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GENERALIZED q-LAGUERRE POLYNOMIALS WITH q-PARTIAL DIFFERENCE EQUATIONS

ZINAH M. HUSSEIN, SAMAHER A. ABDUL-GHANI*

Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq

*Corresponding author: samaheradnanmath@gmail.com

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Abstract. In this paper we define the generalized q-Laguerre polynomials $L_n^{\alpha}(a,b,c,d,e,x,y|q)$. We found that they can be regarded as the solution of a q-partial difference equation. Then, we proved that an analytic function in several variables satisfies a certain system of q-partial difference equations if and only if it can be expanded in terms of generalized q-Laguerre polynomials. As applications of these conclusions, we prove the generating functions and their extensions, Mehler's formula and its extension, the Rogers formula and its extension, and mixed generating functions for generalized q-Laguerre polynomials by the method of q-partial difference equations. We also proved a multilinear generating function for generalized q-Laguerre polynomials.

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1. Introduction

Orthogonal polynomials are frequently found in a variety of classical physics in mathematics (see [22,25]). The Laguerre polynomials, for example, emerge in the propagation of harmonic oscillators, whereas electromagnetic waves. However, the study of q-orthogonal polynomials is also an important study. This topic is covered in relevant literature [6,29,34,38]. In this paper, we will follow common notations and definitions for the q-series that are used in [19]. We assume that $0 < q < 1$. For $\vartheta \in \mathbb{C}$, the q-shifted factorial is defined by [19]

$$(\vartheta;q)_0 = 1, \quad (\vartheta;q)_n = \prod_{j=0}^{n-1}(1-\vartheta q^j), \quad (\vartheta;q)_\infty = \prod_{j=0}^{\infty}(1-\vartheta q^j),$$

and the multiple q-shifted factorials by:

$$(\vartheta_1,\vartheta_2,\ldots,\vartheta_r;q)_m = (\vartheta_1;q)_m(\vartheta_2;q)_m\cdots(\vartheta_r;q)_m,$$

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where $m \in \mathbb{Z}$ or ∞.

The basic hypergeometric series ${}_r\phi_s$ is defined by

$$ {}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}x^n. $$

The q-derivative of $f(\mu)$ with respect to μ is defined by [1,28,36]

$$ D_q\{f(\mu)\}=\frac{f(\mu)-f(q\mu)}{\mu}. $$

According to the above equation, it is not difficult to prove

$$ D_q\{f(\mu)g(\mu)\}=D_q\{f(\mu)\}g(\mu)+f(q\mu)D_q\{g(\mu)\} \tag{1.1} $$

and

$$ \partial_{q,y}^k\{y^n\}=\frac{(q;q)_n}{(q;q)_{n-k}}y^{n-k} \tag{1.2} $$

and the Leibniz rule for the product of two functions

$$ D_q^k\{f(\mu)g(\mu)\}=\sum_{\jmath=0}^{k}\begin{bmatrix}k\\ \jmath\end{bmatrix}q^{\jmath(\jmath-k)}D_q^{\jmath}\{f(\mu)\}D_q^{k-\jmath}\{g(q^{\jmath}\mu)\}, \tag{1.3} $$

where

$$ \begin{bmatrix}\kappa\\ j\end{bmatrix}=\frac{(q;q)_\kappa}{(q;q)_j(q;q)_{\kappa-j}},0\le j\le\kappa,\kappa\in\mathbb{N} \tag{1.4} $$

For any nonnegative integer n, we have the higher-order q-derivative formula:

$$ \partial_{q,x}^n\left\{\frac{1}{(xt;q)_\infty}\right\}=\frac{t^n}{(xt;q)_\infty}, \tag{1.5} $$

which is the case $s=0$ of the following general higher-order q-derivative formula:

$$ \partial_{q,x}^n\left\{\frac{(xs;q)_\infty}{(xt;q)_\infty}\right\}=t^n(s/t;q)_n\frac{(xsq^n;q)_\infty}{(xt;q)_\infty}. \tag{1.6} $$

The q-binomial theorem

$$ \sum_{\jmath=0}^{\infty}\frac{(a;q)_\jmath}{(q;q)_\jmath}x^\jmath=\frac{(ax;q)_\infty}{(x;q)_\infty},\quad |x|<1. \tag{1.7} $$

The following two identities can also be derived from this theorem:

$$ \sum_{\jmath=0}^{\infty}\frac{1}{(q;q)_\jmath}x^\jmath=\frac{1}{(x;q)_\infty},\quad |x|<1, \tag{1.8} $$

$$ \sum_{\jmath=0}^{\infty}\frac{(-1)^\jmath q^{\binom{\jmath}{2}}x^\jmath}{(q;q)_\jmath}=(x;q)_\infty. \tag{1.9} $$

For over a century, the theory of fundamental hypergeometric series has advanced significantly, and there are numerous useful methods to study it, such as transformation, inversion, operator, and the

Wilf-Zeilberger algorithm, for instance (see [3,5,7–9,11,12,14–20,23,39–41]). Liu first presented the q-partial differential equation ten years ago. Many mathematicians have expressed interest in this novel approach. For more information, kindly consult [26–30]. In order to achieve this, we first define the q-partial derivative [32].

Definition 1.1. *The q-partial derivative of a function of multiple variables is its q-derivative with respect to one of those variables, regarding the other variables as constants. The q-partial derivative of a function f in relation to the variable x is represented by $\partial_{q,x}\{f\}$.*

The homogeneous Rogers-Szegő polynomials are defined as follows [37]:

$$h_n(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^k y^{n-k}. \tag{1.10}$$

Proposition 1.2. ([28]) *$h_n(x,y|q)$ satisfies the q-partial differential equation*

$$\partial_{q,x}\{h_n(x,y|q)\}=\partial_{q,y}\{h_n(x,y|q)\} \tag{1.11}$$

The Al-Salam-Carlitz polynomials were introduced by Al-Salam and Carlitz as (see [2,4,6]):

$$\varphi_n^{(a)}(x|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(a;q)_k x^k \tag{1.12}$$

The homogeneous Hahn polynomials are defined as (see [29])

$$\varphi_n^{(a)}(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(a;q)_k x^k y^{n-k}. \tag{1.13}$$

Proposition 1.3. ([29]) *The homogeneous Hahn polynomials satisfy the q-partial differential equation*

$$\partial_{q,x}\left\{\phi_n^{(\alpha)}(x,y|q)\right\}=\partial_{q,y}\left\{\phi_n^{(\alpha)}(x,y|q)-\alpha\phi_n^{(\alpha)}(xq,y|q)\right\}. \tag{1.14}$$

The homogeneous generalized Hahn polynomials are defined as (see [13])

$$\varphi_n^{(a,b,c)}(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(a,b;q)_k}{(c;q)_k}x^k y^{n-k}. \tag{1.15}$$

Proposition 1.4. ([35]) *The generalized homogeneous Hahn polynomials satisfy the q-partial difference equations*

$$\begin{aligned}&\partial_{q,x}\left\{\varphi_n^{(a,b,c)}(x,y|q)-q^{-1}c\varphi_n^{(a,b,c)}(xq,y|q)\right\}\\ =&\partial_{q,y}\left\{\varphi_n^{(a,b,c)}(x,y|q)-(a+b)\varphi_n^{(a,b,c)}(qx,y|q)+ab\varphi_n^{(a,b,c)}(q^2x,y|q)\right\}\end{aligned} \tag{1.16}$$

Koekoek and Swarttouw [24] state that the q-Laguerre polynomials were initially discovered by Hahn [21] and are defined by

$$L_n^{(\alpha)}(x|q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\phi_1\left(\begin{matrix}q^{-n}\\q^{\alpha+1}\end{matrix};q,-q^{n+\alpha+1}x\right),\alpha>-1. \tag{1.17}$$

The homogeneous Laguerre polynomials are defined as (see [31])

$$L_n^{(\alpha)}(x,y)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}\frac{q^{k^2+\alpha k}}{(q^{\alpha+1};q)_k}x^k y^{n-k},\alpha>-1 \tag{1.18}$$

Clearly,

$$L_n^{(\alpha)}(x,y)=\frac{(q;q)_n}{(q^{\alpha+1};q)_n}y^n L_n^{(\alpha)}(x/y|q),\mathbb{L}_n^{(\alpha)}(x,1)=\frac{(q;q)_n}{(q^{\alpha+1};q)_n}L_n^{(\alpha)}(x|q),L_n^{(\alpha)}(0,y)=y^n. \tag{1.19}$$

Proposition 1.5. ([10])*The homogeneous q-Laguerre polynomials satisfy the q-partial differential equation*

$$\partial_{q,x}(1-q^{\alpha}\eta_x)\left\{L_n^{(\alpha)}(x,y)\right\}=-q^{\alpha+1}\eta_x^2\partial_{q,y}\left\{L_n^{(\alpha)}(x,y)\right\} \tag{1.20}$$

The generating functions for the homogeneous q-Laguerre polynomials are as follows (see [10]):

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}L_n^{(\alpha)}(x,y|q)=(ty;q)_\infty\,{}_0\phi_2\left(\begin{matrix}-\\q^{\alpha+1},ty\end{matrix};q,-q^{\alpha+1}tx\right). \tag{1.21}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda;q)_n}{(q;q)_n}L_n^{(\alpha)}(x,y|q)t^n=\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\,{}_1\phi_2\left(\begin{matrix}\lambda\\q^{\alpha+1},\lambda ty\end{matrix};q,-q^{\alpha+1}tx\right). \tag{1.22}$$

$$\sum_{n=0}^{\infty}\frac{1}{(q;q)_n}L_n^{(\alpha)}(x,y|q)t^n=\frac{1}{(yt;q)_\infty}\,{}_0\phi_1\left(\begin{matrix}-\\q^{\alpha+1}\end{matrix};q,-q^{\alpha+1}tx\right). \tag{1.23}$$

The structure of this paper is as follows. In Section 2, it is demonstrated that generalized homogeneous q-Laguerre polynomials and an analytic function satisfies q-partial difference equations, provided that it can be expressed in terms of generalized homogeneous q-Laguerre polynomials (refer to Theorem 2.4). Section 3 is an application of Theorem 2.4, in which we demonstrate the generating functions and some generating functions of q-partial difference equations using generalized homogeneous q-Laguerre polynomials with various weights. In Section 4, we present Rogers-type formulas for the generalized homogeneous q-Laguerre polynomials that satisfy a system of q-partial difference equations (see Theorems 4.1 and 4.2). Mehler's formula and its extension for the generalized homogeneous q-Laguerre polynomials in Section 5 satisfy the q-partial difference equations. In Section 6, we offer mixed generating functions and some results for generalized q-Laguerre polynomials by the method of q-partial difference equations. In Section 7, for generalized q-Laguerre polynomials, we develop a multilinear generating function that satisfies q-partial difference equations.

2. Generalized q-Laguerre Polynomials

In this section, we will begin by introducing the generalized q-Laguerre polynomials and showing that an analytic function in several variables satisfies a certain system of q-partial difference equations if and only if it can be expanded in terms of generalized q-Laguerre polynomials.

Definition 2.1. *We define the generalized homogeneous q-Laguerre polynomials as follows: for* $\alpha > -1$

$$\mathbb{L}_n^{\alpha}(a,b,c;d,e,x,y,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{k^2+\alpha k}(a,b,c;q)_k}{(q^{\alpha+1},d,e;q)_k} x^k y^{n-k}. \tag{2.1}$$

- Setting $a = b = c = d = e = 0$ in equation 2.1, we get the homogeneous q-Laguerre polynomials (equation (1.18)).
- Setting $a = a, b = q, c = d = e = \alpha = 0$, $x = -xq^{-k}, y = 1$ in equation 2.1, we get (equation (1.12)).
- Setting $a = a$, $b = q$, $c = d = e = \alpha = 0$, $x = -xq^{-k}$ in equation 2.1, we get (equation (1.13)).
- Setting $a = a$, $b = b$, $c = q, d = c, e = \alpha = 0$, $x = -xq^{-k}$ in equation 2.1, we get (equation (1.15)).
- Setting $a = q, b = c = d = e = \alpha = 0, x = -xq^{-k}$ in equation 2.1, we get (equation (1.10)).

