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THE FEKETE–SZEGÖ INEQUALITY FOR A CLASS OF STARLIKE FUNCTIONS DEFINED VIA THE q-FIBONACCI SEQUENCE

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Abstract. In this paper, we introduce and study a new subclass of analytic functions defined via the principle of subordination to a generating function associated with the q-Fibonacci sequence. The newly defined class is contained in the class of q-starlike functions and exhibits a close geometric relationship with shell-like starlike domains, which provide a rich structure for capturing non-linear geometric behavior. Building upon the interaction between q-calculus and Fibonacci-type sequences, the proposed subclass offers a unified setting that combines classical analytic function theory with modern discrete-analytic structures. We establish a sufficient condition for functions to belong to this class and derive explicit coefficient bounds for the initial coefficients a_2, a_3, and a_4, which play a fundamental role in determining the geometric properties of the functions. Furthermore, we investigate the Fekete–Szegö functional and obtain sharp estimates that extend several known results in the literature. In addition, a Krushkal-type inequality is derived in the context of the introduced class, providing further insight into higher-order coefficient relations.

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1. Introduction

The function

$$f : \mathscr{O} \to \mathbb{C},$$

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where

$$\mathscr{O} := \{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}| < 1\},$$

is said to be a member of the class $\mathscr{A}$ if it is analytic in the open unit disk $\mathscr{O}$. Throughout this study, we focus specifically on functions $\mathsf{f} \in \mathscr{A}$ that fulfill the following normalization conditions:

$$\mathsf{f}(0) = 0 \qquad \text{and} \qquad \mathsf{f}'(0) = 1.$$

These normalizations are standard in the theory of analytic functions, since they eliminate the freedom associated with rotations and dilations and consequently provide a unified setting for studying the geometric structure and analytic properties of the underlying function classes.

Each function $\mathsf{f} \in \mathscr{A}$ admits a Taylor–Maclaurin series representation centered at the origin, which can be expressed as follows:

$$\mathsf{f}(\mathfrak{z}) = \mathfrak{z} + \sum_{\mathfrak{s}=2}^{\infty} \mathsf{a}_{\mathfrak{s}}\, \mathfrak{z}^{\mathfrak{s}}, \qquad (\mathfrak{z} \in \mathscr{O}). \tag{1}$$

Moreover, an analytic function f in the unit disk $\mathscr{O}$ is referred to as a Schwarz function if it fulfills the following conditions:

$$|\mathsf{f}(\mathfrak{z})| < 1, \qquad \mathfrak{z} \in \mathscr{O},$$

together with the normalization condition

$$\mathsf{f}(0) = 0.$$

Schwarz functions play a fundamental role in geometric function theory (GFT) due to their strong connection with subordination theory, conformal mappings, and coefficient problems for analytic functions.

Moreover, let S denote the class of all functions $\mathsf{f} \in \mathscr{A}$ that are univalent in the open unit disk $\mathscr{O}$. We further consider the Carathéodory class P, which consists of analytic functions $\mathfrak{p}$ defined in $\mathscr{O}$ satisfying

$$\Re\big(\mathfrak{p}(\mathfrak{z})\big) > 0, \qquad \mathfrak{z} \in \mathscr{O},$$

together with the normalization condition

$$\mathfrak{p}(0) = 1.$$

Every function $\mathfrak{p} \in \mathsf{P}$ possesses the Taylor expansion

$$\mathfrak{p}(\mathfrak{z}) = 1 + \sum_{\mathfrak{s}=1}^{\infty} \mathfrak{p}_{\mathfrak{s}} \mathfrak{z}^{\mathfrak{s}} = 1 + \mathfrak{p}_1\mathfrak{z} + \mathfrak{p}_2\mathfrak{z}^2 + \mathfrak{p}_3\mathfrak{z}^3 + \cdots, \qquad (\mathfrak{z} \in \mathscr{O}). \tag{2}$$

Let $\mathsf{f}_1, \mathsf{f}_2 \in \mathscr{A}$. The function f_1 is said to be subordinate to f_2, written as

$$\mathsf{f}_1 \prec \mathsf{f}_2,$$

provided that there exists a Schwarz function $\eta$for which the following relation holds:

$$f_1(\mathfrak{z}) = f_2\big(\eta(\mathfrak{z})\big), \qquad \mathfrak{z} \in \mathscr{O}.$$

Subordination theory occupies a central position in GFT and has been established as a powerful approach for investigating various classes of analytic and univalent functions. In particular, it has played a major role in the introduction and investigation of several subclasses of starlike and convex functions; see [8].

Ma and Minda [21] developed a comprehensive and unified framework encompassing a broad range of subclasses of starlike functions by introducing the class

$$S^*(\Omega) = \left\{ f \in \mathscr{A} : \frac{\mathfrak{z} f'(\mathfrak{z})}{f(\mathfrak{z})} \prec \Omega(\mathfrak{z}), \quad \Omega \in P, \quad \mathfrak{z} \in \mathscr{O} \right\}.$$

This formulation provides a flexible framework through which several well-known subclasses may be recovered by selecting suitable choices of the function Ω.

Among the important special cases are the following:

(1) By choosing $\Omega(\mathfrak{z}) = \dfrac{1+\mathfrak{z}}{1-\mathfrak{z}}$, one obtains the classical family of starlike functions introduced by Janowski [22], namely

$$S^*\left(\frac{1+\mathfrak{z}}{1-\mathfrak{z}}\right) = \left\{ f \in \mathscr{A} : \frac{\mathfrak{z} f'(\mathfrak{z})}{f(\mathfrak{z})} \prec \frac{1+\mathfrak{z}}{1-\mathfrak{z}} \right\}.$$

(2) If $\Omega(\mathfrak{z}) = \dfrac{1+(1-2\varkappa)\mathfrak{z}}{1-\mathfrak{z}}, \qquad 0 \le \varkappa < 1$, then one arrives at the subclass of starlike functions of order $\varkappa$, introduced by Robertson [23], given by

$$S^*(\varkappa) = \left\{ f \in \mathscr{A} : \frac{\mathfrak{z} f'(\mathfrak{z})}{f(\mathfrak{z})} \prec \frac{1+(1-2\varkappa)\mathfrak{z}}{1-\mathfrak{z}} \right\}.$$

(3) Taking $\Omega(\mathfrak{z}) = \dfrac{1+\varkappa^2\mathfrak{z}^2}{1-\varkappa\mathfrak{z}-\varkappa^2\mathfrak{z}^2}, \qquad \varkappa = \frac{1-\sqrt{5}}{2}$, Sokól [25] introduced the class

$$SL(\varkappa) = \left\{ f \in \mathscr{A} : \frac{\mathfrak{z} f'(\mathfrak{z})}{f(\mathfrak{z})} \prec \frac{1+\varkappa^2\mathfrak{z}^2}{1-\varkappa\mathfrak{z}-\varkappa^2\mathfrak{z}^2} \right\},$$

which is closely connected with shell-like curves associated with the Fibonacci sequence.

A further significant topic in GFT is the investigation of the Fekete–Szegö functional, which occupies a central position in the theory of coefficient estimates for analytic and univalent functions. The concept was first introduced by Fekete and Szegö in their classical work [26], and it has subsequently become an essential tool in the study of a wide variety of subclasses of analytic functions, see [15].

The classical Fekete-Szegö functional is defined by the following expression:

$$L_\delta = \left| a_3 - \delta a_2^2 \right|, \qquad 0 \le \delta \le 1.$$

This functional has been extensively studied in connection with coefficient problems for subclasses of univalent functions. In particular, Fekete and Szegö [26] established that, for $\mathsf{f} \in \mathsf{S}$,

$$\mathrm{L}_\delta \leq 1 + 2\exp\left(\frac{-\delta}{1-\delta}\right), \qquad 0 \leq \delta < 1,$$

and the estimate is sharp.

