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# EXISTENCE OF PERIODIC SOLUTIONS OF NONLINEAR SYLVESTER MATRIX VOLTERRA INTEGRO-DYNAMIC SYSTEMS ON PERIODIC TIME SCALES

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Abstract. This paper establishes sufficient conditions for the existence of periodic solutions for scalar and Sylvester matrix Volterra integro-dynamic systems on arbitrary periodic time scales. By employing the vectorization operator and Kronecker product, the original matrix system is transformed into an equivalent scalar Kronecker-product integro-dynamic system. The existence results are established using Schaefer's fixed point theorem together with the Arzelà–Ascoli compactness theorem. Furthermore, appropriate Lyapunov functionals and time-scale Sobolev-type inequalities are constructed to derive uniform a priori bounds and new inequalities for the $\Delta$-derivatives of absolute value functions. The cases corresponding to positive, negative, and vanishing system coefficients are investigated separately to obtain sufficient conditions for the existence of periodic solutions. Finally, a numerical example together with graphical simulations is presented to demonstrate the applicability and effectiveness of the proposed theoretical results.

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## 1. Introduction

In recent decades, the theory of dynamic equations on time scales has attracted considerable attention due to its unification of discrete and continuous analysis, allowing for a more general framework for

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studying dynamical systems [10, 12, 13]. Time scales calculus provides a versatile approach to model processes that evolve over arbitrary time domains, enabling the treatment of difference, differential, and hybrid systems within a single framework.

A particularly important class of systems in this context is the Volterra integro-dynamic system. Such systems naturally generalize classical Volterra integral equations to time scales, incorporating memory effects and hereditary properties in the system evolution [14, 15, 24, 25]. These systems have been widely studied for their mathematical richness and applicability in modeling biological, ecological, and engineering processes [22, 23, 26, 27].

The applications of dynamic systems on time scales are diverse, including population dynamics, epidemiology, economic modeling, and control theory [30–32, 41]. Particularly, integro-dynamic models have been successfully used to describe predator-prey interactions, epidemic spread, and neural network dynamics where the system memory plays a crucial role [16–18]. These applications demonstrate the significance of establishing rigorous existence, stability, and periodicity results.

A key focus in the qualitative analysis of such systems is the study of stability and periodic solutions. Stability results, including Lyapunov-type methods and exponential stability criteria, provide fundamental insights into the long-term behavior of dynamical systems [28, 29, 36, 43]. Several authors have investigated periodic solutions of Volterra integro-dynamic systems using fixed-point methods, measures of noncompactness, and functional analytic techniques [34, 35, 37–39].

Recent investigations have focused on periodic solutions, exponential stability, synchronization, and controllability analysis for nonlinear matrix and integro-dynamic systems on time scales using dichotomy theory, Lyapunov methods, and fixed-point techniques [4–7]. Furthermore, the study of fractional Sylvester matrix equations and delayed neural network systems has attracted considerable attention due to their important applications in control theory, biological modeling, and dynamic network analysis [8, 9].

Another important direction in the study of matrix dynamical systems is the Sylvester matrix system, which naturally arises in the control, signal processing, and networked systems contexts. Sylvester matrix equations provide a compact framework to represent coupled systems, and their dynamic extensions on time scales allow the investigation of multi-dimensional and coupled processes [11, 19, 40, 42]. The combination of Volterra integral operators with Sylvester matrices leads to rich integro-dynamic models capable of capturing complex interactions between components.

Recent advances in delayed reaction-diffusion Nicholson's blowflies equations have established sufficient conditions for the stability and global exponential attractivity of positive periodic solutions under multiple time-varying delays and Neumann boundary conditions, significantly extending existing theoretical results [3]. For generalized non-autonomous delayed Nicholson's blowflies models with Gamma–Ricker-type reproduction rates, new differential inequality methods and fixed-point

techniques have been employed to derive existence and global attractivity criteria for positive periodic solutions, yielding substantial improvements over previous studies [2]. The global dynamics of non-autonomous delay-structured tick metapopulation models with time-varying coefficients and delays have been rigorously analyzed, leading to novel results on the attractivity of positive periodic solutions and providing deeper insights into long-term population behavior [1].

Although periodic solutions of Volterra integro-dynamic systems and stability properties of Sylvester matrix dynamical systems have been extensively investigated, the existence theory for nonlinear Sylvester Volterra integro-dynamic systems on periodic time scales remains largely unexplored. In particular, the combination of matrix-valued nonlinearities, Volterra integral operators, and arbitrary periodic time scales introduces substantial analytical challenges that are not encountered in classical scalar or vector settings.

Despite the extensive research on Volterra integro-dynamic systems and Sylvester matrix systems, several challenges remain unresolved. In particular, most existing studies focus on scalar or vector dynamical systems, whereas the analysis of nonlinear coupled Volterra Sylvester matrix integro-dynamic systems on periodic time scales remains relatively limited [20,21]. Unlike scalar and vector systems, matrix-valued Sylvester systems involve coupled matrix interactions and non-commutative matrix operations, which prevent the direct application of many existing existence theories. Furthermore, the matrix-valued structure, together with Volterra integral operators and time-scale dynamics, introduces substantial analytical difficulties that cannot be addressed directly using techniques developed for scalar or vector systems.

To the best of our knowledge, no general existence theory is currently available for nonlinear Sylvester Volterra integro-dynamic systems on arbitrary periodic time scales. The objective is to establish a general theoretical framework capable of handling matrix-valued nonlinearities, memory effects, and periodic behavior within a unified time-scale setting.

$$X^{\Delta}(\iota) = A(\iota)X(\iota) + X(\iota)B(\iota) + \int_{-\infty}^{\iota} K(\iota,\zeta)X(\zeta)\,\Delta\zeta + F(x(\iota)) + C(\iota), \tag{1}$$

where $A(\iota), B(\iota) \in \mathbb{R}^{n\times n}$ are $\omega$-periodic matrix-valued functions. $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$. $C : \mathbb{T} \to \mathbb{R}^n$, and $K$ is a matrix rd-continuous function $n \times n$.

To achieve this goal, we employ the vectorization operator and Kronecker product techniques to transform the original matrix-valued system into an equivalent vector integro-dynamic system. This transformation overcomes difficulties arising from coupled matrix structures and non-commutative matrix operations, enabling the application of fixed-point methods to establish sufficient conditions for the existence of periodic solutions.

The proposed framework accommodates a broad class of nonlinear Sylvester matrix integro-dynamic systems on periodic time scales and extends several existing results from scalar and vector systems to

the matrix-valued setting. Moreover, such systems arise naturally in coupled neural network models, interconnected control systems, and matrix-valued population dynamics where memory effects and periodic environmental influences play an important role.

The proposed framework is further validated through representative examples, demonstrating its applicability and computational effectiveness. The theoretical findings broaden the understanding of periodic behavior in nonlinear matrix integro-dynamic systems on time scales and provide a rigorous analytical basis for investigating matrix-valued dynamical models.

The remainder of this article is organized as follows. Section 2 introduces the fundamental concepts and preliminary results from time-scale calculus, periodic dynamic systems, matrix exponential functions, and fixed-point theory that are required in the subsequent analysis. Section 3 establishes the existence of $\omega$-periodic solutions for nonlinear Sylvester Volterra integro-dynamic systems by reformulating the original matrix equation into an equivalent Kronecker representation and applying Schaefer's fixed-point theorem. Section 4 derives sufficient conditions for the existence of periodic solutions of scalar Kronecker-reformulated Volterra integro-dynamic systems using Lyapunov functionals, Sobolev-type inequalities, and newly established $\Delta$-derivative inequalities under suitable parameter assumptions. Finally, Section 5 presents a numerical example to validate the theoretical results and demonstrate the effectiveness of the proposed approach.

## 2. Preliminaries

In this section, we present the preliminary material required for the results that follow.

**Definition 1.** A time scale $\mathbb{T}$ is said to be periodic if there exists a positive constant $\omega > 0$ such that $\iota + \omega \in \mathbb{T}$ and $\iota - \omega \in \mathbb{T}$ for all $\iota \in \mathbb{T}$.

**Definition 2.** [11] Let $\mathbb{T}$ be a periodic time scale with period $\omega$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be periodic with period $T$ if there exists a positive integer $n$ such that $T = n\omega$ and

$$f(\iota + T) = f(\iota)$$

for all $\iota \in \mathbb{T}$. The smallest such positive number $T$ is called the period of $f$.