The following theorem highlights a crucial characteristic of generalized homogeneous q-Laguerre polynomials:

Theorem 2.2. *For* $\alpha > -1$, $n \in \mathbb{N}\bigcup\{0\}$, *the homogeneous generalized q-Laguerre polynomials* $L_n^{\alpha}(a,b,c;d,e,x,y)$ *satisfy the q-partial difference equation*

$$\begin{aligned} &\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)L_n^{\alpha}(a,b,c;d,e,x,y,q)\Big\} \\ &\quad = -q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2 L_n^{\alpha}(a,b,c;d,e,x,y,q)\Big\}. \end{aligned} \tag{2.2}$$

Proof. Let L.H.S. denote the left-hand side of equation 2.2. By using the identity $\partial_{q,x}\{x^k\} = (1-q^k)x^{k-1}$, we get

$$\begin{aligned} LHS =& \partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)f(x,y)\Big\} \\ =& \partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)L_n^{\alpha}(a,b,c;d,e,x,y,q)\Big\} \\ =& \partial_{q,x}\Big\{\sum_{k=1}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(-1)^k q^{k^2+\alpha k}(a,b,c;q)_k}{(q^{\alpha+1},d,e;q)_{k-1}}x^k y^{n-k}\Big\} \\ =& \sum_{k=1}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(1-q^k)\frac{(-1)^k q^{k^2+\alpha k}(a,b,c;q)_k}{(q^{\alpha+1},d,e;q)_{k-1}}x^{k-1}y^{n-k}. \end{aligned}$$

Similarly, use R.H.S. to denote the right-hand side of equation 2.2,

$$\begin{aligned} RHS =& -q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2 L_n^{\alpha}(a,b,c;d,e,x,y,q)\Big\} \\ =& -q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(-1)^k q^{k^2+\alpha k}(a,b,c;q)_k}{(q^{\alpha+1},d,e;q)_k}x^k y^{n-k}\Big\}. \\ =& \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(1-q^{n-k})\frac{(-1)^{k+1}q^{(k+1)^2+\alpha(k+1)}(a,b,c;q)_{k+1}}{(q^{\alpha+1},d,e;q)_k}x^k y^{n-k-1} \end{aligned}$$

$$=\sum_{k=1}^{n}\begin{bmatrix} n \\ k-1 \end{bmatrix}(1-q^{n-k+1})\frac{(-1)^k q^{(k)^2+\alpha(k)}(a,b,c;q)_k}{(q^{\alpha+1},d,e;q)_{k-1}}x^{k-1}y^{n-k}.$$

□

From the definition of the q-binomial coefficients, we can prove that

$$\begin{bmatrix} n \\ k \end{bmatrix}(1-q^k)=\begin{bmatrix} n \\ k-1 \end{bmatrix}(1-q^{n-k+1})$$

It follows from 2.2 that LHS=RHS.

- Setting $a=b=c=d=e=0$ in equation 2.2, we get (equation (1.20)).
- Setting $a=a, b=q, c=d=e=\alpha=0, x=-xq^{-k}$ in equation 2.2, we get (equation (1.14)).
- Setting $a=a$, $b=b$, $c=q, d=c, e=\alpha=0, x=-xq^{-k}$ in equation 2.2, we get (equation (1.16)).
- Setting $a=q, b=c=d=e=\alpha=0, x=-xq^{-}k$ in equation 2.2, we get (equation (1.11)).

Definition 2.3. *(see [33]) If $\xi(x_1,x_2,\cdots,x_j)$ is an analytic function at the origin $(0,0,\cdots,0)\in C^j$, then ξ can be expressed as a power series that converges absolutely and uniformly.*

$$\xi(x_1,x_2,\cdots,x_j)=\sum_{n_1,n_2,\cdots,n_j=0}^{\infty}\lambda_{n_1,n_2,\cdots,n_j}x_1^{n_1}x_2^{n_2}\cdots x_j^{n_j}.$$

The main outcome of this section is the following expansion theorem for analytic functions in several variables.

Theorem 2.4. If $\xi^{(a,b,c,d,e)}(x_1,y_1,x_2,y_2,\cdots,x_j,y_j)$ is a $2j$-variable analytic function at $(0,0,\cdots,0)\in C^{2j}$, then ξ can be expanded
$\sum_{n_1,n_2,\cdots,n_j=0}^{\infty}\lambda_{n_1,n_2,\cdots,n_j}L_{n_1}^{(\alpha_1)}(a_1,b_1,c_1;d_1,e_1,x_1,y_1|q)\cdots L_{n_j}^{(\alpha_j)}(a_j,b_j,c_j;d_j,e_j,x_j,y_j|q)$,
where $\lambda_{n_1,n_2,\cdots,n_j}$ are independent of $x_1,y_1,\cdots,x_j,y_j$, if and only if it satisfies the q-partial difference equations.

$$\begin{aligned}&\partial_{q,x_k}\left\{(1-d_kq^{-1}\eta_x)(1-e_kq^{-1}\eta_x)(1-q^{\alpha_k}\eta_x)L_{n_k}^{(\alpha_k)}(a_k,b_k,c_k;d_k,e_k,x_k,y_k|q)\right\}\\&\quad=-q^{\alpha_k+1}\partial_{q,y_k}\left\{(1-a_k\eta_x)(1-b_k\eta_x)(1-c_k\eta_x)\eta_{x_k}^2L_{n_k}^{(\alpha_k)}(a_k,b_k,c_k;d_k,e_k,x_k,y_k|q)\right\}.\end{aligned}\tag{2.3}$$

for $k\in 1,2,\cdots,j$.

Proof. Theorem 2.4 can be proved using mathematical induction. First, we prove the case where $j=1$. It follows from definition (2.3) that ξ can be expanded in an absolutely and uniformly convergent power series in the neighborhood of $(0,0)$. Thus, there exists a sequence $\{\lambda_{m,n}\}$ independent of x_1 and y_1 such that

$$\xi(x_1,y_1)=\sum_{m,n=0}^{\infty}\lambda_{m,n}x_1^my_1^n=\sum_{m=0}^{\infty}x_1^m\{\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n\}.\tag{2.4}$$

Substituting the above equation into the following q-partial differential equation

$$\partial_{q,x_1}\left\{(1-d_1q^{-1}\eta_{x_1})(1-e_1q^{-1}\eta_{x_1})(1-q^{\alpha_1}\eta_{x_1})\{\xi(x_1,y_1)\}\right\}$$
$$=-q^{\alpha_1+1}\partial_{q,y_1}\left\{(1-a_1\eta_{x_1})(1-b_1\eta_{x_1})(1-c_1\eta_{x_1})\eta_{x_1}^2\xi(x_1,y_1)\right\} \tag{2.5}$$

we obtain

$$\sum_{m=1}^{\infty}(1-q^{\alpha_1+m})(1-q^m)(1-d_1q^{m-1})(1-e_1q^{m-1})x_1^{m-1}\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n$$
$$=-q^{\alpha_1+1}\sum_{m=0}^{\infty}(1-a_1q^m)(1-b_1q^m)(1-c_1q^m)x_1^mq^{2m}\partial_{q,y_1}\left\{\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n\right\}$$

$$\sum_{m=1}^{\infty}(1-q^{\alpha_1+m})(1-q^m)(1-d_1q^{m-1})(1-e_1q^{m-1})x_1^{m-1})x_1^{m-1}\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n$$
$$=-q^{\alpha_1+1}\sum_{m=1}^{\infty}(1-a_1q^{m-1})(1-b_1q^{m-1})(1-c_1q^{m-1})x_1^{m-1}q^{2m-2}\partial_{q,y_1}\left\{\sum_{n=0}^{\infty}\lambda_{m-1,n}y_1^n\right\} \tag{2.6}$$

By comparing the coefficients of x^{m-1} on both sides of equation (2.6), we can readily conclude that

$$\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n=\frac{=-q^{\alpha_1+1}(1-a_1q^{m-1})(1-b_1q^{m-1})(1-c_1q^{m-1})x_1^{m-1}q^{2m-2}}{(1-q^{\alpha_1+m})(1-q^m)(1-d_1q^{m-1})(1-e_1q^{m-1})}\partial_{q,y_1}\left\{\sum_{n=0}^{\infty}\lambda_{m-1,n}y_1^n\right\}$$

Then iterating this relation $m-1$ times and using (1.2) yields

$$\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n=\frac{(-q^{\alpha_1+1})^mq^{m(m-1)}(a_1;q)_m(b_1;q)_m(c_1;q)_m}{(q;q)_m(q^{\alpha_1+1};q)_m(d_1;q)_m(e_1;q)_m}\partial_{q,y_1}^m\left\{\sum_{n=0}^{\infty}\lambda_{0,n}y_1^n\right\}$$
$$=\frac{(-1)^mq^{m^2+\alpha_1m}(a_1;q)_m(b_1;q)_m(c_1;q)_m}{(q;q)_m(q^{\alpha_1+1};q)_m(d_1;q)_m(e_1;q)_m}\sum_{n=0}^{\infty}\lambda_{0,n}\frac{(q;q)_n}{(q;q)_{n-m}}y_1^{n-m}$$
$$=\sum_{n=m}^{\infty}\lambda_{0,n}(-1)^m\begin{bmatrix}n\\m\end{bmatrix}\frac{q^{m^2+\alpha_1m}(a_1;q)_m(b_1;q)_m(c_1;q)_m}{(q^{\alpha_1+1};q)_m(d_1;q)_m(e_1;q)_m}y_1^{n-m}.$$

Recognizing that the series $\xi(x_1,y_1)$ in (2.4) converges absolutely, we can substitute the previously mentioned equation into (2.3) and swap the order of summation, leading us to conclude that

$$\xi(x_1,y_1)=\sum_{m=0}^{\infty}x_1^m\sum_{n=m}^{\infty}\lambda_{0,n}(-1)^m\begin{bmatrix}n\\m\end{bmatrix}\frac{q^{m^2+\alpha_1m}(a_1;q)_m(b_1;q)_m(c_1;q)_m}{(q^{\alpha_1+1};q)_m(d_1;q)_m(e_1;q)_m}y_1^{n-m}$$
$$=\sum_{n=0}^{\infty}\lambda_{0,n}\sum_{m=0}^{n}(-1)^m\begin{bmatrix}n\\m\end{bmatrix}\frac{q^{m^2+\alpha_1m}(a_1;q)_m(b_1;q)_m(c_1;q)_m}{(q^{\alpha_1+1};q)_m(d_1;q)_m(e_1;q)_m}x_1^my_1^{n-m}$$
$$=\sum_{n=0}^{\infty}\lambda_{0,n}L_n^{(\alpha_1)}(a_1,b_1,c_1;d_1,e_1,x_1,y_1|q).$$

This suggests that $\xi(x_1,y_1)$ can be represented using $L_n^{(\alpha_1)}(a_1,b_1,c_1;d_1,e_1,x_1,y_1|q)$. On the other hand, if $\xi(x_1,y_1)$ can be expressed in terms of $L_n^{(\alpha_1)}(a_1,b_1,c_1;d_1,e_1,x_1,y_1|q)$ then, according to (2.4), it follows that $\xi(x_1,y_1)$ fulfills condition (2.3). Therefore, we conclude that Theorem 2.4 is valid for $j=1$. Next, we assume

Theorem 2.4 is valid for the case $j-1$ and examine the case j. When we treat $\xi(x_1, y_1, \cdots, x_j, y_j)$ as a function of x_1 and y_1, then ξ is analytic at $(0,0)$ and satisfies (2.4). Thus, there exists a sequence $\{\nu_{n_1}(x_2, y_2, \cdots, x_j, y_j)\}$ independent of x_1 and y_1 such that

$$\xi(x_1, y_1, \cdots, x_j, y_j) = \sum_{n_1=0}^{\infty} \nu_{n_1}(x_2, y_2, \cdots, x_j, y_j) L_{n_1}^{(\alpha_1)}(a_1, b_1, c_1; d_1, e_1, x_1, y_1) \tag{2.7}$$

Setting $x_1 = 0$ in the above equation and using $L_{n_1}^{(\alpha_1)}(a_1, b_1, c_1; d_1, e_1, 0, y_1) = y_1^{n_1}$, we obtain

$$\xi(0, y_1, x_2, y_2, \cdots, x_j, y_j) = \sum_{n_1=0}^{\infty} \nu_{n_1}(x_2, y_2, \cdots, x_j, y_j) y_1^{n_1}$$

By using the Maclaurin expansion theorem, we have

$$\nu_{n_1}(x_2, y_2, \cdots, x_j, y_j) = \frac{\partial^{n_1} \xi(0, y_1, x_2, y_2, \cdots, x_j, y_j)}{n_1! \partial_{y_1}^{n_1}}|_{y_1=0} \tag{2.8}$$