In recent years, $\mathfrak{q}$-calculus has received substantial attention owing to its wide-ranging applications in mathematical analysis, the theory of special functions, and mathematical physics. In contrast to classical calculus, it is formulated through $\mathfrak{q}$-difference operators instead of limiting processes. The basic foundations of the theory were established by Jackson [2, 3] through the introduction of the $\mathfrak{q}$-derivative and $\mathfrak{q}$-integral. Subsequently, Aral and Gupta [4, 5, 7] developed several $\mathfrak{q}$-analogues of classical operators and demonstrated important connections between $\mathfrak{q}$-calculus and GFT. These contributions motivated the study of new subclasses of analytic functions together with their geometric and coefficient properties [9–14, 16–20]. More recently, Alsoboh et al. [6] introduced a $\mathfrak{q}$-extension of starlike functions associated with $\mathfrak{q}$-Fibonacci numbers by means of subordination techniques and obtained several coefficient and geometric results.

Following the method introduced in [6], we define a $\mathfrak{q}$-analogue of the shell-like starlike class by replacing the ordinary derivative with the corresponding $\mathfrak{q}$-difference operator. A function $\mathsf{f} \in \mathscr{A}$ is said to belong to $\mathsf{SL}_\mathfrak{q}$ if

$$\frac{\mathfrak{z}\,\partial_\mathfrak{q}\langle \mathsf{f}(\mathfrak{z})\rangle}{\mathsf{f}(\mathfrak{z})} \prec \aleph(\mathfrak{z};\mathfrak{q}), \qquad \mathfrak{z} \in \mathscr{O}, \tag{3}$$

where

$$\begin{aligned}\aleph(\mathfrak{z};\mathfrak{q}) &= \frac{1+\mathfrak{q}\varkappa_\mathfrak{q}^2\mathfrak{z}^2}{1-\varkappa_\mathfrak{q}\mathfrak{z}-\mathfrak{q}\varkappa_\mathfrak{q}^2\mathfrak{z}^2} \\ &= 1+\varkappa_\mathfrak{q}\mathfrak{z}+(1+2\mathfrak{q})\varkappa_\mathfrak{q}^2\mathfrak{z}^2+(1+3\mathfrak{q})\varkappa_\mathfrak{q}^3\mathfrak{z}^3+(2\mathfrak{q}^2+4\mathfrak{q}+1)\varkappa_\mathfrak{q}^4\mathfrak{z}^4+\cdots,\end{aligned} \tag{4}$$

with

$$\varkappa_\mathfrak{q} = \frac{1-\sqrt{4\mathfrak{q}+1}}{2\mathfrak{q}}. \tag{5}$$

The parameter $\varkappa_\mathfrak{q}$ is the negative root of

$$\mathfrak{q}x^2 - x - 1 = 0,$$

and therefore satisfies

$$\mathfrak{q}\varkappa_\mathfrak{q}^2 - \varkappa_\mathfrak{q} - 1 = 0. \tag{6}$$

We next examine the geometry determined by $\aleph$. For $\mathfrak{z} = e^{i\xi}$, define the associated boundary curve by

$$\Gamma_\mathfrak{q} = \left\{\aleph(e^{i\xi};\mathfrak{q}) : \xi \in [0,2\pi) \setminus \{\pi\}\right\}. \tag{7}$$

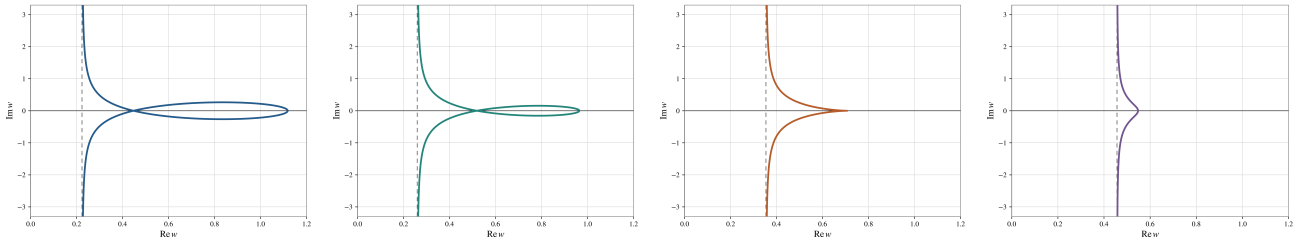
The value $\xi = \pi$ is excluded because $e^{i\pi} = -1$ is a pole of $\aleph$. Indeed, (6) yields

$$1 + \varkappa_\mathfrak{q} - \mathfrak{q}\varkappa_\mathfrak{q}^2 = 0,$$

which is precisely the denominator of $\aleph(-1;\mathfrak{q})$. Moreover,

$$\lim_{\xi\to\pi^-}\operatorname{Im}\aleph(e^{i\xi};\mathfrak{q})=-\infty,\qquad \lim_{\xi\to\pi^+}\operatorname{Im}\aleph(e^{i\xi};\mathfrak{q})=+\infty. \tag{8}$$

Figure 1 shows how the geometry of $\Gamma_{\mathfrak{q}}$ changes with $\mathfrak{q}$.



(a) $q\to 1^-$ (b) $q=0.68$ (c) $q=0.25$ (d) $q=0.05$

FIGURE 1. Boundary curves Γ_q, parameterized by $w(\xi)=G(e^{i\xi};q)$, for selected values of q, with $\xi\in[0,2\pi)\setminus\{\pi\}$. The dashed line represents the vertical asymptote. A loop occurs when $q>\frac{1}{4}$, collapses at the critical value $q=\frac{1}{4}$, and disappears when $0<q<\frac{1}{4}$, where the curve becomes conchoid-like.

The change in the boundary geometry is closely related to the injectivity of $\aleph$. Differentiating (4), we obtain

$$\aleph'(\mathfrak{z};\mathfrak{q})=\frac{\varkappa_{\mathfrak{q}}+4\mathfrak{q}\varkappa_{\mathfrak{q}}^2\mathfrak{z}-\mathfrak{q}\varkappa_{\mathfrak{q}}^3\mathfrak{z}^2}{\left(1-\varkappa_{\mathfrak{q}}\mathfrak{z}-\mathfrak{q}\varkappa_{\mathfrak{q}}^2\mathfrak{z}^2\right)^2}. \tag{9}$$

For $\mathfrak{q}>\frac{1}{4}$, the numerator in (9) vanishes at

$$\mathfrak{z}_{\mathfrak{q}}=\frac{2\mathfrak{q}-\sqrt{\mathfrak{q}(4\mathfrak{q}+1)}}{\mathfrak{q}\varkappa_{\mathfrak{q}}},\qquad 0<\mathfrak{z}_{\mathfrak{q}}<1. \tag{10}$$

Hence, $\aleph'(\mathfrak{z}_{\mathfrak{q}};\mathfrak{q})=0$, so $\aleph$ is not locally univalent, and consequently cannot be univalent, in $\mathscr{O}$. Geometrically, this failure of local univalence is reflected by the loops visible in Figure 1(a) and (b).

For $\mathfrak{q}>\frac{3}{4}$, the failure of injectivity can also be verified directly. In this range,

$$-\frac{1}{2\mathfrak{q}\varkappa_{\mathfrak{q}}}\in\mathscr{O},$$

and a straightforward calculation gives

$$\aleph(0;\mathfrak{q})=\aleph\left(-\frac{1}{2\mathfrak{q}\varkappa_{\mathfrak{q}}};\mathfrak{q}\right)=1.$$

Thus, $\aleph$ assumes the same value at two distinct points of $\mathscr{O}$.

