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OPTIMAL HARVESTING FOR AGE-SIZE-STRUCTURED POPULATION DYNAMICS WITH NONLOCAL DIFFUSION

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Abstract. We study an optimal control problem for a linear population dynamics model structured by age and size. We establish an existence and uniqueness result and prove the existence of optimal control. We also establish the necessary optimality conditions.

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1. Introduction

In this paper, we consider the following linear age-size-dependent population dynamics system:

$$\begin{cases} \partial_t u + \partial_a u + \partial_s (g(s)u) - \Delta u + \mu u = f - uv & \text{in } Q, \\ u(x,t,0,s) = \int_0^A \int_0^S \beta(a,p,s) u(x,t,a,s) dp da & \text{in } Q_{T,S}, \\ u(x,t,a,0) = \int_0^S \int_0^A \lambda(a,p,s) u(x,t,a,s) dp ds & \text{in } Q_{T,A}, \\ u(x,0,a,s) = u_0(x,a,s) & \text{in } Q_{A,S} \\ u(x,t,a,s) = 0 & \text{on } \Sigma \end{cases} \tag{1}$$

where $Q = (0,T) \times (0,A) \times (0,S) \times (0,S) \times \Omega$, $\Sigma = \partial\Omega \times (0,T) \times (0,A)$ and $Q_{X,Y} = (0,X) \times (0,Y) \times \Omega;\ X, Y > 0$. We denote by T a positive real number, A represents the maximum life of an individual, and S the maximum size that this individual can reach during their lifetime. The mortality process is controlled by the age-size-dependent mortality modulus $\mu(a,s)$ and the reproductive process is controlled by the fertility modulus $\beta(a,p,s)$. We denote by $u(x,t,a,s)$ the density of individuals at time t, of age a and size s present at position x. In the model (2), size is viewed as a continuum variable s specific to individuals, such as mass, volume, length, maturity, bacterial or viral load, or other physiologic or demographic property. It is assumed that size increases in the same way for all

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individuals in the population, as controlled by a growth modulus $g(s)$. The interpretation of the growth modulus is that

$$\int_a^b \frac{1}{g(s)} ds$$

is the time required for an individual to increase size from a to b where $0 \leqslant a < b \leqslant S$.
In the sequel we study the following optimal control problem:

$$\text{Find } v^* \in \mathcal{U} \text{ such that } J(v^*) = \max_{v \in \mathcal{U}} J(v) \tag{2}$$

where

$$J(v) = \int_Q w(x,t,a,s)v(x,t,a,s)u(x,t,a,s)dxdtdads$$

and the set of controllers is

$$\mathcal{U} = \left\{v \in L^2(Q) : \zeta_1(x,t,a,s) \leq v(x,t,a,s) \leq \zeta_2(x,t,a,s) \text{ a.e. } (x,t,a,s) \in Q\right\}$$

for some $\zeta_1, \zeta_2 \in L^\infty(Q), 0 \leq \zeta_1(x,t,a,s) \leq \zeta_2(x,t,a,s)$ a.e. in Q.
It is clear that $\mathcal{U}$ is a closed convex subset of $L^2(Q)$. As in ([1]) and ([13]), the normal cone to $\mathcal{U}$ at any point v of $\mathcal{U}$ is

$$N_{\mathcal{U}}(y) = \left\{z \in L^2(\Omega);\ z(x,t,a,s) \leq 0 \text{ for } y(x,t,a,s) = \zeta_1(x,t,a,s) < \zeta_2(x,t,a,s),\right.$$
$$\left.\text{and } \ z(x,t,a,s) \geq 0 \text{ for } y(x,t,a,s) = \zeta_2(x,t,a,s) > \zeta_1(x,t,a,s)\right\}.$$

An element $z \in L^2(Q)$ belongs to $N_{\mathcal{U}}(y)$ if and only if

$$\int_Q z(x,t,a,s)(v(x,t,a,s) - y(x,t,a,s))dxdadsdt \leq 0, \text{ for any } v \in \mathcal{U}. \tag{3}$$

From a biological point of view $w(x,t,a,s) \geq 0$ is a weight (the price of an individual of age a at time t and location x) and $u_0(x,a,s) \geq 0$ is the initial distribution of population.

The same type of problem has been studied by Ainseba and al in [2] and Anita in [1]. In these two works, the author considers population dynamics that do not depend on size. However, he takes into account much more general mortality and fertility rates. In [3], the author studies the same problem on a population dynamic structured by size but not by age. In [4], the authors study the problem of optimal harvesting for a periodic population dynamic dependent on age. The harvesting problem for the age-structured population of linear initial value has been previously studied in [5], [6], [7], [8], [9].

In this article, we consider the optimization problem (2) on a population dynamics structured by age and size with non-local initial conditions in age and size.