Since $\xi(x_1, y_1, \cdots, x_j, y_j)$ is an analytic function near $(x_1, y_1, \cdots, x_j, y_j) = (0, 0, \cdots, 0) \in C^{2j}$, it follows from the above equation that $\nu_{n_1}(x_2, y_2, \cdots, x_j, y_j)$ is analytic near $(x_2, y_2, \cdots, x_j, y_j) = (0, \cdots, 0) \in C^{2j-2}$. Putting (2.7) into (2.3), we find that for $k = 2, \cdots, j$,

$$\begin{aligned}
&\sum_{n_1=0}^{\infty} \partial_{q,x_k} \Big\{ (1 - q^{\alpha_k} \eta_{x_k})(1 - d_k q^{-1} \eta_{x_k})(1 - e_k q^{-1} \eta_{x_k}) \{\nu_{n_1}(x_2, y_2, \cdots, x_j, y_j)\} \\
&\qquad \times L_{n_1}^{(\alpha_1)}(a_1, b_1, c_1; d_1, e_1, x_1, y_1) \Big\} \\
&= \sum_{n_1=0}^{\infty} (-q^{\alpha_k+1} \eta_{x_k}^2) \partial_{q,y_k} \Big\{ (1 - a_k \eta_{x_k})(1 - b_k \eta_{x_k})(1 - c_k \eta_{x_k}) \nu_{n_1}(x_2, y_2, \cdots, x_j, y_j) \\
&\qquad \times L_{n_1}^{(\alpha_1)}(a_1, b_1, c_1; d_1, e_1, x_1, y_1) \Big\}.
\end{aligned}$$

By equating the coefficients of $L_{n_1}^{(\alpha_1)}(a_1, b_1, c_1; d_1, e_1, x_1, y_1)$ in the above equation, we find

$$\begin{aligned}
&\partial_{q,x_k} \Big\{ (1 - q^{\alpha_k} \eta_{x_k})(1 - d_k q^{-1} \eta_{x_k})(1 - e_k q^{-1} \eta_{x_k}) \{\nu_{n_1}(x_2, y_2, \cdots, x_j, y_j)\} \Big\} \\
&= -q^{\alpha_k+1} \eta_{x_k}^2 \partial_{q,y_k} \Big\{ (1 - a_k \eta_{x_k})(1 - b_k \eta_{x_k})(1 - c_k \eta_{x_k}) \nu_{n_1}(x_2, y_2, \cdots, x_j, y_j) \Big\}.
\end{aligned}$$

According to the inductive hypothesis, the sequence $\lambda_{n_1, n_2, \cdots, n_j}$ is independent of $x_2, y_2, \cdots, x_j, y_j$, and of course independent of x_1 and y_1.

$$\nu_{n_1}(x_2, y_2, \cdots, x_j, y_j) = \sum_{n_1, \cdots, n_j}^{\infty} L_{n_2}^{(\alpha_2)}(a_2, b_2, c_2; d_2, e_2, x_2, y_2|q) \cdots L_{n_j}^{(\alpha_j)}(a_j, b_j, c_j; d_j, e_j, x_j, y_j|q)$$

Putting this equation in (2.7), we proved that the theorem is sufficient. However, if ξ can be expanded in terms of $L_{n_1}^{(\alpha_1)}(a_1, b_1, c_1; d_1, e_1, x_1, y_1|q) \cdots L_{n_j}^{(\alpha_j)}(a_j, b_j, c_j; d_j, e_j, x_j, y_j|q)$, then it follows from Definition 2.3 that ξ satisfies (2.3). This completes the proof of the theorem. □

3. The Generating Function for $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$

In this section, we use Theorem 2.2 to derive the generating function and its extension for the generalized q-Laguerre polynomials $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$. A number of special values are supplied for the parameters of the generating function, in addition to its extension for polynomials $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$ to get the generating function and multi generating function for $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$.

Theorem 3.1. (Generating function for $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$). *For $\alpha > -1$, we have*

$$\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)\frac{t^n}{(q;q)_n} = (yt;q)_\infty\, {}_3\phi_5\left(\begin{matrix}a,b,c\\ d,e,q^{\alpha+1},yt,0\end{matrix};q,-xtq^{\alpha+1}\right). \tag{3.1}$$

Proof. *Let us denote the right-hand side of 3.1 by $f(x,y)$. It follows from Theorem 2.4 that $f(x,y)$ is an analytic function of x and y. Thus $f(x,y)$ is analytic at $(0,0)\in\mathbb{C}^2$. On the one hand, by using 2.2, we have*

$$\begin{aligned}
&\partial_{q,x}\left\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^\alpha\eta_x)f(x,y)\right\}\\
&=\partial_{q,x}\left\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^\alpha\eta_x)(yt;q)_\infty\, {}_3\phi_5\left(\begin{matrix}a,b,c\\ d,e,q^{\alpha+1},yt,0\end{matrix};q,-xq^{\alpha+1}\right)\right\}\\
&=(yt;q)_\infty\partial_{q,x}\left\{(1-dq^{-1}\eta x)(1-eq^{-1}\eta x)(1-q^\alpha\eta_x)\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^3(a,b,c;q)_k}{(q^{\alpha+1},d,e,yt;q)_k(q;q)_k}(-xtq^{\alpha+1})^k\right\}\\
&=(yt;q)_\infty\sum_{k=1}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^3(-tq^{\alpha+1})^k(a,b,c;q)_k}{(q^{\alpha+1},d,e;q)_{k-1}(yt;q)_k(q;q)_k}(1-q^k)x^{k-1}\\
&=(yt;q)_\infty\sum_{k=1}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^3(-tq^{\alpha+1})^k(a,b,c;q)_k}{(q^{\alpha+1},d,e;q)_{k-1}(yt;q)_k(q;q)_{k-1}}x^{k-1}\\
&=-tq^{\alpha+1}(yt;q)_\infty\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k+1}{2}}]^3(a,b,c;q)_{k+1}}{(q^{\alpha+1},d,e;q)_k(yt;q)_{k+1}(q;q)_k}(-xtq^{\alpha+1})^k
\end{aligned}$$

On the other hand, by using 2.2,

$-q^{\alpha+1}\partial_{q,y}\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2 f(x,y)\}$

Let us take $\partial_{q,y}\{f(x,y)\}$ and use (1.1):

$$\begin{aligned}
&\partial_{q,y}\{f(x,y)\}\\
=&\partial_{q,y}\left\{(yt;q)_\infty\right\}\, {}_3\phi_5\left(\begin{matrix}a,b,c\\ d,e,q^{\alpha+1},yt,0\end{matrix};q,-xq^{\alpha+1}\right)+\left\{(yqt;q)_\infty\right\}\\
&\times\partial_{q,y}\left\{\, {}_3\phi_5\left(\begin{matrix}a,b,c\\ d,e,q^{\alpha+1},yt,0\end{matrix};q,-xq^{\alpha+1}\right)\right\}\\
=&\frac{-t(yt;q)_\infty}{(1-yt)}\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^3(a,b,c;q)_k}{(q^{\alpha+1},d,e;q)_k(yt;q)_k(q;q)_k}(-xtq^{\alpha+1})^k
\end{aligned}$$

$$+ (yqt;q)_\infty \sum_{k=0}^{\infty} \frac{t(1-q^k)\,[(-1)^k q^{\binom{k}{2}}]^3 (a,b,c;q)_k}{(1-ytq^k)\,(q^{\alpha+1},d,e;q)_k (yt;q)_k (q;q)_k} (-xtq^{\alpha+1})^k$$

$$=(yt;q)_\infty \sum_{k=0}^{\infty} \frac{-tq^k[(-1)^k q^{\binom{k}{2}}]^3 (a,b,c;q)_k}{(q^{\alpha+1},d,e;q)_k (yt;q)_{k+1} (q;q)_k} (-xtq^{\alpha+1})^k$$

Now we obtain,

$$-q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2 f(x,y)\Big\}$$

$$= - q^{\alpha+1}(yt;q)_\infty \sum_{k=0}^{\infty} \frac{-tq^k[(-1)^k q^{\binom{k}{2}}]^3 (a,b,c;q)_{k+1}}{(q^{\alpha+1},d,e;q)_k (yt;q)_{k+1} (q;q)_k} (-xq^2tq^{\alpha+1})^k$$

$$= - tq^{\alpha+1}(yt;q)_\infty \sum_{k=0}^{\infty} \frac{[(-1)^k q^{\binom{k+1}{2}}]^3 (a,b,c;q)_{k+1}}{(q^{\alpha+1},d,e;q)_k (yt;q)_{k+1} (q;q)_k} (-xtq^{\alpha+1})^k$$

Then, by Theorem 2.4, there exists a sequence $\{\nu_n\}$ independent of x and y such that

$$(yt;q)_\infty \, {}_3\phi_5 \left(\begin{matrix} a,b,c \\ d,e,q^{\alpha+1},yt,0 \end{matrix} ; q, -xtq^{\alpha+1} \right) = \sum_{n=0}^{\infty} \nu_n L_n^{(\alpha)}(a,b,c;d,e,x,y|q).$$

Setting $x=0$ in the above equation, using $L_n^{(\alpha)}(a,b,c;d,e,0,y|q) = y^n$ and by Cauchy (1.9), we get

$$\sum_{n=0}^{\infty} \nu_n y^n = (yt;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} (yt)^n$$

which suggests

$$\nu_n = \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} (t)^n$$

□

- Setting $a=b=c=d=e=0$ in equation 3.1, we recover the generating function for the polynomials $L_n^{(\alpha)}(x,y|q)$ in equation (1.21).

Theorem 3.2. For $\alpha > -1$. Another generating function for generalized q-Laguerre polynomials $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$

$$\sum_{n=0}^{\infty} \mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q) \frac{t^n}{(q;q)_n} = \frac{1}{(yt;q)_\infty} \, {}_3\phi_4 \left(\begin{matrix} a,b,c \\ d,e,q^{\alpha+1},0 \end{matrix} ; q, -xtq^{\alpha+1} \right), \tag{3.2}$$

where $|yt| < 1$.

Proof. Let $f(x,y)$ *denote the right-hand side of Theorem 3.2; then, by using Theorem 2.2,*

$$\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^\alpha\eta_x) f(x,y)\Big\}$$

$$= -q^{\alpha+1}\partial_{q,y}\left\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2 f(x,y)\right\}$$

$$\begin{aligned}
&\partial_{q,x}\left\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)f(x,y)\right\}\\
&=\partial_{q,x}\left\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\frac{1}{(yt;q)_\infty}\,{}_3\phi_4\left(\begin{matrix}a,b,c\\ d,e,q^{\alpha+1},0\end{matrix};q,-xtq^{\alpha+1}\right)\right\}\\
&=\frac{1}{(yt;q)_\infty}\partial_{q,x}\left\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2(a,b,c;q)_k}{(q;q)_k(d,e,q^{\alpha+1;q})_k}(-xtq^{\alpha+1})^k\right\}\\
&=\frac{1}{(yt;q)_\infty}\left\{\sum_{k=1}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2(a,b,c;q)_k}{(q;q)_{k-1}(d,e,q^{\alpha+1;q})_{k-1}}(-tq^{\alpha+1})^k x^{k-1}\right\}\\
&=\frac{-tq^{\alpha+1}}{(yt;q)_\infty}\left\{\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2 q^{2k}(a,b,c;q)_{k+1}}{(q;q)_k(d,e,q^{\alpha+1;q})_k}(-xtq^{\alpha+1})^k\right\}
\end{aligned}$$

On the other hand, by using Theorem 2.2,

$$\begin{aligned}
&-q^{\alpha+1}\partial_{q,y}\left\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2 f(x,y)\right\}\\
&=-q^{\alpha+1}\partial_{q,y}\left\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2\left[\frac{1}{(yt;q)_\infty}\,{}_3\phi_4\left(\begin{matrix}a,b,c\\ d,e,q^{\alpha+1},0\end{matrix};q,-xtq^{\alpha+1}\right)\right]\right\}\\
&=-q^{\alpha+1}\partial_{q,y}\left\{\frac{1}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2 q^{2k}(a,b,c;q)_{k+1}}{(q;q)_k(d,e,q^{\alpha+1};q)_k}(-xtq^{\alpha+1})^k\right\}\\
&=\frac{-tq^{\alpha+1}}{(yt;q)_\infty}\left\{\sum_{k=1}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2 q^{2k}(a,b,c;q)_{k+1}}{(q;q)_k(d,e,q^{\alpha+1;q})_k}(-xtq^{\alpha+1})^k\right\}
\end{aligned}$$

Then, by Theorem 2.4, there exists a sequence $\{\nu_n\}$ independent of x and y such that

$$\frac{1}{(yt;q)_\infty}\,{}_3\phi_4\left(\begin{matrix}a,b,c\\ d,e,q^{\alpha+1},0\end{matrix};q,-xtq^{\alpha+1}\right)=\sum_{n=0}^{\infty}\nu_n L_n^{(\alpha)}(a,b,c;d,e,x,y|q).$$

Setting $x=0$ in the above equation, using $L_n^{(\alpha)}(a,b,c;d,e,0,y|q)=y^n$ and by Cauchy (1.8), we get

$$\sum_{n=0}^{\infty}\nu_n y^n=\frac{1}{(yt;q)_\infty}=\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}$$

which suggests

$$\nu_n=\frac{t^n}{(q;q)_n}$$

□

- Setting $a=b=c=d=e=0$ in equation 3.2, we recover the generating function for the polynomials $L_n^{(\alpha)}(x,y|q)$ in equation (1.23).