**Theorem 1.** [12] *Let $f_1, f_2 : \mathbb{T} \to \mathbb{R}$ are $\Delta$-differentiable functions. Then the following statements hold:*

i. *If $\sigma(\iota) > \iota$, then $f_1^{\Delta}(\iota) = \frac{f_1^{\sigma}(\iota) - f_1(\iota)}{\mu(\iota)}$.*
ii. *For every $\iota \in \mathbb{T}$, $f_1^{\sigma}(\iota) = f_1(\iota) + \mu(\iota) f_1^{\Delta}(\iota)$.*
iii. *The product $f_1.f_2 : \mathbb{T} \to \mathbb{R}$ is $\Delta$-differentiable, and*

$$(f_1 f_2)^{\Delta}(\iota) = f_1^{\Delta}(\iota)\, f_2(\iota) + f_1^{\sigma}(\iota)\, f_2^{\Delta}(\iota) = f_1^{\Delta}(\iota)\, f_2^{\sigma}(\iota) + f_1(\iota)\, f_2^{\Delta}(\iota).$$

iv. *If* $f_2(\iota)\, f_2^{\sigma}(\iota) \neq 0$, *then the quotient* $\dfrac{f_1}{f_2}$ *is* $\Delta$*-differentiable with*

$$\left(\frac{f_1}{f_2}\right)^{\Delta}(\iota) = \frac{f_1^{\Delta}(\iota) f_2(\iota) - f_1(\iota) f_2^{\Delta}(\iota)}{f_2(\iota) f_2^{\sigma}(\iota)}.$$

*Lemma* 1. [13] Let $\mathbb{T}$ with period $\omega$. Assume that $f_1 : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ satisfies the conditions stated in [13]. Then the following identity holds:

$$\left[\int_{\iota-T}^{\iota} f_1(\iota, \zeta)\, \Delta\zeta\right]^{\Delta} = f_1(\sigma(\iota), \iota) - f_1(\sigma(\iota), \iota - T) + \int_{\iota-T}^{\iota} f_1^{\Delta}(\iota, \zeta)\, \Delta\zeta,$$

where $T = n_0\omega$ with $n_0 \in \mathbb{N}$ a positive integer.

The following result presents a Jensen-type inequality on time scales, which will be used in the subsequent analysis.

**Theorem 2.** [13] *Suppose that* $a, b \in \mathbb{T}$ *with* $a < b$. *Let* $f_2 : [a, b]_{\mathbb{T}} \to [c, d]$ *be an rd-continuous function, where* $c, d \in \mathbb{R}$, *and let* $F : (c, d) \to \mathbb{R}$ *be a continuous convex function. Then*

$$F\left(\frac{1}{b-a}\int_a^b f_2(t)\, \Delta t\right) \leq \frac{1}{b-a}\int_a^b F(f_2(t))\, \Delta t.$$

**Theorem 3.** [15] *Let* $D : \mathbb{T} \to \mathbb{R}^{n\times n}$ *be a matrix-valued rd-continuous function and let* $h : \mathbb{T} \to \mathbb{R}^n$ *be rd-continuous. For a given* $\iota_0 \in \mathbb{T}$, *the initial value problem*

$$z^{\Delta}(\iota) = D(\iota)z(\iota) + h(\iota), \qquad z(\iota_0) = z_0,$$

*has a unique solution*

$$z : \mathbb{T} \to \mathbb{R}^n.$$

*Moreover,*

$$z(\iota) = \phi_D(\iota, \iota_0)z_0 + \int_{\iota_0}^{\iota} \phi_D(\iota, \sigma(\tau))h(\tau)\Delta\tau,$$

*where* $\phi_D(\iota, \iota_0)$ *denotes the fundamental matrix solution associated with the homogeneous system*

$$z^{\Delta}(\iota) = D(\iota)z(\iota).$$

*Lemma* 2. [15] Let $p \in \mathcal{R}$. Then the following properties hold:

(i) $\Phi^0(\iota, \zeta) = 1, \qquad \Phi^p(\iota, \iota) = 1.$

(ii) $\Phi^p(\sigma(\iota), \zeta) = \big(1 + \mu(\iota)p(\iota)\big)\Phi^p(\iota, \zeta).$

(iii) $\frac{1}{\Phi^p(\iota,\zeta)} = \Phi^{\ominus p}(\iota, \zeta)$, where $(\ominus p)(\iota) = \frac{-p(\iota)}{1+\mu(\iota)p(\iota)}$.

(iv) $\Phi^p(\iota, \zeta) = \frac{1}{\Phi^{\ominus p}(\iota,\zeta)} = \Phi^{\ominus p}(\zeta, \iota).$

(v) $\Phi^p(\iota, \zeta)\Phi^p(\zeta, r) = \Phi^p(\iota, r).$

**Theorem 4** (Schaefer Fixed Point Theorem)**.** [33] *Let* $(M, \|\cdot\|)$ *be a normed linear space and let* $Q : M \to M$ *be a continuous and compact operator. Then one of the following alternatives holds:*

(i) *The equation*

$$z = Qz$$

*has at least one solution in* $M$*;*

(ii) *The set*

$$\{z \in M : \ z = \lambda Qz, \ 0 < \lambda < 1\}$$

*is unbounded.*

## 3. Existence of Periodic Solutions

In this section, we establish the existence of $\omega$-periodic solutions for the nonlinear system (2). We first reformulate the problem as an equivalent integral equation, which naturally leads to the definition of the operator $Q$. We then prove that $Q$ is continuous and compact. Finally, Schaefer's fixed point theorem is applied to establish the existence of an $\omega$-periodic solution of system (2).

To transform the Sylvester matrix Volterra integro-dynamic system (1) into an equivalent vector-valued system, we employ the vectorization (Vec) operator together with the properties of the Kronecker product. Consequently, system (1) can be rewritten as

$$\begin{cases} z^{\Delta}(\iota) = D(\iota)z(\iota) + \displaystyle\int_{-\infty}^{\iota} (I \otimes K)(\iota, \zeta)z(\zeta)\Delta\zeta + h(z(\iota)) + c(\iota), \\ z(\iota_0) = z_0, \end{cases} \tag{2}$$

which is piecewise rd-continuous on $\Omega := \{(\iota, \zeta) \in \mathbb{T}_0 \times \mathbb{T}_0 : \iota_0 \leq \zeta \leq \iota < \infty\}$, and where

$$D(\iota) = B^T(\iota) \otimes I + I \otimes A(\iota)$$

is an $n^2 \times n^2$ matrix. $\mathrm{Vec}(X(\iota)) = z(\iota), \quad h(z(\iota)) = \mathrm{Vec}(F(X(\iota))), \quad c(\iota) = \mathrm{Vec}(C(\iota))$.

*Remark* 1. [15] It follows immediately from the properties of the vectorization operator that, if $X(\iota)$ is the solution of (1) then $VecX(\iota) = z(\iota)$ is the solution of (2) and vice-versa.

In this paper, we impose the following assumptions:

A1. The nonlinear function $h : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ is continuous, and the forcing term $c : \mathbb{T} \to \mathbb{R}^{n^2}$ is rd-continuous and $\omega$-periodic.

A2. The kernel $(I \otimes K) : \mathbb{T} \times \mathbb{T} \to \mathbb{R}^{n^2 \times n^2}$ satisfies the following conditions:

(i) $(I \otimes K)(\iota, \zeta)$ is rd-continuous on

$$\Omega = \{(\iota, \zeta) \in \mathbb{T} \times \mathbb{T} : \zeta \leq \iota\}.$$

(ii)

$$(I \otimes K)(\iota, \zeta) = 0, \qquad \zeta > \iota.$$

(iii) The kernel is $\omega$-periodic, that is,

$$(I \otimes K)(\iota + \omega, \zeta + \omega) = (I \otimes K)(\iota, \zeta),$$

for all $(\iota, \zeta) \in \Omega$.

(iv) The kernel satisfies the boundedness condition

$$\sup_{\iota \in \mathbb{T}} \int_{-\infty}^{\iota} \|(I \otimes K)(\iota, \zeta)\| \, \Delta\zeta < \infty.$$

Here $|\iota|$ represents the matrix norm associated with a norm in $\mathbb{R}^{n^2}$, which we denote by the same symbol. For $a, b \in \mathbb{T}$, we denote

$$[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}.$$

Define

$$Y^{\top} = \{\varphi \in C(\mathbb{T}, \mathbb{R}^{n^2}) : \varphi(\iota + \omega) = \varphi(\iota)\},$$

where $C(\mathbb{T}, \mathbb{R}^{n^2})$ denotes the space of all vector-valued rd-continuous functions on $\mathbb{T}$. Equipped with the norm

$$|z|_0 = \max_{1 \le i \le n^2} \left\{ \sup_{\iota \in [0,\omega]_{\mathbb{T}}} |z_i(\iota)| \right\},$$

the space $Y^{\top}$ becomes a Banach space.

Now, consider system (2) and the associated family of equations

$$\begin{aligned} z^{\Delta}(\iota) &= \lambda\big[-\gamma I + D\big] z(\iota) + \gamma z(\iota) \\ &\quad + \lambda\left\{ h(z(\iota)) + \int_{-\infty}^{\iota} (I \otimes K)(\iota, \zeta)\, z(\zeta)\, \Delta\zeta + c(\iota) \right\}, \end{aligned} \tag{3}$$

where $0 < \lambda \le 1$ and $\gamma \in \mathbb{R}$.

