

CHARACTERIZATIONS OF LENGTH AND MEAN-FUZZY IDEALS IN HILBERT ALGEBRAS OVER HYPERSTRUCTURES

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ABSTRACT. This paper introduces and studies the notions of the length and the mean of a hyperstructure in Hilbert algebras. By employing these notions, we define the concepts of length fuzzy ideals and mean fuzzy ideals of Hilbert algebras and investigate their fundamental properties. Several characterization theorems are established, describing the necessary and sufficient conditions for a hyperstructure to form a length (resp., mean) k -fuzzy ideal for $k \in \{1, 2, 3, 4\}$. Relationships between length (resp., mean) fuzzy ideals and hyperfuzzy ideals are examined in detail. Furthermore, we provide results connecting these classes of fuzzy ideals with their upper-, lower-, and equal-level subsets, and demonstrate that particular combinations of constant and fuzzy structures naturally lead to length- or mean-type fuzzy ideals. Illustrative examples and counterexamples are presented to validate the main results.

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1. INTRODUCTION

The theory of hyperstructures (also known as multialgebras) was first introduced by Marty [14] in 1934 at the 8th Congress of Scandinavian Mathematicians. Since then, hyperstructures have found extensive applications across several branches of pure and applied mathematics, providing a rich framework for generalizing classical algebraic structures. Comprehensive accounts of hyperstructure theory and its applications in Mathematics and Computer Science can be found in [5,6]. Moreover, Corsini [4] presented a detailed historical overview of hyperstructure theory and proposed several new research directions, highlighting its potential connections with fuzzy systems, algebraic logic, and computational structures.

The concept of a Hilbert algebra was introduced by Henkin [11] in the early 1950s as an algebraic framework for investigating implications in intuitionistic and other non-classical logics. Subsequently, Diego [7] developed the algebraic properties of Hilbert algebras and proved that they form a locally finite variety. Further studies by Busneag [1,2] and Jun [12] explored their deductive systems and structural properties, while Dudek [8–10] extended these ideas by introducing the fuzzification of subalgebras, ideals, and deductive systems within Hilbert algebras.

In recent years, the interplay between hyperstructures and fuzzy algebraic systems has gained significant attention, as it allows researchers to model uncertainty and generalization simultaneously within algebraic frameworks. Despite the growing literature on hyperfuzzy sets and hyperfuzzy ideals, the notions that capture the quantitative characteristics of such fuzzy structures—particularly the length and the mean of a hyperstructure—remain unexplored in the context of Hilbert algebras.

Recently, several researchers have extended the study of fuzzy structures by incorporating the notions of length and mean into various algebraic systems. Tacha, Phayapsiang, and Iampan [16] introduced and investigated length and mean fuzzy UP-subalgebras of UP-algebras, providing foundational results that connect algebraic operations with quantitative fuzzy measures. More recently, Rajesh, Oner, Iampan, and Senturk [15] examined the concepts of length and mean fuzzy ideals within the framework of Sheffer stroke Hilbert algebras, further generalizing the classical fuzzy ideal theory. These developments motivate the exploration of similar ideas in broader algebraic contexts. In particular, the present work extends these notions to hyperstructures in Hilbert algebras, establishing new characterizations and relationships with hyperfuzzy ideals.

Motivated by this observation, this paper introduces and studies the notions of the length and the mean of a hyperstructure in Hilbert algebras. Based on these notions, we define and investigate the new concepts of length fuzzy ideals and mean fuzzy ideals. Several characterization theorems are established, providing necessary and sufficient conditions for a hyperstructure to form a length (resp., mean) k -fuzzy ideal for $k \in \{1, 2, 3, 4\}$. Furthermore, we explore the relationships between these new classes of fuzzy ideals and existing hyperfuzzy ideals, as well as their connections with upper-, lower-, and equal-level subsets. The results not only extend the classical theory of fuzzy ideals but also open new perspectives for future studies in the algebraic structures of logic and hyperfuzzification.

2. PRELIMINARIES

In this section, we recall some basic definitions and fundamental results that will be used throughout the paper. For completeness, we briefly review the essential notions related to Hilbert algebras, fuzzy ideals, and hyperstructures. The classical properties of Hilbert algebras and their deductive systems provide the algebraic framework for our study, while the concept of a hyperfuzzy set serves as the

foundation for defining fuzzy and hyperfuzzy ideals. These preliminaries are necessary to establish the concepts of length fuzzy ideals and mean fuzzy ideals in the subsequent sections.

Definition 2.1. [7] A Hilbert algebra is a triplet with the formula $X = (X, *, 1)$, where X is a non-empty set, $*$ is a binary operation, and 1 is a fixed member of X that is true according to the axioms stated below:

$$(\forall x, y \in X)(x * (y * x) = 1) \quad (2.1)$$

$$(\forall x, y, z \in X)((x * (y * z)) * ((x * y) * (x * z)) = 1) \quad (2.2)$$

$$(\forall x, y \in X)(x * y = 1, y * x = 1 \Rightarrow x = y) \quad (2.3)$$

In [8], the following conclusion was established.

Lemma 2.1. Let $X = (X, *, 1)$ be a Hilbert algebra. Then

- (1) $(\forall x \in X)(x * x = 1)$
- (2) $(\forall x \in X)(1 * x = x)$
- (3) $(\forall x \in X)(x * 1 = 1)$
- (4) $(\forall x, y, z \in X)(x * (y * z) = y * (x * z))$
- (5) $(\forall x, y, z \in X)((x * z) * ((z * y) * (x * y)) = 1)$.

In a Hilbert algebra $X = (X, *, 1)$, the binary relation \leq is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 2.2. [13] A non-empty subset S of a Hilbert algebra $X = (X, *, 1)$ is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$.

Definition 2.3. [3] A non-empty subset S of a Hilbert algebra $X = (X, *, 1)$ is called an ideal of X if the following conditions hold:

- (1) $1 \in S$
- (2) $(\forall x, y \in X)(y \in S \Rightarrow x * y \in S)$
- (3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in S \Rightarrow (y_1 * (y_2 * x)) * x \in S)$.

A fuzzy set [17] in a nonempty set X is defined to be a function $\mu : X \rightarrow [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 2.4. Let X be a non-empty set. A mapping $\tilde{f} : X \rightarrow \tilde{P}([0, 1])$ is called a hyperfuzzy set over X , where $\tilde{P}([0, 1])$ is the family of all non-empty subsets of $[0, 1]$. An ordered pair (X, \tilde{f}) is called a hyperstructure over X .

Definition 2.5. Given a hyperstructure (X, \tilde{f}) over a non-empty set X , we define two fuzzy structures (X, \tilde{f}_{\inf}) and (X, \tilde{f}_{\sup}) in X as follows:

$$\tilde{f}_{\inf} : X \rightarrow [0, 1]; x \mapsto \inf\{\tilde{f}(x)\}$$

$$\tilde{f}_{\sup} : X \rightarrow [0, 1]; x \mapsto \sup\{\tilde{f}(x)\}$$

3. ON k -FUZZY IDEALS

In this section, we recall the concept of fuzzy ideals in Hilbert algebras and generalize them into the notion of k -fuzzy ideals, where $k \in \{1, 2, 3, 4\}$. The classification of fuzzy ideals into four types provides a flexible framework for analyzing various levels of algebraic membership and logical strength within a fuzzy structure. These generalized ideals capture the behavior of fuzzy subsets under the algebraic operation of a Hilbert algebra and play a crucial role in formulating the notions of length fuzzy ideals and mean fuzzy ideals developed in the subsequent sections. Moreover, the results presented here serve as fundamental tools for establishing several characterization theorems concerning hyperfuzzy and k -fuzzy structures in Hilbert algebras.

Definition 3.1. A fuzzy structure (X, f) in a Hilbert algebra $X = (X, *, 1)$ is said to be a

(1) fuzzy ideal of X with type 1 (briefly, 1-fuzzy ideal of X) if

$$(\forall x \in X)(f(1) \geq f(x)) \tag{3.1}$$

$$(\forall x, y \in X)(f(x * y) \geq f(y)) \tag{3.2}$$

$$(\forall x, y_1, y_2 \in X)(f((y_1 * (y_2 * x)) * x) \geq \min\{f(y_1), f(y_2)\}) \tag{3.3}$$

(2) fuzzy ideal of X with type 2 (briefly, 2-fuzzy ideal of X) if

$$(\forall x \in X)(f(1) \leq f(x)) \tag{3.4}$$

$$(\forall x, y \in X)(f(x * y) \leq f(y)) \tag{3.5}$$

$$(\forall x, y_1, y_2 \in X)(f((y_1 * (y_2 * x)) * x) \leq \min\{f(y_1), f(y_2)\}) \tag{3.6}$$

(3) fuzzy ideal of X with type 3 (briefly, 3-fuzzy ideal of X) if

$$(\forall x \in X)(f(1) \geq f(x)) \tag{3.7}$$

$$(\forall x, y \in X)(f(x * y) \geq f(y)) \tag{3.8}$$

$$(\forall x, y_1, y_2 \in X)(f((y_1 * (y_2 * x)) * x) \geq \max\{f(y_1), f(y_2)\}) \tag{3.9}$$

(4) fuzzy ideal of X with type 4 (briefly, 4-fuzzy ideal of X) if

$$(\forall x \in X)(f(1) \leq f(x)) \quad (3.10)$$

$$(\forall x, y \in X)(f(x * y) \leq f(y)) \quad (3.11)$$

$$(\forall x, y_1, y_2 \in X)(f((y_1 * (y_2 * x)) * x) \leq \max\{f(y_1), f(y_2)\}). \quad (3.12)$$

Theorem 3.1. For any fuzzy structure (X, f) over a Hilbert algebra $X = (X, *, 1)$, the following assertions are valid:

(1) Every 3-fuzzy ideal of X is a 1-fuzzy ideal of X .

(2) Every 2-fuzzy ideal of X is a 4-fuzzy ideal of X .

Proof. (1) Let (X, f) be a 3-fuzzy ideal of X and $x, y_1, y_2 \in X$. Then

$$\begin{aligned} f((y_1 * (y_2 * x)) * x) &\geq \max\{f(y_1), f(y_2)\} \\ &\geq \min\{f(y_1), f(y_2)\}. \end{aligned}$$

Hence, (X, f) is a 1-fuzzy ideal of X .

(2) Let (X, f) be a 2-fuzzy ideal of X and $x, y_1, y_2 \in X$. Then

$$\begin{aligned} f((y_1 * (y_2 * x)) * x) &\leq \min\{f(y_1), f(y_2)\} \\ &\leq \max\{f(y_1), f(y_2)\}. \end{aligned}$$

Hence, (X, f) is a 4-fuzzy ideal of X . □

The following example shows that the converse of Theorem 3.1 is not true.

Example 3.1. Let $X = \{1, x, y, z, w, 0\}$ with the following Cayley table:

*	1	x	y	z	w	0
1	1	x	y	z	w	0
x	1	1	1	z	w	1
y	1	x	1	z	w	0
z	1	1	1	1	1	1
w	1	x	y	z	1	0
0	1	x	1	z	w	1

Then X is a Hilbert algebra. We define a fuzzy set f on X as follows:

$$f = \begin{pmatrix} 1 & x & y & z & w & 0 \\ 0.9 & 0.3 & 0.6 & 0.2 & 0.6 & 0.6 \end{pmatrix}$$

Then f is an 1-fuzzy ideal of X . Since $f((1 * (z * y)) * y) = f((1 * 1) * y) = f(1 * y) = f(y) = 0.6 \not\geq 0.9 = \max\{0.9, 0.2\} = \max\{f(1), f(z)\}$. Hence, X is not an 3-fuzzy ideal of X .