Theorem 3.3. *Extension of Generating function for* $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$

$$\sum_{n=0}^{\infty}\mathbb{L}_{n+m}^{(\alpha)}(a,b,c,d,e,x,y|q)\frac{t^n}{(q;q)_n}=\frac{1}{(yt;q)_\infty}$$
$$\times\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}t^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k}x^{k+j}\begin{bmatrix}m\\j\end{bmatrix}y^{m-j}(yt;q)_j \tag{3.3}$$

Where $|yt|<1$.

Proof. Let $f(x,y)$ denote the right-hand side of Theorem 3.3. Then, by using Theorem 2.2, the extension of the generating function satisfies the q-partial difference equation as

$$\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^\alpha\eta_x)f(x,y)\Big\}$$
$$=\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^\alpha\eta_x)$$
$$\times\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}t^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k}x^{k+j}\begin{bmatrix}m\\j\end{bmatrix}\frac{y^{m-j}}{(ytq^j;q)_\infty}\Big\}$$
$$=\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^\alpha\eta_x)$$
$$\times\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{(k)^2+\alpha(k)}t^{k-j}}{(d,e,q^{\alpha+1};q)_k(q;q)_{k-j}}x^k\begin{bmatrix}m\\j\end{bmatrix}\frac{y^{m-j}}{(ytq^j;q)_\infty}\Big\}$$
$$=\partial_{q,x}\Big\{\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_k}x^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(q;q)_m}{(q;q)_{m-j}}\frac{y^{m-j}t^{k-j}}{(ytq^j;q)_\infty}\Big\}$$
$$=\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_{k-1}}x^{k-1}\partial_{q,y}^k\Big\{\frac{y^m}{(yt;q)_\infty}\Big\}$$
$$=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k\partial_{q,y}^{k+1}\Big\{\frac{y^m}{(yt;q)_\infty}\Big\}$$
$$=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k$$
$$\times\Big\{\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{t^{k-j}(q;q)_m}{(q;q)_{m-j}}\frac{y^{m-j-1}}{(ytq^j;q)_\infty}[1-q^{(m-j)}+yq^mt]\Big\}$$

Set $m=j=0$

$$=\frac{-tq^{\alpha+1}}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}(q^2xt)^k$$

On the other hand

$$-q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2f(x,y)\Big\}$$
$$=-q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2$$

$$\times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}t^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k}x^{k+j}\begin{bmatrix}m\\ j\end{bmatrix}\frac{y^{m-j}}{(ytq^j;q)_\infty}\Bigg\}$$

$$=-q^{\alpha+1}\partial_{q,y}\Bigg\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{(k)^2+\alpha(k)}t^{k-j}}{(d,e,q^{\alpha+1};q)_k(q;q)_{k-j}}x^k$$

$$\times\begin{bmatrix}m\\ j\end{bmatrix}\frac{y^{m-j}}{(ytq^j;q)_\infty}\Bigg\}$$

$$=-q^{\alpha+1}\partial_{q,y}\Bigg\{\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}\frac{(q;q)_m}{(q;q)_{m-j}}\frac{y^{m-j}t^{k-j}}{(ytq^j;q)_\infty}\Bigg\}$$

$$=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k\partial_{q,y}^{k+1}\Bigg\{\frac{y^m}{(yt;q)_\infty}\Bigg\}$$

$$=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k$$

$$\times\Bigg\{\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}\frac{t^{k-j}(q;q)_m}{(q;q)_{m-j}}\frac{y^{m-j-1}}{(ytq^j;q)_\infty}[1-q^{(m-j)}+yq^mt]\Bigg\}$$

Set $m=j=0$; then

$$=\frac{-tq^{\alpha+1}}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}(q^2xt)^k.$$

Then, by Theorem 2.4, there exists a sequence $\{\nu_n\}$ independent of x and y such that

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}t^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k}x^{k+j}\begin{bmatrix}m\\ j\end{bmatrix}\frac{y^{m-j}}{(ytq^j;q)_\infty}=\sum_{n=0}^{\infty}\nu_nL_{n+m}^{(\alpha)}(a,b,c,d,e,x,y|q).$$

Setting $x=0$ in the above equation, using $L_{n+m}^{(\alpha)}(a,b,c,d,e,0,y|q)=y^{n+m}$ and by (1.7), we get

$$\frac{y^m}{(yt;q)_\infty}=\sum_{n=0}^{\infty}\nu_ny^{n+m}$$

$$\sum_{n=0}^{\infty}\frac{y^m(yt)^n}{(q;q)_n}=\sum_{n=0}^{\infty}\nu_ny^{n+m}$$

$$\nu_n=\frac{t^n}{(q;q)_n}$$

□

- Setting $m=j=0$ in equation 3.3, we regain the generating function for the polynomials $L_n^{(\alpha)}(a,b,c,d,e,x,y|q)$ (equation 3.2)
- Setting $a=b=c=d=e=0$ in equation 3.3, we recover the extension of the generating function for the polynomials $L_n^{(\alpha)}(x,y|q)$ in the following corollary.

Theorem 3.4. (Some generating function for $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$). *For $\alpha > -1$ and $\lambda \in R$, we have*

$$\sum_{n=0}^{\infty} \mathbb{L}_n^{(\alpha)}(a,b,c;d,e,x,y|q)\frac{(\lambda;q)_n t^n}{(q;q)_n} = \frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\, {}_4\phi_5\left(\begin{matrix} a,b,c,\lambda \\ d,e,q^{\alpha+1},\lambda yt,0 \end{matrix};q,-xtq^{\alpha+1}\right) \tag{3.4}$$

where $|yt| < 1$.

Proof. *Let us denote the right-hand side of 3.4 by $f(x,y)$. It follows from Theorem 2.4 that $f(x,y)$ is an analytic function of x and y. Thus $f(x,y)$ is analytic at $(0,0) \in \mathbb{C}^2$. On the one hand, by using 2.2, we have*

$$\begin{aligned}
&\partial_{q,x}\left\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^\alpha\eta_x)f(x,y)\right\} \\
&=\partial_{q,x}\left\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^\alpha\eta_x)\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\, {}_4\phi_5\left(\begin{matrix} a,b,c,\lambda \\ q^{\alpha+1},d,e,\lambda yt,0 \end{matrix};q,-xtq^{\alpha+1}\right)\right\} \\
&=\partial_{q,x}\left\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^\alpha\eta_x)\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2(a,b,c,\lambda;q)_k}{(q;q)_k(q^{\alpha+1},d,e,\lambda yt;q)_k}(-xtq^{\alpha+1})^k\right\} \\
&=\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\partial_{q,x}\left\{\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2(a,b,c,\lambda;q)_k}{(q;q)_k(q^{\alpha+1},d,e;q)_{k-1}(\lambda yt;q)_k}(-tq^{\alpha+1})^k x^k\right\} \\
&=\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\left\{\sum_{k=1}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2(a,b,c,\lambda;q)_k}{(q;q)_{k-1}(q^{\alpha+1},d,e;q)_{k-1}(\lambda yt;q)_k}(-tq^{\alpha+1})^k x^{k-1}\right\} \\
&=\frac{-tq^{\alpha+1}(\lambda yt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2 q^{2k}(a,b,c,\lambda;q)_{k+1}}{(q;q)_k(q^{\alpha+1},d,e;q)_k(\lambda yt;q)_{k+1}}(-xtq^{\alpha+1})^k
\end{aligned}$$

On the other hand, by Theorem 2.2 and (1.1),

$$\begin{aligned}
&\partial_{q,y}\{f(x,y)\} \\
&=\partial_{q,y}\left\{\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\, {}_4\phi_5\left(\begin{matrix} a,b,c,\lambda \\ q^{\alpha+1},d,e,\lambda yt,0 \end{matrix};q,-xtq^{\alpha+1}\right)\right\} \\
&=\partial_{q,y}\left\{[\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2(a,b,c,\lambda;q)_k}{(q;q)_k(q^{\alpha+1},d,e,\lambda yt;q)_k}(-xtq^{\alpha+1})^k]\right\} \\
&=\partial_{q,y}\left\{\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\right\}\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2(a,b,c,\lambda;q)_k}{(q;q)_k(q^{\alpha+1},d,e,\lambda yt;q)_k}(-xtq^{\alpha+1})^k \\
&+\frac{(\lambda ytq;q)_\infty}{(ytq;q)_\infty}\partial_{q,y}\left\{\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2(a,b,c,\lambda;q)_k}{(q;q)_k(q^{\alpha+1},d,e,\lambda yt;q)_k}(-xtq^{\alpha+1})^k\right\}
\end{aligned}$$

by using (1.6)

$$\begin{aligned}
&=\frac{t(1-\lambda)(\lambda yt;q)_\infty}{(1-\lambda yt)(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2(a,b,c,\lambda;q)_k}{(q;q)_k(q^{\alpha+1},d,e,\lambda yt;q)_k}(-xtq^{\alpha+1})^k \\
&+\frac{(1-ty)(\lambda yt;q)_\infty}{(1-\lambda yt)(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{\lambda t(1-q^k)[(-1)^k q^{\binom{k}{2}}]^2(a,b,c,\lambda;q)_k}{(1-\lambda ytq^k)(q;q)_k(q^{\alpha+1},d,e,\lambda yt;q)_k}(-xtq^{\alpha+1})^k
\end{aligned}$$

$$=\frac{t(\lambda yt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2(a,b,c;q)_k(\lambda;q)_{k+1}}{(q;q)_k(q^{\alpha+1},d,e;q)_k(\lambda yt;q)_{k+1}}(-xtq^{\alpha+1})^k,$$

from which we obtain

$$\begin{aligned}
&-q^{\alpha+1}\,\partial_{q,y}\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2 f(x,y)\}\\
&\quad=\frac{-tq^{\alpha+1}(\lambda yt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{[(-1)^k q^{\binom{k}{2}}]^2 q^{2k}(a,b,c;q)_{k+1}(\lambda;q)_{k+1}}{(q;q)_k(q^{\alpha+1},d,e;q)_k(\lambda yt;q)_{k+1}}(-xtq^{\alpha+1})^k\\
&\quad=\partial_{q,x}\left\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^\alpha\eta_x)f(x,y)\right\}
\end{aligned}$$

By Theorem 2.4, there exists a sequence $\{\nu_n\}$ independent of x and y such that

$$\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\,{}_4\phi_5\left(\begin{matrix}a,b,c,\lambda\\ d,e,q^{\alpha+1},\lambda yt,0\end{matrix};q,-xtq^{\alpha+1}\right)=\sum_{n=0}^{\infty}\nu_n L_n^{(\alpha)}(a,b,c;d,e,x,y|q).$$

Setting $x=0$ in the above equation, using $L_n^{(\alpha)}(a,b,c,d,e,0,y|q)=y^n$ and Cauchy (1.7), we find that

$$\sum_{n=0}^{\infty}\nu_n y^n=\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}=\sum_{n=0}^{\infty}\frac{(\lambda;q)_n}{(q;q)_n}(yt)^n$$

by equating which suggests

$$\nu_n=\frac{(\lambda;q)_n}{(q;q)_n}(t)^n$$

□

- Setting $a=b=c=d=e=0$ in equation 3.4, we recover the generating function for the polynomials $L_n^{(\alpha)}(x,y|q)$ in equation (1.22).
- Setting $\lambda=0$ in equation 3.4, we regain the simple generating function for the generalized q-Laguerre polynomials $L_n^{(\alpha)}(a,b,c,d,e,x,y|q)$ in equation 3.2
- Setting $\lambda=0$ and $a=b=c=d=e=0$ in equation 3.4, we regain the simple generating function for the q-Laguerre polynomials $L_n^{(\alpha)}(x,y|q)$ in equation (1.23)

4. Rogers Formula for $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$

In this section, we will present the Rogers formula and its extension for the generalized q-Laguerre polynomials $\mathbb{L}_n^{(\alpha)}(a,b,c;d,e,x,y|q)$ using the method of q-partial difference equations.