**Theorem 5.** *If $z \in Y^{\top}$, then $z$ is a solution of (3) iff*

$$z(\iota) = \lambda\big(1 - \phi_{\gamma}(\iota, \iota - \omega)\big)^{-1} \int_{\iota - \omega}^{\iota} U(\zeta, z(\zeta))\, \phi_{\gamma}(\iota, \sigma(\zeta))\, \Delta\zeta, \tag{4}$$

*where*

$$U(\zeta, z(\zeta)) = \{-\gamma I + D\} z(\zeta) + h(z(\zeta)) + \int_{-\infty}^{\zeta} (I \otimes K)(\zeta, \rho) z(\rho)\, \Delta\rho + c(\zeta).$$

*Proof.* Suppose $z \in Y^{\top}$ is a solution of (3). Then (3) can be expressed as

$$z^{\Delta}(\iota) = \lambda U(\iota, z(\iota)) + \gamma z(\iota). \tag{5}$$

Applying the integrating factor $\phi_{\ominus\gamma}(\sigma(\iota), \iota_0)$ to both sides of equation (5) yields

$$z^{\Delta}(\iota)\phi_{\ominus\gamma}(\sigma(\iota), \iota_0) - \gamma z(\iota)\phi_{\ominus\gamma}(\sigma(\iota), \iota_0) = \lambda U(\iota, z(\iota))\phi_{\ominus\gamma}(\sigma(\iota), \iota_0).$$

By Lemma 2, this becomes

$$\big[z(\iota)\phi_{\ominus\gamma}(\iota,\iota_0)\big]^{\Delta} = \lambda U(\iota, z(\iota))\phi_{\ominus\gamma}(\sigma(\iota),\iota_0).$$

Integrating from $\iota-\omega$ to $\iota$ yields

$$z(\iota)\phi_{\ominus\gamma}(\iota,\iota_0) - z(\iota-\omega)\phi_{\ominus\gamma}(\iota-\omega,\iota_0) = \lambda\int_{\iota-\omega}^{\iota} U(\zeta,z(\zeta))\phi_{\ominus\gamma}(\sigma(\zeta),\iota_0)\,\Delta\zeta.$$

Since $z(\iota) = z(\iota-\omega)$ for $z \in Y^{\top}$, and using

$$\frac{\phi_{\ominus\gamma}(\iota-\omega,\iota_0)}{\phi_{\ominus\gamma}(\iota,\iota_0)} = \phi_\gamma(\iota,\iota-\omega), \frac{\phi_{\ominus\gamma}(\sigma(\zeta),\iota_0)}{\phi_{\ominus\gamma}(\iota,\iota_0)} = \phi_\gamma(\iota,\sigma(\zeta)),$$

we obtain (4). Since all steps are reversible, the equivalence is proved. □

*Remark* 2. In the special case $\mathbb{T} = \mathbb{R}$, the integral representation in Theorem 5 differs from the corresponding representation given in [13].

Since the kernel $(I\otimes K)(\iota,\zeta)$ is $\omega$-periodic, it follows that

$$U(\zeta,z(\zeta)) = U(\zeta+\omega, z(\zeta+\omega)), \qquad z\in Y^{\top}. \tag{6}$$

Define the operator

$$Q : Y^{\top} \to Y^{\top}$$

by

$$(Qz)(\iota) = \big(1-\phi_\gamma(\iota,\iota-\omega)\big)^{-1}\int_{\iota-\omega}^{\iota} U(\zeta,z(\zeta))\phi_\gamma(\iota,\sigma(\zeta))\,\Delta\zeta. \tag{7}$$

Equation (4) is equivalent to

$$z = \lambda Q z. \tag{8}$$

*Lemma* 3. The operator $Q : Y^{\top}\to Y^{\top}$ is $\omega$-periodic. Furthermore,

$$\begin{aligned} Q^{\Delta}z(\iota) = {} & \big(1-\phi_\gamma(\sigma(\iota),\sigma(\iota)-\omega)\big)^{-1}\Big\{U(\iota,z(\iota)) - U(\iota-\omega,z(\iota-\omega))\phi_\gamma(\sigma(\iota),\sigma(\iota-\omega)) \\ & + \gamma\int_{\iota-\omega}^{\iota} U(\zeta,z(\zeta))\phi_\gamma(\iota,\sigma(\zeta))\,\Delta\zeta\Big\} \\ & + \frac{\gamma\phi_\gamma(\iota,\iota-\omega)}{\big(1-\phi_\gamma(\iota,\iota-\omega)\big)\big(1-\phi_\gamma(\sigma(\iota),\sigma(\iota-\omega))\big)}\int_{\iota-\omega}^{\iota} U(\zeta,z(\zeta))\phi_\gamma(\iota,\sigma(\zeta))\,\Delta\zeta \end{aligned} \tag{9}$$

$$= U(\iota,z(\iota)) + \gamma\big(1-\phi_\gamma(\iota,\iota-\omega)\big)^{-1}Qz(\iota), \tag{10}$$

where

$$k_\gamma = 1-\phi_\gamma(\iota,\iota-\omega)$$

is independent of $\iota\in\mathbb{T}$.

*Proof.* Using the periodicity property (6), together with the change of variables $\rho = \zeta + \omega$ and Lemma 1, we obtain

$$(Qz)(\iota + \omega) = (Qz)(\iota),$$

which shows that $Q : Y^{\top} \to Y^{\top}$. Differentiating (7) with the aid of Lemma 1 yields (9). □

**Theorem 6.** *The operator $Q : Y^{\top} \to Y^{\top}$ is continuous and compact.*

*Proof.* Let us consider the set

$$Z := \{z \in Y^{\top} : |z|_0 \leq M\}.$$

Clearly, $Z$ is a closed and bounded subset of $Y^{\top}$.

For $\psi_1, \psi_2 \in Z$, we compute

$$\begin{aligned}
|Q\psi_1(\iota) - Q\psi_2(\iota)| &\leq E_1E_2 \int_{\iota-\omega}^{\iota} |U(\zeta, \psi_1(\zeta)) - U(\zeta, \psi_2(\zeta))| \, \Delta\zeta \\
&\leq E_1E_2 \int_{\iota-\omega}^{\iota} |-\gamma I + D| \, |\psi_1(\zeta) - \psi_2(\zeta)| \, \Delta\zeta \\
&\quad + E_1E_2 \int_{\iota-\omega}^{\iota} |h(\psi_1(\zeta)) - h(\psi_2(\zeta))| \, \Delta\zeta \\
&\quad + E_1E_2 \int_{\iota-\omega}^{\iota} \int_{-\infty}^{\zeta} |(I \otimes K)(\zeta, \rho)| \, |\psi_1(\rho) - \psi_2(\rho)| \, \Delta\rho \, \Delta\zeta.
\end{aligned}$$

where

$$E_1 = \max_{\iota \in [0,\omega]} \left| \big(1 - \phi_\gamma(\iota, \iota - \omega)\big)^{-1} \right|, \qquad E_2 = \max_{\zeta \in [\iota-\omega, \iota]} |\phi_\gamma(\iota, \sigma(\zeta))|.$$

Since $h$ is rd-continuous, it is uniformly continuous on the compact set $[-M, M]^{n^2}$. Therefore, the operator $Q : Y^{\top} \to Y^{\top}$ is continuous.

Next, we show compactness of $Q$. For this we apply the Arzelà–Ascoli theorem. Define

$$W := \{Q\psi_n(\iota) : \psi_n \in Z\} \subset \mathbb{R}^{n^2}.$$

To apply the theorem, we need to verify relative compactness and equicontinuity of $\{Q\psi_n\}$.

Because $[-M, M]^{n^2}$ is compact in $\mathbb{R}^{n^2}$ and $h : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ is continuous, there exists a constant $C > 0$ such that

$$|U(\zeta, z(\zeta))| \leq C, \quad \text{for all } z \in Z, \ \zeta \in [0, \omega]_{\mathbb{T}}. \tag{11}$$

Now fix $\iota \in [0, \omega]_{\mathbb{T}}$ and $\psi_n \in Z$. Then

$$|Q\psi_n(\iota)| \leq E_1E_2 \int_{\iota-\omega}^{\iota} |U(\zeta, \psi_n(\zeta))| \, \Delta\zeta \leq E_1E_2\omega C =: N. \tag{12}$$

Hence the family $\{Q\psi_n\}_{n \in \mathbb{N}}$ is uniformly bounded.

Finally, using (9), together with (11) and (12), we deduce that $\left|Q^{\Delta}\psi_n(\iota)\right|$ is uniformly bounded for $\psi_n \in Z$. Thus $\{Q\psi_n\}$ is equicontinuous. By the Arzelà–Ascoli theorem, the operator $Q$ is compact. □

**Theorem 7.** *Suppose there exists a constant $M > 0$ such that every $\omega$-periodic solution of (3) with $0 < \lambda < 1$ satisfies $|z|_0 \leq M$. Then the nonlinear system (2) admits at least one $\omega$-periodic solution in $Y^\top$.*

*Proof.* Consider the operator $Q$ defined in (7). By Theorem 6, the operator $Q : Y^\top \to Y^\top$ is continuous and compact. Since every $\omega$-periodic solution of (3) corresponding to $0 < \lambda < 1$ satisfies the uniform bound $|z|_0 \leq M$, the second alternative of Schaefer's fixed point theorem is excluded. Therefore, there exists a solution of the equation

$$z = \lambda Q z$$

for $\lambda = 1$.

