for circular intuitionistic fuzzy decision-making [27].

Recent developments in fuzzy decision-making have introduced advanced aggregation techniques for handling uncertainty. Ullah et al. (2026) introduced intuitionistic fuzzy grouping and overlap functions for mobile selection problems [28]. Qawaqneh et al. (2026) proposed an intuitionistic fuzzy Dombi–Archimedean RANCOM–AROMAN framework for AI-enabled telemedicine platform selection [29]. Qawaqneh et al. (2026) investigated Pythagorean fuzzy Dombi–Archimedean operators for network security decision-making [30]. The authors have also developed Dombi–Archimedean aggregation operators, including hybrid formulations, within various extensions of fuzzy set theory, particularly complex q-rung orthopair fuzzy sets and q-rung orthopair fuzzy sets, to address complex multi-attribute decision-making problems [31,32], [33]. Ullah et al. (2025) developed Aczel–Alsina group overlap functions for criminal justice policy selection under intuitionistic fuzzy information [34].

Sugeno (1974) developed the theory of fuzzy measures and fuzzy integrals, providing a mathematical framework for representing non-additive measures and modeling interactions among evaluation criteria [35]. The Choquet integral was introduced by Choquet (1954) within the theory of capacities as a generalized integration framework based on non-additive measures [36]. Grabisch (1996) investigated the application of fuzzy integrals to multicriteria decision-making problems, emphasizing their ability to capture interactions among evaluation criteria [37]. Marichal (2000) further established an axiomatic foundation for the discrete Choquet integral as an aggregation tool for handling interacting criteria in decision-making problems [38].

Shapley (1953) introduced the Shapley value to determine the contribution of individual players in cooperative games [39]. Building on this concept, Meng et al. (2013) developed generalized Shapley–Choquet aggregation operators for interval-valued intuitionistic fuzzy group decision-making [40]. More recently, Ullah et al. (2025) introduced Aczel–Alsina Shapley–Choquet integral aggregation operators within complex intuitionistic fuzzy environments for multicriteria decision-making [41].

Xu (2010) developed Choquet integral-based aggregation operators for weighted intuitionistic fuzzy information, incorporating correlations among evaluation criteria [42]. Wu et al. (2013) investigated the mathematical properties of intuitionistic fuzzy-valued Choquet integrals and their applications to multicriteria decision-making [43]. Jia and Wang (2022) proposed Choquet integral-based intuitionistic fuzzy arithmetic aggregation operators to capture interactions among criteria and improve multicriteria decision-making [44].

Triangular norms and conorms provide another mathematical layer for fuzzy aggregation. Menger's study of statistical metric spaces [45] opened a research direction that later developed into systematic investigations of triangular norms [46]. In this family, the Aczel–Alsina (AA) operations [47] are particularly useful because their parameter regulates the strictness of conjunction and disjunction. Such flexibility allows the decision maker to move between a more compensatory aggregation attitude and a more conservative one.

The present paper develops an IF AA generalized Shapley–Choquet integral (SCI) framework using a consistent notation and an operational structure compatible with IF information. The formulation keeps the closure condition explicit and allows the principal algebraic properties to be established without altering the meanings of the MD and NMD components.

1.1. **Motivation.** The motivation for this work comes from four connected observations. First, uncertainty in MCDM should be modeled in a way that retains support, opposition, and hesitation rather than converting the assessment immediately into a crisp value. IFNs offer this type of representation. Second, criteria in practical decision systems frequently interact, and additive weights alone may fail to express their combined influence. The Shapley–Choquet mechanism responds to this issue by using coalition-based importance. Third, AA operations introduce a tunable parameter, which makes it possible to express different levels of aggregation strictness. Fourth, machine-learning model selection provides a natural application, because prediction accuracy alone is not enough for deployment; calibration, interpretability, and cost are also part of the decision.

1.2. **Main contributions.** The main contributions of this paper are as follows:

(i) The Aczel–Alsina operational structure is integrated with the generalized Shapley–Choquet integral to obtain aggregation tools for IF information.

(ii) Four IF Aczel–Alsina Shapley–Choquet integral operators are developed, covering the WA, WG, OWA, and OWG cases.

(iii) The proposed operators combine interdependent IF decision data while retaining the membership and non-membership structure of IFNs.

(iv) The main mathematical properties of the operator family are proved, including closure, idempotency, boundedness, monotonicity, and relevant limiting cases.

(v) A machine-learning model-selection example is worked out to show how the proposed IF decision-making procedure can be applied.

1.3. **Paper organization.** The paper proceeds as follows. Section 2 presents the preliminaries needed for the subsequent operator construction. Section 3 develops the average and geometric generalized Shapley–Choquet integral operators and proves their basic properties. Section 4 explains the MCDM procedure. Section 5 gives the machine-learning model-selection example, including the numerical tables, rankings, graphs, sensitivity discussion, and comparison with baseline aggregation methods. Section 7 closes the study and identifies possible directions for later work.

2. Preliminaries

This section records the notation and definitions used in the sequel. The symbols are chosen to remain consistent with the later operator construction, so the IF components, FMs, SI, and AA laws can be applied without ambiguity.

Definition 2.1. [5] *Let $\mathcal{U}$ be a non-empty universe. A FSs F over $\mathcal{U}$ is specified through an MD mapping $\vartheta_F : \mathcal{U} \to [0,1]$. For any $u \in \mathcal{U}$, the value $\vartheta_F(u)$ expresses the degree of membership of u in F.*

Definition 2.2. [10] *Let $\mathcal{U}$ be a finite universe. An IFS $\mathfrak{I}$ on $\mathcal{U}$ is expressed as*

$$\mathfrak{I} = \Big\{ \big(u, \xi_{\mathfrak{I}}(u), \zeta_{\mathfrak{I}}(u)\big) : u \in \mathcal{U} \Big\}, \tag{2.1}$$

where $\xi_{\mathfrak{I}} : \mathcal{U} \to [0,1]$ and $\zeta_{\mathfrak{I}} : \mathcal{U} \to [0,1]$ represent the MD and NMDs, respectively, and

$$0 \le \xi_{\mathfrak{I}}(u) + \zeta_{\mathfrak{I}}(u) \le 1, \qquad u \in \mathcal{U}. \tag{2.2}$$

The corresponding hesitation degree is

$$\pi_{\mathfrak{I}}(u) = 1 - \xi_{\mathfrak{I}}(u) - \zeta_{\mathfrak{I}}(u). \tag{2.3}$$

Definition 2.3. [10] *An IFN is denoted by*

$$\mathfrak{a} = \langle \xi_{\mathfrak{a}}, \zeta_{\mathfrak{a}} \rangle, \tag{2.4}$$

where $\xi_{\mathfrak{a}}, \zeta_{\mathfrak{a}} \in [0,1]$ and $0 \le \xi_{\mathfrak{a}} + \zeta_{\mathfrak{a}} \le 1$. The hesitation component is $\pi_{\mathfrak{a}} = 1 - \xi_{\mathfrak{a}} - \zeta_{\mathfrak{a}}$.

Definition 2.4. [10] *For an IFN $\mathfrak{a} = \langle \xi_{\mathfrak{a}}, \zeta_{\mathfrak{a}} \rangle$, the score and accuracy functions are defined as*

$$\mathscr{S}(\mathfrak{a}) = \xi_{\mathfrak{a}} - \zeta_{\mathfrak{a}}, \qquad \mathscr{H}(\mathfrak{a}) = \xi_{\mathfrak{a}} + \zeta_{\mathfrak{a}}. \tag{2.5}$$

When two IFNs have unequal scores, the IFN with the larger score is preferred. If the scores are equal, preference is assigned to the IFN with the larger accuracy value.

Definition 2.5. [35] *Let $\mathcal{N} = \{1, 2, \ldots, n\}$ be a finite set of criteria and let $\mathcal{P}(\mathcal{N})$ denote its power set. A FM is a mapping $\kappa : \mathcal{P}(\mathcal{N}) \to [0, 1]$ satisfying*

$$\kappa(\varnothing) = 0, \qquad \kappa(\mathcal{N}) = 1, \tag{2.6}$$

$$A \subseteq B \subseteq \mathcal{N} \quad \Longrightarrow \quad \kappa(A) \leq \kappa(B). \tag{2.7}$$

Definition 2.6. [35] *A FM κ_λ is said to be a λ-FM when, for all disjoint $A, B \subseteq \mathcal{N}$,*

$$\kappa_\lambda(A \cup B) = \kappa_\lambda(A) + \kappa_\lambda(B) + \lambda\, \kappa_\lambda(A)\kappa_\lambda(B), \qquad \lambda > -1. \tag{2.8}$$

For $A \subseteq \mathcal{N}$ and $\lambda \neq 0$, one obtains

$$\kappa_\lambda(A) = \frac{1}{\lambda}\left\{\prod_{i \in A}\left[1 + \lambda\kappa_\lambda(\{i\})\right] - 1\right\}. \tag{2.9}$$

In the special case $\lambda = 0$, the measure reduces to the additive form:

$$\kappa_0(A) = \sum_{i \in A} \kappa_0(\{i\}). \tag{2.10}$$

The value of λ is obtained from

$$1 + \lambda = \prod_{i=1}^{n}\left[1 + \lambda\kappa_\lambda(\{i\})\right]. \tag{2.11}$$

Definition 2.7. [36] *For a FM κ on $\mathcal{N}$, the SVs associated with criterion $i \in \mathcal{N}$ is*

$$\mathfrak{Sh}_i(\kappa, \mathcal{N}) = \sum_{T \subseteq \mathcal{N}\setminus\{i\}} \frac{(n - |T| - 1)!\,|T|!}{n!}\Big[\kappa(T \cup \{i\}) - \kappa(T)\Big]. \tag{2.12}$$

The term $\mathfrak{Sh}_i(\kappa, \mathcal{N})$ represents the average marginal contribution of criterion i over all coalitions that do not include i.