Theorem 4.1. (Rogers Formula for $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$). *For $\alpha>-1$, we have*

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathbb{L}_{n+m}^{(\alpha)}(a,b,c,d,e,x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}&=\frac{1}{(yt,ys;q)_\infty}\\
&\times\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}t^j s^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_j(q;q)_k}(-xq^{\alpha+k+j})^{k+j}(ys;q)_j,\\
&\max\{|yt|,|ys|\}<1.
\end{aligned}\tag{4.1}$$

Proof. *Let us denote the right-hand side of 4.1 by $f(x,y)$. It follows from Theorem 2.4 that $f(x,y)$ is an analytic function of x and y. Thus $f(x,y)$ is analytic at $(0,0)\in\mathbb{C}^2$. On the one hand, by using 2.2, we have*

$$\begin{aligned}
&\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)f(x,y)\Big\}\\
&=\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\frac{1}{(yt,ys;q)_\infty}\\
&\quad\times\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}t^js^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_j(q;q)_k}(-xq^{\alpha+k+j})^{k+j}(ys;q)_j\Big\}\\
&=\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\frac{1}{(yt,ys;q)_\infty}\\
&\quad\times\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{(a,b,c;q)_{k}t^js^{k-j}}{(d,e,q^{\alpha+1};q)_{k}(q;q)_j(q;q)_{k-j}}(-xq^{\alpha+k})^{k}(ys;q)_j\Big\}\\
&=\partial_{q,x}\Big\{\frac{1}{(yt,ys;q)_\infty}\sum_{k=1}^{\infty}\sum_{j=0}^{k}\frac{(a,b,c;q)_{k}t^js^{k-j}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_j(q;q)_{k-j}}(-xq^{\alpha+k})^{k}(ys;q)_j\Big\}\\
&=\partial_{q,x}\Big\{\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_k}(x)^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{t^js^{k-j}}{(yt,ysq^j;q)_\infty}\Big\}
\end{aligned}$$

By using equation (1.3) we get

$$\begin{aligned}
&=\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_{k-1}}(x)^{k-1}\partial_{q,y}^k\Big\{\frac{1}{(yt,ys;q)_\infty}\Big\}\\
&=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}(x)^k\partial_{q,y}^{k+1}\Big\{\frac{1}{(yt,ys;q)_\infty}\Big\}\\
&=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}(x)^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{t^js^{k-j}[t+(1-yt)sq^j]}{(yt;q)_\infty(ysq^j;q)_\infty}
\end{aligned}$$

Set $j=0$

$$=\frac{-q^{\alpha+1}}{(yt;q)_\infty(ys;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}(xs)^k[t+(1-yt)s]$$

On the other hand

$$\begin{aligned}
&-q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2f(x,y)\Big\}\\
&=-q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2\frac{1}{(yt,ys;q)_\infty}\\
&\quad\times\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}t^js^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_j(q;q)_k}(-xq^{\alpha+k+j})^{k+j}(ys;q)_j\Big\}\\
&=-q^{\alpha+1}\partial_{q,y}\Big\{\frac{1}{(yt,ys;q)_\infty}\times\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{(a,b,c;q)_{k+1}q^{2k}t^js^{k-j}}{(d,e,q^{\alpha+1};q)_k(q;q)_j(q;q)_{k-j}}(-xq^{\alpha+k})^{k}(ys;q)_j\Big\}
\end{aligned}$$

$$= - q^{\alpha+1}\partial_{q,y}\left\{ \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1} q^{2k}(-1)^k q^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k (q;q)_k} x^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{t^j s^{k-j}}{(yt, ysq^j;q)_\infty} \right\}$$

By using equation (1.3) we get

$$= - q^{\alpha+1}\partial_{q,y}\left\{ \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1} q^{2k}(-1)^k q^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k (q;q)_k} x^k \partial_{q,y}^k \left\{ \frac{1}{(yt,ys;q)_\infty} \right\} \right\}$$

$$= - q^{\alpha+1} \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1}(-1)^k q^{2k} q^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k (q;q)_k} (x)^k \partial_{q,y}^{k+1} \left\{ \frac{1}{(yt,ys;q)_\infty} \right\}$$

$$= - q^{\alpha+1} \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1}(-1)^k q^{2k} q^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k (q;q)_k} (x)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{t^j s^{k-j}[t+(1-yt)sq^j]}{(yt;q)_\infty (ysq^j;q)_\infty}$$

Then, by Theorem 2.4, there exists a sequence $\{\nu_n\}$ independent of x and y such that

$$\frac{1}{(yt,ys;q)_\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+j} t^j s^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_j (q;q)_k} (-xq^{\alpha+k+j})^{k+j} (ys;q)_j$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \nu_n L_{n+m}^{(\alpha)}(a,b,c,d,e,x,y|q).$$

Setting $x = 0$ in the above equation, using $L_{n+m}^{(\alpha)}(a,b,c,d,e,0,y|q) = y^{n+m}$ and by (1.7), we get

$$\frac{1}{(yt;q)_\infty}\frac{1}{(ys;q)_\infty} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \nu_n y^{n+m}$$
$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(yt)^n}{(q;q)_n}\frac{(ys)^m}{(q;q)_m} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \nu_n y^{n+m}$$
$$\nu_n = \frac{t^n s^m}{(q;q)_n (q;q)_m}$$

□

Theorem 4.2. (Extension of Rogers formula for $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y,|q)$). *For $\alpha > -1$, we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty} \mathbb{L}_{n+m+i}^{(\alpha)}(a,b,c,d,e,x,y|q) \frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{\ell^i}{(q;q)_i} = \frac{1}{(yt,y\ell;q)_\infty}$$
$$\times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+j} t^j}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_j (q;q)_k} (-xq^{\alpha+k+j})^{k+j} (y\ell;q)_j \sum_{n=i}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \frac{\ell^i s^{k-i}}{(ysq^j q^i;q)_\infty}, \tag{4.2}$$

where $\max\{|yt|,|ys|,|y\ell|\} < 1$.

Proof. *Let us denote the right-hand side of 4.2 by $f(x,y)$. It follows from Theorem 2.4 that $f(x,y)$ is an analytic function of x and y. Thus $f(x,y)$ is analytic at $(0,0) \in \mathbb{C}^2$. On the one hand, by using 2.2, we have*

$$\partial_{q,x}\left\{ (1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^\alpha \eta_x) f(x,y) \right\}$$

$$
\begin{aligned}
&=\partial_{q,x}\Bigg\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\frac{1}{(yt,y\ell;q)_\infty}\\
&\quad\times\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}t^j}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_j(q;q)_k}(-xq^{\alpha+k+j})^{k+j}(y\ell;q)_j\sum_{n=i}^{k}\begin{bmatrix}k\\i\end{bmatrix}\frac{\ell^i s^{k-i}}{(ysq^jq^i;q)_\infty}\Bigg\}\\
&=\partial_{q,x}\Bigg\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\frac{1}{(yt,y\ell;q)_\infty}\\
&\quad\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{(a,b,c;q)_k(t)^j}{(d,e,q^{\alpha+1};q)_k(q;q)_j(q;q)_{k-j}}(-xq^{\alpha+k})^k(y\ell;q)_j\sum_{n=i}^{k-j}\begin{bmatrix}k-j\\i\end{bmatrix}\frac{\ell^i s^{k-j-i}}{(ysq^jq^i;q)_\infty}\Bigg\}\\
&=\partial_{q,x}\Bigg\{\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_k}x^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{t^j}{(yt;q)_\infty}\sum_{n=i}^{k-j}\begin{bmatrix}k-j\\i\end{bmatrix}\frac{\ell^i s^{k-j-i}}{(y\ell q^j;q)_\infty(ysq^jq^i;q)_\infty}\Bigg\}
\end{aligned}
$$

By using equation (1.3) we get

$$
\begin{aligned}
&=\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_{k-1}}x^{k-1}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\partial_{q,y}^j\Bigg\{\frac{1}{(yt;q)_\infty}\Bigg\}\partial_{q,y}^{k-j}\Bigg\{\frac{1}{(y\ell q^j,ysq^j;q)_\infty}\Bigg\}\\
&=\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_{k-1}}x^{k-1}\partial_{q,y}^k\Bigg\{\frac{1}{(yt,ys,y\ell;q)_\infty}\Bigg\}\\
&=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k\partial_{q,y}^{k+1}\Bigg\{\frac{1}{(yt,ys,y\ell;q)_\infty}\Bigg\}\\
&=\frac{-q^{\alpha+1}}{(yt,y\ell;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}t^j(y\ell;q)_j\sum_{i=0}^{k-j}\begin{bmatrix}k-j\\i\end{bmatrix}\frac{\ell^i s^{k-j-i}}{(ysq^jq^i;q)_\infty}\\
&\quad[t+(1-yt)[\ell q^j+(1-y\ell q^j)sq^i+j]]
\end{aligned}
$$

On the other hand, by using Theorem 2.2,

$$
\begin{aligned}
&-q^{\alpha+1}\partial_{q,y}\Bigg\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2f(x,y)\Bigg\}\\
&\quad=-q^{\alpha+1}\partial_{q,y}\Bigg\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2\frac{1}{(yt,y\ell;q)_\infty}\\
&\qquad\times\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}t^j}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_j(q;q)_k}(-xq^{\alpha+k+j})^{k+j}(y\ell;q)_j\sum_{n=i}^{k}\begin{bmatrix}k\\i\end{bmatrix}\frac{\ell^i s^{k-i}}{(ysq^jq^i;q)_\infty}\Bigg\}\\
&\quad=-q^{\alpha+1}\partial_{q,y}\Bigg\{\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{(a,b,c;q)_{k+1}q^{2k}}{(d,e,q^{\alpha+1};q)_k(q;q)_j(q;q)_{k-j}}(-xq^{\alpha+k})^k\frac{t^j}{(yt;q)_\infty}\\
&\qquad\times\sum_{n=i}^{k-j}\begin{bmatrix}k-j\\i\end{bmatrix}\frac{\ell^i s^{k-j-i}}{(y\ell q^j;q)_\infty(ysq^jq^i;q)_\infty}\Bigg\}\\
&\quad=-q^{\alpha+1}\partial_{q,y}\Bigg\{\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}q^{2k}}{(d,e,q^{\alpha+1};q)_k(q;q)_j(q;q)_k}(-xq^{\alpha+k})^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{t^j}{(yt;q)_\infty}
\end{aligned}
$$

$$\times \sum_{n=i}^{k-j} \begin{bmatrix} k-j \\ i \end{bmatrix} \frac{\ell^i s^{k-j-i}}{(y\ell q^j;q)_\infty (ysq^j q^i;q)_\infty} \Bigg\}$$

By using equation (1.3) and (1.5) we get

$$\begin{aligned}
&= -q^{\alpha+1}\partial_{q,y} \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1} q^{2k}(-xq^{\alpha+k})^k}{(d,e,q^{\alpha+1};q)_k (q;q)_k} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \partial_{q,y}^j \left\{ \frac{1}{(yt;q)_\infty} \right\} \partial_{q,y}^{k-j} \left\{ \frac{1}{(y\ell q^j, ysq^j;q)_\infty} \right\} \\
&= -q^{\alpha+1}\partial_{q,y} \left\{ \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1} (-1)^k q^{2k} q^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k (q;q)_k} x^k \partial_{q,y}^k \left\{ \frac{1}{(yt,ys,y\ell;q)_\infty} \right\} \right\} \\
&= -q^{\alpha+1} \left\{ \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1} (-1)^k q^{2k} q^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k (q;q)_k} x^k \partial_{q,y}^{k+1} \left\{ \frac{1}{(yt,ys,y\ell;q)_\infty} \right\} \right\} \\
&= \frac{-q^{\alpha+1}}{(yt,y\ell;q)_\infty} \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1} (-1)^k q^{2k} q^{k^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k (q;q)_k} x^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} t^j (y\ell;q)_j \sum_{i=0}^{k-j} \begin{bmatrix} k-j \\ i \end{bmatrix} \frac{\ell^i s^{k-j-i}}{(ysq^j q^i;q)_\infty} \\
&\quad \times \left[t + (1-yt)[\ell q^j + (1-y\ell q^j) sq^i + j] \right].
\end{aligned}$$

□

- Letting $\ell = 0$, $i = 0$ in equation 4.2, we get the Rogers formula for the polynomials $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$ (equation 4.1).