Finally, as $\mathfrak{q}\to 1^-$, the $\mathfrak{q}$-difference operator approaches the ordinary derivative, while

$$\varkappa_{\mathfrak{q}}\longrightarrow\varkappa=\frac{1-\sqrt{5}}{2}.$$

Accordingly, the defining condition in (3) reduces to

$$\frac{\mathfrak{z}\mathsf{f}'(\mathfrak{z})}{\mathsf{f}(\mathfrak{z})}\prec\frac{1+\varkappa^2\mathfrak{z}^2}{1-\varkappa\mathfrak{z}-\varkappa^2\mathfrak{z}^2}. \tag{11}$$

This is the classical shell-like starlike class introduced by Raina and Sokól [24]. The family $\mathsf{SL}_{\mathfrak{q}}$ therefore provides a natural $\mathfrak{q}$-extension of the classical shell-like model associated with Fibonacci-type structures.

Lemma 1.1. [1] *Let $\mathfrak{p} \in \mathsf{P}(0)$ be of the form*

$$\mathfrak{p}(\mathfrak{z}) = 1 + \epsilon_1 \mathfrak{z} + \epsilon_2 \mathfrak{z}^2 + \cdots .$$

Then the coefficients of $\mathfrak{p}$ satisfy

$$|\epsilon_{\mathsf{s}}| \leq 2, \qquad (\mathsf{s} \geq 1). \tag{12}$$

Moreover, equality in (12) is attained if and only if

$$\mathfrak{p}(\mathfrak{z}) = \mathfrak{p}_*(\mathfrak{z}) = \frac{1 + x\mathfrak{z}}{1 - x\mathfrak{z}}, \qquad |x| = 1,$$

where

$$x = \frac{\epsilon_1}{2}.$$

Conversely, if

$$\mathfrak{p}(\mathfrak{z}) = \mathfrak{p}_*(\mathfrak{z})$$

for some complex number x satisfying $|x| = 1$, then

$$\epsilon_1 = 2x.$$

In addition, the following sharp estimate holds:

$$\left| \epsilon_2 - \frac{\epsilon_1^2}{2} \right| \leq 2 - \frac{|\epsilon_1|^2}{2}. \tag{13}$$

Lemma 1.2. *Let $\mathfrak{p} \in \mathsf{P}(0)$ be represented by*

$$\mathfrak{p}(\mathfrak{z}) = 1 + \epsilon_1 \mathfrak{z} + \epsilon_2 \mathfrak{z}^2 + \epsilon_3 \mathfrak{z}^3 + \cdots .$$

Assume that the real parameters M and N satisfy

$$0 \leq M \leq 1, \qquad M(2M - 1) \leq N \leq M.$$

Then the following coefficient inequality is valid:

$$\left| \epsilon_3 - 2M\epsilon_1\epsilon_2 + N\epsilon_1^3 \right| \leq 2.$$

2. The Main Results

With the essential preliminary notions and supporting results in place, we are now in a position to state the main results of the present study. The following theorem constitutes the first main result of the paper. Throughout the following, we assume that $\chi_n = [n]_{\mathfrak{q}}, n = 2, 3, 4$.

Theorem 2.1. *Let the function $\mathfrak{p}(\mathfrak{z})$ be defined by (2) and suppose that*

$$\mathfrak{p}(\mathfrak{z}) \prec \aleph(\mathfrak{z}; \mathfrak{q}).$$

Then the following coefficient estimates hold:

$$|\mathfrak{p}_1| \leq |\varkappa_{\mathfrak{q}}|,$$

$$|\mathfrak{p}_2| \leq (1 + 2\mathfrak{q})\varkappa_{\mathfrak{q}}^2,$$

and

$$|\mathfrak{p}_3| \leq |\varkappa_{\mathfrak{q}}| + (1 + 3\mathfrak{q})|\varkappa_{\mathfrak{q}}|^3.$$

Furthermore, each of the above inequalities is sharp.

Proof. Since $\mathfrak{p} \prec \aleph(\mathfrak{z}; \mathfrak{q})$, there exists an analytic function ω defined in $\mathscr{O}$ such that

$$|\omega(\mathfrak{z})| \leq |\mathfrak{z}|, \qquad (\mathfrak{z} \in \mathscr{O}),$$

and

$$\mathfrak{p}(\mathfrak{z}) = \aleph\big(\omega(\mathfrak{z}); \mathfrak{q}\big), \qquad (\mathfrak{z} \in \mathscr{O}).$$

That is, the function $\mathfrak{p}$ is subordinate to the function $\aleph$ through the Schwarz function ω.

Therefore, we have

$$\mathfrak{h}(\mathfrak{z}) = \frac{1 + \omega(\mathfrak{z})}{1 - \omega(\mathfrak{z})} = 1 + \epsilon_1\mathfrak{z} + \epsilon_2\mathfrak{z}^2 + \epsilon_3\mathfrak{z}^3 + \epsilon_4\mathfrak{z}^4 + \cdots, \qquad \mathfrak{z} \in \mathscr{O},$$

is in the class $\mathfrak{p}(0)$İt follows that

$$\omega(\mathfrak{z}) = \frac{\epsilon_1}{2}\mathfrak{z} + \frac{1}{2}\left(\epsilon_2 - \frac{\epsilon_1^2}{2}\right)\mathfrak{z}^2 + \frac{1}{2}\left(\epsilon_3 - \epsilon_1\epsilon_2 + \frac{\epsilon_1^3}{4}\right)\mathfrak{z}^3 + \cdots, \tag{14}$$

and

$$\begin{aligned}
\aleph(\omega(\mathfrak{z}); \mathfrak{q}) = 1 &+ \widehat{\mathfrak{p}_1}\left(\frac{\epsilon_1}{2}\mathfrak{z} + \frac{1}{2}\left(\epsilon_2 - \frac{\epsilon_1^2}{2}\right)\mathfrak{z}^2 + \frac{1}{2}\left(\epsilon_3 - \epsilon_1\epsilon_2 + \frac{\epsilon_1^3}{4}\right)\mathfrak{z}^3 + \cdots\right) \\
&+ \widehat{\mathfrak{p}_2}\left(\frac{\epsilon_1}{2}\mathfrak{z} + \frac{1}{2}\left(\epsilon_2 - \frac{\epsilon_1^2}{2}\right)\mathfrak{z}^2 + \frac{1}{2}\left(\epsilon_3 - \epsilon_1\epsilon_2 + \frac{\epsilon_1^3}{4}\right)\mathfrak{z}^3 + \cdots\right)^2 \\
&+ \widehat{\mathfrak{p}_3}\left(\frac{\epsilon_1}{2}\mathfrak{z} + \frac{1}{2}\left(\epsilon_2 - \frac{\epsilon_1^2}{2}\right)\mathfrak{z}^2 + \frac{1}{2}\left(\epsilon_3 - \epsilon_1\epsilon_2 + \frac{\epsilon_1^3}{4}\right)\mathfrak{z}^3 + \cdots\right)^3 + \cdots \\
&= 1 + \frac{\widehat{\mathfrak{p}_1}\,\epsilon_1}{2}\mathfrak{z} + \left[\frac{\widehat{\mathfrak{p}_1}}{2}\left(\epsilon_2 - \frac{\epsilon_1^2}{2}\right) + \frac{\widehat{\mathfrak{p}_2}\,\epsilon_1^2}{4}\right]\mathfrak{z}^2 \\
&+ \left[\frac{\widehat{\mathfrak{p}_1}}{2}\left(\epsilon_3 - \epsilon_1\epsilon_2 + \frac{\epsilon_1^3}{4}\right) + \frac{\widehat{\mathfrak{p}_2}}{2}\epsilon_1\left(\epsilon_2 - \frac{\epsilon_1^2}{2}\right) + \frac{\widehat{\mathfrak{p}_3}\,\epsilon_1^3}{8}\right]\mathfrak{z}^3 + \cdots.
\end{aligned} \tag{15}$$