Definition 2.8. [40] *For a coalition $Q \subseteq \mathcal{N}$, the GSI is defined by*

$$\mathfrak{Sh}_Q(\kappa, \mathcal{N}) = \sum_{T \subseteq \mathcal{N}\setminus Q} \frac{(n - |Q| - |T|)!\,|T|!}{(n - |Q| + 1)!}\Big[\kappa(Q \cup T) - \kappa(T)\Big]. \tag{2.13}$$

When $Q = \{i\}$, this formula coincides with the classical SVs.

Definition 2.9. [36] *Let $g : \mathcal{N} \to \mathbb{R}_+$, and let $g_{(1)} \leq g_{(2)} \leq \cdots \leq g_{(n)}$ denote the increasing rearrangement of g. Set $B_{(j)} = \{(j), (j + 1), \ldots, (n)\}$ and $B_{(n+1)} = \varnothing$. The DCI of g with respect to κ is then*

$$C_\kappa(g) = \sum_{j=1}^{n} g_{(j)}\Big[\kappa\big(B_{(j)}\big) - \kappa\big(B_{(j+1)}\big)\Big]. \tag{2.14}$$

Definition 2.10. [46, 47] *For $x, y \in [0, 1]$ and $\rho > 0$, the AA TNs and TCNs are given, respectively, by*

$$T_\rho^{AA}(x, y) = \exp\left\{-\left[(-\ln x)^\rho + (-\ln y)^\rho\right]^{1/\rho}\right\}, \tag{2.15}$$

$$S_\rho^{AA}(x, y) = 1 - \exp\left\{-\left[(-\ln(1 - x))^\rho + (-\ln(1 - y))^\rho\right]^{1/\rho}\right\}. \tag{2.16}$$

3. Proposed Generalized Shapley–Choquet Integral Operators

This section develops the proposed operator family in a unified symbolic form. Throughout the construction, an IFN is denoted by $\mathfrak{a} = \langle \xi, \zeta \rangle$, where ξ represents the MD and ζ represents the NMD. The AA parameter is written as $\rho > 0$. For notational compactness, define

$$U_j = -\ln(1-\xi_j), \qquad V_j = -\ln(\zeta_j), \qquad P_j = -\ln(\xi_j), \qquad Q_j = -\ln(1-\zeta_j). \tag{3.1}$$

Let $\mathcal{N} = \{1, 2, \ldots, n\}$ denote the criterion set, and let κ be a fuzzy measure on $\mathcal{N}$. For the ordered coalition $\mathcal{B}_{(j)}$, set

$$\theta_j = \mathfrak{Sh}_{\mathcal{B}_{(j)}}(\kappa, \mathcal{N}) - \mathfrak{Sh}_{\mathcal{B}_{(j-1)}}(\kappa, \mathcal{N}), \qquad j = 1, 2, \ldots, n, \tag{3.2}$$

where $\mathcal{B}_{(0)} = \varnothing$, $\theta_j \geq 0$, and $\sum_{j=1}^{n} \theta_j = 1$. Hence the coefficients θ_j serve as GSC increments within the aggregation procedure.

3.1. **GIFAAWAGSCI operator.**

Definition 3.1 (GIFAAWAGSCI operator)**.** *Let $\mathfrak{a}_j = \langle \xi_j, \zeta_j \rangle$, $j = 1, 2, \ldots, n$, be IFNs, and let θ_j be as in (3.2). The GIFAAWAGSCI operator is defined by*

$$\mathrm{GIFAAWAGSCI}_\rho(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n) = \overset{n}{\underset{j=1}{\boxplus_\rho}} \left(\theta_j \odot_\rho \mathfrak{a}_j\right). \tag{3.3}$$

Here $\boxplus_\rho$ denotes the AA sum, and $\theta_j \odot_\rho \mathfrak{a}_j$ denotes the AA scalar multiplication of an IFN.

Theorem 3.1 (Explicit form of the GIFAAWAGSCI operator)**.** *For IFNs $\mathfrak{a}_j = \langle \xi_j, \zeta_j \rangle$, $j = 1, 2, \ldots, n$, the operator introduced in Definition 3.1 has the explicit form*

$$\begin{aligned} \mathrm{GIFAAWAGSCI}_\rho(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) = \Bigg\langle & 1 - \exp\left[-\left\{ \sum_{j=1}^{n} \theta_j \left[-\ln(1-\xi_j)\right]^\rho \right\}^{1/\rho} \right], \\ & \exp\left[-\left\{ \sum_{j=1}^{n} \theta_j \left[-\ln(\zeta_j)\right]^\rho \right\}^{1/\rho} \right] \Bigg\rangle. \end{aligned} \tag{3.4}$$

Proof. The proof is carried out by induction on n.

Step 1: Case $n = 1$. Since $\theta_1 = 1$, Definition 3.1 yields

$$\mathrm{GIFAAWAGSCI}_\rho(\mathfrak{a}_1) = 1 \odot_\rho \mathfrak{a}_1. \tag{3.5}$$

Applying the Aczel–Alsina scalar law gives

$$1 \odot_\rho \mathfrak{a}_1 = \left\langle 1 - \exp\left[-\{1(-\ln(1-\xi_1))^\rho\}^{1/\rho} \right], \exp\left[-\{1(-\ln \zeta_1)^\rho\}^{1/\rho} \right] \right\rangle. \tag{3.6}$$

Therefore

$$1 \odot_\rho \mathfrak{a}_1 = \langle 1 - \exp\{\ln(1-\xi_1)\}, \exp\{\ln \zeta_1\} \rangle = \langle \xi_1, \zeta_1 \rangle. \tag{3.7}$$

This coincides with (3.4) when $n = 1$.

Step 2: Case $n = 2$**.** By (3.3),

$$\mathrm{GIFAAWAGSCI}_\rho(\mathfrak{a}_1, \mathfrak{a}_2) = (\theta_1 \odot_\rho \mathfrak{a}_1) \boxplus_\rho (\theta_2 \odot_\rho \mathfrak{a}_2). \tag{3.8}$$

For $j = 1, 2$, scalar multiplication gives

$$\theta_j \odot_\rho \mathfrak{a}_j = \left\langle 1 - \exp\left[-\{\theta_j U_j^\rho\}^{1/\rho}\right], \exp\left[-\{\theta_j V_j^\rho\}^{1/\rho}\right]\right\rangle. \tag{3.9}$$

The MD component of the AA sum is

$$1 - \exp\left[-\left\{\left[-\ln(1-(1-e^{-(\theta_1 U_1^\rho)^{1/\rho}}))\right]^\rho + \left[-\ln(1-(1-e^{-(\theta_2 U_2^\rho)^{1/\rho}}))\right]^\rho\right\}^{1/\rho}\right] \tag{3.10}$$

$$= 1 - \exp\left[-\{\theta_1 U_1^\rho + \theta_2 U_2^\rho\}^{1/\rho}\right]. \tag{3.11}$$

In the same manner, the NMD component is

$$\exp\left[-\left\{\left[-\ln(e^{-(\theta_1 V_1^\rho)^{1/\rho}})\right]^\rho + \left[-\ln(e^{-(\theta_2 V_2^\rho)^{1/\rho}})\right]^\rho\right\}^{1/\rho}\right] \tag{3.12}$$

$$= \exp\left[-\{\theta_1 V_1^\rho + \theta_2 V_2^\rho\}^{1/\rho}\right]. \tag{3.13}$$

Hence (3.4) holds for $n = 2$.

Step 3: Induction assumption. Assume that (3.4) is valid for $n = k$. Then

$$\mathrm{GIFAAWAGSCI}_\rho(\mathfrak{a}_1, \ldots, \mathfrak{a}_k) = \left\langle 1 - e^{-\left(\sum_{j=1}^k \theta_j U_j^\rho\right)^{1/\rho}}, e^{-\left(\sum_{j=1}^k \theta_j V_j^\rho\right)^{1/\rho}}\right\rangle. \tag{3.14}$$

Let

$$M_k = 1 - e^{-\left(\sum_{j=1}^k \theta_j U_j^\rho\right)^{1/\rho}}, \qquad N_k = e^{-\left(\sum_{j=1}^k \theta_j V_j^\rho\right)^{1/\rho}}. \tag{3.15}$$

Hence $\mathrm{GIFAAWAGSCI}_\rho(\mathfrak{a}_1, \ldots, \mathfrak{a}_k) = \langle M_k, N_k\rangle$.