5. Mehler's Formula for $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$

In this section, we construct Mehler's formula and its extension for the generalized Laguerre polynomials $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$ and prove that Mehler's formula and its extension for the generalized q-Laguerre polynomials satisfy the q-partial differential equation according to Theorem 2.2.

Theorem 5.1. (Mehler's formula for $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$). *For $\alpha > -1, n \in \mathbb{N}$ and $\beta > -1$, we have*

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q) \mathbb{L}_n^{(\beta)}(f,g,h,r,s,w,z|q) \frac{t^n}{(q;q)_n} &= \frac{1}{(ytz;q)_\infty} \\
&\times \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+j} (-1)^{k+j} q^{(k+j)^2+\alpha(k+j)} (tz)^k}{(d,e,q^{\alpha+1};q)_{k+j} (q;q)_k (q;q)_j} x^{k+j} \\
&\times \sum_{l=0}^{\infty} \frac{(f,g,h;q)_l [(-1)^l q^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l (q;q)_{l-j}} (-wtq^{\beta+1})^l y^{l-j} (ytz;q)_j
\end{aligned} \tag{5.1}$$

where $|ytz| < 1$.

Let $f(x,y)$ denote the right-hand side of Mehler's formula 5.1. Then, by using Theorem 2.2, Mehler's formula for the generalized Laguerre polynomials $\mathbb{L}_n^{(\alpha)}(a,b,c;d,e,x,y|q)$ satisfies the q-partial differential equation as

Proof.

$$\partial_{q,x} \Big\{ (1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^\alpha \eta_x) f(x,y) \Big\}$$

$$
\begin{aligned}
=&\partial_{q,x}\Bigg\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}(tz)^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k(q;q)_j}\\
&\times\ x^{k+j}\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^lq^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l(q;q)_{l-j}}(-wtq^{\beta+1})^l\frac{y^{l-j}}{(ytzq^j;q)_\infty}\Bigg\}\\
=&\partial_{q,x}\Bigg\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{(k)^2+\alpha(k)}(tz)^{k-j}}{(d,e,q^{\alpha+1};q)_k(q;q)_{k-j}(q;q)_j}x^k\\
&\times\ \sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^lq^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l(q;q)_{l-j}}(-wtq^{\beta+1})^l\frac{y^{l-j}}{(ytzq^j;q)_\infty}\Bigg\}\\
=&\partial_{q,x}\Bigg\{\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_k}x^k\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^lq^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l\\
&\times\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(q;q)_l}{(q;q)_{l-j}}y^{l-j}\frac{(tz)^{k-j}}{(ytzq^j;q)_\infty}\Bigg\}
\end{aligned}
$$

By using equation (1.3) we get

$$
\begin{aligned}
=&\ \partial_{q,x}\Bigg\{\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_k}x^k\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^lq^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l\\
&\times\ \partial_{q,y}^k\{\frac{y^l}{(ytz;q)_\infty}\}\Bigg\}\\
=&-q^{\alpha+1}\Bigg\{\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}q^{2k}(-1)^kq^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^lq^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l\\
&\times\ \partial_{q,y}^{k+1}\{\frac{y^l}{(ytz;q)_\infty}\}\Bigg\}\\
=&-q^{\alpha+1}\Bigg\{\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}q^{2k}(-1)^kq^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^lq^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l\\
&\times\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(q;q)_l(tz)^{k-j}}{(q;q)_{l-j}}\frac{y^{l-j-1}}{(ytzq^j;q)_\infty}[1-q^{l-j}+yq^ltz]\Bigg\}
\end{aligned}
$$

Putting $l=j=0$ in the above equation, we get

$$
=\frac{-tzq^{\alpha+1}}{(ytz;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{(k)^2+\alpha(k)}(tz)^k}{(d,e,q^{\alpha+1};q)_k(q;q)_k}(xq^2)^k
$$

On the other hand

$$
\begin{aligned}
&-q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2f(x,y)\Big\}\\
&=-q^{\alpha+1}\partial_{q,y}\Bigg\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}(tz)^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k(q;q)_j}\\
&\times\ x^{k+j}\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^lq^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l(q;q)_{l-j}}(-wtq^{\beta+1})^l\frac{y^{l-j}}{(ytzq^j;q)_\infty}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= - q^{\alpha+1}\partial_{q,y}\Bigg\{ \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} \frac{(a,b,c;q)_{k+1}q^{2(k)}(-1)^k q^{(k)^2+\alpha(k)}(tz)^{k-j}}{(d,e,q^{\alpha+1};q)_k(q;q)_{k-j}(q;q)_j}x^k \\
&\quad \times \sum_{l=0}^{\infty} \frac{(f,g,h;q)_l[(-1)^l q^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l(q;q)_{l-j}}(-wtq^{\beta+1})^l \frac{y^{l-j}}{(ytzq^j;q)_\infty}\Bigg\} \\
&= - q^{\alpha+1}\partial_{q,y}\Bigg\{ \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1}(-1)^k q^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k \sum_{l=0}^{\infty} \frac{(f,g,h;q)_l[(-1)^l q^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l \\
&\quad \times \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(q;q)_l}{(q;q)_{l-j}} y^{l-j} \frac{(tz)^{k-j}}{(ytzq^j;q)_\infty}\Bigg\}
\end{aligned}
$$

By using equation (1.3) we get

$$
\begin{aligned}
&= - q^{\alpha+1}\partial_{q,y}\Bigg\{ \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1}(-1)^k q^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k \sum_{l=0}^{\infty} \frac{(f,g,h;q)_l[(-1)^l q^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l \\
&\quad \times \partial_{q,y}^k\{\frac{y^l}{(ytz;q)_\infty}\} \\
&= - q^{\alpha+1}\Bigg\{ \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1}q^{2k}(-1)^k q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k \sum_{l=0}^{\infty} \frac{(f,g,h;q)_l[(-1)^l q^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l \\
&\quad \times \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(q;q)_l(tz)^{k-j}}{(q;q)_{l-j}} \frac{y^{l-j-1}}{(ytzq^j;q)_\infty}[1-q^{l-j}+yq^l tz]\Bigg\}
\end{aligned}
$$

Put $l = j = 0$

$$
= \frac{-tzq^{\alpha+1}}{(ytz;q)_\infty} \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1}(-1)^k q^{(k)^2+\alpha(k)}(tz)^k}{(d,e,q^{\alpha+1};q)_k(q;q)_k}(xq^2)^k
$$

□

Theorem 5.2. (Extension of Mehler's formula for $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$). *For $\alpha,\beta > -1$ and $n,m \in \mathbb{N}$, we have*

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \mathbb{L}_{n+m}^{(\alpha)}(a,b,c,d,e,x,y|q)\mathbb{L}_n^{(\beta)}(f,g,h,r,s,w,z|q)\frac{t^n}{(q;q)_n} \\
&= \frac{1}{(ytz;q)_\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}(tz)^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k)(q;q)_j}x^{k+j} \\
&\quad \times \sum_{l=0}^{\infty} \frac{(f,g,h;q)_l[(-1)^l q^{\binom{l}{2}}]^2(q;q)_{m+l}}{(r,s,q^{\beta+1};q)_l(q;q)_l(q;q)_{m+l-j}}(-wtq^{\beta+1})^l y^{m+l-j}(ytz;q)_j
\end{aligned} \tag{5.2}
$$

where $|ytz| < 1$.

Proof. Let $f(x,y)$ denote the right-hand side of the extension of Mehler's formula 5.2. Then, by using Theorem 2.2, the extension of Mehler's formula for the generalized q-Laguerre polynomials

$\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$ satisfies the q-partial differential equation as

$$\begin{aligned}
&\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)f(x,y)\Big\}\\
&=\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}(tz)^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k(q;q)_j}\\
&\quad\times\ x^{k+j}\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^lq^{\binom{l}{2}}]^2(q;q)_{m+l}}{(r,s,q^{\beta+1};q)_l(q;q)_l(q;q)_{m+l-j}}(-wtq^{\beta+1})^l\frac{y^{m+l-j}}{(ytzq^j;q)_\infty}\Big\}\\
&=\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{(k)^2+\alpha(k)}(tz)^{k-j}}{(d,e,q^{\alpha+1};q)_k(q;q)_{k-j}(q;q)_j}x^k\\
&\quad\times\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^lq^{\binom{l}{2}}]^2(q;q)_{m+l}}{(r,s,q^{\beta+1};q)_l(q;q)_l(q;q)_{m+l-j}}(-wtq^{\beta+1})^l\frac{y^{m+l-j}}{(ytzq^j;q)_\infty}\Big\}\\
&=\partial_{q,x}\Big\{\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_k}x^k\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^lq^{\binom{l}{2|2}}]}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l\\
&\quad\times\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(q;q)_{m+l}}{(q;q)_{m+l-j}}\frac{y^{m+l-j}(tz)^{k-j}}{(ytzq^j;q)_\infty}\Big\}
\end{aligned}$$

By using equation (1.3) we get

$$\begin{aligned}
&=\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_{k-1}}x^{k-1}\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^lq^{\binom{l}{2|2}}]}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l\\
&\quad\times\ \partial_{q,y}^k\Big\{\frac{y^{m+l}}{(ytz;q)_\infty}\Big\}\\
&=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^lq^{\binom{l}{2|2}}]}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l\\
&\quad\times\ \partial_{q,y}^{k+1}\Big\{\frac{y^{m+l}}{(ytz;q)_\infty}\Big\}\\
&=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^lq^{\binom{l}{2|2}}]}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l\\
&\quad\times\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(q;q)_{m+l}(tz)^{k-j}}{(q;q)_{m+l-j}}\frac{y^{m+l-j-1}}{(ytzq^j;q)_\infty}[1-q^{m+l-j}+yq^{m+l}tz]
\end{aligned}$$

Put $m=j=l=0$

$$=\frac{-tzq^{\alpha+1}}{(ytz;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}q^{2(k)}(-1)^kq^{(k)^2+\alpha(k)}(tz)^k}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k$$

On the other hand, by Theorem 2.2,

$$
\begin{aligned}
&-q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2 f(x,y)\Big\}\\
&\quad=-q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}(tz)^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k(q;q)_j}\\
&\qquad\times\ x^{k+j}\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^l q^{\binom{l}{2}}]^2(q;q)_{m+l}}{(r,s,q^{\beta+1};q)_l(q;q)_l(q;q)_{m+l-j}}(-wtq^{\beta+1})^l\frac{y^{m+l-j}}{(ytzq^j;q)_\infty}\Big\}\\
&\quad=-q^{\alpha+1}\partial_{q,y}\Big\{\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{(a,b,c;q)_{k+1}q^{2(k)}(-1)^k q^{(k)^2+\alpha(k)}(tz)^{k-j}}{(d,e,q^{\alpha+1};q)_k(q;q)_{k-j}(q;q)_j}x^k\\
&\qquad\times\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^l q^{\binom{l}{2}}]^2(q;q)_{m+l}}{(r,s,q^{\beta+1};q)_l(q;q)_l(q;q)_{m+l-j}}(-wtq^{\beta+1})^l\frac{y^{m+l-j}}{(ytzq^j;q)_\infty}\Big\}\\
&\quad=-q^{\alpha+1}\partial_{q,y}\Big\{\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}q^{2(k)}(-1)^k q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k\\
&\qquad\times\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^l q^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l\sum_{j=0}^{\infty}\begin{bmatrix}k\\j\end{bmatrix}\frac{(q;q)_{m+l}}{(q;q)_{m+l-j}}\frac{y^{m+l-j}(tz)^{k-j}}{(ytzq^j;q)_\infty}\Big\}
\end{aligned}
$$

By using equation (1.3) we get

$$
\begin{aligned}
&=-q^{\alpha+1}\partial_{q,y}\Big\{\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}q^{2(k)}(-1)^k q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k\\
&\qquad\times\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^l q^{\binom{l}{2}}]^2}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l\partial_{q,y}^k\{\frac{y^{m+l}}{(ytz;q)_\infty}\}\Big\}\\
&=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^k q^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^l q^{\binom{l}{2}}]}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l\\
&\qquad\times\ \partial_{q,y}^{k+1}\Big\{\frac{y^{m+l}}{(ytz;q)_\infty}\Big\}\\
&=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^k q^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k\sum_{l=0}^{\infty}\frac{(f,g,h;q)_l[(-1)^l q^{\binom{l}{2}}]}{(r,s,q^{\beta+1};q)_l(q;q)_l}(-wtq^{\beta+1})^l\\
&\qquad\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(q;q)_{m+l}(tz)^{k-j}}{(q;q)_{m+l-j}}\frac{y^{m+l-j-1}}{(ytzq^j;q)_\infty}[1-q^{m+l-j}+yq^{m+l}tz]
\end{aligned}
$$

Put $m=j=l=0$

$$
=\frac{-tzq^{\alpha+1}}{(ytz;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}q^{2(k)}(-1)^k q^{(k)^2+\alpha(k)}(tz)^k}{(d,e,q^{\alpha+1};q)_k(q;q)_k}x^k
$$

□

6. Mixed Generating Functions for $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$

In this section, we create mixed generating functions for the generalized q-Laguerre polynomials $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$ using the method of q-partial difference equations.