Example 3.2. Let $X = \{1, x, y, z, w, 0\}$ with the following Cayley table:

*	1	x	y	z	w	0
1	1	x	y	z	w	0
x	1	1	1	1	1	0
y	1	x	1	z	w	0
z	1	x	1	1	w	0
w	1	x	1	1	1	0
0	1	x	1	1	w	1

Then X is a Hilbert algebra. We define a fuzzy set f on X as follows:

$$f = \begin{pmatrix} 1 & x & y & z & w & 0 \\ 0.3 & 0.7 & 0.5 & 0.6 & 0.7 & 0.7 \end{pmatrix}$$

Then f is an 4-fuzzy ideal of X . Since $f((1 * (x * w)) * w) = f((1 * 1) * w) = f(1 * w) = f(w) = 0.7 \not\leq 0.3 = \min\{0.3, 0.7\} = \min\{f(1), f(x)\}$. Hence, X is not an 2-fuzzy ideal of X .

4. LENGTH OF A HYPERSTRUCTURE IN HILBERT ALGEBRAS

In this section, we introduce the notion of the length of a hyperstructure in Hilbert algebras. The notions of length fuzzy ideals of Hilbert algebras are introduced, and related properties are investigated. Relations between length fuzzy ideals and hyperfuzzy ideals are established. Moreover, we discuss the relationships among the length fuzzy ideals and upper-, lower-, and equal-level subsets of the length of a hyperstructure in Hilbert algebras.

Definition 4.1. Given a hyperstructure (X, \tilde{f}) over a non-empty set X , we define a fuzzy structures (X, \tilde{f}_l) in X as follows:

$$\tilde{f}_l : X \rightarrow [0, 1]; x \mapsto \tilde{f}_{\text{sup}}(x) - \tilde{f}_{\text{inf}}(x)$$

which is called the length of \tilde{f} .

Definition 4.2. A hyperstructure (X, \tilde{f}) over a Hilbert algebra X is called a length 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, 4-fuzzy) ideal of X if the fuzzy structure (X, \tilde{f}_l) is a 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, 4-fuzzy) ideal of X .

Example 4.1. Let $X = \{1, x, y, z, w, 0\}$ with the following Cayley table:

*	1	x	y	z	w	0
1	1	x	y	z	w	0
x	1	1	y	z	1	1
y	1	1	1	z	1	1
z	1	1	1	1	1	1
w	1	x	y	z	1	0
0	1	x	y	z	w	1

Then X is a Hilbert algebra. Let (X, \tilde{f}) be a hyperstructure over X in which \tilde{f} is given as follows:

$$\tilde{f} : X \rightarrow \tilde{P}([0, 1]); x \mapsto \begin{cases} (0.1, 1) & \text{if } x = 1 \\ [0.2, 0.3) \cup (0.7, 0.8] & \text{if } x = x \\ [0.3, 0.8] & \text{if } x = y \\ [0.2, 0.3) & \text{if } x = z \\ (0.1, 0.8] & \text{if } x = w \\ [0.1, 0.4) \cup (0.4, 0.9) & \text{if } x = 0. \end{cases}$$

Thus, we get a fuzzy structures (X, \tilde{f}_1) by define a fuzzy set on X as follows:

$$\tilde{f}_1 = \begin{pmatrix} 1 & x & y & z & w & 0 \\ 0.9 & 0.6 & 0.5 & 0.1 & 0.7 & 0.8 \end{pmatrix}$$

Then \tilde{f}_1 is an 1-fuzzy ideal of X . Hence, a hyperstructure (X, \tilde{f}) over a Hilbert algebra X is a length 1-fuzzy ideal of X .

Proposition 4.1. Given a hyperstructure (X, \tilde{f}) over a Hilbert algebra $X = (X, *, 1)$, we have the following assertions:

(1) If (X, \tilde{f}) is a length k -fuzzy ideal of X for $k \in \{1, 3\}$, then

$$(\forall x, y \in X)(x \leq y \Rightarrow \tilde{f}_1(x) \leq \tilde{f}_1(y)).$$

(2) If (X, \tilde{f}) is a length k -fuzzy ideal of X for $k \in \{2, 4\}$, then

$$(\forall x, y \in X)(x \leq y \Rightarrow \tilde{f}_1(x) \geq \tilde{f}_1(y)).$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$. If (X, \tilde{f}) is a length k -fuzzy ideal of X for $k \in \{1, 3\}$, then

$$\begin{aligned} \tilde{f}_1(y) &= \tilde{f}_1(1 * y) \\ &= \tilde{f}_1(((x * y) * (x * y)) * y) \\ &\geq \min\{\tilde{f}_1(x * y), \tilde{f}_1(x)\} \\ &= \min\{\tilde{f}_1(1), \tilde{f}_1(x)\} \\ &= \tilde{f}_1(x), \end{aligned}$$

$$\begin{aligned} \tilde{f}_1(y) &= \tilde{f}_1(1 * y) \\ &= \tilde{f}_1(((x * y) * (x * y)) * y) \\ &\geq \max\{\tilde{f}_1(x * y), \tilde{f}_1(x)\} \\ &= \max\{\tilde{f}_1(1), \tilde{f}_1(x)\} \\ &\geq \tilde{f}_1(x). \end{aligned}$$

If (X, \tilde{f}) is a length k -fuzzy ideal of X for $k \in \{2, 4\}$, then

$$\begin{aligned}
 \tilde{f}_I(y) &= \tilde{f}_I(1 * y) \\
 &= \tilde{f}_I(((x * y) * (x * y)) * y) \\
 &\leq \min\{\tilde{f}_I(x * y), \tilde{f}_I(x)\} \\
 &= \min\{\tilde{f}_I(1), \tilde{f}_I(x)\} \\
 &\leq \tilde{f}_I(x), \\
 \tilde{f}_I(y) &= \tilde{f}_I(1 * y) \\
 &= \tilde{f}_I(((x * y) * (x * y)) * y) \\
 &\leq \max\{\tilde{f}_I(x * y), \tilde{f}_I(x)\} \\
 &= \max\{\tilde{f}_I(1), \tilde{f}_I(x)\} \\
 &= \tilde{f}_I(x).
 \end{aligned}$$

□

Theorem 4.1. For any hyperstructure (X, \tilde{f}) over a Hilbert algebra $X = (X, *, 1)$, the following assertions are valid:

- (1) Every length 3-fuzzy ideal of X is a length 1-fuzzy ideal of X .
- (2) Every length 2-fuzzy ideal of X is a length 4-fuzzy ideal of X .

Proof. The result follows directly from Theorem 3.1. □

The following example shows that the converse of Theorem 4.1 is not true.

Example 4.2. Let $X = \{1, x, y, z, w, 0\}$ with the following Cayley table:

$*$	1	x	y	z	w	0
1	1	x	y	z	w	0
x	1	1	y	z	w	0
y	1	x	1	z	w	1
z	1	1	1	1	1	1
w	1	1	1	z	1	1
0	1	x	y	z	w	1

Then X is a Hilbert algebra. Let (X, \tilde{f}) be a hyperstructure over X in which \tilde{f} is given as follows:

$$\tilde{f} : X \rightarrow \tilde{P}([0, 1]); x \mapsto \begin{cases} (0.2, 0.9] & \text{if } x = 1 \\ [0.2, 0.3] \cup (0.3, 0.4] & \text{if } x = x \\ (0.1, 0.2) \cup (0.4, 0.5) & \text{if } x = y \\ [0.7, 0.8) & \text{if } x = z \\ (0.8, 0.9] & \text{if } x = w \\ [0.4, 0.7) \cup (0.7, 0.9) & \text{if } x = 0. \end{cases}$$

Thus, we get a fuzzy structures (X, \tilde{f}_1) by define a fuzzy set on X as follows:

$$\tilde{f}_1 = \begin{pmatrix} 1 & x & y & z & w & 0 \\ 0.7 & 0.2 & 0.4 & 0.1 & 0.1 & 0.5 \end{pmatrix}$$

Then \tilde{f}_1 is an 1-fuzzy ideal of X . Thus, a hyperstructure (X, \tilde{f}) over a Hilbert algebra X is a length 1-fuzzy ideal of X . Since $\tilde{f}_1((y * (1 * 0)) * 0) = \tilde{f}_1((y * 0) * 0) = \tilde{f}_1(1 * 0) = \tilde{f}_1(0) = 0.5 \not\geq 0.7 = \max\{0.4, 0.7\} = \max\{\tilde{f}_1(y), \tilde{f}_1(1)\}$. Then \tilde{f}_1 is not an 3-fuzzy ideal of X . Hence, (X, \tilde{f}) over a Hilbert algebra X is not a length 3-fuzzy ideal of X .

Example 4.3. Let $X = \{1, x, y, z, w, 0\}$ with the following Cayley table:

	*	1	x	y	z	w	0
1	1	x	y	z	w	0	
x	1	1	y	z	1	0	
y	1	x	1	z	1	1	
z	1	1	y	1	1	0	
w	1	x	y	z	1	0	
0	1	x	y	z	w	1	

Then X is a Hilbert algebra. Let (X, \tilde{f}) be a hyperstructure over X in which \tilde{f} is given as follows:

$$\tilde{f} : X \rightarrow \tilde{P}([0, 1]); x \mapsto \begin{cases} (0.8, 1] & \text{if } x = 1 \\ [0.2, 0.8] & \text{if } x = x \\ (0.1, 0.9) & \text{if } x = y \\ [0, 0.7) & \text{if } x = z \\ (0.2, 0.7] & \text{if } x = w \\ [0.5, 0.7) \cup (0.7, 0.9) & \text{if } x = 0. \end{cases}$$

Thus, we get a fuzzy structures (X, \tilde{f}_1) by define a fuzzy set on X as follows:

$$\tilde{f}_1 = \begin{pmatrix} 1 & x & y & z & w & 0 \\ 0.2 & 0.6 & 0.8 & 0.7 & 0.5 & 0.4 \end{pmatrix}$$

Then \tilde{f}_1 is an 4-fuzzy ideal of X . Thus, a hyperstructure (X, \tilde{f}) over a Hilbert algebra X is a length 4-fuzzy ideal of X . Since $\tilde{f}_1((0 * (z * x)) * x) = \tilde{f}_1((0 * 1) * x) = \tilde{f}_1(1 * x) = \tilde{f}_1(x) = 0.6 \not\geq 0.4 = \min\{0.4, 0.7\} = \min\{\tilde{f}_1(0), \tilde{f}_1(z)\}$. Then \tilde{f}_1 is not an 2-fuzzy ideal of X . Hence, (X, \tilde{f}) over a Hilbert algebra X is not a length 2-fuzzy ideal of X .

Theorem 4.2. Given an ideal S of a Hilbert algebra $X = (X, *, 1)$ and $B_1, B_2 \in \tilde{P}([0, 1])$, let (X, \tilde{f}) be a hyperstructure over X given by

$$\tilde{f} : X \rightarrow \tilde{P}([0, 1]); x \mapsto \begin{cases} B_2 & \text{if } x \in S, \\ B_1 & \text{otherwise.} \end{cases}$$

(1) If $B_1 \subset B_2$, then (X, \tilde{f}) is a length 1-fuzzy ideal of X .

(2) If $B_2 \subset B_1$, then (X, \tilde{f}) is a length 4-fuzzy ideal of X .

Proof. If $x \in S$, then $\tilde{f}(x) = B_2$ and so $\tilde{f}_l(x) = \tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x) = \sup \tilde{f}(x) - \inf \tilde{f}(x) = \sup B_2 - \inf B_2$. If $x \notin S$, then $\tilde{f}(x) = B_1$ and so $\tilde{f}_l(x) = \tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x) = \sup \tilde{f}(x) - \inf \tilde{f}(x) = \sup B_1 - \inf B_1$.