To determine the coefficients $\widehat{\mathfrak{p}}_{\mathfrak{s}}$ of the function $\aleph$, we introduce the substitution

$$\vartheta = \varkappa_{\mathfrak{q}}\mathfrak{z}, \qquad |\vartheta| < |\varkappa_{\mathfrak{q}}|.$$

$$\begin{aligned}
\aleph(\mathfrak{z};\mathfrak{q}) = \frac{1+\mathfrak{q}\vartheta^2}{1-\vartheta-\mathfrak{q}\vartheta^2} &= \left(\mathfrak{q}\vartheta+\frac{1}{\vartheta}\right)\frac{\vartheta}{1-\vartheta-\mathfrak{q}\vartheta^2} \\
&= \left(\mathfrak{q}\vartheta+\frac{1}{\vartheta}\right)\frac{1}{\sqrt{4\mathfrak{q}+1}}\left(\frac{\varkappa_{\mathfrak{q}}}{\vartheta+\varkappa_{\mathfrak{q}}}+\frac{\mathfrak{q}\varkappa_{\mathfrak{q}}-1}{q\vartheta+(1-\mathfrak{q}\varkappa_{\mathfrak{q}})}\right) \\
&= \left(\mathfrak{q}\vartheta+\frac{1}{\vartheta}\right)\frac{1}{\sqrt{4\mathfrak{q}+1}}\left(\frac{1}{1+\frac{\vartheta}{\varkappa_{\mathfrak{q}}}}-\frac{1}{1+\frac{q\vartheta}{1-\mathfrak{q}\varkappa_{\mathfrak{q}}}}\right) \\
&= \left(\mathfrak{q}\vartheta+\frac{1}{\vartheta}\right)\frac{1}{\sqrt{4\mathfrak{q}+1}}\sum_{\mathfrak{s}=1}^{\infty}(-1)^{\mathfrak{s}}\left[\left(\frac{\vartheta}{\varkappa_{\mathfrak{q}}}\right)^{\mathfrak{s}}-\left(\frac{q\vartheta}{1-\mathfrak{q}\varkappa_{\mathfrak{q}}}\right)^{\mathfrak{s}}\right] \\
&= \left(\mathfrak{q}\vartheta+\frac{1}{\vartheta}\right)\sum_{\mathfrak{s}=1}^{\infty}\frac{(1-\mathfrak{q}\varkappa_{\mathfrak{q}})^{\mathfrak{s}}-(\varkappa_{\mathfrak{q}})^{\mathfrak{s}}}{\sqrt{4\mathfrak{q}+1}}t^{\mathfrak{s}} \\
&= \left(\mathfrak{q}\vartheta+\frac{1}{\vartheta}\right)\sum_{\mathfrak{s}=1}^{\infty}\varphi_{\mathfrak{s}}(\mathfrak{q})\vartheta^{\mathfrak{s}} = 1+\sum_{\mathfrak{s}=1}^{\infty}\left(\varphi_{\mathfrak{s}+1}(\mathfrak{q})+q\varphi_{\mathfrak{s}-1}(\mathfrak{q})\right)\varkappa_{\mathfrak{q}}^{\mathfrak{s}}\mathfrak{z}^{\mathfrak{s}},
\end{aligned} \tag{16}$$

where

$$\varphi_{\mathfrak{s}}(\mathfrak{q}) = \frac{(1-\mathfrak{q}\varkappa_{\mathfrak{q}})^{\mathfrak{s}}-(\varkappa_{\mathfrak{q}})^{\mathfrak{s}}}{\sqrt{4\mathfrak{q}+1}}, \qquad \varkappa_{\mathfrak{q}} = \frac{1-\sqrt{4\mathfrak{q}+1}}{2\mathfrak{q}} \qquad \mathfrak{s}\in\mathbb{N}\cup\{0\}. \tag{17}$$

This shows that the function $\aleph$ is closely related to the sequence of $\mathfrak{q}$-analogues of the Fibonacci numbers

$$\{\varphi_{\mathfrak{s}}(\mathfrak{q})\}_{\mathfrak{s}\geq 0},$$

defined by

$$\varphi_0(\mathfrak{q}) = 0, \qquad \varphi_1(\mathfrak{q}) = 1,$$

and

$$\varphi_{\mathfrak{s}+2}(\mathfrak{q}) = \varphi_{\mathfrak{s}+1}(\mathfrak{q}) + \mathfrak{q}\,\varphi_{\mathfrak{s}}(\mathfrak{q}), \qquad (\mathfrak{s}\geq 0).$$

Therefore, by using (17), the expression in (15) can be rewritten as

$$\begin{aligned}
\aleph(\omega(\mathfrak{z});\mathfrak{q}) = 1 &+ \frac{\varkappa_{\mathfrak{q}}\epsilon_1}{2}\mathfrak{z} + \left[\frac{\varkappa_{\mathfrak{q}}}{2}\epsilon_2+\frac{\left((1+2\mathfrak{q})\varkappa_{\mathfrak{q}}^2-\varkappa_{\mathfrak{q}}\right)\epsilon_1^2}{4}\right]\mathfrak{z}^2 \\
&+ \left[\frac{\varkappa_{\mathfrak{q}}}{2}\epsilon_3+\frac{\left((1+2\mathfrak{q})\varkappa_{\mathfrak{q}}^2-\varkappa_{\mathfrak{q}}\right)}{2}\epsilon_1\epsilon_2+\frac{\varkappa_{\mathfrak{q}}\left(1-2(1+2\mathfrak{q})\varkappa_{\mathfrak{q}}+(1+3\mathfrak{q})\varkappa_{\mathfrak{q}}^2\right)}{8}\epsilon_1^3\right]\mathfrak{z}^3+\cdots.
\end{aligned} \tag{18}$$

Thus from (15) and (18), we find

$$\mathfrak{p}_1 = \frac{\varkappa_{\mathfrak{q}}\epsilon_1}{2} \tag{19}$$

$$\mathfrak{p}_2 = \frac{\varkappa_{\mathfrak{q}}}{2}\left(\epsilon_2-\frac{\epsilon_1^2}{2}\right)+\frac{(1+2\mathfrak{q})\varkappa_{\mathfrak{q}}^2\epsilon_1^2}{4} \tag{20}$$

$$\mathfrak{p}_3 = \frac{\varkappa_{\mathfrak{q}}}{2}\left[\epsilon_3+\left((1+2\mathfrak{q})\varkappa_{\mathfrak{q}}-1\right)\epsilon_1\epsilon_2+\frac{1-2(1+2\mathfrak{q})\varkappa_{\mathfrak{q}}+(1+3\mathfrak{q})\varkappa_{\mathfrak{q}}^2}{4}\epsilon_1^3\right] \tag{21}$$

From (19) and Lemma 1.1, we immediately obtain

$$|\mathfrak{p}_1| \leq |\varkappa_\mathfrak{q}|.$$

Furthermore, by applying (20) together with the coefficient estimate given in (13), we derive

$$\begin{aligned} |\mathfrak{p}_2| &= \left| \frac{\varkappa_\mathfrak{q}}{2}\left(\epsilon_2 - \frac{\epsilon_1^2}{2}\right) + \frac{(1+2\mathfrak{q})\varkappa_\mathfrak{q}^2\epsilon_1^2}{4} \right| \\ &\leq \frac{|\varkappa_\mathfrak{q}|}{2}\left(\epsilon_2 - \frac{\epsilon_1^2}{2}\right) + \frac{(1+2\mathfrak{q})\varkappa_\mathfrak{q}^2}{4}\left|\epsilon_1^2\right| \\ &\leq \frac{|\varkappa_\mathfrak{q}|}{2}\left(2 - \frac{|\epsilon_1|^2}{2}\right) + \frac{(1+2\mathfrak{q})\varkappa_\mathfrak{q}^2}{4}\left|\epsilon_1^2\right| \\ &\leq (1+2\mathfrak{q})\varkappa_\mathfrak{q}^2. \end{aligned} \tag{22}$$