Step 4: Case $n = k + 1$**.** Using Definition 3.1,

$$\mathrm{GIFAAWAGSCI}_\rho(\mathfrak{a}_1, \ldots, \mathfrak{a}_k, \mathfrak{a}_{k+1}) = \langle M_k, N_k\rangle \boxplus_\rho (\theta_{k+1} \odot_\rho \mathfrak{a}_{k+1}). \tag{3.16}$$

The final scalar term is

$$\theta_{k+1} \odot_\rho \mathfrak{a}_{k+1} = \left\langle 1 - e^{-(\theta_{k+1} U_{k+1}^\rho)^{1/\rho}}, e^{-(\theta_{k+1} V_{k+1}^\rho)^{1/\rho}}\right\rangle. \tag{3.17}$$

The MD component of the resulting sum is therefore

$$1 - \exp\left[-\left\{[-\ln(1-M_k)]^\rho + \theta_{k+1} U_{k+1}^\rho\right\}^{1/\rho}\right] \tag{3.18}$$

$$= 1 - \exp\left[-\left\{\sum_{j=1}^k \theta_j U_j^\rho + \theta_{k+1} U_{k+1}^\rho\right\}^{1/\rho}\right] \tag{3.19}$$

$$= 1 - \exp\left[-\left\{\sum_{j=1}^{k+1} \theta_j U_j^\rho\right\}^{1/\rho}\right]. \tag{3.20}$$

The NMD component is

$$\exp\left[-\left\{[-\ln(N_k)]^\rho + \theta_{k+1}V_{k+1}^\rho\right\}^{1/\rho}\right] \tag{3.21}$$

$$= \exp\left[-\left\{\sum_{j=1}^{k}\theta_j V_j^\rho + \theta_{k+1}V_{k+1}^\rho\right\}^{1/\rho}\right] \tag{3.22}$$

$$= \exp\left[-\left\{\sum_{j=1}^{k+1}\theta_j V_j^\rho\right\}^{1/\rho}\right]. \tag{3.23}$$

Therefore the formula is valid for $k+1$. Since the result holds for $n=1$ and $n=2$, and validity at k implies validity at $k+1$, the theorem follows for every positive integer n. □

Theorem 3.2 (Idempotency of GIFAAWAGSCI). *If $\mathfrak{a}_j = \mathfrak{a} = \langle \xi, \zeta\rangle$ for every $j = 1, 2, \ldots, n$, then*

$$\mathrm{GIFAAWAGSCI}_\rho(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n) = \mathfrak{a}. \tag{3.24}$$

Proof. Substitute $\xi_j = \xi$ and $\zeta_j = \zeta$ into (3.4). This gives

$$\mathrm{GIFAAWAGSCI}_\rho(\mathfrak{a}, \ldots, \mathfrak{a}) = \left\langle 1 - \exp\left[-\left\{\sum_{j=1}^{n}\theta_j[-\ln(1-\xi)]^\rho\right\}^{1/\rho}\right], \tag{3.25}$$

$$\exp\left[-\left\{\sum_{j=1}^{n}\theta_j[-\ln\zeta]^\rho\right\}^{1/\rho}\right]\right\rangle. \tag{3.26}$$

Since $\sum_{j=1}^{n}\theta_j = 1$, the MD component reduces to

$$1 - \exp\left[-\{[-\ln(1-\xi)]^\rho\}^{1/\rho}\right] = 1 - \exp\{\ln(1-\xi)\} = \xi. \tag{3.27}$$

The NMD component similarly becomes

$$\exp\left[-\{[-\ln\zeta]^\rho\}^{1/\rho}\right] = \exp\{\ln\zeta\} = \zeta. \tag{3.28}$$

Hence the aggregate is $\langle \xi, \zeta\rangle = \mathfrak{a}$, and idempotency is established. □

Theorem 3.3 (Boundedness of GIFAAWAGSCI). *Let $\mathfrak{a}_j = \langle \xi_j, \zeta_j\rangle$ be IFNs, and define*

$$\mathfrak{a}^- = \left\langle \min_j \xi_j, \max_j \zeta_j \right\rangle, \qquad \mathfrak{a}^+ = \left\langle \max_j \xi_j, \min_j \zeta_j \right\rangle. \tag{3.29}$$

Then

$$\mathfrak{a}^- \preceq \mathrm{GIFAAWAGSCI}_\rho(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \preceq \mathfrak{a}^+. \tag{3.30}$$

Proof. Let $\xi^- = \min_j \xi_j$ and $\xi^+ = \max_j \xi_j$. Because $\xi^- \le \xi_j \le \xi^+$, the increasing mapping $x \mapsto -\ln(1-x)$ yields

$$-\ln(1-\xi^-) \le -\ln(1-\xi_j) \le -\ln(1-\xi^+). \tag{3.31}$$

After raising the terms to ρ, multiplying by $\theta_j \geq 0$, and summing over j, one obtains

$$[-\ln(1-\xi^-)]^\rho \leq \sum_{j=1}^{n} \theta_j [-\ln(1-\xi_j)]^\rho \leq [-\ln(1-\xi^+)]^\rho. \tag{3.32}$$

Applying the map $u \mapsto 1 - e^{-u^{1/\rho}}$ gives

$$\xi^- \leq \xi_{\text{GIFAAWAGSCI}} \leq \xi^+. \tag{3.33}$$

Now set $\zeta^- = \min_j \zeta_j$ and $\zeta^+ = \max_j \zeta_j$. Since $\zeta^- \leq \zeta_j \leq \zeta^+$, the order induced by the decreasing behaviour of $-\ln(x)$ gives

$$-\ln(\zeta^+) \leq -\ln(\zeta_j) \leq -\ln(\zeta^-). \tag{3.34}$$

Hence

$$[-\ln(\zeta^+)]^\rho \leq \sum_{j=1}^{n} \theta_j [-\ln(\zeta_j)]^\rho \leq [-\ln(\zeta^-)]^\rho. \tag{3.35}$$

Applying $u \mapsto e^{-u^{1/\rho}}$ gives the corresponding bound

$$\zeta^- \leq \zeta_{\text{GIFAAWAGSCI}} \leq \zeta^+. \tag{3.36}$$

Consequently, the aggregate has its MD between the minimum and maximum MDs and its NMD between the minimum and maximum NMDs. Under the IF ordering $\preceq$, this gives $\mathfrak{a}^- \preceq \text{GIFAAWAGSCI}_\rho \preceq \mathfrak{a}^+$. □

Theorem 3.4 (Monotonicity of GIFAAWAGSCI)**.** *Let $\mathfrak{a}_j = \langle \xi_j, \zeta_j \rangle$ and $\mathfrak{a}'_j = \langle \xi'_j, \zeta'_j \rangle$ be two collections of IFNs. If $\xi_j \leq \xi'_j$ and $\zeta_j \geq \zeta'_j$ for each j, then*

$$\text{GIFAAWAGSCI}_\rho(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \preceq \text{GIFAAWAGSCI}_\rho(\mathfrak{a}'_1, \ldots, \mathfrak{a}'_n). \tag{3.37}$$

Proof. Since $\xi_j \leq \xi'_j$,

$$-\ln(1-\xi_j) \leq -\ln(1-\xi'_j). \tag{3.38}$$

Hence

$$\sum_{j=1}^{n} \theta_j [-\ln(1-\xi_j)]^\rho \leq \sum_{j=1}^{n} \theta_j [-\ln(1-\xi'_j)]^\rho. \tag{3.39}$$

The function $u \mapsto 1 - e^{-u^{1/\rho}}$ is increasing; therefore, the aggregated MD of the primed collection cannot be smaller. Moreover, $\zeta_j \geq \zeta'_j$ implies

$$-\ln(\zeta_j) \leq -\ln(\zeta'_j). \tag{3.40}$$

Therefore

$$\sum_{j=1}^{n} \theta_j [-\ln(\zeta_j)]^\rho \leq \sum_{j=1}^{n} \theta_j [-\ln(\zeta'_j)]^\rho. \tag{3.41}$$

Since $u \mapsto e^{-u^{1/\rho}}$ is decreasing, the aggregated NMD of the primed collection is not larger. Hence the primed aggregate is at least as preferable as the unprimed one, and monotonicity follows. □

3.2. **Ordered weighted average, geometric, and ordered weighted geometric operators.**

Definition 3.2 (GIFAAOWAGSCI operator)**.** *Let* $(\mathfrak{a}_{\pi(1)}, \mathfrak{a}_{\pi(2)}, \ldots, \mathfrak{a}_{\pi(n)})$ *be a permutation of* $(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$*, arranged in descending order according to the score function. For ordered weights* $\lambda_j \geq 0$ *satisfying* $\sum_{j=1}^{n} \lambda_j = 1$*, define*

$$\mathrm{GIFAAOWAGSCI}_\rho(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) = \overset{n}{\underset{j=1}{\boxplus_\rho}} \left(\lambda_j \odot_\rho \mathfrak{a}_{\pi(j)}\right). \tag{3.42}$$