(1) Assume that $B_1 \subset B_2$. Then $\sup B_2 - \inf B_2 \geq \sup B_1 - \inf B_1$. Since $1 \in S$, we have $\tilde{f}_l(1) = \tilde{f}_{\sup}(1) - \tilde{f}_{\inf}(1) = \sup B_2 - \inf B_2 \geq \tilde{f}_l(x)$ for all $x \in X$.

Case 1: Let $x \in X$ and $y \in S$. Then $\tilde{f}_l(y) = \sup B_2 - \inf B_2$. Since S is an ideal of X , we have $x * y \in S$ and so $\tilde{f}_l(x * y) = \sup B_2 - \inf B_2$. Thus, $\tilde{f}_l(x * y) = \sup B_2 - \inf B_2 = \tilde{f}_l(y)$.

Case 2: Let $x \in X$ and $y \notin S$. Then $\tilde{f}_l(y) = \sup B_1 - \inf B_1$. Since S is an ideal of X , we have $x * y \in S$ or $x * y \notin S$ and so $\tilde{f}_l(x * y) = \sup B_2 - \inf B_2$ or $\tilde{f}_l(x * y) = \sup B_1 - \inf B_1$. Thus, $\tilde{f}_l(x * y) \geq \sup B_2 - \inf B_2 = \tilde{f}_l(y)$.

Case 1: Let $x \in X$ and $y_1, y_2 \in S$. Then $\tilde{f}_l(y_1) = \sup B_2 - \inf B_2$ and $\tilde{f}_l(y_2) = \sup B_2 - \inf B_2$. Thus, $\min\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\} = \sup B_2 - \inf B_2$. Since S is an ideal of X , we have $(y_1 * (y_2 * x)) * x \in S$ and so $\tilde{f}_l((y_1 * (y_2 * x)) * x) = \sup B_2 - \inf B_2$. Thus, $\tilde{f}_l((y_1 * (y_2 * x)) * x) = \sup B_2 - \inf B_2 = \min\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\}$.

Case 2: Let $x \in X$ and $y_1, y_2 \notin S$. Then $\tilde{f}_l(y_1) = \sup B_1 - \inf B_1$ and $\tilde{f}_l(y_2) = \sup B_1 - \inf B_1$, so $\min\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\} = \sup B_1 - \inf B_1$. Thus, $\tilde{f}_l((y_1 * (y_2 * x)) * x) \geq \sup B_1 - \inf B_1 = \min\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\}$.

Case 3: Let $x \in X$, $y_1 \notin S$ and $y_2 \in S$. Then $\tilde{f}_l(y_1) = \sup B_1 - \inf B_1$ and $\tilde{f}_l(y_2) = \sup B_2 - \inf B_2$, so $\min\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\} = \sup B_1 - \inf B_1$. Thus, $\tilde{f}_l((y_1 * (y_2 * x)) * x) \geq \sup B_1 - \inf B_1 = \min\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\}$.

Case 4: Let $x \in X$, $y_1 \in S$ and $y_2 \notin S$. Then $\tilde{f}_l(y_1) = \sup B_2 - \inf B_2$ and $\tilde{f}_l(y_2) = \sup B_1 - \inf B_1$, so $\min\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\} = \sup B_1 - \inf B_1$. Thus, $\tilde{f}_l((y_1 * (y_2 * x)) * x) \geq \sup B_1 - \inf B_1 = \min\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\}$.

Hence, \tilde{f}_l is a 1-fuzzy ideal of X and so (X, \tilde{f}) is a length 1-fuzzy ideal of X .

(2) Assume that $B_2 \subset B_1$. Then $\sup B_2 - \inf B_2 \leq \sup B_1 - \inf B_1$. Since $1 \in S$, we have $\tilde{f}_l(1) = \tilde{f}_{\sup}(1) - \tilde{f}_{\inf}(1) = \sup B_2 - \inf B_2 \leq \tilde{f}_l(x)$ for all $x \in X$.

Case 1: Let $x \in X$ and $y \in S$. Then $\tilde{f}_l(y) = \sup B_2 - \inf B_2$. Since S is an ideal of X , we have $x * y \in S$ and so $\tilde{f}_l(x * y) = \sup B_2 - \inf B_2$. Thus, $\tilde{f}_l(x * y) = \sup B_2 - \inf B_2 = \tilde{f}_l(y)$.

Case 2: Let $x \in X$ and $y \notin S$. Then $\tilde{f}_l(y) = \sup B_1 - \inf B_1$. Since S is an ideal of X , we have $x * y \in S$ or $x * y \notin S$ and so $\tilde{f}_l(x * y) = \sup B_2 - \inf B_2$ or $\tilde{f}_l(x * y) = \sup B_1 - \inf B_1$. Thus, $\tilde{f}_l(x * y) \leq \sup B_2 - \inf B_2 = \tilde{f}_l(y)$.

Case 1: Let $x \in X$ and $y_1, y_2 \in S$. Then $\tilde{f}_l(y_1) = \sup B_2 - \inf B_2$ and $\tilde{f}_l(y_2) = \sup B_2 - \inf B_2$. Thus, $\max\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\} = \sup B_2 - \inf B_2$. Since S is an ideal of X , we have $(y_1 * (y_2 * x)) * x \in S$ and so $\tilde{f}_l((y_1 * (y_2 * x)) * x) = \sup B_2 - \inf B_2$. Thus, $\tilde{f}_l((y_1 * (y_2 * x)) * x) = \sup B_2 - \inf B_2 = \max\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\}$.

Case 2: Let $x \in X$ and $y_1, y_2 \notin S$. Then $\tilde{f}_l(y_1) = \sup B_1 - \inf B_1$ and $\tilde{f}_l(y_2) = \sup B_1 - \inf B_1$, so $\max\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\} = \sup B_1 - \inf B_1$. Thus, $\tilde{f}_l((y_1 * (y_2 * x)) * x) \leq \sup B_1 - \inf B_1 = \max\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\}$.

Case 3: Let $x \in X$, $y_1 \notin S$ and $y_2 \in S$. Then $\tilde{f}_l(y_1) = \sup B_1 - \inf B_1$ and $\tilde{f}_l(y_2) = \sup B_2 - \inf B_2$, so $\max\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\} = \sup B_1 - \inf B_1$. Thus, $\tilde{f}_l((y_1 * (y_2 * x)) * x) \leq \sup B_1 - \inf B_1 = \max\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\}$.

Case 4: Let $x \in X, y_1 \in S$ and $y_2 \notin S$. Then $\tilde{f}_1(y_1) = \sup B_2 - \inf B_2$ and $\tilde{f}_1(y_2) = \sup B_1 - \inf B_1$, so $\max\{\tilde{f}_1(y_1), \tilde{f}_1(y_2)\} = \sup B_1 - \inf B_1$. Thus, $\tilde{f}_1((y_1 * (y_2 * x)) * x) \leq \sup B_1 - \inf B_1 = \max\{\tilde{f}_1(y_1), \tilde{f}_1(y_2)\}$.

Hence, \tilde{f}_1 is a 4-fuzzy ideal of X and so (X, \tilde{f}) is a length 4-fuzzy ideal of X . \square

Definition 4.3. Let (X, f) be a fuzzy structure in a non-empty set X . For any $t \in [0, 1]$, the sets

$$\mathcal{U}(f, t) = \{x \in X \mid f(x) \geq t\},$$

$$\mathcal{L}(f, t) = \{x \in X \mid f(x) \leq t\},$$

$$\mathcal{E}(f, t) = \{x \in X \mid f(x) = t\}$$

are called an upper t -level subset, a lower t -level subset, and an equal t -level subset of f , respectively.

Theorem 4.3. A hyperstructure (X, \tilde{f}) over a Hilbert algebra $X = (X, *, 1)$ is a length 1-fuzzy ideal of X if and only if the set $\mathcal{U}(\tilde{f}_1, t)$ is an ideal of X for all $t \in [0, 1]$ with $\mathcal{U}(\tilde{f}_1, t) \neq \emptyset$.

Proof. Assume that a hyperstructure (X, \tilde{f}) over X is a length 1-fuzzy ideal of X and let $t \in [0, 1]$ be such that $\mathcal{U}(\tilde{f}_1, t)$ is non-empty. Obviously, $1 \in \mathcal{U}(\tilde{f}_1, t)$. Let $x, y \in X$ be such that $y \in \mathcal{U}(\tilde{f}_1, t)$. Then $\tilde{f}_1(y) \geq t$, which imply from (3.2) that $\tilde{f}_1(x * y) \geq \tilde{f}_1(y) \geq t$. Hence, $x * y \in \mathcal{U}(\tilde{f}_1, t)$. Let $x, y_1, y_2 \in X$ be such that $y_1 \in \mathcal{U}(\tilde{f}_1, t)$ and $y_2 \in \mathcal{U}(\tilde{f}_1, t)$. Then $\tilde{f}_1(y_1) \geq t$ and $\tilde{f}_1(y_2) \geq t$, which imply from (3.3) that $\tilde{f}_1((y_1 * (y_2 * x)) * x) \geq \min\{\tilde{f}_1(y_1), \tilde{f}_1(y_2)\} \geq t$. Hence, $(y_1 * (y_2 * x)) * x \in \mathcal{U}(\tilde{f}_1, t)$, and therefore $\mathcal{U}(\tilde{f}_1, t)$ is an ideal of X .

Conversely, suppose that $\mathcal{U}(\tilde{f}_1, t)$ is an ideal of X . If $\tilde{f}_1(1) < \tilde{f}_1(a)$ for some $a \in X$, then $a \in \mathcal{U}(\tilde{f}_1, \tilde{f}_1(a))$ and hence $\mathcal{U}(\tilde{f}_1, \tilde{f}_1(a))$ is an ideal of X . Thus, $1 \in \mathcal{U}(\tilde{f}_1, \tilde{f}_1(a))$, and so $\tilde{f}_1(1) \geq \tilde{f}_1(a)$. This is a contradiction, and thus $\tilde{f}_1(1) \geq \tilde{f}_1(x)$ for all $x \in X$. Assume that there exist $a, b \in X$ such that $\tilde{f}_1(a * b) < \tilde{f}_1(b)$. Taking $t = \tilde{f}_1(b)$ implies that $b \in \mathcal{U}(\tilde{f}_1, t)$. Since $\mathcal{U}(\tilde{f}_1, t)$ is an ideal of X , we have $a * b \in \mathcal{U}(\tilde{f}_1, t)$. Hence, $\tilde{f}_1(a * b) \geq t = \tilde{f}_1(b)$, which is a contradiction. Hence, $\tilde{f}_1(x * y) \geq \tilde{f}_1(y)$ for all $x, y \in X$. Assume that there exist $a, b_1, b_2 \in X$ such that $\tilde{f}_1((b_1 * (b_2 * a)) * a) < \min\{\tilde{f}_1(b_1), \tilde{f}_1(b_2)\}$. Taking $t = \min\{\tilde{f}_1(b_1), \tilde{f}_1(b_2)\}$ implies that $(b_1 * (b_2 * a)) * a \in \mathcal{U}(\tilde{f}_1, t)$. Since $\mathcal{U}(\tilde{f}_1, t)$ is an ideal of X , we have $a \in \mathcal{U}(\tilde{f}_1, t)$. Hence, $\tilde{f}_1(a) \geq t = \min\{\tilde{f}_1(a * b), \tilde{f}_1(b)\}$, which is a contradiction. Hence, $\tilde{f}_1(x) \geq \min\{\tilde{f}_1(x * y), \tilde{f}_1(y)\}$ for all $x, y \in X$. Therefore, (X, \tilde{f}) is a length 1-fuzzy ideal of X . \square

Corollary 4.1. If (X, \tilde{f}) is a length 3-fuzzy ideal of a Hilbert algebra $X = (X, *, 1)$, then the set $\mathcal{U}(\tilde{f}_1, t)$ is an ideal of X for all $t \in [0, 1]$ with $\mathcal{U}(\tilde{f}_1, t) \neq \emptyset$.