Theorem 6.1. (Mixed generating function for $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$). *For $\alpha > -1, \beta > -1, n \in \mathbb{N}$, we have*

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)\mathbb{L}_n^{(\beta)}(w,z|q)\frac{t^n}{(q;q)_n} &= \frac{1}{(ytz;q)_\infty}\\ &\times\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}(tz)^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k(q;q)_j}x^{(k+j)}\\ &\times\sum_{m=0}^{\infty}\frac{q^{m^2+\beta m}}{(q^{\beta+1};q)_m(q;q)_{m-j}}(-wt)^m y^{m-j}(ytz;q)_j \end{aligned}\tag{6.1}$$

provided $\{|ytz|\} < 1$.

Proof. Let $f(x,y)$ denote the right-hand side of the mixed generating function 6.1. Then, by using Theorem 2.2, $f(x,y)$ satisfies the q-partial differential equation as

$$\begin{aligned}&\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)f(x,y)\Big\}\\ &=\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\frac{1}{(ytz;q)_\infty}\\ &\quad\times\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}(tz)^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k(q;q)_j}x^{(k+j)}\\ &\quad\times\sum_{m=0}^{\infty}\frac{q^{m^2+\beta m}}{(q^{\beta+1};q)_m(q;q)_{m-j}}(-wt)^m y^{m-j}(ytz;q)_j\Big\}\\ &=\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{(a,b,c;q)_k(-1)^k q^{(k)^2+\alpha k}(tz)^{k-j}}{(d,e,q^{\alpha+1};q)_k(q;q)_{k-j}(q;q)_j}x^{(k)}\\ &\quad\times\sum_{m=0}^{\infty}\frac{q^{m^2+\beta m}}{(q^{\beta+1};q)_m(q;q)_{m-j}}(-wt)^m\frac{y^{m-j}}{(ytzq^j;q)_\infty}\Big\}\\ &=\partial_{q,x}\Big\{\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^k q^{(k)^2+\alpha k}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_k}x^k\\ &\qquad\times\sum_{m=0}^{\infty}\frac{q^{m^2+\beta m}}{(q^{\beta+1};q)_m(q;q)_m}(-wt)^m\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}\frac{(q;q)_m}{(q;q)_{m-j}}\frac{y^{m-j}(tz)^{k-j}}{(ytzq^j;q)_\infty}\Big\}.\end{aligned}$$

By using equation (1.3) we get

$$=\partial_{q,x}\Big\{\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^k q^{(k)^2+\alpha k}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_k}x^{(k)}\times\sum_{m=0}^{\infty}\frac{q^{m^2+\beta m}}{(q^{\beta+1};q)_m(q;q)_m}(-wt)^m\partial_{q,y}^k\{\frac{y^m}{(ytz;q)_\infty}\}\Big\}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{(a,b,c;q)_k(-1)^k q^{(k)^2+\alpha k}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_{k-1}} x^{(k-1)} \times \sum_{m=0}^{\infty} \frac{q^{m^2+\beta m}}{(q^{\beta+1};q)_m(q;q)_m}(-wt)^m \partial_{q,y}^k \{\frac{y^m}{(ytz;q)_\infty}\} \\
&= -q^{\alpha+1} \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1}(-1)^k q^{2k} q^{(k)^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k(q;q)_k} x^{(k)} \times \sum_{m=0}^{\infty} \frac{q^{m^2+\beta m}}{(q^{\beta+1};q)_m(q;q)_m}(-wt)^m \partial_{q,y}^{k+1} \{\frac{y^m}{(ytz;q)_\infty}\} \\
&= -q^{\alpha+1} \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1}(-1)^k q^{2k} q^{(k)^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k(q;q)_k} x^{(k)} \times \sum_{m=0}^{\infty} \frac{q^{m^2+\beta m}}{(q^{\beta+1};q)_m(q;q)_m}(-wt)^m \\
&\quad \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(q;q)_m}{(q;q)_{m-j}} \frac{y^{m-j-1}(tz)^{k-j}}{(ytzq^j;q)_\infty}[1-q^{m-j}+ytzq^m].
\end{aligned}
$$

On the other hand, by Theorem 2.2,

$$
\begin{aligned}
&-q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2 f(x,y)\Big\} \\
&\quad = -q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2 \frac{1}{(ytz;q)_\infty} \\
&\qquad \times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+j}(-1)^{k+j} q^{(k+j)^2+\alpha(k+j)}(tz)^k}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k(q;q)_j} x^{(k+j)} \\
&\qquad \times \sum_{m=0}^{\infty} \frac{q^{m^2+\beta m}}{(q^{\beta+1};q)_m(q;q)_{m-j}}(-wt)^m y^{m-j}(ytz;q)_j \Big\} \\
&\quad = -q^{\alpha+1}\partial_{q,y}\Big\{ \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} \frac{(a,b,c;q)_{k+1}(-1)^k q^{2k} q^{(k)^2+\alpha(k)}(tz)^{k-j}}{(d,e,q^{\alpha+1};q)_k(q;q)_{k-j}(q;q)_j} x^{(k)} \\
&\qquad \times \sum_{m=0}^{\infty} \frac{q^{m^2+\beta m}}{(q^{\beta+1};q)_m(q;q)_{m-j}}(-wt)^m \frac{y^{m-j}}{(ytzq^j;q)_\infty} \Big\} \\
&\quad = -q^{\alpha+1}\partial_{q,y}\Big\{ \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1}(-1)^k q^{2k} q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k} x^{(k)} \\
&\qquad \times \sum_{m=0}^{\infty} \frac{q^{m^2+\beta m}}{(q^{\beta+1};q)_m(q;q)_m}(-wt)^m \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(q;q)_m}{(q;q)_{m-j}} \frac{y^{m-j}(tz)^{k-j}}{(ytzq^j;q)_\infty} \Big\}
\end{aligned}
$$

By using equation (1.3) we get

$$
\begin{aligned}
&= -q^{\alpha+1} \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1}(-1)^k q^{2k} q^{(k)^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k(q;q)_k} x^{(k)} \times \sum_{m=0}^{\infty} \frac{q^{m^2+\beta m}}{(q^{\beta+1};q)_m(q;q)_m}(-wt)^m \partial_{q,y}^{k+1} \{\frac{y^m}{(ytz;q)_\infty}\} \\
&= -q^{\alpha+1} \sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+1}(-1)^k q^{2k} q^{(k)^2+\alpha k}}{(d,e,q^{\alpha+1};q)_k(q;q)_k} x^{(k)} \times \sum_{m=0}^{\infty} \frac{q^{m^2+\beta m}}{(q^{\beta+1};q)_m(q;q)_m}(-wt)^m \\
&\quad \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(q;q)_m}{(q;q)_{m-j}} \frac{y^{m-j-1}(tz)^{k-j}}{(ytzq^j;q)_\infty}[1-q^{m-j}+ytzq^m].
\end{aligned}
$$

□

Theorem 6.2. (Another mixed generating function for $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$). *For $\alpha > -1, \beta > -1$, we have*

$$\sum_{n=0}^{\infty} \mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)\varphi_n^{\beta}(z|q)\frac{t^n}{(q;q)_n} = \frac{(yz\beta t;q)_\infty}{(yt,yzt;q)_\infty}$$
$$\times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k(q;q)_j}(xt)^{k+j}z^j\frac{(yt,\beta;q)_j}{(yz\beta t;q)_j} \tag{6.2}$$

provided $|yt|, |yzt| < 1$.

Proof. Let $f(x,y)$ denote the right-hand side of another mixed generating function 6.2. Then, by using Theorem 2.2, $f(x,y)$ satisfies the q-partial difference equation as

$$\begin{aligned}
&\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)f(x,y)\Big\}\\
&=\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\frac{(yz\beta t;q)_\infty}{(yt,yzt;q)_\infty}\\
&\quad\times\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k(q;q)_j}(xt)^{k+j}z^j\frac{(yt,\beta;q)_j}{(yz\beta t;q)_j}\Big\}\\
&=\partial_{q,x}\Big\{(1-dq^{-1}\eta_x)(1-eq^{-1}\eta_x)(1-q^{\alpha}\eta_x)\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_{k-j}(q;q)_j}(xt)^kz^j\\
&\qquad\qquad\times\frac{(yz\beta tq^j;q)_\infty(\beta;q)_j}{(yzt;q)_\infty(ytq^j;q)_\infty}\Big\}\\
&=\partial_{q,x}\Big\{\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_k}(x)^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(yz\beta tq^j;q)_\infty(\beta;q)_jt^kz^j}{(yzt;q)_\infty(ytq^j;q)_\infty}\Big\}
\end{aligned}$$

By using equation (1.3) we get

$$\begin{aligned}
&=\sum_{k=1}^{\infty}\frac{(a,b,c;q)_k(-1)^kq^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_{k-1}(q;q)_{k-1}}(x)^{k-1}\partial_{q,y}^k\Big\{\frac{(yz\beta t;q)_\infty}{(yt,yzt;q)_\infty}\Big\}\\
&=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}(x)^k\partial_{q,y}^{k+1}\Big\{\frac{(yz\beta t;q)_\infty}{(yt,yzt;q)_\infty}\Big\}\\
&=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^kq^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}(tx)^k\\
&\quad\times\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{z^j(\beta;q)_j(yz\beta tq^{j+1};q)_\infty}{(ytq^j,yzt;q)_\infty}[tz+(1-ytz)tq^j(1-\beta z)]
\end{aligned}$$

On the other hand

$$\begin{aligned}
&-q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2f(x,y)\Big\}\\
&\quad=-q^{\alpha+1}\partial_{q,y}\Big\{(1-a\eta_x)(1-b\eta_x)(1-c\eta_x)\eta_x^2\frac{(yz\beta t;q)_\infty}{(yt,yzt;q)_\infty}
\end{aligned}$$

$$\times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+j}(-1)^{k+j}q^{(k+j)^2+\alpha(k+j)}}{(d,e,q^{\alpha+1};q)_{k+j}(q;q)_k(q;q)_j}(xt)^{k+j}z^j\frac{(yt,\beta;q)_j}{(yz\beta t;q)_j}\Bigg\}$$

$$=-q^{\alpha+1}\partial_{q,y}\Bigg\{\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^k q^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_{k-j}(q;q)_j}(xt)^k z^j\frac{(yz\beta tq^j;q)_\infty(\beta;q)_j}{(yzt,ytq^j;q)_\infty}\Bigg\}$$

$$=-q^{\alpha+1}\partial_{q,y}\Bigg\{\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^k q^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}(x)^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(yz\beta tq^j;q)_\infty(\beta;q)_j t^k z^j}{(yzt;q)_\infty(ytq^j;q)_\infty}\Bigg\}$$

By using equation (1.3) we get

$$=-q^{\alpha+1}\Bigg\{\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^k q^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}(x)^k\partial_{q,y}^{k+1}\{\frac{(yz\beta t;q)_\infty}{(yt,yzt;q)_\infty}\}\Bigg\}$$

$$=-q^{\alpha+1}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_{k+1}(-1)^k q^{2k}q^{(k)^2+\alpha(k)}}{(d,e,q^{\alpha+1};q)_k(q;q)_k}(tx)^k$$

$$\times\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{z^j(\beta;q)_j(yz\beta tq^{j+1};q)_\infty}{(ytq^j,yzt;q)_\infty}[tz+(1-ytz)tq^j(1-\beta z)]$$

□

7. Multilinear Generating Function for Generalized q-Laguerre Polynomials $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$

In this section, we obtain the following multilinear generating function for the generalized q-Laguerre polynomials using the technique of q-partial difference equations and some special cases.