By using (21), taking absolute values on both sides, and then applying Lemma 1.2, we obtain

$$\begin{aligned} |\mathfrak{p}_3| &\leq \frac{|\varkappa_\mathfrak{q}|}{2}\left|\epsilon_3 - \epsilon_1\epsilon_2 + \frac{1}{4}\epsilon_1^3 + (1+2\mathfrak{q})\varkappa_\mathfrak{q}\epsilon_1\epsilon_2 - \frac{(1+2\mathfrak{q})\varkappa_\mathfrak{q}}{2}\epsilon_1^3\right| + (1+3\mathfrak{q})|\varkappa_\mathfrak{q}|^3 \\ &\leq \frac{|\varkappa_\mathfrak{q}|}{2}\left|\epsilon_3 - \epsilon_1\epsilon_2 + \frac{1}{4}\epsilon_1^3\right| + \left|(1+2\mathfrak{q})\varkappa_\mathfrak{q}\epsilon_1\epsilon_2 - \frac{(1+2\mathfrak{q})\varkappa_\mathfrak{q}}{2}\epsilon_1^3\right| + (1+3\mathfrak{q})|\varkappa_\mathfrak{q}|^3 \\ &\leq |\varkappa_\mathfrak{q}| + (1+3\mathfrak{q})|\varkappa_\mathfrak{q}|^3. \end{aligned}$$

□

Lemma 2.2. *Let the function* f *be given by* (1) *and assume that* $\mathsf{f} \in \mathsf{SL}_\mathfrak{q}$. *Then, for every* $\mathfrak{s} \geq 2$, *the Taylor coefficients of* f *satisfy the estimate*

$$|\mathsf{a}_\mathfrak{s}| \leq |\varkappa_\mathfrak{q}|^{\mathfrak{s}-1}\varphi_\mathfrak{s}(\mathfrak{q}), \tag{23}$$

where the quantities $\varphi_\mathfrak{s}(\mathfrak{q})$ *and* $\varkappa_\mathfrak{q}$ *are defined in* (17).

Furthermore, the obtained estimate is sharp. The equality case is realized by the extremal function

$$\aleph_0(\mathfrak{z};\mathfrak{q}) = \frac{\mathfrak{z}}{1 - \varkappa_\mathfrak{q}\mathfrak{z} - \mathfrak{q}\varkappa_\mathfrak{q}^2\mathfrak{z}^2} = \mathfrak{z} + \varkappa_\mathfrak{q}\mathfrak{z}^2 + (1+\mathfrak{q})\varkappa_\mathfrak{q}^2\mathfrak{z}^3 + (1+2\mathfrak{q})\varkappa_\mathfrak{q}^3\mathfrak{z}^4 + \mathcal{O}(\mathfrak{z}^5). \tag{24}$$

Theorem 2.3. *Let the function* f *be represented by* (1) *and suppose that* $\mathsf{f} \in \mathsf{SL}_\mathfrak{q}$. *Then*

$$\begin{aligned} |\mathsf{a}_2| &\leq \frac{|\varkappa_\mathfrak{q}|}{\mathfrak{q}}, \\ |\mathsf{a}_3| &\leq \frac{\varkappa_\mathfrak{q}^2}{\mathfrak{q}^2\chi_2} + \frac{(1+2\mathfrak{q})|\varkappa_\mathfrak{q}|^2}{\mathfrak{q}\chi_2}, \\ |\mathsf{a}_4| &\leq \frac{(3\mathfrak{q}^4 + 4\mathfrak{q}^3 + 6\mathfrak{q}^2 + 1 + 2\mathfrak{q})\,|\varkappa_\mathfrak{q}|^3}{\mathfrak{q}^3(1+\mathfrak{q})\chi_3}. \end{aligned}$$

Proof. Let $\mathsf{f}(\mathfrak{z}) \in \mathsf{SL}_\mathfrak{q}$. Then, by the definition of the class $\mathsf{SL}_\mathfrak{q}$, there exists a function

$$\mathfrak{p}(\mathfrak{z}) = 1 + \mathfrak{p}_1\mathfrak{z} + \mathfrak{p}_2\mathfrak{z}^2 + \mathfrak{p}_3\mathfrak{z}^3 + \cdots$$

such that

$$\frac{\mathfrak{z}\,\partial_{\mathfrak{q}}\mathfrak{f}(\mathfrak{z})}{\mathfrak{f}(\mathfrak{z})} = \aleph\big(\omega(\mathfrak{z})\big),$$

where ω denotes a Schwarz function fulfilling the following condition:

$$|\omega(\mathfrak{z})| \leq |\mathfrak{z}|, \qquad (\mathfrak{z} \in \mathscr{O}).$$

Expanding both sides in power series form, we obtain

$$\begin{aligned}
&\mathfrak{z} + \chi_2 \mathfrak{a}_2 \mathfrak{z}^2 + \chi_3 \mathfrak{a}_3 \mathfrak{z}^3 + \chi_4 \mathfrak{a}_4 \mathfrak{z}^4 + \cdots \\
&\quad = \Big(\mathfrak{z} + \mathfrak{a}_2 \mathfrak{z}^2 + \mathfrak{a}_3 \mathfrak{z}^3 + \mathfrak{a}_4 \mathfrak{z}^4 + \cdots\Big)\Big(1 + \mathfrak{p}_1 \mathfrak{z} + \mathfrak{p}_2 \mathfrak{z}^2 + \mathfrak{p}_3 \mathfrak{z}^3 + \cdots\Big) \\
&\quad = \mathfrak{z} + (\mathfrak{p}_1 + \mathfrak{a}_2)\mathfrak{z}^2 + (\mathfrak{p}_2 + \mathfrak{a}_2\mathfrak{p}_1 + \mathfrak{a}_3)\mathfrak{z}^3 + (\mathfrak{p}_3 + \mathfrak{a}_2\mathfrak{p}_2 + \mathfrak{a}_3\mathfrak{p}_1 + \mathfrak{a}_4)\mathfrak{z}^4 + \cdots.
\end{aligned}$$

By equating coefficients, we obtain

$$\mathfrak{a}_2 = \frac{\mathfrak{p}_1}{\mathfrak{q}}, \tag{25}$$

$$\mathfrak{a}_3 = \frac{\mathfrak{p}_2}{\mathfrak{q}\chi_2} + \frac{\mathfrak{p}_1^2}{\mathfrak{q}^2\chi_2}, \tag{26}$$

$$\mathfrak{a}_4 = \frac{1}{\mathfrak{q}\chi_3}\left[\mathfrak{p}_3 + \frac{1+\chi_2}{\mathfrak{q}\chi_2}\mathfrak{p}_1\mathfrak{p}_2 + \frac{\mathfrak{p}_1^3}{\mathfrak{q}^2\chi_2}\right]. \tag{27}$$

From (4), we have

$$\mathfrak{p}_1 = \varkappa_{\mathfrak{q}}, \qquad \mathfrak{p}_2 = (1+2\mathfrak{q})\varkappa_{\mathfrak{q}}^2, \qquad \mathfrak{p}_3 = (1+3\mathfrak{q})\varkappa_{\mathfrak{q}}^3.$$