Proof. This follows directly from Theorems 4.1 (1) and 4.3. \square

Theorem 4.4. A hyperstructure (X, \tilde{f}) over a Hilbert algebra $X = (X, *, 1)$ is a length 4-fuzzy ideal of X if and only if the set $\mathcal{L}(\tilde{f}_1, t)$ is an ideal of X for all $t \in [0, 1]$ with $\mathcal{L}(\tilde{f}_1, t) \neq \emptyset$.

Proof. Assume that a hyperstructure (X, \tilde{f}) over X is a length 4-fuzzy ideal of X and let $t \in [0, 1]$ be such that $\mathcal{L}(\tilde{f}_t, t)$ is non-empty. Obviously, $1 \in \mathcal{L}(\tilde{f}_t, t)$. Let $x, y \in X$ be such that $y \in \mathcal{L}(\tilde{f}_t, t)$. Then $\tilde{f}_t(y) \leq t$, which imply from (3.11) that $\tilde{f}_t(x * y) \leq \tilde{f}_t(y) \leq t$. Hence, $x * y \in \mathcal{L}(\tilde{f}_t, t)$. Let $x, y_1, y_2 \in X$ be such that $y_1 \in \mathcal{L}(\tilde{f}_t, t)$ and $y_2 \in \mathcal{L}(\tilde{f}_t, t)$. Then $\tilde{f}_t(y_1) \leq t$ and $\tilde{f}_t(y_2) \leq t$, which imply from (3.12) that $\tilde{f}_t((y_1 * (y_2 * x)) * x) \leq \min\{\tilde{f}_t(y_1), \tilde{f}_t(y_2)\} \leq t$. Hence, $(y_1 * (y_2 * x)) * x \in \mathcal{L}(\tilde{f}_t, t)$, and therefore $L_\ell(\tilde{f}, t)$ is an ideal of X .

Conversely, suppose that $\mathcal{L}(\tilde{f}_t, t)$ is an ideal of X . If $\tilde{f}_t(1) > \tilde{f}_t(a)$ for some $a \in X$, then $a \in \mathcal{L}(\tilde{f}_t, \tilde{f}_t(a))$ and hence $\mathcal{L}(\tilde{f}_t, \tilde{f}_t(a))$ is an ideal of X . Thus, $1 \in \mathcal{L}(\tilde{f}_t, \tilde{f}_t(a))$, and so $\tilde{f}_t(1) \leq \tilde{f}_t(a)$. This is a contradiction, and thus $\tilde{f}_t(1) \leq \tilde{f}_t(x)$ for all $x \in X$. Assume that there exist $a, b \in X$ such that $\tilde{f}_t(a * b) > \tilde{f}_t(b)$. Taking $t = \tilde{f}_t(b)$ implies that $b \in \mathcal{L}(\tilde{f}_t, t)$. Since $\mathcal{L}(\tilde{f}_t, t)$ is an ideal of X , we have $a * b \in \mathcal{L}(\tilde{f}_t, t)$. Hence, $\tilde{f}_t(a * b) \leq t = \tilde{f}_t(b)$, which is a contradiction. Hence, $\tilde{f}_t(x * y) \leq \tilde{f}_t(y)$ for all $x, y \in X$. Assume that there exist $a, b_1, b_2 \in X$ such that $\tilde{f}_t((b_1 * (b_2 * a)) * a) > \max\{\tilde{f}_t(b_1), \tilde{f}_t(b_2)\}$. Taking $t = \max\{\tilde{f}_t(b_1), \tilde{f}_t(b_2)\}$ implies that $(b_1 * (b_2 * a)) * a \in \mathcal{L}(\tilde{f}_t, t)$. Since $\mathcal{L}(\tilde{f}_t, t)$ is an ideal of X , we have $a \in \mathcal{L}(\tilde{f}_t, t)$. Hence, $\tilde{f}_t(a) \leq t = \max\{\tilde{f}_t(a * b), \tilde{f}_t(b)\}$, which is a contradiction. Hence, $\tilde{f}_t(x) \leq \max\{\tilde{f}_t(x * y), \tilde{f}_t(y)\}$ for all $x, y \in X$. Therefore, (X, \tilde{f}) is a length 4-fuzzy ideal of X . \square

Corollary 4.2. *If (X, \tilde{f}) is a length 2-fuzzy ideal of a Hilbert algebra $X = (X, *, 1)$, then the set $\mathcal{L}(\tilde{f}_t, t)$ is an ideal of X for all $t \in [0, 1]$ with $\mathcal{L}(\tilde{f}_t, t) \neq \emptyset$.*

Proof. This follows directly from Theorems 4.1 (2) and 4.4. \square

Theorem 4.5. *If (X, \tilde{f}) is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{inf}})$ is constant and $(X, \tilde{f}_{\text{sup}})$ is a 1-fuzzy ideal of X , then (X, \tilde{f}) is a length 1-fuzzy ideal of X .*

Proof. Assume that (X, \tilde{f}) is a hyperstructure over X in which $(X, \tilde{f}_{\text{inf}})$ is constant and $(X, \tilde{f}_{\text{sup}})$ is a 1-fuzzy ideal of X . Let $x, y \in X$. Since $(X, \tilde{f}_{\text{inf}})$ is constant, we have $\tilde{f}_{\text{inf}}(x) = \tilde{f}_{\text{inf}}(1)$ for all $x \in X$. Since $(X, \tilde{f}_{\text{sup}})$ is a 1-fuzzy ideal of X , we have

$$\begin{aligned} (\forall x \in X)(\tilde{f}_{\text{sup}}(1) &\geq \tilde{f}_{\text{sup}}(x)) \\ (\forall x, y \in X)(\tilde{f}_{\text{sup}}(x * y) &\geq \tilde{f}_{\text{sup}}(y)) \\ (\forall x, y_1, y_2 \in X)(\tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) &\geq \min\{\tilde{f}_{\text{sup}}(y_1), \tilde{f}_{\text{sup}}(y_2)\}). \end{aligned}$$

Let $x \in X$. Then

$$\begin{aligned} \tilde{f}_t(1) &= \tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}(1) \\ &\geq \tilde{f}_{\text{sup}}(x) - \tilde{f}_{\text{inf}}(1) \\ &= \tilde{f}_{\text{sup}}(x) - \tilde{f}_{\text{inf}}(x) \\ &= \tilde{f}_t(x). \end{aligned}$$

Let $x, y \in X$. Then

$$\begin{aligned}\tilde{f}_l(x * y) &= \tilde{f}_{\text{sup}}(x * y) - \tilde{f}_{\text{inf}}(x * y) \\ &\geq \tilde{f}_{\text{sup}}(y) - \tilde{f}_{\text{inf}}(1) \\ &= \tilde{f}_{\text{sup}}(y) - \tilde{f}_{\text{inf}}(y) \\ &= \tilde{f}_l(y).\end{aligned}$$

Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}\tilde{f}_l((y_1 * (y_2 * x)) * x) &= \tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) - \tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x) \\ &= \tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) - \tilde{f}_{\text{inf}}(1) \\ &\geq \min\{\tilde{f}_{\text{sup}}(y_1), \tilde{f}_{\text{sup}}(y_2)\} - \tilde{f}_{\text{inf}}(1) \\ &= \min\{\tilde{f}_{\text{sup}}(y_1) - \tilde{f}_{\text{inf}}(1), \tilde{f}_{\text{sup}}(y_2) - \tilde{f}_{\text{inf}}(1)\} \\ &= \min\{\tilde{f}_{\text{sup}}(y_1) - \tilde{f}_{\text{inf}}(y_1), \tilde{f}_{\text{sup}}(y_2) - \tilde{f}_{\text{inf}}(y_2)\} \\ &= \min\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\}.\end{aligned}$$

Hence, (X, \tilde{f}_l) is a 1-fuzzy ideal of X , that is, (X, \tilde{f}) is a length 1-fuzzy ideal of X . \square

Corollary 4.3. *If (X, \tilde{f}) is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{inf}})$ is constant and $(X, \tilde{f}_{\text{sup}})$ is a 3-fuzzy ideal of X , then (X, \tilde{f}) is a length 3-fuzzy ideal of X .*

Proof. This follows directly from Theorems 3.1 (1) and 4.5. \square

Corollary 4.4. *For $j \in \{1, 3\}$, every $(2(3), j)$ -hyperfuzzy ideal of a Hilbert algebra $X = (X, *, 1)$ is a length 1-fuzzy ideal of X .*

Proof. This follows directly from Theorem 4.5 together with Corollary 4.3. \square

Theorem 4.6. *If (X, \tilde{f}) is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{inf}})$ is constant and $(X, \tilde{f}_{\text{sup}})$ is a 4-fuzzy ideal of X , then (X, \tilde{f}) is a length 4-fuzzy ideal of X .*

Proof. Assume that (X, \tilde{f}) is a hyperstructure over X in which $(X, \tilde{f}_{\text{inf}})$ is constant and $(X, \tilde{f}_{\text{sup}})$ is a 4-fuzzy ideal of X . Let $x, y \in X$. Since $(X, \tilde{f}_{\text{inf}})$ is constant, we have $\tilde{f}_{\text{inf}}(x) = \tilde{f}_{\text{inf}}(1)$ for all $x \in X$. Since $(X, \tilde{f}_{\text{sup}})$ is a 4-fuzzy ideal of X , we have

$$\begin{aligned}(\forall x \in X)(\tilde{f}_{\text{sup}}(1) \leq \tilde{f}_{\text{sup}}(x)) \\ (\forall x, y \in X)(\tilde{f}_{\text{sup}}(x * y) \leq \tilde{f}_{\text{sup}}(y)) \\ (\forall x, y_1, y_2 \in X)(\tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) \leq \max\{\tilde{f}_{\text{sup}}(y_1), \tilde{f}_{\text{sup}}(y_2)\}).\end{aligned}$$

Let $x \in X$. Then

$$\begin{aligned}\tilde{f}_l(1) &= \tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}(1) \\ &\leq \tilde{f}_{\text{sup}}(x) - \tilde{f}_{\text{inf}}(1) \\ &= \tilde{f}_{\text{sup}}(x) - \tilde{f}_{\text{inf}}(x) \\ &= \tilde{f}_l(x).\end{aligned}$$

Let $x, y \in X$. Then

$$\begin{aligned}\tilde{f}_l(x * y) &= \tilde{f}_{\text{sup}}(x * y) - \tilde{f}_{\text{inf}}(x * y) \\ &\leq \tilde{f}_{\text{sup}}(y) - \tilde{f}_{\text{inf}}(1) \\ &= \tilde{f}_{\text{sup}}(y) - \tilde{f}_{\text{inf}}(y) \\ &= \tilde{f}_l(y).\end{aligned}$$

Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}\tilde{f}_l((y_1 * (y_2 * x)) * x) &= \tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) - \tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x) \\ &= \tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) - \tilde{f}_{\text{inf}}(1) \\ &\leq \max\{\tilde{f}_{\text{sup}}(y_1), \tilde{f}_{\text{sup}}(y_2)\} - \tilde{f}_{\text{inf}}(1) \\ &= \max\{\tilde{f}_{\text{sup}}(y_1) - \tilde{f}_{\text{inf}}(1), \tilde{f}_{\text{sup}}(y_2) - \tilde{f}_{\text{inf}}(1)\} \\ &= \max\{\tilde{f}_{\text{sup}}(y_1) - \tilde{f}_{\text{inf}}(y_1), \tilde{f}_{\text{sup}}(y_2) - \tilde{f}_{\text{inf}}(y_2)\} \\ &= \max\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\}.\end{aligned}$$