Hence, the transformed Kronecker-product system (2) possesses an $\omega$-periodic solution. By the equivalence between the original Sylvester matrix system (1) and the transformed system (2) established through the Vec operator and the Kronecker product, it follows that the original nonlinear Sylvester Volterra integro-dynamic system (1) also admits an $\omega$-periodic solution. □

## 4. Sufficient Conditions for Periodic Solutions

This section investigates the existence of $\omega$-periodic solutions for the scalar Kronecker-product (KP) Volterra integro-dynamic system on periodic time scales. By constructing suitable Lyapunov functionals together with a Sobolev-type inequality on periodic time scales, we derive a priori bounds for periodic solutions and establish sufficient conditions for their existence under different assumptions on the parameter $a$.

Let

$$z^\Delta(\iota) = az(\iota) + h(z(\iota)) + \int_{-\infty}^{\iota} (I \otimes K)(\iota, \zeta) z(\zeta)\, \Delta\zeta + c(\iota), \quad \iota \in \mathbb{T}, \tag{13}$$

together with the associated family of equations

$$z_\lambda^\Delta(\iota) = \big[\lambda(\alpha + a) - \alpha\big] z_\lambda(\iota) + \lambda h(z_\lambda(\iota)) + \lambda \int_{-\infty}^{\iota} (I \otimes K)(\iota, \zeta) z_\lambda(\zeta)\, \Delta\zeta + \lambda c(\iota), \tag{14}$$

for $\iota \in \mathbb{T}$, where $\mathbb{T}$ is an $\omega$-periodic time scale satisfying $0 < \lambda < 1$.

Besides assumptions (A.1) and (A.2), suppose that there exist a function $J : \mathbb{R}_+ \to \mathbb{R}_+$ and a positive constant $N$ such that

$$|(I \otimes K)(\iota, \iota - \rho)| \leq J(\rho), \qquad \int_0^\infty J(\rho)\, \Delta\rho = N, \tag{15}$$

and

$$\sup_{\iota \in \mathbb{T}} \int_{-\infty}^{\iota} \int_{\iota}^{\infty} |(I \otimes K)(\rho, \zeta)|\, \Delta\rho \Delta\zeta < \infty. \tag{16}$$

From (15), it follows that

$$\sup_{\iota \in \mathbb{T}} \int_{\iota}^{\infty} |(I \otimes K)(\rho, \iota)|\, \Delta\rho \leq N.$$

The analysis of equation (13) is carried out separately for the cases $a > 0$, $a < 0$, and $a = 0$. Based on this classification, we obtain sufficient criteria for the existence of periodic solutions. To facilitate the subsequent arguments, we first define the Lyapunov functionals $V_1$ and $V_2$ as

$$V_1(\iota, z_\lambda(\iota)) = |z_\lambda(\iota)| + \lambda \int_{-\infty}^{\iota} \int_{\iota}^{\infty} |(I \otimes K)(\rho, \zeta)|\, |z_\lambda(\zeta)|\, \Delta\rho\Delta\zeta, \tag{17}$$

and

$$V_2(\iota, z_\lambda(\iota)) = |z_\lambda(\iota)| - \lambda \int_{-\infty}^{\iota} \int_{\iota}^{\infty} |(I \otimes K)(\rho, \zeta)|\, |z_\lambda(\zeta)|\, \Delta\rho\Delta\zeta. \tag{18}$$

Next, using the periodicity of $V_1$ and $V_2$ together with a Sobolev's inequality on periodic time scales, we derive a priori bounds for solutions of (14). Combining these bounds with Theorem 7 then yields the desired existence results.

Unlike the classical analysis of KP Volterra integro-dynamic system periodic time scale for the cases $a > 0$, $a < 0$, and $a = 0$. our approach requires additional care due to the structure of the $\Delta$-derivative. In particular, obtaining estimates for the derivatives of $V_1$ and $V_2$ is more delicate because differentiating $|z_\lambda(\iota)|$ on periodic time scale differs from the real-valued case.

In the standard setting, one easily computes

$$\frac{d}{d\iota}|z(\iota)| = \frac{z(\iota)}{|z(\iota)|}\, z'(\iota)$$

By exploiting the equality $z^2(\iota) = |z(\iota)|^2$ and applying the product rule, the following relation is obtained. It should be noted that, on $\omega$-periodic time scales, the product rule takes the form

$$(hr)^\Delta = h^\Delta r^\sigma + hr^\Delta,$$

which leads to

$$|z|^\Delta = \frac{z + z^\sigma}{|z| + |z^\sigma|}\, z^\Delta, \qquad z \neq 0. \tag{19}$$

Thus, the coefficient in front of $z^\Delta$ depends not only on the sign of $z(\iota)$ but also on $z^\sigma(\iota)$. Consequently, the simplified relation

$$|z|^\Delta = \frac{z}{|z|}\, z^\Delta$$

holds only when $z(\iota)z^\sigma(\iota) \geq 0$ and $z \neq 0$.

For a given $z$, we distinguish this case by splitting the periodic time scale $\mathbb{T}$ into two disjoint subsets:

$$\mathbb{T}_- := \{\zeta \in \mathbb{T} : z(\zeta)z^\sigma(\zeta) < 0\}, \qquad \mathbb{T}_+ := \{\zeta \in \mathbb{T} : z(\zeta)z^\sigma(\zeta) \geq 0\}.$$

*Lemma* 4. [12] Let $z \neq 0$ be $\Delta$-differentiable. Then

$$|z(\iota)|^\Delta = \begin{cases} \dfrac{z(\iota)}{|z(\iota)|} z^\Delta(\iota), & \iota \in \mathbb{T}_+, \\ -\dfrac{2}{\mu(\iota)}|z(\iota)| - \dfrac{z(\iota)}{|z(\iota)|} z^\Delta(\iota), & \iota \in \mathbb{T}_-. \end{cases} \tag{20}$$

*Remark* 3. [12] If $zz^{\sigma} \neq 0$, then for all $\iota \in \mathbb{T}$,

$$\frac{z}{|z|} z^{\Delta} \;\leq\; |z|^{\Delta} \;\leq\; \frac{z^{\sigma}}{|z^{\sigma}|} z^{\Delta}. \tag{21}$$

Next, we establish a Sobolev's inequality on periodic time scale. This inequality plays a key role in obtaining a priori bounds for all possible periodic solutions of (14).

*Corollary* 1. Let $z \in C_{rd}([0,\omega]_{\mathbb{T}}, \mathbb{R})$. Then

$$|z|_1 + \sigma(\omega)\,|z^{\Delta}|_1 \;\geq\; \omega\,|z|_0,$$

where

$$|z|_1 = \int_0^{\omega} |z(\zeta)|\,\Delta\zeta, \qquad |z^{\Delta}|_1 = \int_0^{\omega} |z^{\Delta}(\zeta)|\,\Delta\zeta, \qquad |z|_0 = \sup_{\zeta\in[0,\omega]_{\mathbb{T}}} |z(\zeta)|.$$

*Proof.* From Remark 3, we have

$$\big||z(\iota)|^{\Delta}\big| \leq |z^{\Delta}(\iota)|, \quad \forall \iota \in \mathbb{T}.$$

Applying the integration by parts formula on time scales, we obtain

$$\int_0^{\iota} |z(\zeta)|\,\Delta\zeta = \iota\,|z(\iota)| - \int_0^{\iota} \sigma(\zeta)\,|z(\zeta)|^{\Delta}\,\Delta\zeta \;\geq\; \iota\,|z(\iota)| - \sigma(\iota)\int_0^{\iota} |z^{\Delta}(\zeta)|\,\Delta\zeta.$$

Taking the supremum over $\iota \in [0,\omega]_{\mathbb{T}}$ gives

$$|z|_1 + \sigma(\omega)\,|z^{\Delta}|_1 \;\geq\; \omega\,|z|_0.$$

□

We begin by setting $\phi(z) = -h(z)$ and establishing a uniform a priori estimate for the family of solutions to (14) corresponding to $\lambda \in (0,1)$. With this estimate available, Theorem 7 is applied to conclude that system (13) possesses an $\omega$-periodic solution in the case $a < 0$.

For convenience, define

$$\psi(z) = -h(z), \qquad \phi(z) = -h(z), \qquad r(z) = z.$$

With these notations, we establish the following existence result for the case $a < 0$.