Substituting into (25), we obtain

$$|\mathfrak{a}_2| = \frac{|\varkappa_{\mathfrak{q}}|}{\mathfrak{q}}.$$

Next, using (26) and the triangle inequality, we get

$$|\mathfrak{a}_3| \leq \frac{|\mathfrak{p}_2|}{\mathfrak{q}\chi_2} + \frac{|\mathfrak{p}_1|^2}{\mathfrak{q}^2\chi_2} = \frac{(1+2\mathfrak{q})|\varkappa_{\mathfrak{q}}|^2}{\mathfrak{q}\chi_2} + \frac{|\varkappa_{\mathfrak{q}}|^2}{\mathfrak{q}^2\chi_2}.$$

Finally, from (27), we have

$$|\mathfrak{a}_4| \leq \frac{1}{\mathfrak{q}\chi_3}\left[|\mathfrak{p}_3| + \frac{1+\chi_2}{\mathfrak{q}\chi_2}|\mathfrak{p}_1\mathfrak{p}_2| + \frac{|\mathfrak{p}_1|^3}{\mathfrak{q}^2\chi_2}\right].$$

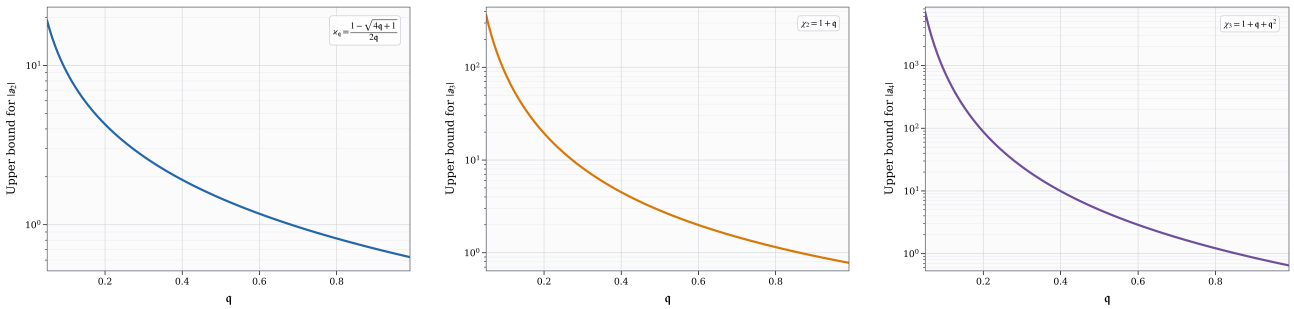
Substituting the values of $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$, we obtain

$$|\mathfrak{a}_4| \leq \frac{|\varkappa_{\mathfrak{q}}|^3}{\mathfrak{q}\chi_3}\left[(1+3\mathfrak{q}) + \frac{(1+\chi_2)(1+2\mathfrak{q})}{\mathfrak{q}\chi_2} + \frac{1}{\mathfrak{q}^2\chi_2}\right].$$

Combining into a single fraction yields

$$|\mathfrak{a}_4| \leq \frac{\left(3\mathfrak{q}^4 + 4\mathfrak{q}^3 + 6\mathfrak{q}^2 + 1 + 2\mathfrak{q}\right)|\varkappa_{\mathfrak{q}}|^3}{\mathfrak{q}^3(1+\mathfrak{q})\chi_3}.$$

The proof is thereby completed. □



(a) Upper bound for $|a_2|$ (b) Upper bound for $|a_3|$ (c) Upper bound for $|a_4|$

FIGURE 2. Dependence of the upper bounds for $|a_2|$, $|a_3|$, and $|a_4|$ in Theorem 2.3 on the quantum parameter q over the interval $0.05 \leq q \leq 0.99$.

Figure 2 illustrates the dependence of the coefficient estimates obtained in Theorem 2.3 on the quantum parameter q. All three bounds decrease as q increases. Their growth becomes particularly pronounced as $q \to 0^+$, with respective asymptotic orders $O(q^{-1})$, $O(q^{-2})$, and $O(q^{-3})$ for $|a_2|$, $|a_3|$, and $|a_4|$. This behavior indicates that higher-order coefficient estimates are more sensitive to small values of the quantum parameter. As $q \to 1^-$, the bounds approach finite values associated with the corresponding classical case. These numerical profiles illustrate the effect of the quantum deformation on the coefficient estimates in Theorem 2.3.

Theorem 2.4. *Let the function $\mathsf{f}(\mathfrak{z})$ be given by (1) and assume that $\mathsf{f} \in \mathsf{SL}_\mathfrak{q}$. Then, for every complex number $\delta \in \mathbb{C}$, the following Fekete–Szegö type inequality holds:*

$$|\mathsf{a}_3 - \delta\, \mathsf{a}_2| \leq (2\mathfrak{q} + |\delta|)\, \frac{\varkappa_\mathfrak{q}^2}{\mathfrak{q}^2}. \tag{28}$$

Moreover, the estimate in (28) is sharp. In the case $\delta > 0$, equality is attained by the extremal function $\aleph_0(\mathfrak{z};\mathfrak{q})$ defined in (24), whereas for $\delta < 0$, the equality case is achieved by the function

$$-\aleph_0(-\mathfrak{z};\mathfrak{q}).$$

Proof. Let $\mathsf{f}(\mathfrak{z}) \in \mathsf{SL}_\mathfrak{q}$, from (25) and (26), we consider the Fekete–Szegö functional $\mathsf{a}_3 - \delta\, \mathsf{a}_2^2$. A straightforward computation gives

$$\left|\mathsf{a}_3 - \delta\, \mathsf{a}_2^2\right| \leq \frac{|\varkappa_\mathfrak{q}|}{\mathfrak{q}\chi_2} + \frac{1}{\mathfrak{q}^2\chi_2}\left|\mathfrak{q}\big((1+2\mathfrak{q})\varkappa_\mathfrak{q}^2 - \varkappa_\mathfrak{q}\big) + \varkappa_\mathfrak{q}^2\big(1-\delta\,\chi_2\big)\right|.$$

Applying the triangle inequality and using the well-known bounds

$$|\mathfrak{p}_1| \leq \varkappa_\mathfrak{q}, \qquad |\mathfrak{p}_2| \leq (1+2\mathfrak{q})\varkappa_\mathfrak{q}^2,$$

we obtain

$$\begin{aligned}\left|\mathfrak{a}_3-\delta\,\mathfrak{a}_2^2\right| &\leq \frac{|\mathfrak{p}_1\mathfrak{a}_2|+|\mathfrak{p}_2|}{\chi_3-1}+\frac{|\delta|\,|\mathfrak{p}_1|^2}{(\chi_2-1)^2}\\ &\leq \frac{\varkappa_{\mathfrak{q}}^2+(1+2\mathfrak{q})\varkappa_{\mathfrak{q}}^2}{\chi_3-1}+\frac{|\delta|\varkappa_{\mathfrak{q}}^2}{(\chi_2-1)^2}\\ &= \left(\frac{2}{\mathfrak{q}}+\frac{|\delta|}{\mathfrak{q}^2}\right)\varkappa_{\mathfrak{q}}^2=(2\mathfrak{q}+|\delta|)\,\frac{\varkappa_{\mathfrak{q}}^2}{\mathfrak{q}^2}.\end{aligned}$$

□

To illustrate the dependence of the estimate in Theorem 2.4 on the parameters $\mathfrak{q}$ and δ, define

$$\mathcal{U}(\mathfrak{q},\delta)=(2\mathfrak{q}+|\delta|)\,\frac{\varkappa_{\mathfrak{q}}^2}{\mathfrak{q}^2}. \tag{29}$$

Since $\mathcal{U}(\mathfrak{q},\delta)$ depends on δ only through its modulus, the argument of the complex parameter δ does not affect the upper bound. It is therefore sufficient to consider $|\delta|\geq 0$.