Hence, (X, \tilde{f}_l) is a 4-fuzzy ideal of X , that is, (X, \tilde{f}) is a length 4-fuzzy ideal of X . \square

Corollary 4.5. *If (X, \tilde{f}) is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{inf}})$ is constant and $(X, \tilde{f}_{\text{sup}})$ is a 2-fuzzy ideal of X , then (X, \tilde{f}) is a length 4-fuzzy ideal of X .*

Proof. This follows directly from Theorems 4.1 and 4.6. \square

Corollary 4.6. *For $j \in \{2, 4\}$, every $(2(3), j)$ -hyperfuzzy ideal of a Hilbert algebra $X = (X, *, 1)$ is a length 4-fuzzy ideal of X .*

Proof. This follows directly from Theorem 4.6 together with Corollary 4.5. \square

Theorem 4.7. *If (X, \tilde{f}) is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{sup}})$ is constant and $(X, \tilde{f}_{\text{inf}})$ is a 4-fuzzy ideal of X , then (X, \tilde{f}) is a length 1-fuzzy ideal of X .*

Proof. Assume that (X, \tilde{f}) is a hyperstructure over X in which $(X, \tilde{f}_{\text{sup}})$ is constant and $(X, \tilde{f}_{\text{inf}})$ is a 4-fuzzy ideal of X . Let $x, y \in X$. Since $(X, \tilde{f}_{\text{sup}})$ is constant, we have $\tilde{f}_{\text{sup}}(x) = \tilde{f}_{\text{sup}}(1)$ for all $x \in X$. Since $(X, \tilde{f}_{\text{inf}})$ is a 4-fuzzy ideal of X , we have

$$\begin{aligned}(\forall x \in X)(\tilde{f}_{\text{inf}}(1) &\leq \tilde{f}_{\text{inf}}(x)), \\ (\forall x, y \in X)(\tilde{f}_{\text{inf}}(x * y) &\leq \tilde{f}_{\text{inf}}(y)), \\ (\forall x, y_1, y_2 \in X)(\tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x) &\leq \max\{\tilde{f}_{\text{inf}}(y_1), \tilde{f}_{\text{inf}}(y_2)\}).\end{aligned}$$

Let $x \in X$. Then

$$\begin{aligned}\tilde{f}_l(1) &= \tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}(1) \\ &\geq \tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}(x) \\ &= \tilde{f}_{\text{sup}}(x) - \tilde{f}_{\text{inf}}(x) \\ &= \tilde{f}_l(x).\end{aligned}$$

Let $x, y \in X$. Then

$$\begin{aligned}\tilde{f}_l(x * y) &= \tilde{f}_{\text{sup}}(x * y) - \tilde{f}_{\text{inf}}(x * y) \\ &\geq \tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}(y) \\ &= \tilde{f}_{\text{sup}}(y) - \tilde{f}_{\text{inf}}(y) \\ &= \tilde{f}_l(y).\end{aligned}$$

Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}\tilde{f}_l((y_1 * (y_2 * x)) * x) &= \tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) - \tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x) \\ &= \tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x) \\ &\geq \tilde{f}_{\text{sup}}(1) - \max\{\tilde{f}_{\text{inf}}(y_1), \tilde{f}_{\text{inf}}(y_2)\} \\ &= \min\{\tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}(y_1), \tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}(y_2)\} \\ &= \min\{\tilde{f}_{\text{sup}}(y_1) - \tilde{f}_{\text{inf}}(y_1), \tilde{f}_{\text{sup}}(y_2) - \tilde{f}_{\text{inf}}(y_2)\} \\ &= \min\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\}.\end{aligned}$$

Hence, (X, \tilde{f}_l) is a 1-fuzzy ideal of X , that is, (X, \tilde{f}) is a length 1-fuzzy ideal of X . \square

Corollary 4.7. *If (X, \tilde{f}) is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{sup}})$ is constant and $(X, \tilde{f}_{\text{inf}})$ is a 2-fuzzy ideal of X , then (X, \tilde{f}) is a length 1-fuzzy ideal of X .*

Proof. This follows directly from Theorems 4.1 and 4.7. \square

Corollary 4.8. *For $i \in \{2, 4\}$, every $(i, 2(3))$ -hyperfuzzy ideal of a Hilbert algebra $X = (X, *, 1)$ is a length 1-fuzzy ideal of X .*

Proof. This follows directly from Theorem 4.7 together with Corollary 4.7. \square

Theorem 4.8. *If (X, \tilde{f}) is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{sup}})$ is constant and $(X, \tilde{f}_{\text{inf}})$ is a 1-fuzzy ideal of X , then (X, \tilde{f}) is a length 4-fuzzy ideal of X .*

Proof. Assume that (X, \tilde{f}) is a hyperstructure over X in which $(X, \tilde{f}_{\text{sup}})$ is constant and $(X, \tilde{f}_{\text{inf}})$ is a 1-fuzzy ideal of X . Let $x, y \in X$. Since $(X, \tilde{f}_{\text{sup}})$ is constant, we have $\tilde{f}_{\text{sup}}(x) = \tilde{f}_{\text{sup}}(1)$ for all $x \in X$. Since $(X, \tilde{f}_{\text{inf}})$ is a 1-fuzzy ideal of X , we have

$$\begin{aligned}(\forall x \in X)(\tilde{f}_{\text{inf}}(1) \geq \tilde{f}_{\text{inf}}(x)) \\ (\forall x, y \in X)(\tilde{f}_{\text{inf}}(x * y) \geq \tilde{f}_{\text{inf}}(y)) \\ (\forall x, y_1, y_2 \in X)(\tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x) \geq \min\{\tilde{f}_{\text{inf}}(y_1), \tilde{f}_{\text{inf}}(y_2)\}).\end{aligned}$$

Let $x \in X$. Then

$$\begin{aligned}\tilde{f}_l(1) &= \tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}(1) \\ &\leq \tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}(x) \\ &= \tilde{f}_{\text{sup}}(x) - \tilde{f}_{\text{inf}}(x) \\ &= \tilde{f}_l(x).\end{aligned}$$

Let $x, y \in X$. Then

$$\begin{aligned}\tilde{f}_l(x * y) &= \tilde{f}_{\text{sup}}(x * y) - \tilde{f}_{\text{inf}}(x * y) \\ &\leq \tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}(y) \\ &= \tilde{f}_{\text{sup}}(y) - \tilde{f}_{\text{inf}}(y) \\ &= \tilde{f}_l(y).\end{aligned}$$

Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}\tilde{f}_l((y_1 * (y_2 * x)) * x) &= \tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) - \tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x) \\ &= \tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x) \\ &\leq \tilde{f}_{\text{sup}}(1) - \min\{\tilde{f}_{\text{inf}}(y_1), \tilde{f}_{\text{inf}}(y_2)\} \\ &= \max\{\tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}(y_1), \tilde{f}_{\text{sup}}(1) - \tilde{f}_{\text{inf}}(y_2)\} \\ &= \max\{\tilde{f}_{\text{sup}}(y_1) - \tilde{f}_{\text{inf}}(y_1), \tilde{f}_{\text{sup}}(y_2) - \tilde{f}_{\text{inf}}(y_2)\} \\ &= \max\{\tilde{f}_l(y_1), \tilde{f}_l(y_2)\}.\end{aligned}$$

Hence, (X, \tilde{f}_l) is a 4-fuzzy ideal of X , that is, (X, \tilde{f}) is a length 4-fuzzy ideal of X . \square

Corollary 4.9. *If (X, \tilde{f}) is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{sup}})$ is constant and $(X, \tilde{f}_{\text{inf}})$ is a 3-fuzzy ideal of X , then (X, \tilde{f}) is a length 4-fuzzy ideal of X .*

Proof. This follows directly from Theorems 3.1 (1) and 4.8. \square

Corollary 4.10. *For $i \in \{1, 3\}$, every $(i, 2(3))$ -hyperfuzzy ideal of a Hilbert algebra $X = (X, *, 1)$ is a length 4-fuzzy ideal of X .*

Proof. This follows directly from Theorem 4.8 together with Corollary 4.9. \square

5. MEAN OF A HYPERSTRUCTURE IN HILBERT ALGEBRAS

In this section, we introduce the notion of the mean of a hyperstructure in Hilbert algebras. The notions of mean fuzzy ideals of Hilbert algebras are introduced, and related properties are investigated. Relations between mean fuzzy ideals and hyper fuzzy ideals are established. Moreover, we discuss the relationships among the mean fuzzy ideals and upper-, lower-, and equal-level subsets of the mean of a hyperstructure in Hilbert algebras.

Definition 5.1. *Given a hyperstructure (X, \tilde{f}) over a non-empty set X , we define a fuzzy structures (X, \tilde{f}_m) in X as follows:*

$$\tilde{f}_m : X \rightarrow [0, 1]; x \mapsto \frac{\tilde{f}_{\text{sup}}(x) + \tilde{f}_{\text{inf}}(x)}{2}$$

which is called the mean of \tilde{f} .

Definition 5.2. A hyperstructure (X, \tilde{f}) over a Hilbert algebra X is called a mean 1-fuzzy (resp., 2-fuzzy, 3-fuzzy and 4-fuzzy) ideal of X if the fuzzy structure (X, \tilde{f}_m) is a 1-fuzzy (resp., 2-fuzzy, 3-fuzzy and 4-fuzzy) ideal of X .

Example 5.1. Let $X = \{1, x, y, z, w, 0\}$ with the following Cayley table:

*	1	x	y	z	w	0
1	1	x	y	z	w	0
x	1	1	y	z	w	1
y	1	1	1	1	1	1
z	1	x	y	1	w	1
w	1	x	y	z	1	1
0	1	x	y	z	w	1

Then X is a Hilbert algebra. Let (X, \tilde{f}) be a hyperstructure over X in which \tilde{f} is given as follows:

$$\tilde{f} : X \rightarrow \tilde{P}([0, 1]); x \mapsto \begin{cases} (0.8, 1) & \text{if } x = 1 \\ [0.4, 0.5] \cup (0.5, 0.9) & \text{if } x = x \\ [0, 0.2] & \text{if } x = y \\ (0.3, 0.6) \cup (0.6, 0.7) & \text{if } x = z \\ (0.1, 0.5] & \text{if } x = w \\ [0.6, 0.9) \cup (0.9, 1] & \text{if } x = 0. \end{cases}$$

Thus, we get a fuzzy structures (X, \tilde{f}_m) by define a fuzzy set on X as follows:

$$\tilde{f}_m = \begin{pmatrix} 1 & x & y & z & w & 0 \\ 0.9 & 0.7 & 0.1 & 0.5 & 0.3 & 0.8 \end{pmatrix}$$

Then \tilde{f}_m is an 1-fuzzy ideal of X . Hence, (X, \tilde{f}) over a Hilbert algebra X is a mean 1-fuzzy ideal of X .