We assume the following hypotheses:

(H1) The fertility rate β and λ satisfy

$$\beta \in L^\infty([0,A)^2 \times [0,S]),\ \beta(a,p,s) \geq 0 \text{ a.e. } (a,p,s) \in [0,A)^2 \times [0,S]$$
$$\lambda \in L^\infty([0,A) \times [0,S)^2),\ \lambda(a,p,s) \geq 0 \text{ a.e. } (a,p,s) \in [0,A) \times [0,S)^2$$

(H2) The mortality rate satisfies

$$\mu \in L^{\infty}_{\text{loc}}\left(\bar{\Omega} \times [0,T] \times [0,A) \times [0,S]\right), \mu(x,t,a,s) \geq \mu_0(a,t,s) \geq 0 \text{ a.e. } (x,t,a,x) \in Q,$$
$$\text{where } \mu_0 \in L^{\infty}_{\text{loc}}\left([0,T] \times [0,A) \times ([0,S]\ \right) \text{ and}$$

$$\int_0^A \mu_0\left(t+a-A,a,s\right) da = +\infty, \qquad \text{a.e. } t \in (0,T), \qquad s \in (0,S)$$

(H3) $u_0 \in L^2\left(Q_{A,S}\right), u_0(x,a,s) \geq 0$ a.e. $(x,a,s) \in \Omega \times (0,A) \times (0,S)$.

(H4) $f \in L^2(Q), w \in L^{\infty}(Q), f(x,t,a,s), w(x,t,a,s) \geq 0$ a.e. $(x,t,a,s) \in Q$.

(H5) $g : [0,S] \to [0,\infty)$ is continuously differentiable and $|\partial_s g(s)| \leq L_g$ for some constant $L_g > 0$, $g(s) > 0$ if $s \in [0,S)$ and $g\left(S\right) = 0$.

In Section 2, we will prove the existence and uniqueness of solutions to the model (1) using Banach's fixed point theorem. The Section 3 is devoted to the existence of solutions to the problem (2). We will conclude in Section 4 with the characterization of the control.

2. Existence, uniqueness of solution

Let us consider here the following auxiliary model.

$$\begin{cases} \partial_t u + \partial_a u + \partial_s\left(g(s)u\right) - \Delta u + \mu u = f & \text{in } Q, \\ u(x,t,0,s) = \int_0^A \int_0^S \beta(a,p,s)u(x,t,a,s)dpda & \text{in } Q_{T,S}, \\ u(x,t,a,0) = \int_0^S \int_0^A \lambda(a,p,s)u(x,t,a,s)dpds & \text{in } Q_{T,A}, \\ u(x,0,a,s) = u_0(x,a,s) & \text{in } Q_{A,S} \\ u(x,t,a,s) = 0 & \text{on } \Sigma \end{cases} \tag{4}$$

Theorem 2.1. *Under the hypotheses* **(*H1*)** - **(*H5*)**, *there exists a unique positive solution* $u \in L^{\infty}(0,T;L^2(Q_{A,S})$ *to* (4) *and we have*

$$\|u(.,t,.,.)\|^2_{L^2(Q_{A,S})} \leqslant C_T\left(\|u_0\|^2_{L^2(Q_{A,S})} + \|f\|^2_{L^2(Q)}\right), \quad t \in [0,T]. \tag{5}$$

Proof of theorem 2.1

We write the equation (4) following the characteristic curves $\tau \to (\tau + c_1, \tau + c_2, G^{-1}(\tau + G(c_3))$ where c_1, c_2 and c_3 are constants at our disposal and $\frac{ds}{d\tau} = g(s)$. To do this, we set $\overline{m}(\tau) = u(\tau + c_1, \tau + c_2, G^{-1}(\tau + G(c_3)))$, then $\overline{m}$ satisfied the following ordinary differential equation

$$\begin{cases} \frac{d\overline{m}(\tau)}{d\tau} + \triangle\overline{m}(\tau) = F(u)(\tau), \quad t \in \mathbb{R} \\ \overline{m}(0) = u(c_1,c_2,c_3) \end{cases} \tag{6}$$

where $F(u)(\tau) = -g'(G^{-1}(\tau + G(c_3)))u(x,\tau+c_1,\tau+c_2,G^{-1}(\tau+G(c_3))) - \mu(\tau+c_2,G^{-1}(\tau+G(c_3)))u(x,\tau+c_1,\tau+c_2,G^{-1}(\tau+G(c_3))) + f(x,\tau+c_1,\tau+c_2,G^{-1}(\tau+G(c_3))) + f(x,\tau+c_1,\tau+$

$c_2, G^{-1}(\tau + G(c_3)))$.

Thus,

$$\overline{m}(\tau) = T(\tau)\overline{m}(0) + \int_0^{\tau} T(\tau-\sigma)F(u)(\sigma)d\sigma, \tag{7}$$

where $\{T(t), t \geqslant 0\}$ is the semigroup of Δ in $L^2(\Omega)$ (see [10]). By choosing appropriate values for the constants τ, c_1, c_2, and c_3 in (7), we obtain

$$u(x,t,a;s) = \begin{cases} T(t)u(0, a-t, G^{-1}(G(s)-t)) + \int_0^t T(t-\sigma)F(u)(\sigma) & \text{a.e. } a > t \\ T(a)u(t-a, 0, G^{-1}(G(s)-a)) + \int_0^a T(a-\sigma)F(u)(\sigma) & \text{a.e. } a < t \\ T(G(s))u(t-G(s), a-G(s), 0) + \int_0^{G(s)} T(G(s)-\sigma)F(u)(\sigma(\sigma) & \text{a.e. } a > G(s),\ t > G(s). \end{cases} \tag{8}$$