Theorem 3.5 (Explicit form of GIFAAOWAGSCI)**.** *The operator stated in Definition 3.2 satisfies*

$$\mathrm{GIFAAOWAGSCI}_\rho(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) = \left\langle 1 - e^{-\left(\sum_{j=1}^{n} \lambda_j [-\ln(1-\xi_{\pi(j)})]^\rho\right)^{1/\rho}}, e^{-\left(\sum_{j=1}^{n} \lambda_j [-\ln(\zeta_{\pi(j)})]^\rho\right)^{1/\rho}} \right\rangle. \tag{3.43}$$

Proof. For $n = 1$, $\lambda_1 = 1$, and the result reduces to $\mathfrak{a}_{\pi(1)}$. For $n = 2$, the two scalar terms $\lambda_1 \odot_\rho \mathfrak{a}_{\pi(1)}$ and $\lambda_2 \odot_\rho \mathfrak{a}_{\pi(2)}$ are inserted into the AA sum. The logarithmic simplification used in Theorem 3.1 then gives the two-term form of (3.43). Assume now that the formula is true for $n = k$. When the resulting k-argument aggregate is combined with $\lambda_{k+1} \odot_\rho \mathfrak{a}_{\pi(k+1)}$, the terms $\lambda_{k+1}[-\ln(1-\xi_{\pi(k+1)})]^\rho$ and $\lambda_{k+1}[-\ln(\zeta_{\pi(k+1)})]^\rho$ are added. Hence the formula is valid for $k+1$, and the induction is complete. □

Definition 3.3 (GIFAAWGGSCI operator)**.** *The generalized IF AA weighted geometric generalized SC integral operator is defined as*

$$\mathrm{GIFAAWGGSCI}_\rho(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) = \overset{n}{\underset{j=1}{\boxtimes_\rho}} \left(\mathfrak{a}_j^{\wedge_\rho \theta_j}\right). \tag{3.44}$$

Theorem 3.6 (Explicit form of GIFAAWGGSCI)**.** *For IFNs* $\mathfrak{a}_j = \langle \xi_j, \zeta_j \rangle$*,*

$$\mathrm{GIFAAWGGSCI}_\rho(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) = \left\langle e^{-\left(\sum_{j=1}^{n} \theta_j [-\ln(\xi_j)]^\rho\right)^{1/\rho}}, 1 - e^{-\left(\sum_{j=1}^{n} \theta_j [-\ln(1-\zeta_j)]^\rho\right)^{1/\rho}} \right\rangle. \tag{3.45}$$

Proof. For $n = 1$, $\theta_1 = 1$, and (3.45) reduces to $\langle \xi_1, \zeta_1 \rangle$. For $n = 2$, write $\mathfrak{a}_j^{\wedge_\rho \theta_j} = \langle e^{-(\theta_j P_j^\rho)^{1/\rho}}, 1 - e^{-(\theta_j Q_j^\rho)^{1/\rho}} \rangle$. Applying the AA product to these two powered IFNs yields

$$\left\langle e^{-(\theta_1 P_1^\rho + \theta_2 P_2^\rho)^{1/\rho}}, 1 - e^{-(\theta_1 Q_1^\rho + \theta_2 Q_2^\rho)^{1/\rho}} \right\rangle. \tag{3.46}$$

Assume that the formula is valid for $n = k$. Multiplying the k-argument aggregate by $\mathfrak{a}_{k+1}^{\wedge_\rho \theta_{k+1}}$ adds $\theta_{k+1} P_{k+1}^\rho$ to the MD expression and $\theta_{k+1} Q_{k+1}^\rho$ to the NMD expression. Therefore the formula is valid for $k+1$, and induction establishes it for all n. □

Definition 3.4 (GIFAAOWGGSCI operator)**.** *With the same ordered permutation* π *and ordered weights* λ_j*, the ordered weighted geometric operator is written as*

$$\mathrm{GIFAAOWGGSCI}_\rho(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) = \overset{n}{\underset{j=1}{\boxtimes_\rho}} \left(\mathfrak{a}_{\pi(j)}^{\wedge_\rho \lambda_j}\right). \tag{3.47}$$

Theorem 3.7 (Explicit form of GIFAAOWGGSCI)**.** *The operator in Definition 3.4 satisfies*

$$\mathrm{GIFAAOWGGSCI}_\rho(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) = \left\langle e^{-\left(\sum_{j=1}^{n} \lambda_j [-\ln(\xi_{\pi(j)})]^\rho\right)^{1/\rho}}, 1 - e^{-\left(\sum_{j=1}^{n} \lambda_j [-\ln(1-\zeta_{\pi(j)})]^\rho\right)^{1/\rho}} \right\rangle. \tag{3.48}$$

Proof. For $n = 1$, the assertion is immediate. For $n = 2$, applying the Aczel–Alsina product to $\mathfrak{a}_{\pi(1)}^{\wedge_\rho \lambda_1}$ and $\mathfrak{a}_{\pi(2)}^{\wedge_\rho \lambda_2}$ gives the two-term version of (3.48). Suppose that (3.48) is valid for $n = k$. After the $(k+1)$th ordered IFN is incorporated, the product contributes $\lambda_{k+1}[-\ln(\xi_{\pi(k+1)})]^\rho$ to the MD part and $\lambda_{k+1}[-\ln(1-\zeta_{\pi(k+1)})]^\rho$ to the NMD part. Hence the closed form holds for $k+1$, and the induction is complete. □

The idempotency, boundedness, and monotonicity of GIFAAOWAGSCI$_\rho$, GIFAAWGGSCI$_\rho$, and GIFAAOWGGSCI$_\rho$ follow from the same component-wise logarithmic arguments. Since the positional and Shapley coefficients are non-negative and normalized, identical IFNs remain unchanged, the aggregate stays within the component-wise IF bounds, and a component-wise improvement of the input collection leads to a non-worse aggregate.

4. MCDM Procedure Based on the Proposed Operators

Consider an MCDM problem in which $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ denotes the set of alternatives and $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ denotes the set of criteria. The assessment of A_i under criterion C_j is expressed as the IFN $\mathfrak{a}_{ij} = \langle \xi_{ij}, \zeta_{ij} \rangle$. The decision procedure based on the proposed operators is summarized in Table 1.

Table 1. Algorithmic structure of the proposed MCDM method

Step	Operation
1	Identify the learning alternatives, criteria, and decision objective.
2	Construct the expert-wise IF decision matrices.
3	Combine expert assessments into a collective IF decision matrix.
4	Compute the FMs κ and the GSI.
5	Arrange the IFNs according to their score values and obtain ordered coalitions.
6	Apply GIFAAWAGSCI$_\rho$, GIFAAOWAGSCI$_\rho$, GIFAAWGGSCI$_\rho$, or GIFAAOWGGSCI$_\rho$.
7	Compute score values and rank the alternatives.
8	Compare the result with existing IF aggregation operators and inspect stability through graphs.