Proposition 5.1. Given a hyperstructure (X, \tilde{f}) over a Hilbert algebra $X = (X, *, 1)$, we have the following assertions:

(1) If (X, \tilde{f}) is a mean k -fuzzy ideal of X for $k \in \{1, 3\}$, then

$$(\forall x \in X)(\tilde{f}_m(1) \geq \tilde{f}_m(x)).$$

(2) If (X, \tilde{f}) is a mean k -fuzzy ideal of X for $k \in \{2, 4\}$, then

$$(\forall x \in X)(\tilde{f}_m(1) \leq \tilde{f}_m(x)).$$

Proof. (1) Let (X, \tilde{f}) be a mean k -fuzzy ideal of X for $k \in \{1, 3\}$. Then

$$\begin{aligned} \tilde{f}_m(1) &= \frac{\tilde{f}_{\text{sup}}(1) + \tilde{f}_{\text{inf}}(1)}{2} \\ &\geq \frac{\tilde{f}_{\text{sup}}(x) + \tilde{f}_{\text{inf}}(x)}{2} \\ &= \tilde{f}_m(x). \end{aligned}$$

(2) Let (X, \tilde{f}) be a mean k -fuzzy ideal of X for $k \in \{2, 4\}$. Then

$$\begin{aligned}\tilde{f}_m(1) &= \frac{\tilde{f}_{\text{sup}}(1) + \tilde{f}_{\text{inf}}(1)}{2} \\ &\leq \frac{\tilde{f}_{\text{sup}}(x) + \tilde{f}_{\text{inf}}(x)}{2} \\ &= \tilde{f}_m(x).\end{aligned}$$

□

Theorem 5.1. For any hyperstructure (X, \tilde{f}) over a Hilbert algebra $X = (X, *, 1)$, the following assertions are valid:

- (1) Every mean 3-fuzzy ideal of X is a mean 1-fuzzy ideal of X .
- (2) Every mean 2-fuzzy ideal of X is a mean 4-fuzzy ideal of X .

Proof. The result follows directly from Theorem 3.1. □

Example 5.2. Let $X = \{1, x, y, z, w, 0\}$ with the following Cayley table:

$*$	1	x	y	z	w	0
1	1	x	y	z	w	0
x	1	1	y	1	w	1
y	1	1	1	1	w	1
z	1	x	y	1	w	0
w	1	x	y	1	1	1
0	1	x	y	1	w	1

Then X is a Hilbert algebra. Let (X, \tilde{f}) be a hyperstructure over X in which \tilde{f} is given as follows:

$$\tilde{f} : X \rightarrow \tilde{P}([0, 1]); x \mapsto \begin{cases} (0.6, 0.8) & \text{if } x = 1 \\ [0.3, 0.4] \cup (0.4, 0.5] & \text{if } x = x \\ [0.1, 0.5] & \text{if } x = y \\ [0.2, 1) & \text{if } x = z \\ (0.3, 0.7] & \text{if } x = w \\ [0.1, 0.7] \cup (0.7, 0.9) & \text{if } x = 0. \end{cases}$$

Thus, we get a fuzzy structures (X, \tilde{f}_m) by define a fuzzy set on X as follows:

$$\tilde{f}_m = \begin{pmatrix} 1 & x & y & z & w & 0 \\ 0.7 & 0.4 & 0.3 & 0.6 & 0.5 & 0.5 \end{pmatrix}$$

Then \tilde{f}_m is an 1-fuzzy ideal of X . Thus, a hyperstructure (X, \tilde{f}) over a Hilbert algebra X is a mean 1-fuzzy ideal of X . Since $\tilde{f}_m((z * (w * w)) * w) = \tilde{f}_m((z * 1) * w) = \tilde{f}_m(1 * w) = \tilde{f}_m(w) = 0.5 \not\geq 0.6 = \max\{0.6, 0.5\} = \max\{\tilde{f}_m(z), \tilde{f}_m(w)\}$. Then \tilde{f}_m is not an 3-fuzzy ideal of X . Hence, (X, \tilde{f}) over a Hilbert algebra X is not a mean 3-fuzzy ideal of X .

Example 5.3. Let $X = \{1, x, y, z, w, 0\}$ with the following Cayley table:

*	1	x	y	z	w	0
1	1	x	y	z	w	0
x	1	1	y	z	w	0
y	1	1	1	z	1	0
z	1	x	y	1	w	0
w	1	1	y	z	1	0
0	1	x	y	1	w	1

Then X is a Hilbert algebra. Let (X, \tilde{f}) be a hyperstructure over X in which \tilde{f} is given as follows:

$$\tilde{f} : X \rightarrow \tilde{P}([0, 1]); x \mapsto \begin{cases} (0, 0.2) & \text{if } x = 1 \\ (0.1, 0.3) & \text{if } x = x \\ [0.3, 0.5] & \text{if } x = y \\ [0, 1] & \text{if } x = z \\ [0.2, 0.4] & \text{if } x = w \\ [0.8, 0.9) \cup (0.9, 0.1) & \text{if } x = 0. \end{cases}$$

Thus, we get a fuzzy structures (X, \tilde{f}_m) by define a fuzzy set on X as follows:

$$\tilde{f}_m = \begin{pmatrix} 1 & x & y & z & w & 0 \\ 0.1 & 0.2 & 0.4 & 0.5 & 0.3 & 0.9 \end{pmatrix}$$

Then \tilde{f}_m is an 4-fuzzy ideal of X . Thus, a hyperstructure (X, \tilde{f}) over a Hilbert algebra X is a mean 4-fuzzy ideal of X . Since $\tilde{f}_m((w * (1 * w)) * w) = \tilde{f}_m((w * w) * w) = \tilde{f}_m(1 * w) = \tilde{f}_m(w) = 0.3 \not\leq 0.1 = \min\{0.3, 0.1\} = \min\{\tilde{f}_m(w), \tilde{f}_m(1)\}$. Then \tilde{f}_m is not an 2-fuzzy ideal of X . Hence, (X, \tilde{f}) over a Hilbert algebra X is not a mean 2-fuzzy ideal of X .

Theorem 5.2. Given an ideal S of a Hilbert algebra $X = (X, *, 1)$ and $B_1, B_2 \in \tilde{P}([0, 1])$, let (X, \tilde{f}) be a hyperstructure over X given by

$$\tilde{f} : X \rightarrow \tilde{P}([0, 1]); x \mapsto \begin{cases} B_2 & \text{if } x \in S \\ B_1 & \text{otherwise.} \end{cases}$$

- (1) If $\sup B_2 \geq \sup B_1$ and $\inf B_2 \geq \inf B_1$, then (X, \tilde{f}) is a mean 1-fuzzy ideal of X .
- (2) If $\sup B_2 \leq \sup B_1$ and $\inf B_2 \leq \inf B_1$, then (X, \tilde{f}) is a mean 4-fuzzy ideal of X .

Proof. If $x \in S$, then $\tilde{f}(x) = B_2$ and so

$$\tilde{f}_m(x) = \frac{\tilde{f}_{\sup}(x) + \tilde{f}_{\inf}(x)}{2} = \frac{\sup \tilde{f}(x) + \inf \tilde{f}(x)}{2} = \frac{\sup B_2 + \inf B_2}{2}.$$

If $x \notin S$, then $\tilde{f}(x) = B_1$ and so

$$\tilde{f}_m(x) = \frac{\tilde{f}_{\sup}(x) + \tilde{f}_{\inf}(x)}{2} = \frac{\sup \tilde{f}(x) + \inf \tilde{f}(x)}{2} = \frac{\sup B_1 + \inf B_1}{2}.$$

(1) Assume that $\sup B_2 \geq \sup B_1$ and $\inf B_2 \geq \inf B_1$. Then

$$\frac{\sup B_2 + \inf B_2}{2} \geq \frac{\sup B_1 + \inf B_1}{2}.$$

Case 1: Let $x, y \in S$. Then $\tilde{f}_m(x) = \frac{\sup B_2 + \inf B_2}{2}$ and $\tilde{f}_m(y) = \frac{\sup B_2 + \inf B_2}{2}$. Thus, $\min\{\tilde{f}_m(x), \tilde{f}_m(y)\} = \frac{\sup B_2 + \inf B_2}{2}$. Since S is an ideal of X , we have $x * y \in S$ and so $\tilde{f}_m(x * y) = \frac{\sup B_2 + \inf B_2}{2}$. Thus, $\tilde{f}_m(x * y) = \frac{\sup B_2 + \inf B_2}{2} = \min\{\tilde{f}_m(x), \tilde{f}_m(y)\}$.

Case 2: Let $x, y \notin S$. Then $\tilde{f}_m(x) = \frac{\sup B_1 + \inf B_1}{2}$ and $\tilde{f}_m(y) = \frac{\sup B_1 + \inf B_1}{2}$, so $\min\{\tilde{f}_m(x), \tilde{f}_m(y)\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus, $\tilde{f}_m(x * y) \geq \frac{\sup B_1 + \inf B_1}{2} = \min\{\tilde{f}_m(x), \tilde{f}_m(y)\}$.

Case 3: Let $x \notin S$ and $y \in S$. Then $\tilde{f}_m(x) = \frac{\sup B_1 + \inf B_1}{2}$ and $\tilde{f}_m(y) = \frac{\sup B_2 + \inf B_2}{2}$, so $\min\{\tilde{f}_m(x), \tilde{f}_m(y)\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus, $\tilde{f}_m(x * y) \geq \frac{\sup B_1 + \inf B_1}{2} = \min\{\tilde{f}_m(x), \tilde{f}_m(y)\}$.

Case 4: Let $x \in S$ and $y \notin S$. Then $\tilde{f}_m(x) = \frac{\sup B_2 + \inf B_2}{2}$ and $\tilde{f}_m(y) = \frac{\sup B_1 + \inf B_1}{2}$, so $\min\{\tilde{f}_m(x), \tilde{f}_m(y)\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus, $\tilde{f}_m(x * y) \geq \frac{\sup B_1 + \inf B_1}{2} = \min\{\tilde{f}_m(x), \tilde{f}_m(y)\}$.

Hence, \tilde{f}_m is a 1-fuzzy ideal of X and so (X, \tilde{f}) is a mean 1-fuzzy ideal of X .

(2) Assume that $\sup B_2 \leq \sup B_1$ and $\inf B_2 \leq \inf B_1$. Then $\frac{\sup B_2 + \inf B_2}{2} \leq \frac{\sup B_1 + \inf B_1}{2}$.

Case 1: Let $x, y \in S$. Then $\tilde{f}_m(x) = \frac{\sup B_2 + \inf B_2}{2}$ and $\tilde{f}_m(y) = \frac{\sup B_2 + \inf B_2}{2}$, so $\max\{\tilde{f}_m(x), \tilde{f}_m(y)\} = \frac{\sup B_2 + \inf B_2}{2}$. Since S is an ideal of X , we have $x * y \in S$ and so $\tilde{f}_m(x * y) = \frac{\sup B_2 + \inf B_2}{2}$. Thus, $\tilde{f}_m(x * y) = \frac{\sup B_2 + \inf B_2}{2} = \max\{\tilde{f}_m(x), \tilde{f}_m(y)\}$.

Case 2: Let $x, y \notin S$. Then $\tilde{f}_m(x) = \frac{\sup B_1 + \inf B_1}{2}$ and $\tilde{f}_m(y) = \frac{\sup B_1 + \inf B_1}{2}$, so $\max\{\tilde{f}_m(x), \tilde{f}_m(y)\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus, $\tilde{f}_m(x * y) \leq \frac{\sup B_1 + \inf B_1}{2} = \max\{\tilde{f}_m(x), \tilde{f}_m(y)\}$.

Case 3: Let $x \notin S$ and $y \in S$. Then $\tilde{f}_m(x) = \frac{\sup B_1 + \inf B_1}{2}$ and $\tilde{f}_m(y) = \frac{\sup B_2 + \inf B_2}{2}$, so $\max\{\tilde{f}_m(x), \tilde{f}_m(y)\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus, $\tilde{f}_m(x * y) \leq \frac{\sup B_1 + \inf B_1}{2} = \max\{\tilde{f}_m(x), \tilde{f}_m(y)\}$.