Theorem 7.1. *Multilinear Generating function for generalized q-Laguerre polynomials* $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$. *If max* $\{|\delta|,|y_1t_1|,\cdots,|y_kt_k|\}<1$, *then we have*

$$\sum_{n_1,n_2,\cdots,n_k=0}^{\infty}\frac{(\beta;q)_{n_1+n_2+\cdots+n_k}L_{n_1}^{(\alpha_1)}(a_1,b_1,c_1,d_1,e_1,x_1,y_1|q)\cdots L_{n_k}^{(\alpha_k)}(a_k,b_k,c_k,d_k,e_k,x_k,y_k|q)}{(\delta;q)_{n_1+n_2+\cdots+n_k}(q;q)_{n_1}\cdots(q;q)_{n_k}}$$

$$\times t_1^{n_1}t_2^{n_2}\cdots t_k^{n_k}$$

$$=\frac{(\beta;q)_\infty}{(\delta;q)_\infty}\sum_{m=0}^{\infty}\frac{(\delta/\beta;q)_m\beta^m}{(q;q)_m}\frac{1}{\prod_{i=1}^{k}(y_it_iq^m;q)_\infty}\prod_{i=1}^{k}{}_3\phi_4\left(\begin{matrix}a_i,b_i,c_i\\d_i,e_i,q^{\alpha_i+1},0\end{matrix};q,-x_it_iq^mq^{\alpha_i+1}\right),\tag{7.1}$$

where $max\{|\delta|,|y_1t_1|,\cdots,|y_kt_k|\}<1$.

Proof. *Let* $f(x_i,y_i)$ *denote the right-hand side of Theorem 3.3; then, by using Theorem 2.2,*

$$\partial_{q,x_i}\Big\{(1-d_iq^{-1}\eta_{x_i})(1-e_iq^{-1}\eta_{x_i})(1-q^{\alpha_i}\eta_{x_i})f(x_i,y_i)\Big\}$$

$$=-q^{\alpha_i+1}\partial_{q,y_i}\Big\{(1-a_i\eta_{x_i})(1-b_i\eta_{x_i})(1-c_i\eta_{x_i})\eta_{x_i}^2f(x_i,y_i)\Big\}$$

$$
\begin{aligned}
LHS =&\partial_{q,x_i}\left\{(1-d_iq^{-1}\eta_{x_i})(1-e_iq^{-1}\eta_{x_i})(1-q^{\alpha_i}\eta_{x_i})f(x_i,y_i)\right\}\\
=&\partial_{q,x_i}\left\{(1-d_iq^{-1}\eta_{x_i})(1-e_iq^{-1}\eta_{x_i})(1-q_i^{\alpha}\eta_{x_i})\frac{(\beta;q)_\infty}{(\delta;q)_\infty}\sum_{m=0}^{\infty}\frac{(\delta/\beta;q)_m\beta^m}{(q;q)_m}\right.\\
&\left.\times\frac{1}{\prod_{i=1}^{k}(y_it_iq^m;q)_\infty}\prod_{i=1}^{k}{}_3\phi_4\left(\begin{matrix}a_i,b_i,c_i\\ d_i,e_i,q^{\alpha_i+1},0\end{matrix};q,-x_it_iq^mq^{\alpha_i+1}\right)\right\}\\
=&\frac{(\beta;q)_\infty}{(\delta;q)_\infty}\partial_{q,x_i}\left\{\sum_{m=0}^{\infty}\frac{(\delta/\beta;q)_m\beta^m}{(q;q)_m}\frac{1}{\prod_{i=1}^{k}(y_it_iq^m;q)_\infty}\right.\\
&\left.\times\prod_{i=1}^{k}\sum_{n_i=1}^{\infty}\frac{(a_i,b_i,c_i;q)_{n_i}q^{{n_i}^2+\alpha_in_i}}{(d_i,e_i,q^{\alpha_i+1};q)_{n_i-1}(q;q)_{n_i}}(-x_it_iq^m)^{n_i}\right\}\\
=&\frac{(\beta;q)_\infty}{(\delta;q)_\infty}\sum_{m=0}^{\infty}\frac{(\delta/\beta;q)_m\beta^m}{(q;q)_m}\frac{1}{\prod_{i=1}^{k}(y_it_iq^m;q)_\infty}\\
&\times\prod_{i=1}^{k}\sum_{n_i=1}^{\infty}\frac{(a_i,b_i,c_i;q)_{n_i}q^{{n_i}^2+\alpha_in_i}}{(d_i,e_i,q^{\alpha_i+1};q)_{n_i-1}(q;q)_{n_{i-1}}}x^{n_i-1}(-t_iq^m)^{n_i}\\
=&\frac{-q^{\alpha_i+1}t_iq^m(\beta;q)_\infty}{(\delta;q)_\infty}\sum_{m=0}^{\infty}\frac{(\delta/\beta;q)_m\beta^m}{(q;q)_m}\frac{1}{\prod_{i=1}^{k}(y_it_iq^m;q)_\infty}\\
&\times\prod_{i=1}^{k}\sum_{n_i=0}^{\infty}\frac{(a_i,b_i,c_i;q)_{n_i+1}q^{2n_i}q^{{n_i}^2+\alpha_in_i}}{(d_i,e_i,q^{\alpha_i+1};q)_{n_i}(q;q)_{n_i}}(-xt_iq^m)^{n_i}
\end{aligned}
$$

$$
\begin{aligned}
RHS =&-q^{\alpha_i+1}\partial_{q,y_i}\left\{(1-a_i\eta_{x_i})(1-b_i\eta_{x_i})(1-c_i\eta_{x_i})\eta_{x_i}^2f(x_i,y_i)\right\}\\
=&-q^{\alpha_i+1}\partial_{q,y_i}\left\{(1-a_i\eta_{x_i})(1-b_i\eta_{x_i})(1-c_i\eta_{x_i})\eta_{x_i}^2\frac{(\beta;q)_\infty}{(\delta;q)_\infty}\sum_{m=0}^{\infty}\frac{(\delta/\beta;q)_m\beta^m}{(q;q)_m}\right.\\
&\left.\times\frac{1}{\prod_{i=1}^{k}(y_it_iq^m;q)_\infty}\prod_{i=1}^{k}{}_3\phi_4\left(\begin{matrix}a_i,b_i,c_i\\ d_i,e_i,q^{\alpha_i+1},0\end{matrix};q,-x_it_iq^mq^{\alpha_i+1}\right)\right\}\\
=&-q^{\alpha_i+1}\frac{(\beta;q)_\infty}{(\delta;q)_\infty}\partial_{q,y_i}\left\{\sum_{m=0}^{\infty}\frac{(\delta/\beta;q)_m\beta^m}{(q;q)_m}\frac{1}{\prod_{i=1}^{k}(y_it_iq^m;q)_\infty}\right.\\
&\left.\times\prod_{i=1}^{k}\sum_{n_i=0}^{\infty}\frac{(a_i,b_i,c_i;q)_{n_i+1}q^{2n_i}q^{{n_i}^2+\alpha_in_i}}{(d_i,e_i,q^{\alpha_i+1};q)_{n_i}(q;q)_{n_i}}(-xt_iq^m)^{n_i}\right\}\\
=&-t_iq^mq^{\alpha_i+1}\frac{(\beta;q)_\infty}{(\delta;q)_\infty}\left\{\sum_{m=0}^{\infty}\frac{(\delta/\beta;q)_m\beta^m}{(q;q)_m}\frac{1}{\prod_{i=1}^{k}(y_it_iq^m;q)_\infty}\right.\\
&\left.\times\prod_{i=1}^{k}\sum_{n_i=0}^{\infty}\frac{(a_i,b_i,c_i;q)_{n_i+1}q^{2n_i}q^{{n_i}^2+\alpha_in_i}}{(d_i,e_i,q^{\alpha_i+1};q)_{n_i}(q;q)_{n_i}}(-xt_iq^m)^{n_i}\right\}
\end{aligned}
$$

Then, by Theorem 2.4, there exists a sequence $\{\nu_{n_1,n_2,\cdots,n_k}\}$ independent of $a_1,b_1,c_1,d_1,e_1,x_1,y_1,\cdots,a_k,b_k,c_k,d_k,e_k,x_k,y_k$ such that

$$\frac{(\beta;q)_\infty}{(\delta;q)_\infty}\sum_{m=0}^{\infty}\frac{(\delta/\beta;q)_m\beta^m}{(q;q)_m}\frac{1}{\prod_{i=1}^{k}(y_it_iq^m;q)_\infty}\prod_{i=1}^{k}{}_3\phi_4\left(\begin{matrix}a_i,b_i,c_i\\ d_i,e_i,q^{\alpha_i+1},0\end{matrix};q,-x_it_iq^mq^{\alpha_i+1}\right)$$

$$=\sum_{n_1,n_2,\cdots,n_k}^{\infty}\nu_{n_1,n_2,\cdots,n_k}L_{n_1}^{(\alpha_1)}(a_1,b_1,c_1,d_1,e_1,x_1,y_1|q)\cdots L_{n_k}^{(\alpha_k)}(a_k,b_k,c_k,d_k,e_k,x_k,y_k|q). \tag{7.2}$$

Setting $x_1=x_2=\cdots=x_k=0$ in the above equation, using $L_n^{(\alpha)}(a,b,c,d,e,0,y|q)=y^n$, we immediately obtain

$$\sum_{n_1,n_2,\cdots,n_k}^{\infty}\nu_{n_1,n_2,\cdots,n_k}y_1^{n_1}y_2^{n_2}\cdots y_k^{n_k}=\frac{(\beta;q)_\infty}{(\delta;q)_\infty}\sum_{m=0}^{\infty}\frac{(\delta/\beta;q)_m\beta^m}{(q;q)_m}\frac{1}{\prod_{i=1}^{k}(y_it_iq^m;q)_\infty}$$

$$=\frac{(\beta;q)_\infty}{(\delta;q)_\infty}\sum_{m=0}^{\infty}\frac{(\delta/\beta;q)_m\beta^m}{(q;q)_m}\sum_{n_1,n_2,\cdots,n_k}^{\infty}\frac{(y_1t_1q^m)^{n_1}(y_2t_2q^m)^{n_2}\cdots(y_kt_kq^m)^{n_k}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_k}}$$

Equating the coefficients of the above equation, we have

$$\nu_{n_1,n_2,\cdots,n_k}=\frac{(\beta;q)_\infty}{(\delta;q)_\infty}\sum_{m=0}^{\infty}\frac{(\delta/\beta;q)_m\beta^m}{(q;q)_m}\frac{(t_1q^m)^{n_1}(t_2q^m)^{n_2}\cdots(t_kq^m)^{n_k}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_k}}$$

$$=\frac{(\beta;q)_\infty}{(\delta;q)_\infty}\sum_{m=0}^{\infty}\frac{(\delta/\beta;q)_m\beta^m}{(q;q)_m}\frac{1}{\prod_{i=1}^{k}(t_iq^m;q)_\infty} \tag{7.3}$$

Substituting equation (7.3) into (7.2), we deduce equation (7.1). The proof of Theorem 7.1 is complete. □

- Setting $m=0$, $\beta=\delta=0$ and $i=1$ in equation 7.1, we obtain the simple generating function for generalized q-Laguerre polynomials $\mathbb{L}_n^{(\alpha)}(a,b,c,d,e,x,y|q)$

8. Conclusions

In this study, we established the generalized q-Laguerre polynomials $L_n^{\alpha}(a,b,c,d,e,x,y|q)$. By using the method of q-partial difference equations, we demonstrated that an analytic function in many variables satisfies a specific set of q-partial difference equations if and only if it can be expanded in terms of generalized q-Laguerre polynomials. As applications of these findings, we proved the generating functions and their extensions, such as the Rogers formula, Mehler's formula, and mixed generating functions, for generalized q-Laguerre polynomials using q-partial difference equations. Additionally, we demonstrated a multilinear generating function for generalized q-Laguerre polynomials.

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