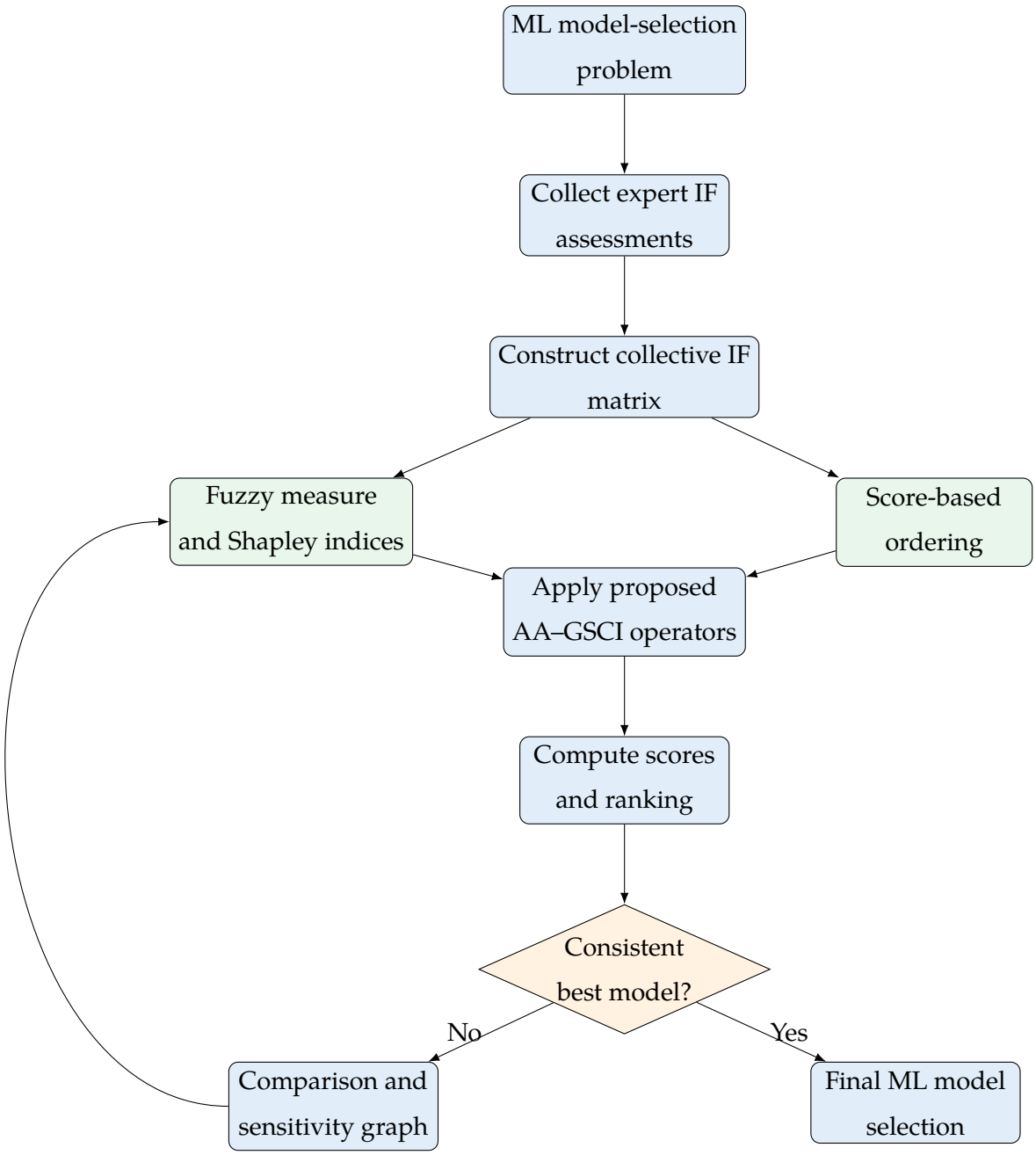

FIGURE 1. Flowchart of the proposed IF AA-GSCI decision process.

5. MACHINE-LEARNING CASE STUDY USING THE NUMERICAL DATA

5.1. **Problem description.** The numerical illustration is based on a machine-learning model-selection problem. Four candidate learning models are considered:

A_1 =convolutional neural network,

A_2 =random forest,

A_3 =support vector machine,

A_4 =XGBoost.

The assessment is performed with respect to the following four criteria: C_1 = predictive accuracy, C_2 = calibration reliability, C_3 = interpretability, C_4 = deployment economy. Three experts, denoted by D^1, D^2, D^3, evaluate each model using IFNs. The weight vector used in the calculation is $(0.40, 0.30, 0.15, 0.15)$.

TABLE 2. IF decision matrix provided by expert D^1 using the numerical data

Alternative	C_1	C_2	C_3	C_4
A_1	$\langle 0.6, 0.2\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.7, 0.1\rangle$	$\langle 0.5, 0.4\rangle$
A_2	$\langle 0.6, 0.4\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.6, 0.2\rangle$
A_3	$\langle 0.5, 0.4\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.4, 0.4\rangle$
A_4	$\langle 0.7, 0.1\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.7, 0.2\rangle$

TABLE 3. IF decision matrix provided by expert D^2 using the numerical data

Alternative	C_1	C_2	C_3	C_4
A_1	$\langle 0.5, 0.4\rangle$	$\langle 0.7, 0.1\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.6, 0.2\rangle$
A_2	$\langle 0.6, 0.1\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.7, 0.1\rangle$	$\langle 0.6, 0.3\rangle$
A_3	$\langle 0.5, 0.3\rangle$	$\langle 0.4, 0.4\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.6, 0.2\rangle$
A_4	$\langle 0.8, 0.1\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.6, 0.1\rangle$

TABLE 4. IF decision matrix provided by expert D^3 using the numerical data

Alternative	C_1	C_2	C_3	C_4
A_1	$\langle 0.6, 0.2\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.5, 0.2\rangle$	$\langle 0.6, 0.1\rangle$
A_2	$\langle 0.4, 0.4\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.7, 0.1\rangle$	$\langle 0.6, 0.3\rangle$
A_3	$\langle 0.6, 0.1\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.4, 0.4\rangle$
A_4	$\langle 0.5, 0.3\rangle$	$\langle 0.5, 0.6\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.7, 0.2\rangle$

5.2. **Expert decision matrices.** The expert assessments are then combined to form the collective matrix shown in Table 5.

TABLE 5. Collective IF decision matrix obtained from the computed values

Alternative	C_1	C_2	C_3	C_4
A_1	$\langle 0.6602, 0.10743\rangle$	$\langle 0.1531, 0.7823\rangle$	$\langle 0.7632, 0.1873\rangle$	$\langle 0.7832, 0.1543\rangle$
A_2	$\langle 0.5679, 0.3327\rangle$	$\langle 0.7832, 0.1543\rangle$	$\langle 0.4583, 0.4092\rangle$	$\langle 0.3214, 0.4526\rangle$
A_3	$\langle 0.4583, 0.4092\rangle$	$\langle 0.3214, 0.4526\rangle$	$\langle 0.4583, 0.4092\rangle$	$\langle 0.1531, 0.7823\rangle$
A_4	$\langle 0.3214, 0.6234\rangle$	$\langle 0.3321, 0.4923\rangle$	$\langle 0.3214, 0.6234\rangle$	$\langle 0.7832, 0.1543\rangle$

5.3. **Fuzzy measure and generalized Shapley values.** The singleton fuzzy densities are taken as $\kappa(\{C_1\}) = 0.18$, $\kappa(\{C_2\}) = 0.20$, $\kappa(\{C_3\}) = 0.24$, and $\kappa(\{C_4\}) = 0.25$. The parameter value is $\lambda = 0.50$. The resulting FM values are listed in Table 6.

TABLE 6. Fuzzy-measure values used in the computation

Coalition	Value	Coalition	Value
$\kappa(\{C_1, C_2\})$	0.39	$\kappa(\{C_1, C_3\})$	0.44
$\kappa(\{C_1, C_4\})$	0.45	$\kappa(\{C_2, C_3\})$	0.46
$\kappa(\{C_2, C_4\})$	0.47	$\kappa(\{C_3, C_4\})$	0.52
$\kappa(\{C_1, C_2, C_3\})$	0.68	$\kappa(\{C_1, C_2, C_4\})$	0.69
$\kappa(\{C_1, C_3, C_4\})$	0.74	$\kappa(\{C_2, C_3, C_4\})$	0.77
$\kappa(\{C_1, C_2, C_3, C_4\})$	1.00	–	–

TABLE 7. Generalized Shapley indices used in the computation

Coalition	Value	Coalition	Value
$\mathfrak{Sh}_{\varnothing}$	0.00	$\mathfrak{Sh}_{C_1}$	0.20
$\mathfrak{Sh}_{C_2}$	0.23	$\mathfrak{Sh}_{C_3}$	0.27
$\mathfrak{Sh}_{C_4}$	0.28	$\mathfrak{Sh}_{C_1,C_2}$	0.27
$\mathfrak{Sh}_{C_1,C_3}$	0.51	$\mathfrak{Sh}_{C_1,C_4}$	0.56
$\mathfrak{Sh}_{C_2,C_3}$	0.51	$\mathfrak{Sh}_{C_2,C_4}$	0.51
$\mathfrak{Sh}_{C_3,C_4}$	0.56	$\mathfrak{Sh}_{C_1,C_2,C_3}$	0.71
$\mathfrak{Sh}_{C_1,C_2,C_4}$	0.72	$\mathfrak{Sh}_{C_2,C_3,C_4}$	0.79
$\mathfrak{Sh}_{C_1,C_3,C_4}$	0.77	$\mathfrak{Sh}_{C_1,C_2,C_3,C_4}$	1.00

TABLE 8. Ordered fuzzy-measure coalitions for alternatives using the computed values

Alternative	Ordering	Fuzzy-measure sequence
A_1	$v_{14} > v_{12} > v_{11} > v_{13}$	$0.25, 0.47, 0.69, 1.00$
A_2	$v_{12} > v_{13} > v_{11} > v_{14}$	$0.20, 0.46, 0.68, 1.00$
A_3	$v_{12} > v_{11} > v_{13} > v_{14}$	$0.20, 0.39, 0.68, 1.00$
A_4	$v_{13} > v_{14} > v_{11} > v_{12}$	$0.24, 0.52, 0.74, 1.00$

TABLE 9. Ordered generalized Shapley sequences for alternatives using the computed values

Alternative	Ordering	Generalized Shapley sequence
A_1	$v_{14} > v_{12} > v_{11} > v_{13}$	$0.28, 0.51, 0.72, 1.00$
A_2	$v_{12} > v_{13} > v_{14} > v_{11}$	$0.23, 0.50, 0.79, 1.00$
A_3	$v_{12} > v_{11} > v_{13} > v_{14}$	$0.23, 0.57, 0.79, 1.00$
A_4	$v_{14} > v_{12} > v_{11} > v_{13}$	$0.27, 0.56, 0.77, 1.00$

5.4. **Aggregated IF evaluations.**

TABLE 10. Overall evaluations under GIFAAWAGSCI with $\rho = 2$, used in the computation