Case 4: Let $x \in S$ and $y \notin S$. Then $\tilde{f}_m(x) = \frac{\sup B_2 + \inf B_2}{2}$ and $\tilde{f}_m(y) = \frac{\sup B_1 + \inf B_1}{2}$, so $\max\{\tilde{f}_m(x), \tilde{f}_m(y)\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus, $\tilde{f}_m(x * y) \leq \frac{\sup B_1 + \inf B_1}{2} = \max\{\tilde{f}_m(x), \tilde{f}_m(y)\}$.

Hence, \tilde{f}_m is a 4-fuzzy ideal of X and so (X, \tilde{f}) is a mean 4-fuzzy ideal of X . \square

Theorem 5.3. A hyperstructure (X, \tilde{f}) over a Hilbert algebra $X = (X, *, 1)$ is a mean 1-fuzzy ideal of X if and only if the set $\mathcal{U}(\tilde{f}_m, t)$ is an ideal of X for all $t \in [0, 1]$ with $\mathcal{U}(\tilde{f}_m, t) \neq \emptyset$.

Proof. Assume that a hyperstructure (X, \tilde{f}) over X is a mean 1-fuzzy ideal of X and let $t \in [0, 1]$ be such that $\mathcal{U}(\tilde{f}_m, t)$ is non-empty. Obviously, $1 \in \mathcal{U}(\tilde{f}_m, t)$. Let $x, y \in X$ be such that $y \in \mathcal{U}(\tilde{f}_m, t)$. Then

$\tilde{f}_m(y) \geq t$, which imply from (3.2) that $\tilde{f}_m(x * y) \geq \tilde{f}_m(y) \geq t$. Hence, $x * y \in \mathcal{U}(\tilde{f}_m, t)$. Let $x, y_1, y_2 \in X$ be such that $y_1 \in \mathcal{U}(\tilde{f}_m, t)$ and $y_2 \in \mathcal{U}(\tilde{f}_m, t)$. Then $\tilde{f}_m(y_1) \geq t$ and $\tilde{f}_m(y_2) \geq t$, which imply from (3.3) that $\tilde{f}_m((y_1 * (y_2 * x)) * x) \geq \min\{\tilde{f}_m(y_1), \tilde{f}_m(y_2)\} \geq t$. Hence, $(y_1 * (y_2 * x)) * x \in \mathcal{U}(\tilde{f}_m, t)$, and therefore $\mathcal{U}(\tilde{f}_m, t)$ is an ideal of X .

Conversely, suppose that $\mathcal{U}(\tilde{f}_m, t)$ is an ideal of X . If $\tilde{f}_m(1) < \tilde{f}_m(a)$ for some $a \in X$, then $a \in \mathcal{U}(\tilde{f}_m, \tilde{f}_m(a))$ and hence $\mathcal{U}(\tilde{f}_m, \tilde{f}_m(a))$ is an ideal of X . Thus, $1 \in \mathcal{U}(\tilde{f}_m, \tilde{f}_m(a))$, and so $\tilde{f}_m(1) \geq \tilde{f}_m(a)$. This is a contradiction, and thus $\tilde{f}_m(1) \geq \tilde{f}_m(x)$ for all $x \in X$. Assume that there exist $a, b \in X$ such that $\tilde{f}_m(a * b) < \tilde{f}_m(b)$. Taking $t = \tilde{f}_m(b)$ implies that $b \in \mathcal{U}(\tilde{f}_m, t)$. Since $\mathcal{U}(\tilde{f}_m, t)$ is an ideal of X , we have $a * b \in \mathcal{U}(\tilde{f}_m, t)$. Hence, $\tilde{f}_m(a * b) \geq t = \tilde{f}_m(b)$, which is a contradiction. Hence, $\tilde{f}_m(x * y) \geq \tilde{f}_m(y)$ for all $x, y \in X$. Assume that there exist $a, b_1, b_2 \in X$ such that $\tilde{f}_m((b_1 * (b_2 * a)) * a) < \min\{\tilde{f}_m(b_1), \tilde{f}_m(b_2)\}$. Taking $t = \min\{\tilde{f}_m(b_1), \tilde{f}_m(b_2)\}$ implies that $(b_1 * (b_2 * a)) * a \in \mathcal{U}(\tilde{f}_m, t)$. Since $\mathcal{U}(\tilde{f}_m, t)$ is an ideal of X , we have $a \in \mathcal{U}(\tilde{f}_m, t)$. Hence, $\tilde{f}_m(a) \geq t = \min\{\tilde{f}_m(a * b), \tilde{f}_m(b)\}$, which is a contradiction. Hence, $\tilde{f}_m(x) \geq \min\{\tilde{f}_m(x * y), \tilde{f}_m(y)\}$ for all $x, y \in X$. Therefore, (X, \tilde{f}) is a mean 1-fuzzy ideal of X . \square

Corollary 5.1. *If (X, \tilde{f}) is a mean 3-fuzzy ideal of X , then $\mathcal{U}(\tilde{f}_m, t)$ is an ideal of X for all $t \in [0, 1]$ with $\mathcal{U}(\tilde{f}_m, t) \neq \emptyset$.*

Proof. This follows directly from Theorems 5.1 (1) and 5.3. \square

Theorem 5.4. *A hyperstructure (X, \tilde{f}) over a Hilbert algebra $X = (X, *, 1)$ is a mean 4-fuzzy ideal of X if and only if the set $\mathcal{L}(\tilde{f}_m, t)$ is an ideal of X for all $t \in [0, 1]$ with $\mathcal{L}(\tilde{f}_m, t) \neq \emptyset$.*

Proof. Assume that a hyperstructure (X, \tilde{f}) over X is a mean 4-fuzzy ideal of X and let $t \in [0, 1]$ be such that $\mathcal{L}(\tilde{f}_m, t)$ is non-empty. Obviously, $1 \in \mathcal{L}(\tilde{f}_m, t)$. Let $x, y \in X$ be such that $y \in \mathcal{L}(\tilde{f}_m, t)$. Then $\tilde{f}_m(y) \leq t$, which imply from (3.11) that $\tilde{f}_m(x * y) \leq \tilde{f}_m(y) \leq t$. Hence, $x * y \in \mathcal{L}(\tilde{f}_m, t)$. Let $x, y_1, y_2 \in X$ be such that $y_1 \in \mathcal{L}(\tilde{f}_m, t)$ and $y_2 \in \mathcal{L}(\tilde{f}_m, t)$. Then $\tilde{f}_m(y_1) \leq t$ and $\tilde{f}_m(y_2) \leq t$, which imply from (3.12) that $\tilde{f}_m((y_1 * (y_2 * x)) * x) \leq \min\{\tilde{f}_m(y_1), \tilde{f}_m(y_2)\} \leq t$. Hence, $(y_1 * (y_2 * x)) * x \in \mathcal{L}(\tilde{f}_m, t)$, and therefore $\mathcal{L}(\tilde{f}_m, t)$ is an ideal of X .

Conversely, suppose that $\mathcal{L}(\tilde{f}_m, t)$ is an ideal of X . If $\tilde{f}_m(1) > \tilde{f}_m(a)$ for some $a \in X$, then $a \in \mathcal{L}(\tilde{f}_m, \tilde{f}_m(a))$ and hence $\mathcal{L}(\tilde{f}_m, \tilde{f}_m(a))$ is an ideal of X . Thus, $1 \in \mathcal{L}(\tilde{f}_m, \tilde{f}_m(a))$, and so $\tilde{f}_m(1) \leq \tilde{f}_m(a)$. This is a contradiction, and thus $\tilde{f}_m(1) \leq \tilde{f}_m(x)$ for all $x \in X$. Assume that there exist $a, b \in X$ such that $\tilde{f}_m(a * b) > \tilde{f}_m(b)$. Taking $t = \tilde{f}_m(b)$ implies that $b \in \mathcal{L}(\tilde{f}_m, t)$. Since $\mathcal{L}(\tilde{f}_m, t)$ is an ideal of X , we have $a * b \in \mathcal{L}(\tilde{f}_m, t)$. Hence, $\tilde{f}_m(a * b) \leq t = \tilde{f}_m(b)$, which is a contradiction. Hence, $\tilde{f}_m(x * y) \leq \tilde{f}_m(y)$ for all $x, y \in X$. Assume that there exist $a, b_1, b_2 \in X$ such that $\tilde{f}_m((b_1 * (b_2 * a)) * a) > \max\{\tilde{f}_m(b_1), \tilde{f}_m(b_2)\}$. Taking $t = \max\{\tilde{f}_m(b_1), \tilde{f}_m(b_2)\}$ implies that $(b_1 * (b_2 * a)) * a \in \mathcal{L}(\tilde{f}_m, t)$. Since $\mathcal{L}(\tilde{f}_m, t)$ is an ideal of X , we have $a \in \mathcal{L}(\tilde{f}_m, t)$. Hence, $\tilde{f}_m(a) \leq t = \max\{\tilde{f}_m(a * b), \tilde{f}_m(b)\}$, which is a contradiction. Hence, $\tilde{f}_m(x) \leq \max\{\tilde{f}_m(x * y), \tilde{f}_m(y)\}$ for all $x, y \in X$. Therefore, (X, \tilde{f}) is a mean 4-fuzzy ideal of X . \square

Corollary 5.2. *If (X, \tilde{f}) is a mean 2-fuzzy ideal of X , then $\mathcal{L}(\tilde{f}_m, t)$ is an ideal of X for all $t \in [0, 1]$ with $\mathcal{L}(\tilde{f}_m, t) \neq \emptyset$.*

Proof. This follows directly from Theorems 5.1 (2) and 5.4. □

Theorem 5.5. *If (X, \tilde{f}) is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{inf}})$ is constant and $(X, \tilde{f}_{\text{sup}})$ is a 1-fuzzy ideal of X , then (X, \tilde{f}) is a mean 1-fuzzy ideal of X .*

Proof. Assume that (X, \tilde{f}) is a hyperstructure over X in which $(X, \tilde{f}_{\text{inf}})$ is constant and $(X, \tilde{f}_{\text{sup}})$ is a 1-fuzzy ideal of X . Let $x, y \in X$. Since $(X, \tilde{f}_{\text{inf}})$ is constant, we have $\tilde{f}_{\text{inf}}(x) = \tilde{f}_{\text{inf}}(1)$ for all $x \in X$. Since $(X, \tilde{f}_{\text{sup}})$ is a 1-fuzzy ideal of X , we have $\tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) \geq \min\{\tilde{f}_{\text{sup}}(y_1), \tilde{f}_{\text{sup}}(y_2)\}$. Thus,

$$\begin{aligned} \tilde{f}_m((y_1 * (y_2 * x)) * x) &= \frac{\tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) + \tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x)}{2} \\ &= \frac{\tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x)}{2} + \frac{\tilde{f}_{\text{inf}}(1)}{2} \\ &\geq \min \left\{ \frac{\tilde{f}_{\text{sup}}(y_1)}{2} + \frac{\tilde{f}_{\text{inf}}(y_2)}{2} \right\} + \frac{\tilde{f}_{\text{sup}}(1)}{2} \\ &= \min \left\{ \frac{\tilde{f}_{\text{sup}}(y_1)}{2} + \frac{\tilde{f}_{\text{inf}}(1)}{2}, \frac{\tilde{f}_{\text{sup}}(y_2)}{2} + \frac{\tilde{f}_{\text{inf}}(1)}{2} \right\} \\ &= \min \left\{ \frac{\tilde{f}_{\text{sup}}(y_1) + \tilde{f}_{\text{inf}}(x)}{2}, \frac{\tilde{f}_{\text{sup}}(y_2) + \tilde{f}_{\text{inf}}(y_2)}{2} \right\} \\ &= \min\{\tilde{f}_m(y_1), \tilde{f}_m(y_2)\}. \end{aligned}$$

Hence, (X, \tilde{f}_m) is a 1-fuzzy ideal of X , that is, (X, \tilde{f}) is a mean 1-fuzzy ideal of X . □

Corollary 5.3. *If (X, \tilde{f}) be a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{inf}})$ is constant and $(X, \tilde{f}_{\text{sup}})$ is a 3-fuzzy ideal of X , then (X, \tilde{f}) is a mean 1-fuzzy ideal of X .*

Proof. This follows directly from Theorems 3.1 (1) and 5.5. □

Corollary 5.4. *For $j \in \{1, 3\}$, every $(2(3), j)$ -hyperfuzzy ideal of a Hilbert algebra $X = (X, *, 1)$ is a mean 1-fuzzy ideal of X .*

Proof. This follows directly from Theorem 5.5 together with Corollary 5.3. □

Theorem 5.6. *If (X, \tilde{f}) is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{inf}})$ is constant and $(X, \tilde{f}_{\text{sup}})$ is a 4-fuzzy ideal of X , then (X, \tilde{f}) is a mean 4-fuzzy ideal of X .*

Proof. Assume that (X, \tilde{f}) is a hyperstructure over X in which $(X, \tilde{f}_{\text{inf}})$ is constant and $(X, \tilde{f}_{\text{sup}})$ is a 4-fuzzy ideal of X . Let $x, y \in X$. Since $(X, \tilde{f}_{\text{inf}})$ is constant, we have $\tilde{f}_{\text{inf}}(x) = \tilde{f}_{\text{inf}}(1)$ for all $x \in X$.