**Theorem 8.** *Assume $a < 0$ and $z^{\sigma}(\iota)\,\psi(z(\iota)) > 0$ for all $z \neq 0$ and $\iota \in \mathbb{T}$. Suppose there exist positive constants $\eta, \beta, \gamma$ such that*

$$|1 + \mu(\iota)(a - \eta)| \leq 1, \tag{22}$$

*and*

$$-|\phi(z(\iota))| + N|r(z(\iota))| \leq -\beta|r(z(\iota))| + \gamma \tag{23}$$

*for all $\iota \in \mathbb{T}$ and $z \in Y^{\top}$, where $N$ is defined in (15). Then, (13) admits a solution in $Y^{\top}$.*

*Proof.* Let $z_\lambda \in Y^\top$ be a solution of (14), and set $\alpha = -a$. Then (14) can be rewritten as

$$z_\lambda^\Delta(\iota) = az_\lambda(\iota) - \lambda\psi(z_\lambda(\iota)) + \lambda \int_{-\infty}^{\iota} (I \otimes K)(\iota, \zeta) z_\lambda(\zeta)\, \Delta\zeta + \lambda c(\iota). \tag{24}$$

If $\iota \in \mathbb{T}_-$, then $zz^\sigma < 0$, so

$$\frac{z_\lambda^\sigma(\iota)}{|z_\lambda^\sigma(\iota)|} = -\frac{z_\lambda(\iota)}{|z_\lambda(\iota)|},$$

and $z\psi(z) < 0$ because $z^\sigma\psi(z) > 0$. The condition $|1 + \mu(\iota)(a - \eta)| \leq 1$ ensures that $-\frac{2}{\mu(\iota)} - a \leq -\eta$. Using (20) and (24), we obtain

$$\begin{aligned} |z_\lambda(\iota)|^\Delta &= -\frac{2}{\mu(\iota)}|z_\lambda(\iota)| - \frac{z_\lambda(\iota)}{|z_\lambda(\iota)|} z_\lambda^\Delta(\iota) \\ &\leq -\eta|z_\lambda(\iota)| - \lambda|\psi(z_\lambda(\iota))| + \lambda \int_{-\infty}^{\iota} |(I \otimes K)(\iota, \zeta)||z_\lambda(\zeta)|\Delta\zeta + |c|_0. \end{aligned} \tag{25}$$

If $\iota \in \mathbb{T}_+$, then $z\psi(z) > 0$, and by (20) and (24),

$$\begin{aligned} |z_\lambda(\iota)|^\Delta &= \frac{z_\lambda(\iota)}{|z_\lambda(\iota)|} z_\lambda^\Delta(\iota) \\ &\leq a|z_\lambda(\iota)| - \lambda|\psi(z_\lambda(\iota))| + \lambda \int_{-\infty}^{\iota} |(I \otimes K)(\iota, \zeta)||z_\lambda(\zeta)|\Delta\zeta + |c|_0. \end{aligned} \tag{26}$$

Combining (25) and (26), we deduce that for all $\iota \in \mathbb{T}$,

$$|z_\lambda(\iota)|^\Delta \leq -\eta^*|z_\lambda(\iota)| - \lambda|\psi(z_\lambda(\iota))| + \lambda \int_{-\infty}^{\iota} |(I \otimes K)(\iota, \zeta)||z_\lambda(\zeta)|\Delta\zeta + |c|_0, \tag{27}$$

where

$$\eta^* = \min\{\eta, -a\} > 0. \tag{28}$$

Define the Lyapunov functional $V_1$ as in (17). Since $z_\lambda \in Y^\top$, it is $\omega$-periodic:

$$V_1(\iota + \omega, z_\lambda(\iota)) = V_1(\iota, z_\lambda(\iota)).$$

Differentiating the Lyapunov functional $V_1$ defined in (17), and applying inequality (27) together with the kernel estimate (15), we obtain

$$\begin{aligned} V_1^\Delta(\iota, z_\lambda(\iota)) &\leq -\eta^*|z_\lambda(\iota)| - \lambda|\psi(z_\lambda(\iota))| + \lambda N|r(z_\lambda(\iota))| + |c|_0 \\ &\leq -\eta^*|z_\lambda(\iota)| + L, \end{aligned} \tag{29}$$

where $L = \gamma + |c|_0$.

Integrating over $[0, \omega]_\mathbb{T}$ and using the periodicity of $V_1$, we get

$$0 = \int_0^\omega V_1^\Delta(\zeta, z_\lambda(\iota))\Delta\zeta \leq -\eta^* \int_0^\omega |z_\lambda(\zeta)|\Delta\zeta + \omega L,$$

which leads to the following a priori estimate:

$$|z_\lambda|_1 \leq R := \frac{\omega L}{\eta^*}.$$

It follows that the established a priori estimate remains valid for every $\lambda \in (0,1)$. Therefore, the hypotheses required in Theorem 7 are verified. By directly applying Theorem 7, we deduce that system (13) has at least one $\omega$-periodic solution belonging to $Y^{\top}$. □

**Theorem 9.** *Assume that $a > 0$ and*

$$z(\iota)\, h(z(\iota)) > 0, \qquad \forall\, \iota \in \mathbb{T},\ z(\iota) \neq 0.$$

*Assume that there exist positive constants $\beta$ and $\gamma$ satisfying*

$$|h(z)| - N|r(z)| \geq \beta|r(z)| - \gamma,$$

*where $N$ is defined in (15). Then, (13) admits a solution in $Y^{\top}$.*

*Proof.* The proof follows the same strategy as that of Theorem 8. For completeness, we present only the main differences. Set $\alpha = -a$ and rewrite (14) as in (24).

From (19) and (21), we have

$$|z_\lambda(\iota)|^{\Delta} \geq \frac{z_\lambda(\iota)}{|z_\lambda(\iota)|} z_\lambda^{\Delta}(\iota), \quad \forall \iota \in \mathbb{T}. \tag{30}$$

Using (18), (24), and (30), it follows that

$$\begin{aligned} V_2^{\Delta}(\iota, z_\lambda(\iota)) &\geq a|z_\lambda(\iota)| + \lambda|h(z_\lambda(\iota))| - \lambda N|r(z_\lambda(\iota))| - |c|_0 \\ &\geq a|z_\lambda(\iota)| - L. \end{aligned} \tag{31}$$

where $L = \gamma + |c|_0$.

Integrating over $[0,\omega]_{\mathbb{T}}$ and using the periodicity of $V_2$, we obtain an a priori bound

$$|z_\lambda|_1 \leq R := \frac{\omega L}{a}.$$

The obtained a priori bound is independent of $\lambda \in (0,1)$. Therefore, the hypotheses of Theorem 7 are satisfied. Hence, by Theorem 7, system (13) admits at least one $\omega$-periodic solution in $Y^{\top}$.

□

**Theorem 10.** *Suppose that $a = 0$ and all the assumptions of Theorem 9 are satisfied. Additionally, assume that there exists a constant $\theta > 0$ such that $|r(z)| \geq \theta r(|z|) \geq 0$, where $r(|z|)$ is concave downward and satisfies $r(u) \to \infty$ as $u \to \infty$. Then system (13) admits at least one $\omega$-periodic solution in $Y^{\top}$.*

*Proof.* Set $\alpha = -1$ and rewrite (14) as

$$z_\lambda^{\Delta}(\iota) = (1-\lambda) z_\lambda(\iota) + \lambda h(z_\lambda(\iota)) + \lambda \int_{-\infty}^{\iota} (I \otimes K)(\iota, \zeta) z_\lambda(\zeta)\, \Delta\zeta + \lambda c(\iota).$$

Differentiating the Lyapunov functional $V_2$ defined in (18) and arguing as in the proof of Theorem 9, we obtain

$$V_2^{\Delta}(\iota, z_\lambda(\iota)) \geq (1-\lambda)|z_\lambda(\iota)| + \lambda|h(z_\lambda(\iota))| - \lambda N|r(z_\lambda(\iota))| - |c|_0$$

$$\geq (1-\lambda)|z_\lambda(\iota)| + \lambda\beta|r(z_\lambda(\iota))| - L. \tag{32}$$

For $0 < \lambda \leq 1/2$, (32) gives

$$V_2^{\Delta}(\iota, z_\lambda(\iota)) \geq \frac{1}{2}|z_\lambda(\iota)| - L,$$

Accordingly, the norm $\|z\|_1$ is bounded above by some positive constant $R_1$, that is,

$$\|z\|_1 \leq R_1.$$

For $1/2 < \lambda < 1$, we have

$$V_2^{\Delta}(\iota, z_\lambda(\iota)) \geq \frac{1}{2}\beta|r(z_\lambda(\iota))| - L \geq \frac{1}{2}\beta\theta r(|z_\lambda(\iota)|) - L.$$

Since $r$ is nonnegative and concave on $[0, \infty)$, Jensen's inequality yields

$$\begin{aligned} 0 &= \int_0^\omega V_2^{\Delta}(\iota, z_\lambda(\iota))\,\Delta\iota \\ &\geq \frac{1}{2}\beta\theta\omega r\left(\frac{1}{\omega}\int_0^\omega |z_\lambda(\iota)|\,\Delta\iota\right) - L\omega. \end{aligned} \tag{33}$$

Since $r(u) \to \infty$ as $u \to \infty$, it follows from (32) that there exists a positive constant $R_2$ satisfying

$$\|z_\lambda\|_1 \leq R_2.$$

Define

$$R = \max\{R_1, R_2\}.$$

Then,

$$\|z_\lambda\|_1 \leq R, \qquad \forall\, \lambda \in (0,1).$$

The above estimate is uniform with respect to $\lambda \in (0,1)$. Consequently, all the assumptions required by Theorem 7 are fulfilled. An application of Theorem 7 therefore shows that system (13) possesses at least one $\omega$-periodic solution in $Y^\top$. □

**Theorem 11.** *Assume that $a > 0$. Furthermore, let $\beta$ be a positive constant with $\beta < a$ such that*

$$|h(z)| + N|r(z)| \leq \beta|z|, \tag{34}$$

*where $N$ is given by (15). Then system (13) possesses at least one $\omega$-periodic solution in $Y^\top$.*

*Proof.* The proof follows the same strategy as that of Theorem 8. Set $\alpha = -a$ and rewrite (14) as in (24).