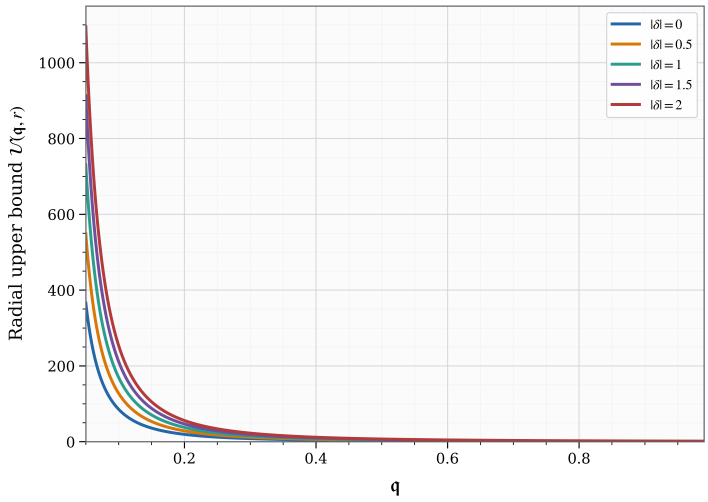


FIGURE 3. Two-dimensional sections of the upper-bound function $\mathcal{U}(\mathfrak{q},\delta)$ for $|\delta| = 0, 0.5, 1, 1.5$, and 2, where $0.05\leq\mathfrak{q}\leq 0.99$.

Figure 3 shows that, for every fixed value of $|\delta|$, the upper bound decreases as $\mathfrak{q}$ increases. Moreover, increasing $|\delta|$ raises the bound at every fixed value of $\mathfrak{q}$. Indeed, (29) is affine with respect to $|\delta|$, with positive slope

$$\frac{\varkappa_{\mathfrak{q}}^2}{\mathfrak{q}^2}.$$

The separation between the curves is therefore more pronounced for small values of $\mathfrak{q}$, whereas the curves become closer as $\mathfrak{q}\to 1^-$.

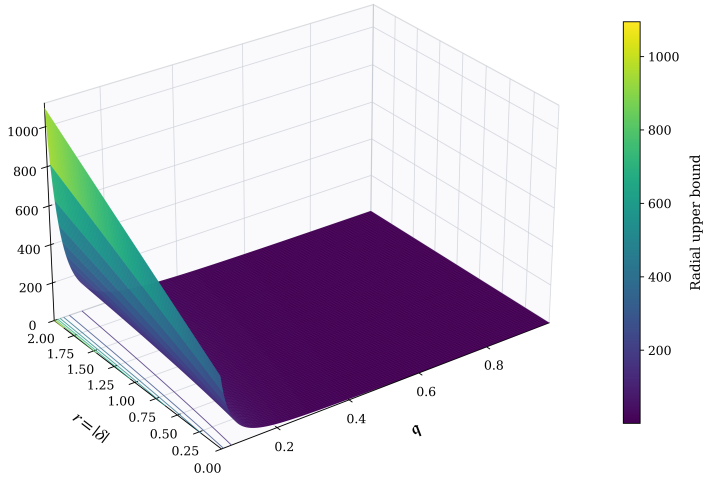


FIGURE 4. Three-dimensional representation of $\mathcal{U}(\mathfrak{q}, \delta)$ over $0.05 \leq \mathfrak{q} \leq 0.99$ and $0 \leq |\delta| \leq 2$. The surface describes the joint influence of the quantum parameter $\mathfrak{q}$ and the modulus of the Fekete–Szegö parameter δ.

The surface in Figure 4 confirms that the estimate is largest when $\mathfrak{q}$ is small and $|\delta|$ is large. Since

$$\varkappa_{\mathfrak{q}} \longrightarrow -1 \qquad \text{as} \qquad \mathfrak{q} \to 0^{+},$$

the leading behavior of the bound is

$$\mathcal{U}(\mathfrak{q}, \delta) \sim \begin{cases} \dfrac{2}{\mathfrak{q}}, & \delta = 0, \\ \dfrac{|\delta|}{\mathfrak{q}^2}, & |\delta| > 0, \end{cases} \qquad \mathfrak{q} \to 0^{+}.$$

This explains the rapid growth of the surface near the lower endpoint of the $\mathfrak{q}$-interval. In contrast, as $\mathfrak{q} \to 1^{-}$, one has

$$\varkappa_{\mathfrak{q}} \longrightarrow \frac{1-\sqrt{5}}{2},$$

and hence

$$\mathcal{U}(\mathfrak{q}, \delta) \longrightarrow (2 + |\delta|)\, \frac{3-\sqrt{5}}{2}.$$

Thus, the Fekete–Szegö estimate approaches a finite classical limit as the quantum parameter tends to 1.

Corollary 2.5. *If*

$$\mathfrak{p}(\mathfrak{z}) = 1 + \mathfrak{p}_1 \mathfrak{z} + \mathfrak{p}_2 \mathfrak{z}^2 + \cdots$$

and

$$\mathfrak{p}(\mathfrak{z}) \prec \aleph(\mathfrak{z}; 1),$$

then, for every $\delta \in \mathbb{C}$, the following estimate holds:

$$|\mathsf{a}_3 - \delta\, \mathsf{a}_2| \leq (2 + |\delta|)\, \varkappa^2. \tag{30}$$

The estimate in (30) is sharp. More precisely, when $\delta > 0$, the equality is realized by the function $\aleph_0(\mathfrak{z}; 1)$ defined in (24); whereas, for $\delta < 0$, equality is attained by the function

$$-\aleph_0(-\mathfrak{z}; 1).$$

Theorem 2.6. *If $\mathsf{f}(\mathfrak{z})$ of the form (1) belongs to the class $\mathsf{SL}_\mathfrak{q}$, Then*

$$\left|\mathsf{a}_2\mathsf{a}_3 - \mathsf{a}_4\right| \leq \frac{|\varkappa_\mathfrak{q}|}{\chi_4 - 1}.$$

Proof. From (25), (26) and (27), we have

$$\begin{aligned}
\mathsf{a}_2\mathsf{a}_3 - \mathsf{a}_4 &= \frac{\mathfrak{p}_1}{\mathfrak{q}^2\chi_2}\left(\mathfrak{p}_2 + \frac{\mathfrak{p}_1^2}{\mathfrak{q}}\right) - \frac{1}{\mathfrak{q}\,\chi_3}\left[\mathfrak{p}_3 + \frac{1+\chi_2}{\mathfrak{q}\,\chi_2}\,\mathfrak{p}_1\mathfrak{p}_2 + \frac{\mathfrak{p}_1^3}{\mathfrak{q}^2\chi_2}\right] \\
&= \frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{q}^2\chi_2} - \frac{1+\chi_2}{\mathfrak{q}^2\,\chi_2\chi_3}\,\mathfrak{p}_1\mathfrak{p}_2 + \frac{\mathfrak{p}_1^3}{\mathfrak{q}^3\chi_2} - \frac{\mathfrak{p}_1^3}{\mathfrak{q}^3\chi_2\chi_3} - \frac{\mathfrak{p}_3}{\mathfrak{q}\,\chi_3} \\
&= \frac{\chi_3 - (1+\chi_2)}{\mathfrak{q}^2\,\chi_2\chi_3}\,\mathfrak{p}_1\mathfrak{p}_2 + \frac{(\chi_3 - 1)\mathfrak{p}_1^3}{\mathfrak{q}^3\chi_2\chi_3} - \frac{\mathfrak{p}_3}{\mathfrak{q}\,\chi_3} \\
&= \frac{\mathfrak{q}^2 - 1}{\mathfrak{q}^2\,\chi_2\chi_3}\,\mathfrak{p}_1\mathfrak{p}_2 + \frac{\mathfrak{p}_1^3}{\mathfrak{q}^2\chi_3} - \frac{\mathfrak{p}_3}{\mathfrak{q}\,\chi_3}
\end{aligned}$$