Alternative	C_1	C_2	C_3	C_4
A_1	$\langle 0.3841, 0.2694\rangle$	$\langle 0.4471, 0.3831\rangle$	$\langle 0.3982, 0.5271\rangle$	$\langle 0.4457, 0.3588\rangle$
A_2	$\langle 0.4120, 0.3761\rangle$	$\langle 0.4891, 0.3167\rangle$	$\langle 0.3562, 0.3091\rangle$	$\langle 0.4095, 0.2981\rangle$
A_3	$\langle 0.3489, 0.2764\rangle$	$\langle 0.4095, 0.3452\rangle$	$\langle 0.5156, 0.2071\rangle$	$\langle 0.5239, 0.1987\rangle$
A_4	$\langle 0.5374, 0.2143\rangle$	$\langle 0.5763, 0.2093\rangle$	$\langle 0.5589, 0.2453\rangle$	$\langle 0.6089, 0.3092\rangle$

TABLE 11. Overall evaluations under IFAAAGSCI with $\rho = 1$, used in the computation

Alternative	C_1	C_2	C_3	C_4
A_1	$\langle 0.4098, 0.3450\rangle$	$\langle 0.4532, 0.3087\rangle$	$\langle 0.4352, 0.2801\rangle$	$\langle 0.4941, 0.3754\rangle$
A_2	$\langle 0.4315, 0.3567\rangle$	$\langle 0.4546, 0.3290\rangle$	$\langle 0.4366, 0.4123\rangle$	$\langle 0.4280, 0.6790\rangle$
A_3	$\langle 0.4696, 0.4579\rangle$	$\langle 0.4842, 0.3908\rangle$	$\langle 0.3567, 0.3089\rangle$	$\langle 0.4280, 0.2769\rangle$
A_4	$\langle 0.5218, 0.4089\rangle$	$\langle 0.5108, 0.4382\rangle$	$\langle 0.4843, 0.3897\rangle$	$\langle 0.5198, 0.3789\rangle$

TABLE 12. Overall evaluations under GIFAAWGGSCI with $\rho = 2$, used in the computation

Alternative	C_1	C_2	C_3	C_4
A_1	$\langle 0.3098, 0.3941\rangle$	$\langle 0.3421, 0.4532\rangle$	$\langle 0.3003, 0.4352\rangle$	$\langle 0.3693, 0.4941\rangle$
A_2	$\langle 0.3776, 0.5332\rangle$	$\langle 0.3506, 0.5629\rangle$	$\langle 0.3016, 0.4294\rangle$	$\langle 0.2023, 0.5478\rangle$
A_3	$\langle 0.3202, 0.5297\rangle$	$\langle 0.4014, 0.5681\rangle$	$\langle 0.4515, 0.5173\rangle$	$\langle 0.3547, 0.5122\rangle$
A_4	$\langle 0.2258, 0.5839\rangle$	$\langle 0.2913, 0.5217\rangle$	$\langle 0.3602, 0.6001\rangle$	$\langle 0.4088, 0.4611\rangle$

TABLE 13. Overall evaluations under IFAAGGSCI with $\rho = 1$, used in the computation

Alternative	C_1	C_2	C_3	C_4
A_1	$\langle 0.3694, 0.4941\rangle$	$\langle 0.3913, 0.4532\rangle$	$\langle 0.4003, 0.4352\rangle$	$\langle 0.3693, 0.4941\rangle$
A_2	$\langle 0.2863, 0.5850\rangle$	$\langle 0.3615, 0.5037\rangle$	$\langle 0.4262, 0.4677\rangle$	$\langle 0.5440, 0.3122\rangle$
A_3	$\langle 0.5476, 0.3053\rangle$	$\langle 0.5131, 0.4226\rangle$	$\langle 0.4274, 0.3686\rangle$	$\langle 0.4306, 0.3607\rangle$
A_4	$\langle 0.5053, 0.4413\rangle$	$\langle 0.5815, 0.3965\rangle$	$\langle 0.4333, 0.3617\rangle$	$\langle 0.5347, 0.4829\rangle$

TABLE 14. Score values obtained from the four computational cases

Alternative	Case (a): GIFAAWAGSCI, $\rho = 2$	Case (b): IFAAAGSCI	Case (c): GIFAAWGGSCI, $\rho = 2$	Case (d): IFAAGGSCI
A_1	0.4172	0.3863	0.3783	0.3572
A_2	0.4423	0.4129	0.4092	0.3897
A_3	0.3897	0.3589	0.3452	0.3120
A_4	0.3540	0.3342	0.3091	0.2789

5.5. **Score values and ranking.** The four computational cases lead to the same ranking:

$$A_2 \succ A_1 \succ A_3 \succ A_4. \tag{5.1}$$

Accordingly, the random forest model A_2 is identified as the best compromise alternative for the given IF decision data.

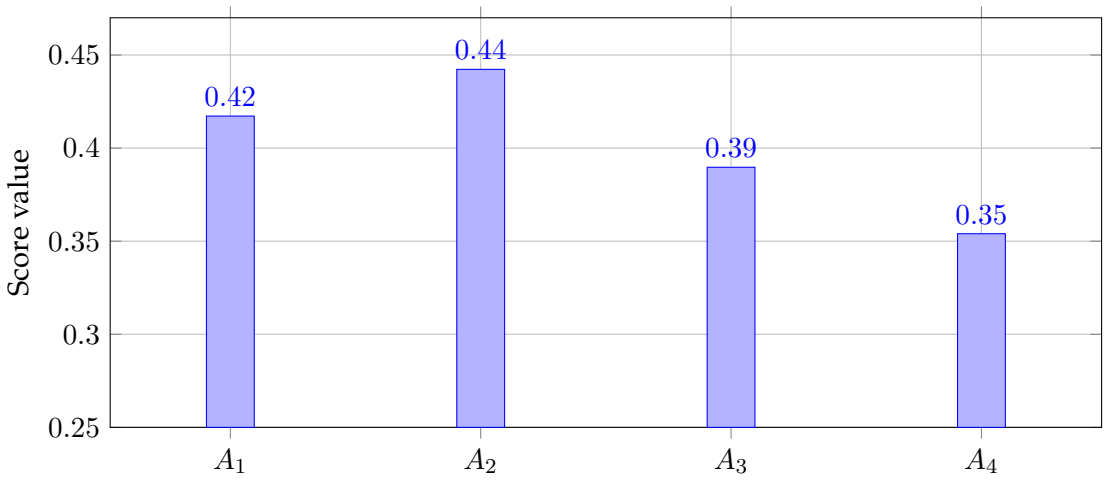

FIGURE 2. Score values of the four machine-learning alternatives under Case (a).

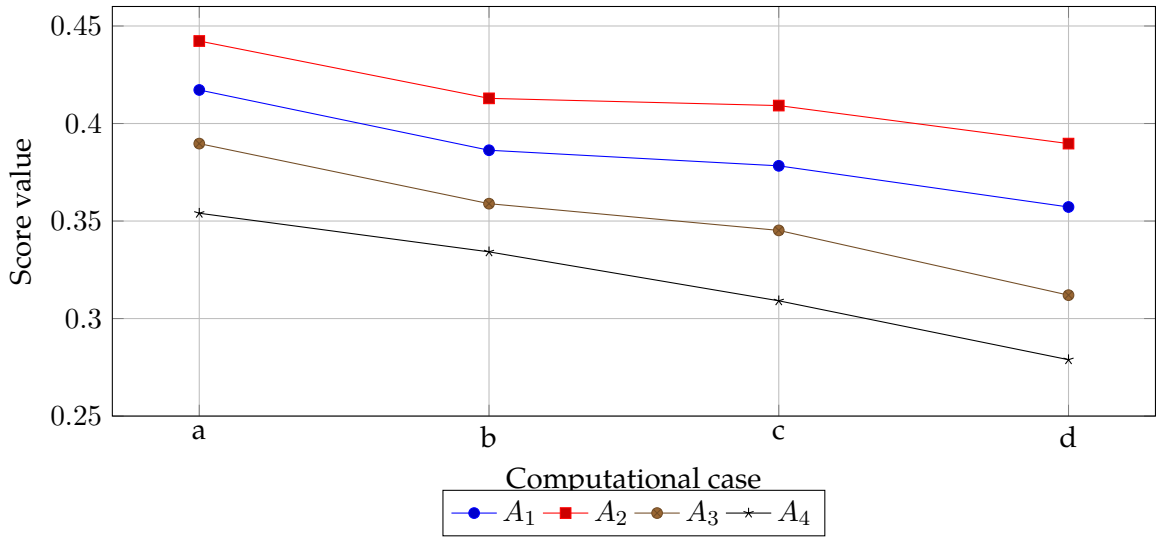

FIGURE 3. Sensitivity of alternative scores across the four computational cases.

6. Comparative Analysis

The comparative analysis is presented in Table 15. The proposed operators select A_2 as the best machine-learning model, whereas the classical IFWA, IFOWA, IFWG, and IFOWG baselines rank A_1 first. This difference is reasonable because the proposed operators include GSC interaction information, while the classical methods depend on additive or purely ordered aggregation mechanisms.