Consider the Lyapunov functional $V_2$ defined in (18). Using (34), we obtain

$$\begin{aligned} V_2^{\Delta}(\iota, z_\lambda(\iota)) &\geq a|z_\lambda(\iota)| - \lambda|h(z_\lambda(\iota))| - \lambda N|r(z_\lambda(\iota))| - |c|_0 \\ &\geq (a-\beta)|z_\lambda(\iota)| - |c|_0. \end{aligned}$$

Integrating over one period $[0,\omega]_{\mathbb{T}}$ and using the periodicity of $V_2$, we obtain

$$\begin{aligned} 0 &= \int_0^\omega V_2^{\Delta}(\iota, z_\lambda(\iota))\,\Delta\iota \\ &\geq (a-\beta)\int_0^\omega |z_\lambda(\iota)|\,\Delta\iota - \omega|c|_0. \end{aligned}$$

Hence,

$$|z_\lambda|_1 \leq \frac{\omega|c|_0}{a-\beta}.$$

Furthermore, by (34),

$$\begin{aligned} \lambda\int_0^\omega |h(z_\lambda(\iota))|\,\Delta\iota &\leq \beta|z_\lambda|_1, \\ \lambda\int_0^\omega N|r(z_\lambda(\iota))|\,\Delta\iota &\leq \beta|z_\lambda|_1, \end{aligned}$$

which shows that these quantities are uniformly bounded.

Therefore, the obtained a priori bound is independent of $\lambda \in (0,1)$. Hence, the hypotheses of Theorem 7 are satisfied. Consequently, by Theorem 7, system (13) admits at least one $\omega$-periodic solution in $Y^\top$. □

**Theorem 12.** *Let $a<0$. Assume that there exist positive constants $\beta$ and $\eta$ such that*

$$0<\beta<-a, \qquad \eta>\beta,$$

*and*

$$|h(z)|+N|r(z)|\leq\beta|z|, \tag{35}$$

*together with*

$$|1+\mu(\iota)(a-\eta)|\leq 1, \tag{36}$$

*for every $\iota\in\mathbb{T}$, where the constant $N$ is defined in (15). Under these assumptions, system (13) possesses at least one $\omega$-periodic solution in $Y^\top$.*

*Proof.* The proof follows the same strategy as that of Theorem 8. Set $\alpha=-a$ and rewrite (14) as in (24).

From the estimates obtained in (25) and (26), we have

$$|z_\lambda(\iota)|^{\Delta}\leq S(\iota)|z_\lambda(\iota)|+\lambda|h(z_\lambda(\iota))|+\lambda\int_{-\infty}^{\iota}|(I\otimes K)(\iota,\zeta)||r(z_\lambda(\zeta))|\,\Delta\zeta+|c|_0,$$

where

$$S(\iota)=\begin{cases} -\eta, & \iota\in\mathbb{T}_-, \\ a, & \iota\in\mathbb{T}_+. \end{cases}$$

Consider the Lyapunov functional $V_1$ defined in (17). Using assumption (35), we obtain

$$V_1^{\Delta}(\iota, z_\lambda(\iota)) \leq \big(S(\iota) + \beta\big)|z_\lambda(\iota)| + |c|_0.$$

Define

$$S^* = \max_{\iota \in \mathbb{T}} \{S(\iota) + \beta\}.$$

Since

$$S(\iota) \leq a < 0, \qquad \beta < -a,$$

it follows that

$$S^* < 0.$$

Integrating over one period $[0, \omega]_{\mathbb{T}}$ and using the periodicity of $V_1$, we obtain

$$\begin{aligned} 0 &= \int_0^\omega V_1^{\Delta}(\iota, z_\lambda(\iota))\, \Delta\iota \\ &\leq S^* \int_0^\omega |z_\lambda(\iota)|\, \Delta\iota + \omega|c|_0. \end{aligned}$$

Hence,

$$|z_\lambda|_1 \leq \frac{\omega|c|_0}{-S^*}.$$

Furthermore, by (35),

$$\begin{aligned} \lambda \int_0^\omega |h(z_\lambda(\iota))|\, \Delta\iota &\leq \beta|z_\lambda|_1, \\ \lambda \int_0^\omega N|r(z_\lambda(\iota))|\, \Delta\iota &\leq \beta|z_\lambda|_1, \end{aligned}$$

which shows that these quantities are uniformly bounded.

Therefore, the obtained a priori bound is independent of $\lambda \in (0, 1)$. Hence, the hypotheses of Theorem 7 are satisfied. Consequently, by Theorem 7, system (13) admits at least one $\omega$-periodic solution in $Y^\top$. □

**Theorem 13.** *Suppose that $a$ is negative and that every point of the periodic time scale $\mathbb{T}$ is right-scattered. Let $\beta, \eta > 0$ satisfy*

$$\eta > \beta,$$

*and assume that*

$$|h(z)| + N|r(z)| \leq \beta|z|, \tag{37}$$

*as well as*

$$|1 + \mu(\iota)a| \geq 1 + \eta\mu(\iota), \tag{38}$$

*for all $\iota \in \mathbb{T}$, where $N$ denotes the constant defined in (15). Under these assumptions, system (13) has an $\omega$-periodic solution in $Y^\top$.*

*Proof.* The proof follows the same strategy as that of Theorem 8. Set $\alpha = -a$ and rewrite (14) as in (24).

Since every point of $\mathbb{T}$ is right-scattered, Theorem 1(i) yields

$$|z_\lambda(\iota)|^\Delta = \frac{|z_\lambda^\sigma(\iota)| - |z_\lambda(\iota)|}{\mu(\iota)}.$$

Using condition (38), we obtain

$$\begin{aligned} |z_\lambda(\iota)|^\Delta \geq \eta|z_\lambda(\iota)| - \lambda|h(z_\lambda(\iota))| \\ - \lambda \int_{-\infty}^{\iota} |(I \otimes K)(\iota, \zeta)||r(z_\lambda(\zeta))|\, \Delta\zeta - |c|_0. \end{aligned}$$

Consider the Lyapunov functional $V_2$ defined in (18). By (37),

$$V_2^\Delta(\iota, z_\lambda(\iota)) \geq (\eta - \beta)|z_\lambda(\iota)| - |c|_0.$$

Integrating over one period $[0, \omega]_\mathbb{T}$ and using the periodicity of $V_2$, we obtain

$$\begin{aligned} 0 &= \int_0^\omega V_2^\Delta(\iota, z_\lambda(\iota))\, \Delta\iota \\ &\geq (\eta - \beta) \int_0^\omega |z_\lambda(\iota)|\, \Delta\iota - \omega|c|_0. \end{aligned}$$

Hence,

$$|z_\lambda|_1 \leq \frac{\omega|c|_0}{\eta - \beta}.$$

Furthermore, by (37),

$$\begin{aligned} \lambda \int_0^\omega |h(z_\lambda(\iota))|\, \Delta\iota \leq \beta|z_\lambda|_1, \\ \lambda \int_0^\omega N|r(z_\lambda(\iota))|\, \Delta\iota \leq \beta|z_\lambda|_1, \end{aligned}$$

which are uniformly bounded.

Therefore, the obtained a priori bound is independent of $\lambda \in (0, 1)$. Hence, the hypotheses of Theorem 7 are satisfied. Consequently, by Theorem 7, system (13) admits at least one $\omega$-periodic solution in $Y^\top$. □

## 5. Illustrative Example

In this section, we present an example to verify the applicability of Theorem 8 for the Volterra integro-dynamic system (2).

*Example* 1. Consider the nonlinear Volterra integro-dynamic system on the periodic time scale

$$\mathbb{T} = \mathbb{R},$$

given by

$$z^{\Delta}(t) = Dz(t) + \int_{-\infty}^{t} (I \otimes K)(t,s)z(s)\,ds + h(z(t)) + c(t), \tag{39}$$

where $z(t) \in \mathbb{R}^4$ and

$$D = (B^T \otimes I) + (I \otimes A).$$

Choose

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$D = (B^T \otimes I) + (I \otimes A) = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

The kernel is chosen as

$$K(t,s) = e^{-(t-s)}I,$$

so that

$$(I \otimes K)(t,s) = e^{-(t-s)}I_4.$$

Let

$$h(z) = 0.2z, \qquad c(t) = e^{-t}\mathbf{1},$$

where

$$\mathbf{1} = (1,1,1,1)^T.$$

Hence, system (39) becomes

$$z^{\Delta}(t) = Dz(t) + \int_{-\infty}^{t} e^{-(t-s)}z(s)\,ds + 0.2z(t) + e^{-t}\mathbf{1}. \tag{40}$$

**Verification of Theorem 8.**

(i) The eigenvalues of the matrix $D$ are

$$-2,\ -3,\ -3,\ -4.$$

Hence we may choose

$$a = -2 < 0,$$

which satisfies the first assumption of Theorem 8.

(ii) Since

$$\mathbb{T} = \mathbb{R},$$

the graininess function satisfies

$$\mu(t) = 0,$$

and therefore

$$|1 + \mu(t)(a - \eta)| = 1 \leq 1,$$

so condition (22) holds automatically.