Since $(X, \tilde{f}_{\text{sup}})$ is a 4-fuzzy ideal of X , we have $\tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) \leq \max\{\tilde{f}_{\text{sup}}(y_1), \tilde{f}_{\text{sup}}(y_2)\}$. Thus,

$$\begin{aligned} \tilde{f}_{\text{m}}((y_1 * (y_2 * x)) * x) &= \frac{\tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) + \tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x)}{2} \\ &= \frac{\tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x)}{2} + \frac{\tilde{f}_{\text{inf}}(1)}{2} \\ &\leq \max\left\{\frac{\tilde{f}_{\text{sup}}(y_1)}{2} + \frac{\tilde{f}_{\text{sup}}(y_2)}{2}\right\} + \frac{\tilde{f}_{\text{inf}}(1)}{2} \\ &= \max\left\{\frac{\tilde{f}_{\text{sup}}(y_1)}{2} + \frac{\tilde{f}_{\text{inf}}(1)}{2}, \frac{\tilde{f}_{\text{sup}}(y_2)}{2} + \frac{\tilde{f}_{\text{inf}}(1)}{2}\right\} \\ &= \max\left\{\frac{\tilde{f}_{\text{sup}}(y_1) + \tilde{f}_{\text{inf}}(y_1)}{2}, \frac{\tilde{f}_{\text{sup}}(y_2) + \tilde{f}_{\text{inf}}(y_2)}{2}\right\} \\ &= \max\{\tilde{f}_{\text{m}}(y_1), \tilde{f}_{\text{m}}(y_2)\}. \end{aligned}$$

Hence, $(X, \tilde{f}_{\text{m}})$ is a 4-fuzzy ideal of X , that is, (X, \tilde{f}) is a mean 4-fuzzy ideal of X . \square

Corollary 5.5. *If (X, \tilde{f}) be a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{inf}})$ is constant and $(X, \tilde{f}_{\text{sup}})$ is a 2-fuzzy ideal of X , then (X, \tilde{f}) is a mean 4-fuzzy ideal of X .*

Proof. This follows directly from Theorems 3.1 (2) and 5.6. \square

Corollary 5.6. *For $j \in \{2, 4\}$, every $(2(3), j)$ -hyperfuzzy ideal of a Hilbert algebra $X = (X, *, 1)$ is a mean 4-fuzzy ideal of X .*

Proof. This follows directly from Theorem 5.6 together with Corollary 5.5. \square

Theorem 5.7. *If (X, \tilde{f}) is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{sup}})$ is constant and $(X, \tilde{f}_{\text{inf}})$ is a 4-fuzzy ideal of X , then (X, \tilde{f}) is a mean 4-fuzzy ideal of X .*

Proof. Assume that (X, \tilde{f}) is a hyperstructure over X in which $(X, \tilde{f}_{\text{sup}})$ is constant and $(X, \tilde{f}_{\text{inf}})$ is a 4-fuzzy ideal of X . Let $x, y \in X$. Since $(X, \tilde{f}_{\text{sup}})$ is constant, we have $\tilde{f}_{\text{sup}}(x) = \tilde{f}_{\text{sup}}(1)$ for some $x \in X$. Since $(X, \tilde{f}_{\text{inf}})$ is a 4-fuzzy ideal of X , we have $\tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x) \leq \max\{\tilde{f}_{\text{inf}}(y_1), \tilde{f}_{\text{inf}}(y_2)\}$. Thus,

$$\begin{aligned} \tilde{f}_{\text{m}}((y_1 * (y_2 * x)) * x) &= \frac{\tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) + \tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x)}{2} \\ &= \frac{\tilde{f}_{\text{sup}}(1) + \tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x)}{2} \\ &= \frac{\tilde{f}_{\text{sup}}(1)}{2} + \frac{\tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x)}{2} \\ &\leq \frac{\tilde{f}_{\text{sup}}(1)}{2} + \max\left\{\frac{\tilde{f}_{\text{inf}}(y_1)}{2}, \frac{\tilde{f}_{\text{inf}}(y_2)}{2}\right\} \\ &= \max\left\{\frac{\tilde{f}_{\text{sup}}(1)}{2} + \frac{\tilde{f}_{\text{inf}}(y_1)}{2}, \frac{\tilde{f}_{\text{sup}}(1)}{2} + \frac{\tilde{f}_{\text{inf}}(y_2)}{2}\right\} \\ &= \max\left\{\frac{\tilde{f}_{\text{sup}}(y_1) + \tilde{f}_{\text{inf}}(y_1)}{2}, \frac{\tilde{f}_{\text{sup}}(y_2) + \tilde{f}_{\text{inf}}(y_2)}{2}\right\} \\ &= \max\{\tilde{f}_{\text{m}}(y_1), \tilde{f}_{\text{m}}(y_2)\}. \end{aligned}$$

Hence, (X, \tilde{f}_m) is a 4-fuzzy ideal of X , that is, (X, \tilde{f}) is a mean 4-fuzzy ideal of X . \square

Corollary 5.7. *If (X, \tilde{f}) be a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{sup}})$ is constant and $(X, \tilde{f}_{\text{inf}})$ is a 2-fuzzy ideal of X , then (X, \tilde{f}) is a mean 4-fuzzy ideal of X .*

Proof. This follows directly from Theorems 3.1 (2) and 5.7. \square

Corollary 5.8. *For $i \in \{2, 4\}$, every $(i, 2(3))$ -hyperfuzzy ideal of a Hilbert algebra $X = (X, *, 1)$ is a mean 4-fuzzy ideal of X .*

Proof. This follows directly from Theorem 5.7 together with Corollary 5.7. \square

Theorem 5.8. *If (X, \tilde{f}) is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{sup}})$ is constant and $(X, \tilde{f}_{\text{inf}})$ is a 1-fuzzy ideal of X , then (X, \tilde{f}) is a mean 1-fuzzy ideal of X .*

Proof. Assume that (X, \tilde{f}) is a hyperstructure over X in which $(X, \tilde{f}_{\text{sup}})$ is constant and $(X, \tilde{f}_{\text{inf}})$ is a 1-fuzzy ideal of X . Let $x, y \in X$. Since $(X, \tilde{f}_{\text{sup}})$ is constant, we have $\tilde{f}_{\text{sup}}(x) = \tilde{f}_{\text{sup}}(1)$ for all $x \in X$. Since $(X, \tilde{f}_{\text{inf}})$ is a 1-fuzzy ideal of X , we obtain $\tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x) \geq \min\{\tilde{f}_{\text{inf}}(y_1), \tilde{f}_{\text{inf}}(y_2)\}$. Thus,

$$\begin{aligned} \tilde{f}_m((y_1 * (y_2 * x)) * x) &= \frac{\tilde{f}_{\text{sup}}((y_1 * (y_2 * x)) * x) + \tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x)}{2} \\ &= \frac{\tilde{f}_{\text{sup}}(1) + \tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x)}{2} \\ &= \frac{\tilde{f}_{\text{sup}}(1)}{2} + \frac{\tilde{f}_{\text{inf}}((y_1 * (y_2 * x)) * x)}{2} \\ &\geq \frac{\tilde{f}_{\text{sup}}(1)}{2} + \min\left\{\frac{\tilde{f}_{\text{inf}}(y_1)}{2}, \frac{\tilde{f}_{\text{inf}}(y_2)}{2}\right\} \\ &= \min\left\{\frac{\tilde{f}_{\text{sup}}(1)}{2} + \frac{\tilde{f}_{\text{inf}}(y_1)}{2}, \frac{\tilde{f}_{\text{sup}}(1)}{2} + \frac{\tilde{f}_{\text{inf}}(y_2)}{2}\right\} \\ &= \min\left\{\frac{\tilde{f}_{\text{sup}}(y_1) + \tilde{f}_{\text{inf}}(y_1)}{2}, \frac{\tilde{f}_{\text{sup}}(y_2) + \tilde{f}_{\text{inf}}(y_2)}{2}\right\} \\ &= \min\{\tilde{f}_m(y_1), \tilde{f}_m(y_2)\}. \end{aligned}$$

Hence, (X, \tilde{f}_m) is a 1-fuzzy ideal of X , that is, (X, \tilde{f}) is a mean 1-fuzzy ideal of X . \square

Corollary 5.9. *If (X, \tilde{f}) is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\text{sup}})$ is constant and $(X, \tilde{f}_{\text{inf}})$ is a 3-fuzzy ideal of X , then (X, \tilde{f}) is a mean 1-fuzzy ideal of X .*

Proof. This follows directly from Theorems 3.1 (1) and 5.8. \square

Corollary 5.10. *For $i \in \{1, 3\}$, every $(i, 2(3))$ -hyperfuzzy ideal of a Hilbert algebra $X = (X, *, 1)$ is a mean 1-fuzzy ideal of X .*

Proof. This follows directly from Theorem 5.8 together with Corollary 5.9. \square

CONCLUSION

In this paper, we have introduced and investigated the notions of the length and the mean of a hyperstructure in Hilbert algebras. Based on these concepts, we defined two new classes of fuzzy ideals, namely, length fuzzy ideals and mean fuzzy ideals, and studied their fundamental properties. The four types of k -fuzzy ideals for $k \in \{1, 2, 3, 4\}$ were employed as a general framework to characterize these structures and to examine their algebraic behaviors.

Several characterization theorems have been established to provide necessary and sufficient conditions for a hyperstructure to form a length (resp., mean) k -fuzzy ideal. Moreover, we explored in detail the interrelations between length (resp., mean) fuzzy ideals and hyperfuzzy ideals, as well as their connections with upper-, lower-, and equal-level subsets. These results extend and generalize classical theories of fuzzy ideals in Hilbert algebras.

The study reveals that specific combinations of constant and fuzzy structures naturally yield length- or mean-based fuzzy ideals, thereby strengthening the link between hyperfuzzification and the underlying algebraic operations. The examples and counterexamples provided illustrate the validity and the non-reversibility of some implications, offering deeper insight into the logical and algebraic structure of Hilbert algebras.

The concepts developed in this work may serve as a foundation for further investigations into fuzzy and hyperfuzzy algebraic systems, particularly in exploring new types of hyperfuzzy filters and homomorphisms, and their applications in non-classical logic and information systems.

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