TABLE 15. Comparison with existing IF aggregation methods using the computed values

Aggregation operator	Alternative score values	Ranking order
IFWA [10]	0.3897, 0.3472, 0.3145, 0.2879	$A_1 \succ A_2 \succ A_3 \succ A_4$
IFOWA [10]	0.3675, 0.3256, 0.2783, 0.2112	$A_1 \succ A_2 \succ A_3 \succ A_4$
IFWG [10]	0.2074, 0.1462, 0.1086, 0.1016	$A_1 \succ A_2 \succ A_3 \succ A_4$
IFOWG [10]	0.3091, 0.3356, 0.2987, 0.3245	$A_1 \succ A_2 \succ A_3 \succ A_4$
GIFAAWAGSCI	0.4172, 0.4423, 0.3897, 0.3540	$A_2 \succ A_1 \succ A_3 \succ A_4$
IFAAAGSCI	0.3863, 0.4129, 0.3589, 0.3342	$A_2 \succ A_1 \succ A_3 \succ A_4$
GIFAAWGGSCI	0.3783, 0.4092, 0.3452, 0.3091	$A_2 \succ A_1 \succ A_3 \succ A_4$
IFAAGGSCI	0.3572, 0.3897, 0.3120, 0.2789	$A_2 \succ A_1 \succ A_3 \succ A_4$

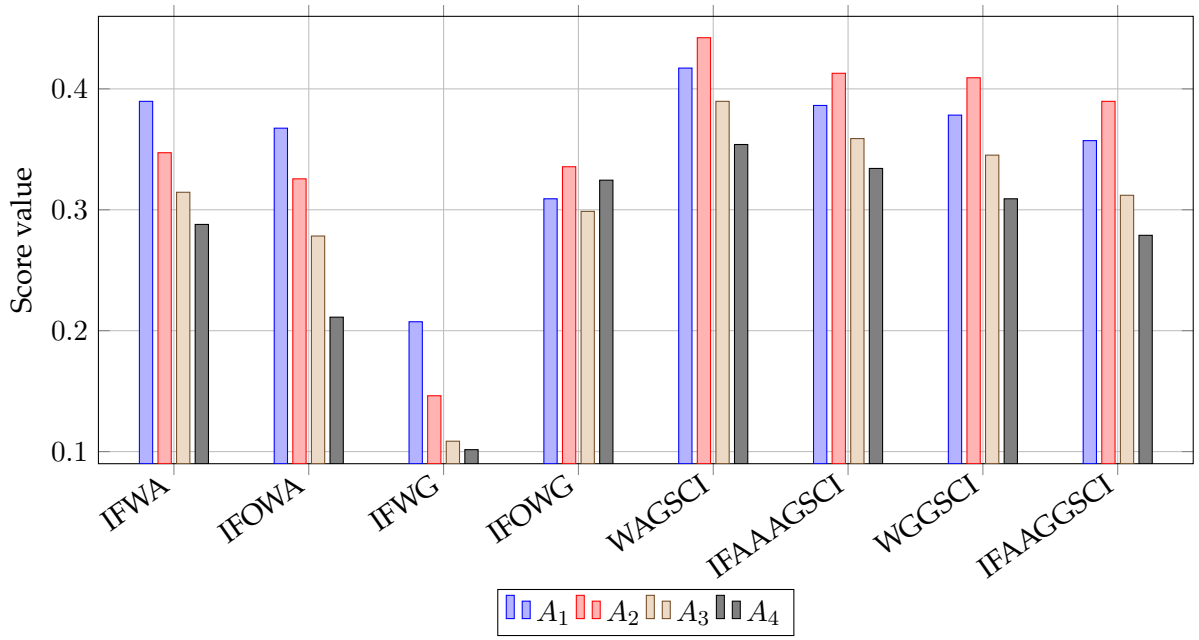

FIGURE 4. Comparison of computed alternative scores under existing and proposed methods.

The comparison shows that criterion interaction can alter the final ranking. When GSC increments are incorporated, A_2 becomes the leading alternative; under the classical IF aggregation baselines, A_1 remains in the first position. This shift reflects the effect of non-additive criterion behaviour on the model-selection result.

7. CONCLUSION

This study has developed a generalized intuitionistic fuzzy Aczel–Alsina Shapley–Choquet integral framework for multi-criteria decision-making problems involving uncertainty and interactions among

evaluation criteria. By combining intuitionistic fuzzy information, Aczel–Alsina operations, and generalized Shapley–Choquet aggregation, the proposed approach provides a mathematical structure for evaluating alternatives without discarding the membership, non-membership, and hesitation information. Four aggregation operators, namely GIFAAWAGSCI, GIFAAOWAGSCI, GIFAAWGGSCI, and GIFAAOWGGSCI, have been formulated. Their explicit expressions and fundamental properties, including idempotency, boundedness, and monotonicity, have been established. The Aczel–Alsina parameter further allows the aggregation behaviour to be adjusted according to different decision-making requirements.

The applicability of the proposed approach has been examined through a machine-learning model-selection problem involving four alternatives: convolutional neural network, random forest, support vector machine, and XGBoost. The alternatives were evaluated using predictive accuracy, calibration reliability, interpretability, and deployment economy. The numerical results consistently produced the ranking

$$A_2 \succ A_1 \succ A_3 \succ A_4,$$

identifying random forest as the preferred alternative under the considered intuitionistic fuzzy assessments. In particular, the GIFAAWAGSCI operator yielded a score of 0.4423 for the random forest model at $\rho = 2$. The sensitivity results showed that although the aggregated score values varied across the four computational cases, the ordering of the alternatives remained unchanged.

The comparative analysis also revealed differences between the proposed operators and conventional intuitionistic fuzzy aggregation approaches. These differences highlight the importance of considering criterion interactions rather than relying exclusively on independent criterion weights. The findings suggest that the proposed framework can support model-selection decisions in which predictive performance must be considered alongside reliability, interpretability, and practical deployment requirements. Nevertheless, the numerical investigation is limited to a specific set of expert assessments and four learning alternatives, and broader empirical validation is needed before generalizing the observed ranking stability.

Future research may extend the proposed operators to interval-valued intuitionistic fuzzy sets, Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, and other generalized fuzzy environments. Further investigations could also consider larger collections of learning models, data-driven estimation of fuzzy measures, and validation using real-world machine-learning benchmarks. Such developments would help establish the practical usefulness and limitations of the proposed aggregation framework in more complex decision-making applications.

Authors Contributions. All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

Data Availability Statement. The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A. Mardani, A. Jusoh, E.K. Zavadskas, Fuzzy Multiple Criteria Decision-Making Techniques and Applications – Two Decades Review from 1994 to 2014, Expert Syst. Appl. 42 (2015), 4126–4148. https://doi.org/10.1016/j.eswa.2015.01.003.

[2] G. Wei, Some Cosine Similarity Measures for Picture Fuzzy Sets and Their Applications to Strategic Decision Making, Informatica 28 (2017), 547–564. https://doi.org/10.15388/Informatica.2017.144.

[3] F. Herrera, L. Martínez, A 2-Tuple Fuzzy Linguistic Representation Model for Computing with Words, IEEE Trans. Fuzzy Syst. 8 (2000), 746–752. https://doi.org/10.1109/91.890332.

[4] M.S.A. Khan, F. Anjum, I. Ullah, T. Senapati, S. Moslem, Priority Degrees and Distance Measures of Complex Hesitant Fuzzy Sets with Application to Multi-Criteria Decision Making, IEEE Access 11 (2023), 13647–13666. https://doi.org/10.1109/ACCESS.2022.3232371.

[5] L.A. Zadeh, Fuzzy Sets, Inf. Control 8 (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[6] L.A. Zadeh, Probability Measures of Fuzzy Events, J. Math. Anal. Appl. 23 (1968), 421–427. https://doi.org/10.1016/0022-247X(68)90078-4.

[7] L.A. Zadeh, Similarity Relations and Fuzzy Orderings, Inf. Sci. 3 (1971), 177–200. https://doi.org/10.1016/S0020-0255(71)80005-1.

[8] L.A. Zadeh, The Concept of a Linguistic Variable and Its Application to Approximate Reasoning—I, Inf. Sci. 8 (1975), 199–249. https://doi.org/10.1016/0020-0255(75)90036-5.

[9] L.A. Zadeh, Fuzzy Sets as a Basis for a Theory of Possibility, Fuzzy Sets Syst. 1 (1978), 3–28. https://doi.org/10.1016/0165-0114(78)90029-5.

[10] K.T. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 20 (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3.

[11] K. Atanassov, G. Gargov, Interval Valued Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 31 (1989), 343–349. https://doi.org/10.1016/0165-0114(89)90205-4.

[12] K.T. Atanassov, More on Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 33 (1989), 37–45. https://doi.org/10.1016/0165-0114(89)90215-7.

[13] K.T. Atanassov, New Operations Defined over the Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 61 (1994), 137–142. https://doi.org/10.1016/0165-0114(94)90229-1.

[14] E. Szmidt, J. Kacprzyk, Distances between Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 114 (2000), 505–518. https://doi.org/10.1016/S0165-0114(98)00244-9.

[15] E. Szmidt, J. Kacprzyk, Entropy for Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 118 (2001), 467–477. https://doi.org/10.1016/S0165-0114(98)00402-3.