(iii) The kernel satisfies

$$|(I \otimes K)(t, s)| = e^{-(t-s)},$$

which is continuous, positive, and exponentially decaying. Consequently, the kernel assumptions (15) and (16) are satisfied.

(iv) The nonlinear function $h$ is globally Lipschitz continuous with Lipschitz constant $0.2$. Moreover, the forcing term $c(t) = e^{-t}\mathbf{1}$ is continuous and bounded on $\mathbb{R}$.

Therefore, the hypotheses required in Theorem 8 are fulfilled.

Hence, by Theorem 8, system (40) admits at least one $\omega$-periodic solution in $Y^{\top}$.

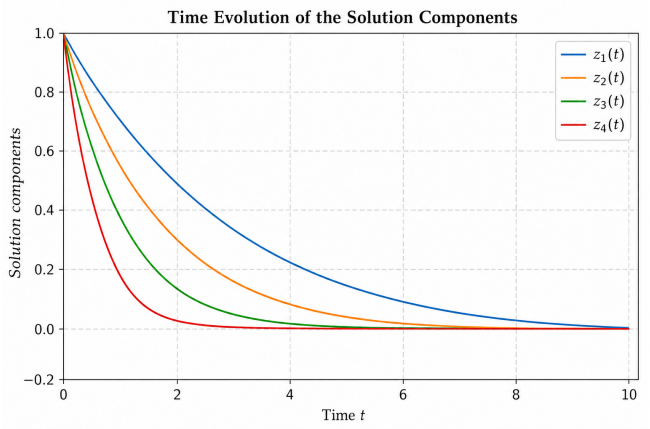


FIGURE 1. The trajectories of the four solution components remain bounded and exhibit exponential decay over the simulation interval.

Figures 1 and 2 present the numerical simulation of system (40). Figure 1 shows that all four solution components remain bounded and decay exponentially throughout the simulation interval. Figure 2 illustrates that the Euclidean norm of the solution decreases monotonically toward zero, confirming the boundedness of the solution. These numerical observations are consistent with the theoretical existence result established in Theorem 8 and demonstrate the effectiveness of the proposed approach.

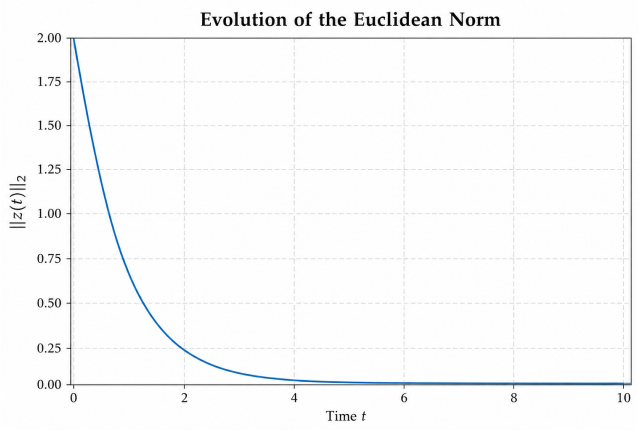


FIGURE 2. The Euclidean norm of the solution decreases monotonically and remains bounded, illustrating the behavior predicted by Theorem 8.

*Example* 2. Consider the nonlinear Sylvester Volterra integro-dynamic system

$$z^{\Delta}(\iota) = Dz(\iota) + \int_{-\infty}^{\iota} (I \otimes K)(\iota, \zeta) z(\zeta)\, \Delta\zeta + h(z(\iota)) + c(\iota), \tag{41}$$

defined on the periodic time scale

$$\mathbb{T} = \mathbb{P}_{1,1} = \bigcup_{k \in \mathbb{Z}} [2k, 2k+1].$$

Choose

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which yields

$$D = (B^T \otimes I) + (I \otimes A) = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

Let

$$K(\iota, \zeta) = e^{-(\iota - \zeta)} I,$$

and define

$$h(z) = 0.2 \tanh(z),$$

together with the forcing function

$$c(\iota) = 0.1 \begin{bmatrix} \sin(\pi\iota) \\ \cos(\pi\iota) \\ \sin(\pi\iota) \\ \cos(\pi\iota) \end{bmatrix}.$$

Hence, system (41) becomes

$$z^{\Delta}(\iota) = Dz(\iota) + \int_{-\infty}^{\iota} e^{-(\iota-\zeta)} z(\zeta)\,\Delta\zeta + 0.2\tanh(z(\iota)) + 0.1 \begin{bmatrix} \sin(\pi\iota) \\ \cos(\pi\iota) \\ \sin(\pi\iota) \\ \cos(\pi\iota) \end{bmatrix}. \tag{42}$$

**Verification of Theorem 8.**

(i) The time scale

$$\mathbb{P}_{1,1}$$

is periodic with period

$$\omega = 2.$$

(ii) The eigenvalues of the matrix

$$D$$

are

$$-2,\ -3,\ -3,\ -4,$$

and therefore we may choose

$$a = -2 < 0.$$

(iii) The kernel

$$K(\iota,\zeta) = e^{-(\iota-\zeta)} I$$

is continuous and exponentially decaying. Hence, the corresponding Volterra integral operator is well defined.

(iv) The nonlinear function

$$h(z) = 0.2\tanh(z)$$

is continuously differentiable and globally Lipschitz continuous since

$$|h'(z)| = 0.2\ \mathrm{sech}^2(z) \le 0.2,$$

for all $z \in \mathbb{R}^4$.

(v) The forcing function satisfies

$$c(\iota+2) = c(\iota),$$

and is therefore $\omega$-periodic on $\mathbb{P}_{1,1}$.

The preceding arguments verify all the hypotheses of Theorem 8. Consequently, Theorem 8 can be applied to conclude that system (42) possesses an $\omega$-periodic solution in $Y^{\top}$.

## 6. Conclusion

In this paper, we investigated the existence of periodic solutions for a class of nonlinear Sylvester Volterra integro-dynamic systems on periodic time scales. By employing the vectorization operator and Kronecker product techniques, the original matrix-valued system was transformed into an equivalent vector integro-dynamic system, which enabled the application of Schaefer's fixed point theorem together with the Arzelà–Ascoli compactness theorem to establish the existence of periodic solutions. Furthermore, suitable Lyapunov-type functionals, Sobolev-type inequalities on periodic time scales, and new inequalities involving the $\Delta$-derivatives of absolute value functions were developed to derive uniform a priori bounds. These analytical tools allowed us to establish sufficient conditions for the existence of periodic solutions under different parameter regimes, including the cases $a > 0$, $a < 0$, and $a = 0$. The obtained results extend several existing existence results for scalar and vector Volterra integro-dynamic systems to the more general setting of nonlinear Sylvester matrix systems on periodic time scales. Finally, a numerical example together with graphical simulations was presented to illustrate the applicability and effectiveness of the proposed theoretical results. Future research may focus on the qualitative analysis of nonlinear Sylvester Volterra integro-dynamic systems on time scales, including positivity, stability, global attractivity, controllability, observability, impulsive and delayed systems, stochastic perturbations, fractional-order models, and optimal control problems. These directions may further broaden the applicability of the proposed framework to engineering, biological, and networked dynamical systems with memory effects.

**Authors' Contributions.** All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

**Conflicts of Interest.** The authors declare that there are no conflicts of interest regarding the publication of this paper.

## References

[1] C. Huang, B. Liu, J. Li, Global Dynamics of a Patch-Constructed Tick Metapopulation Model with Time-Varying Delays, Discret. Contin. Dyn. Syst. Ser. B 38 (2026), 351–378. https://doi.org/10.3934/dcdsb.2026075.

[2] B. Liu, C. Huang, Global Periodic Attractivity for a Generalized Nicholson's Blowflies Equation with Delays, Appl. Math. Lett. 177 (2026), 109876. https://doi.org/10.1016/j.aml.2026.109876.

[3] C. Huang, B. Liu, Periodic Dynamics in a Delayed Diffusive Nicholson's Blowflies Equation, J. Appl. Math. Comput. 72 (2026), 149. https://doi.org/10.1007/s12190-026-02797-w.

[4] C. Harisha, B.V. Appa Rao, A. Sreenivasulu, Periodic Matrix Sylvester Volterra Integro-Dynamic System on Time Scales, J. Control Decis. (2025), 1–10. https://doi.org/10.1080/23307706.2025.2537183.

[5] I. Karafyllis, M. Krstic, On the Existence of Periodic Solutions with Applications to Extremum-Seeking, arXiv:2602.13615, 2026. https://doi.org/10.48550/arXiv.2602.13615.

[6] M.B. Mesmouli, L.F. Iambor, T.S. Hassan, Periodic Solutions to Matrix Delay Difference Systems Under Exponential Dichotomy Conditions, Mathematics 14 (2026), 1101. https://doi.org/10.3390/math14071101.

[7] A. Reinfelds, S. Christian, Exponential Stability of Volterra Integro Dynamic Equations on Time Scales, Mathematics 13 (2025), 3918. https://doi.org/10.3390/math13243918.