Let $\mathsf{f}(\mathfrak{z}) \in \mathsf{SL}_\mathfrak{q}$, then

$$\frac{\mathfrak{z}\,\partial_\mathfrak{q}\mathsf{f}(\mathfrak{z})}{\mathsf{f}(\mathfrak{z})} = \aleph(\omega(\mathfrak{z}); \mathfrak{q})$$

we have

$$\mathsf{a}_2 = \frac{\varkappa_\mathfrak{q}\epsilon_1}{2\,\mathfrak{q}}, \tag{31}$$

$$\mathsf{a}_3 = \frac{\varkappa_\mathfrak{q}}{2\mathfrak{q}\chi_2}\left[\epsilon_2 + \frac{1}{2}\left((1+2\mathfrak{q})\varkappa_\mathfrak{q} - 1 + \frac{\varkappa_\mathfrak{q}}{\mathfrak{q}}\right)\epsilon_1^2\right], \tag{32}$$

and

$$\begin{aligned}
\mathsf{a}_4 = \frac{1}{\chi_4 - 1}\Bigg[&\frac{\varkappa_\mathfrak{q}}{2}\epsilon_3 + \left(\frac{(1+2\mathfrak{q})\varkappa_\mathfrak{q}^2 - \varkappa_\mathfrak{q}}{2} + \frac{(\chi_2 + \chi_3 - 2)\varkappa_\mathfrak{q}^2}{4\mathfrak{q}^2\chi_2}\right)\epsilon_1\epsilon_2 \\
&+ \left(\frac{\varkappa_\mathfrak{q}\big(1 - 2(1+2\mathfrak{q})\varkappa_\mathfrak{q} + (1+3\mathfrak{q})\varkappa_\mathfrak{q}^2\big)}{8} + \frac{(\chi_2 + \chi_3 - 2)\varkappa_\mathfrak{q}^2}{8\mathfrak{q}^2\chi_2}\left((1+2\mathfrak{q})\varkappa_\mathfrak{q} - 1 + \frac{\varkappa_\mathfrak{q}}{\mathfrak{q}}\right) - \frac{\varkappa_\mathfrak{q}^3}{8\mathfrak{q}^2}\right)\epsilon_1^3\Bigg].
\end{aligned} \tag{33}$$

Therefore

$$\begin{aligned}
\mathsf{a}_2\mathsf{a}_3 - \mathsf{a}_4 = &-\frac{\varkappa_\mathfrak{q}}{2(\chi_4 - 1)}\,\epsilon_3 \\
&+ \frac{1}{\chi_4 - 1}\left[\frac{\varkappa_\mathfrak{q}}{2} - \frac{(1+2\mathfrak{q})\varkappa_\mathfrak{q}^2}{2} + \frac{\mathfrak{q}\,\varkappa_\mathfrak{q}^2}{4\chi_2}\right]\epsilon_1\epsilon_2 \\
&+ \frac{1}{\chi_4 - 1}\left[\frac{\mathfrak{q}\,\varkappa_\mathfrak{q}^2}{8\chi_2}\left((1+2\mathfrak{q})\varkappa_\mathfrak{q} - 1 + \frac{\varkappa_\mathfrak{q}}{\mathfrak{q}}\right) - \frac{\varkappa_\mathfrak{q}}{8}\big(1 - 2(1+2\mathfrak{q})\varkappa_\mathfrak{q} + (1+3\mathfrak{q})\varkappa_\mathfrak{q}^2\big) + \frac{\varkappa_\mathfrak{q}^3}{8\mathfrak{q}^2}\right]\epsilon_1^3.
\end{aligned}$$

$$\mathsf{a}_2\mathsf{a}_3 - \mathsf{a}_4 = \frac{\varkappa_{\mathfrak{q}}}{2(\chi_4 - 1)}\left[-\epsilon_3 + A\,\epsilon_1\epsilon_2 + B\,\epsilon_1^3\right],$$

where

$$A = 1 - (1+2\mathfrak{q})\varkappa_{\mathfrak{q}} + \frac{\mathfrak{q}\varkappa_{\mathfrak{q}}}{2\chi_2},$$

and

$$B = \frac{1}{4}\,\Xi(\mathfrak{q}, \varkappa_{\mathfrak{q}}),$$

with

$$\Xi(\mathfrak{q}, \varkappa_{\mathfrak{q}}) = \frac{2\mathfrak{q}\varkappa_{\mathfrak{q}}}{\chi_2}\left((1+2\mathfrak{q})\varkappa_{\mathfrak{q}} - 1 + \frac{\varkappa_{\mathfrak{q}}}{\mathfrak{q}}\right) - \left(1 - 2(1+2\mathfrak{q})\varkappa_{\mathfrak{q}} + (1+3\mathfrak{q})\varkappa_{\mathfrak{q}}^2\right) + \frac{2\varkappa_{\mathfrak{q}}^2}{\mathfrak{q}^2}.$$

Thus,

$$\begin{aligned}\left|\mathsf{a}_2\mathsf{a}_3 - \mathsf{a}_4\right| &= \frac{|\varkappa_{\mathfrak{q}}|}{2(\chi_4 - 1)}\left|\epsilon_3 - A\epsilon_1\epsilon_2 - B\epsilon_1^3\right| \\ &= \frac{|\varkappa_{\mathfrak{q}}|}{2(\chi_4 - 1)}\left|\epsilon_3 - 2M\epsilon_1\epsilon_2 + N\epsilon_1^3\right|,\end{aligned}$$

where

$$M = \frac{1}{2}\left(1 - (1+2\mathfrak{q})\varkappa_{\mathfrak{q}} + \frac{\mathfrak{q}\varkappa_{\mathfrak{q}}}{2\chi_2}\right),$$

and

$$N = -\frac{1}{4}\,\Xi(\mathfrak{q}, \varkappa_{\mathfrak{q}}).$$

Therefore, by Lemma 2.2, if

$$0 \le M \le 1, \qquad M(2M-1) \le N \le M,$$

then

$$\left|\mathsf{a}_2\mathsf{a}_3 - \mathsf{a}_4\right| \le \frac{|\varkappa_{\mathfrak{q}}|}{\chi_4 - 1}.$$

□

Conclusion

In conclusion, this study develops and investigates a novel subclass of starlike functions associated with shell-like curves within the framework of $\mathfrak{q}$-Fibonacci numbers. By employing subordination techniques together with tools from $\mathfrak{q}$-calculus, several geometric properties of this class have been established. In particular, a sufficient condition for inclusion in the class has been derived, and a detailed analysis of the corresponding Fekete–Szegö problem has been carried out. These findings provide additional insight into the relationship between $\mathfrak{q}$-Fibonacci structures and GFT.

The results obtained in this work further emphasize the strong interaction between GFT and quantum calculus. The developed approach illustrates how $\mathfrak{q}$-Fibonacci sequences may be effectively incorporated into the study of analytic and univalent functions in order to derive meaningful geometric and coefficient-related properties. Consequently, the present study contributes to a deeper theoretical understanding of both $\mathfrak{q}$-calculus and Fibonacci-type structures within analytic function theory.

Moreover, the techniques and results presented here open several possible directions for future investigations. It is anticipated that the methods developed in this article will stimulate further research concerning subclasses of analytic functions associated with generalized Fibonacci-type sequences, special functions, and other q-operators. The interplay between sequential structures and geometric properties of analytic functions continues to be a rich and promising area of research with potential applications extending beyond pure mathematics into related scientific disciplines.

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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