[16] S.K. De, R. Biswas, A.R. Roy, An Application of Intuitionistic Fuzzy Sets in Medical Diagnosis, Fuzzy Sets Syst. 117 (2001), 209–213. https://doi.org/10.1016/S0165-0114(98)00235-8.

[17] Z. Xu, R.R. Yager, Some Geometric Aggregation Operators Based on Intuitionistic Fuzzy Sets, Int. J. Gen. Syst. 35 (2006), 417–433. https://doi.org/10.1080/03081070600574353.

[18] Z. Xu, Intuitionistic Fuzzy Aggregation Operators, IEEE Trans. Fuzzy Syst. 15 (2007), 1179–1187. https://doi.org/10.1109/TFUZZ.2006.890678.

[19] Z. Xu, Intuitionistic Preference Relations and Their Application in Group Decision Making, Inf. Sci. 177 (2007), 2363–2379. https://doi.org/10.1016/j.ins.2006.12.019.

[20] F.E. Boran, S. Genç, M. Kurt, D. Akay, A Multi-Criteria Intuitionistic Fuzzy Group Decision Making for Supplier Selection with TOPSIS Method, Expert Syst. Appl. 36 (2009), 11363–11368. https://doi.org/10.1016/j.eswa.2009.03.039.

[21] Z. Xu, N. Zhao, Information Fusion for Intuitionistic Fuzzy Decision Making: An Overview, Inf. Fusion 28 (2016), 10–23. https://doi.org/10.1016/j.inffus.2015.07.001.

[22] K.T. Atanassov, Circular Intuitionistic Fuzzy Sets, J. Intell. Fuzzy Syst. 39 (2020), 5981–5986. https://doi.org/10.3233/JIFS-189072.

[23] T. Senapati, G. Chen, R. Mesiar, R.R. Yager, Intuitionistic Fuzzy Geometric Aggregation Operators in the Framework of Aczel-Alsina Triangular Norms and Their Application to Multiple Attribute Decision Making, Expert Syst. Appl. 212 (2023), 118832. https://doi.org/10.1016/j.eswa.2022.118832.

[24] M. Wasim, A. Yousaf, H. Alolaiyan, M.A. Akbar, A. Alburaikan, et al., Optimizing Decision-Making with Aggregation Operators for Generalized Intuitionistic Fuzzy Sets and Their Applications in the Tech Industry, Sci. Rep. 14 (2024), 16538. https://doi.org/10.1038/s41598-024-57461-9.

[25] A. Fahmi, A. Khan, Z. Maqbool, T. Abdeljawad, Circular Intuitionistic Fuzzy Hamacher Aggregation Operators for Multi-Attribute Decision-Making, Sci. Rep. 15 (2025), 5618. https://doi.org/10.1038/s41598-025-88845-0.

[26] Y. Kumar, S. Ramalingam, G.B. Zegeye, A Novel Intuitionistic Fuzzy Yager Aggregation Framework for Decision Making in Green Supply Chains, Sci. Rep. 16 (2026), 8779. https://doi.org/10.1038/s41598-026-37890-4.

[27] D. Pratama, B. Yusoff, J.C.R. Alcantud, L. Abdullah, F. Al-Sharqi, Improved Ordering and Aggregation Operators for Circular Intuitionistic Fuzzy Sets in Multi-Criteria Decision-Making, Complex Intell. Syst. 12 (2026), 205. https://doi.org/10.1007/s40747-026-02320-5.

[28] I. Ullah, M.S.A. Khan, F. Hussain, C. Jana, N. Ivković, et al., Novel Intuitionistic Fuzzy Decision Making Method for Mobile Selection Problems Based on Grouping and Overlap Function, Int. J. Comput. Intell. Syst. (2026). https://doi.org/10.1007/s44196-026-01538-y.

[29] H. Qawaqneh, A. Shihadeh, W.M.M. Salameh, W. Abdelfattah, I. Ullah, et al., An Intuitionistic Fuzzy Dombi–Archimedean RANCOM–AROMAN Framework for AI-Enabled Telemedicine Platform Selection, J. Niger. Soc. Phys. Sci. (2026), 3619. https://doi.org/10.46481/jnsps.2026.3619.

[30] H. Qawaqneh, A. Shihadeh, W.M.M. Salameh, W. Abdelfattah, I. Ullah, et al., Strengthening the Digital Fortress: A Pythagorean Fuzzy Dombi–Archimedean Approach to Network Security Decisions, J. Niger. Soc. Phys. Sci. (2026), 3617. https://doi.org/10.46481/jnsps.2026.3617.

[31] I. Ullah, Kamran, M.S.A. Khan, F. Hussain, M. Khan, et al., A Robust Decision-Making Framework for Recruitment Optimization via Hybrid Dombi–Archimedean Operators in Complex q-Rung Orthopair Fuzzy Sets, Eur. J. Pure Appl. Math. 18 (2025), 6891. https://doi.org/10.29020/nybg.ejpam.v18i4.6891.

[32] T. Senapati, G. Chen, I. Ullah, M.S.A. Khan, F. Hussain, A Novel Approach Towards Multiattribute Decision Making Using q-Rung Orthopair Fuzzy Dombi–Archimedean Aggregation Operators, Heliyon 10 (2024), e27969. https://doi.org/10.1016/j.heliyon.2024.e27969.

[33] I. Ullah, Z. Khan, F. Hussain, R. Jan, I. Ahmad, et al., Circular Intuitionistic Fuzzy Power Muirhead-Mean-Based Aczel–Alsina Aggregation Operators and Their Application to a MADM Problem, Int. J. Model. Simul. (2025), 1–20. https://doi.org/10.1080/02286203.2025.2608097.

[34] I. Ullah, M.S.A. Khan, F. Hussain, M. Khan, I.-L. Popa, Innovative Aczel–Alsina Group Overlap Functions for AI-Based Criminal Justice Policy Selection under Intuitionistic Fuzzy Set, Comput. Model. Eng. Sci. 144 (2025), 2123–2164. https://doi.org/10.32604/cmes.2025.064832.

[35] M. Sugeno, Theory of Fuzzy Integrals and Its Applications, PhD Thesis, Tokyo Institute of Technology, 1974. https://cir.nii.ac.jp/crid/1572261549724574208.

[36] G. Choquet, Theory of Capacities, Ann. Inst. Fourier 5 (1954), 131–295. https://doi.org/10.5802/aif.53.

[37] M. Grabisch, The Application of Fuzzy Integrals in Multicriteria Decision Making, Eur. J. Oper. Res. 89 (1996), 445–456. https://doi.org/10.1016/0377-2217(95)00176-X.

[38] J.-L. Marichal, An Axiomatic Approach of the Discrete Choquet Integral as a Tool to Aggregate Interacting Criteria, IEEE Trans. Fuzzy Syst. 8 (2000), 800–807. https://doi.org/10.1109/91.890347.

[39] L.S. Shapley, A Value for n-Person Games, in: Contributions to the Theory of Games (AM-28), Volume II, Princeton University Press, pp. 307–318, (1953). https://doi.org/10.1515/9781400881970-018.

[40] F. Meng, Q. Zhang, H. Cheng, Approaches to Multiple-Criteria Group Decision Making Based on Interval-Valued Intuitionistic Fuzzy Choquet Integral with Respect to the Generalized λ-Shapley Index, Knowl.-Based Syst. 37 (2013), 237–249. https://doi.org/10.1016/j.knosys.2012.08.007.

[41] I. Ullah, M.S.A. Khan, Kamran, F. Hussain, M. Khan, et al., Aczel–Alsina Shapley Choquet Integral Operators for Multi-Criteria Decision Making in Complex Intuitionistic Fuzzy Environments, Symmetry 17 (2025), 868. https://doi.org/10.3390/sym17060868.

[42] Z. Xu, Choquet Integrals of Weighted Intuitionistic Fuzzy Information, Inf. Sci. 180 (2010), 726–736. https://doi.org/10.1016/j.ins.2009.11.011.

[43] J. Wu, F. Chen, C. Nie, Q. Zhang, Intuitionistic Fuzzy-Valued Choquet Integral and Its Application in Multicriteria Decision Making, Inf. Sci. 222 (2013), 509–527. https://doi.org/10.1016/j.ins.2012.07.056.

[44] X. Jia, Y. Wang, Choquet Integral-Based Intuitionistic Fuzzy Arithmetic Aggregation Operators in Multi-Criteria Decision-Making, Expert Syst. Appl. 191 (2022), 116242. https://doi.org/10.1016/j.eswa.2021.116242.

[45] K. Menger, Statistical Metrics, Proc. Natl. Acad. Sci. USA 28 (1942), 535–537. https://doi.org/10.1073/pnas.28.12.535.

[46] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Springer Netherlands, 2000. https://doi.org/10.1007/978-94-015-9540-7.

[47] C. Alsina, J. Aczél, Characterizations of Some Classes of Quasilinear Functions with Applications to Triangular Norms and to Synthesizing Judgements, Aequationes Math. 25 (1982), 313–315. https://doi.org/10.1007/bf02189626.