[8] L. Sadek, Control Theory for Fractional Differential Sylvester Matrix Equations with Caputo Fractional Derivative, J. Vib. Control 31 (2025), 1586–1602. https://doi.org/10.1177/10775463241246430.

[9] V. Kumar, R. Guglielmi, Exponential Synchronization of Neutral-Type Neural Networks with Leakage and Mixed Delays on Time Scales, J. Franklin Inst. 363 (2026), 108400. https://doi.org/10.1016/j.jfranklin.2025.108400.

[10] S.R. Christian, Volterra Integral Equations on Time Scales, PhD Thesis, University of Latvia, 2020. https://dspace.lu.lv/handle/7/53363.

[11] H. Chintamaneni, V.A.R. Bhogapurapu, S. Ayyalappagari, Periodic Solution for Almost Linear Volterra Integro-dynamic Matrix Sylvester System on Measure Chains, Discontin. Nonlinearity Complex. 14 (2025), 259–267. https://doi.org/10.5890/dnc.2025.06.001.

[12] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser Boston, 2001. https://doi.org/10.1007/978-1-4612-0201-1.

[13] M. Bohner, A. Peterson (Eds.), Advances in Dynamic Equations on Time Scales, Birkhäuser Boston, 2003. https://doi.org/10.1007/978-0-8176-8230-9.

[14] T.A. Burton, P.W. Eloe, M.N. Islam, Nonlinear Integrodifferential Equations and a Priori Bounds on Periodic Solutions, Ann. Mat. Pura Appl. 161 (1992), 271–283. https://doi.org/10.1007/bf01759641.

[15] B.V.A. Rao, K.A.S.N.V. Prasad, Existence of $\Psi$-Bounded Solutions for Sylvester Matrix Dynamical Systems on Time Scales, Filomat 32 (2018), 4209–4219. https://doi.org/10.2298/fil1812209r.

[16] W. Li, Periodic Solution of a General State-Dependent Predator-Prey Model with Harvesting, J. Math. Anal. Appl. 554 (2026), 130006. https://doi.org/10.1016/j.jmaa.2025.130006.

[17] Z. Cheng, C. Kong, Q. Yuan, Nonlinear Dynamics of Periodic Solutions in Electrostatic MEMS: Theory and Numerical Bifurcation Analysis, Phys. D 482 (2025), 134887. https://doi.org/10.1016/j.physd.2025.134887.

[18] M. El Hajji, Periodic Solutions for an "SVIQR" Epidemic Model in a Seasonal Environment with General Incidence Rate, Int. J. Biomath. 18 (2025), 2450033. https://doi.org/10.1142/s1793524524500335.

[19] C. Harisha, B.V. Appa Rao, A. Sreenivasulu, Periodic and Pseudo Periodic Solutions for Matrix Sylvester Dynamic System on Measure Chains, Mo. J. Math. Sci. 37 (2025), 31–46. https://doi.org/10.35834/2025/3701031.

[20] B. Bordj, A. Ardjouni, Positive Periodic Solutions of Nonautonomous Lotka–Volterra Dynamic Systems with a General Attack Rate on Time Scales, Rocky Mt. J. Math. 53 (2023), 1371–1385. https://doi.org/10.1216/rmj.2023.53.1371.

[21] B. Bordj, A. Ardjouni, Periodic and Asymptotically Periodic Solutions in Nonlinear Coupled Volterra Integro-Dynamic Systems with Infinite Delay on Time Scales, Adv. Theory Nonlinear Anal. Appl. 5 (2021), 180–192. https://doi.org/10.31197/atnaa.691860.

[22] A. Younus, G. ur Rahman, Controllability, Observability, and Stability of a Volterra Integro-Dynamic System on Time Scales, J. Dyn. Control Syst. 20 (2014), 383–402. https://doi.org/10.1007/s10883-014-9226-7.

[23] R.P. Agarwal, A.S. Awan, D. O'Regan, A. Younus, Linear Impulsive Volterra Integro-Dynamic System on Time Scales, Adv. Differ. Equ. 2014 (2014), 6. https://doi.org/10.1186/1687-1847-2014-6.

[24] S.G. Georgiev, Volterra Integro-Differential Equations on Time Scales, Int. J. Appl. Comput. Math. 3 (2017), 1577–1587. https://doi.org/10.1007/s40819-016-0207-2.

[25] A. Reinfelds, S. Christian, Nonlinear Volterra Integrodifferential Equations from Above on Unbounded Time Scales, Mathematics 11 (2023), 1760. https://doi.org/10.3390/math11071760.

[26] U. Ostaszewska, E. Schmeidel, M. Zdanowicz, Exponential Stability of Integro-Differential Volterra Equation on Time Scales, Tatra Mt. Math. Publ. 84 (2023), 77–86. https://doi.org/10.2478/tmmp-2023-0017.

[27] A. Guerfi, A. Ardjouni, Periodic Solutions of Almost Linear Volterra Integro-Dynamic Systems, Malaya J. Mat. 8 (2020), 1427–1433. https://doi.org/10.26637/mjm0804/0015.

[28] S. Ayyalappagari, V.A.R. Bhogapurapu, Stability Criteria for Nonlinear Volterra Integro-Dynamic Matrix Sylvester Systems on Measure Chains, Adv. Differ. Equ. 2021 (2021), 514. https://doi.org/10.1186/s13662-021-03665-6.

[29] A. Sreenivasulu, B.V.A. Rao, Stability and Controllability for Volterra Integro-Dynamical Matrix Sylvester Impulsive System on Time Scales, J. Appl. Math. Comput. 68 (2022), 3705–3720. https://doi.org/10.1007/s12190-021-01688-6.

[30] I.L.D. Dos Santos, S. Tikare, On Reachability and Controllability for a Volterra Integro-Dynamic System on Time Scales, Stud. Univ. Babeş-Bolyai Math. 70 (2025), 115–126. https://doi.org/10.24193/subbmath.2025.1.08.

[31] M.I. Noori, A.H. Mahmood, On a Nonlinear Volterra-Fredholm Integrodifferential Equation on Time Scales, OALib 7 (2020), 1–10. https://doi.org/10.4236/oalib.1106103.

[32] M. Adıvar, Principal Matrix Solutions and Variation of Parameters for Volterra Integro-Dynamic Equations on Time Scales, Glasg. Math. J. 53 (2011), 463–480. https://doi.org/10.1017/s0017089511000073.

[33] H. Schaefer, Über die Methode der a priori-Schranken, Math. Ann. 129 (1955), 415–416. https://doi.org/10.1007/bf01362380.

[34] M. Adıvar, Y.N. Raffoul, Existence Results for Periodic Solutions of Integro-Dynamic Equations on Time Scales, Ann. Mat. Pura Appl. 188 (2009), 543–559. https://doi.org/10.1007/s10231-008-0088-z.

[35] Y.N. Raffoul, Periodic Solutions of Almost Linear Volterra Integro-Dynamic Equations on Periodic Time Scales, Can. Math. Bull. 58 (2015), 174–181. https://doi.org/10.4153/cmb-2014-046-4.

[36] M.N. Islam, Three Fixed Point Theorems: Periodic Solutions of a Volterra Type Integral Equation with Infinite Heredity, Can. Math. Bull. 56 (2013), 80–91. https://doi.org/10.4153/cmb-2011-123-5.

[37] K.S. Chiu, Periodic Solutions for Nonlinear Integro-Differential Systems with Piecewise Constant Argument, Sci. World J. 2014 (2014), 1–14. https://doi.org/10.1155/2014/514854.

[38] Y. Xing, W. Ding, M. Han, Periodic Boundary Value Problems of Integro-Differential Equation of Volterra Type on Time Scales, Nonlinear Anal. Theory Methods Appl. 68 (2008), 127–138. https://doi.org/10.1016/j.na.2006.10.036.

[39] M.N. Islam, Asymptotically Periodic Solutions of Volterra Integral Equations, Electron. J. Differ. Equ. 2016 (2016), 83. https://ejde.math.txstate.edu/Volumes/2016/83/abstr.html.

[40] Y. Li, H. Zhang, Extremal Solutions of Periodic Boundary Value Problems for First-Order Impulsive Integrodifferential Equations of Mixed-Type on Time Scales, Bound. Value Probl. 2007 (2007), 1–16. https://doi.org/10.1155/2007/73176.

[41] V. Kumar, M. Malik, Existence, Stability and Controllability Results of Fractional Dynamic System on Time Scales with Application to Population Dynamics, Int. J. Nonlinear Sci. Numer. Simul. 22 (2021), 741–766. https://doi.org/10.1515/ijnsns-2019-0199.

[42] A. Zada, B. Pervaiz, S.O. Shah, Existence, Uniqueness and Stability of Semilinear Nonautonomous Impulsive Systems on Time Scales, Int. J. Comput. Math. 100 (2023), 304–320. https://doi.org/10.1080/00207160.2022.2118523.

[43] X. Wu, T. Sun, Lyapunov-Type Inequality for Higher Order Nonlinear Time Scale Systems with Anti-Periodic Boundary Conditions, J. Appl. Math. Comput. 70 (2024), 6393–6410. https://doi.org/10.1007/s12190-024-02221-1.