We know that:

$$\|u(.,t,.,.)\|^2_{L^2(Q)} = \int_0^t \int_0^S \int_{\Omega} |u(x,t,a,s)|^2 dxdsda + \int_t^A \int_0^S \int_{\Omega} |u(x,t,a,s)|^2 dxdsda$$

. Using Young's inequality, we get

$$\int_t^A \int_0^S \int_{\Omega} |u(x,t,a,s)|^2 dxdsda \leqslant C\left(K_1 + K_2 + K_3 + K_4 + K_5 + K_6\right)$$

where C is a positive constant; K_1, K_2, K_3, K_4 and K_5 are defined as follows:

$$K_1 = \int_t^A \int_0^S \int_{\Omega} \left|T(t)u(x,0,a-t,G^{-1}(G(s)-t))\right|^2 dxdsda \leqslant C_1 \|u_0\|^2_{L^2(Q_{A,S})},$$

$$K_2 = \int_t^A \int_0^S \int_{\Omega} \left(\int_0^t |T(t-\sigma)g'(G^{-1}(G(s)-t))u(x,\sigma,\sigma+a-t,G^{-1}(\sigma+G(s)-t))|d\sigma \right)^2 dxdsda$$
$$\leqslant C_2(T) \int_0^t \|u(.,\sigma,.,.)\|^2_{L^2(Q_{A,S})} d\sigma,$$

$$K_3 = \int_t^A \int_0^S \int_{\Omega} \left(\int_0^t |T(t-\sigma)\mu(a-t,G^{-1}(G(s)-t))u(x,\sigma,\sigma+a-t,G^{-1}(\sigma+G(s)-t))|d\sigma \right)^2 dxdsda$$
$$\leqslant C_3(T) \int_0^t \|u(.,\sigma,.,.)\|^2_{L^2(Q_{A,S})} d\sigma,$$

$$K_4 = \int_t^A \int_0^S \int_{\Omega} \left(\int_0^t |T(t-\sigma)| \int_{\Omega} J(x-y)u(y,\sigma,\sigma+a-t,G^{-1}(\sigma+G(s)-t))dyd\sigma \right)^2 dxdsda$$
$$\leqslant C_4(T,\Omega) \int_0^t \|u(.,\sigma,.,.)\|^2_{L^2(Q_{A,S})} d\sigma,$$

$$K_5 = \int_t^A \int_0^S \int_{\Omega} \left(\int_0^t |T(t-\sigma)u(x,\sigma,\sigma+a-t,G^{-1}(\sigma+G(s)-t))d\sigma \right)^2 dxdsda$$
$$\leqslant C_5(T) \int_0^t \|u(.,\sigma,.,.)\|^2_{L^2(Q_{A,S})} d\sigma$$

and

$$K_6 = \|f\|^2_{L^2(Q)}.$$

So, we have

$$\int_t^A \int_0^S \int_\Omega |u(x,t,a,s)|^2 dxdsda \leqslant C\left(\|u_0\|^2_{L^2(Q_{A,S})} + \|f\|^2_{L^2(Q)} + \int_0^t \|u(.,\sigma,.,.)\|^2_{L^2(Q_{A,S})} d\sigma \right). \quad (9)$$

We show similarly that

$$\int_0^t \int_0^S \int_\Omega |u(x,t,a,s)|^2 dxdsda \leqslant C\left(\|f\|^2_{L^2(Q)} + \int_0^t \|u(.,\sigma,.,.)\|^2_{L^2(Q_{A,S})} d\sigma \right). \quad (10)$$

Combining (9) and (10), we then have that

$$\|u(.,t,.,.)\|^2_{L^2(Q_{A,S})} \leqslant C\left(\|u_0\|^2_{L^2(Q_{A,S})} + \|f\|^2_{L^2(Q)} + \int_0^t \|u(.,\sigma,.,.)\|^2_{L^2(Q_{A,S})} d\sigma \right). \quad (11)$$

Using Gronwall's lemma, we obtain from (11) the following inequality

$$\|u(.,t,.,.)\|^2_{L^2(Q_{A,S})} \leqslant C\left(\|u_0\|^2_{L^2(Q_{A,S})} + \|f\|^2_{L^2(Q)} \right) \exp\{Ct\}, \quad t \in [0,T]. \quad (12)$$

Thus, starting from (12), the estimate (5) of theorem 2.1 can be easily derived.

We now consider the following norm :

$$\|p\|_\gamma := \text{ess} \sup_{t\in(0,T)} e^{-\gamma t} \|p(\cdot,t,\cdot)\|_{L^2(Q_{A,S})}$$

in $L^\infty\left(0,T; L^2(Q_{A,S})\right)$ for $\gamma > 0$.

Let's define the application $\Phi : L^\infty_+\left(0,T; L^2(Q_{A,S})\right) \longrightarrow L^\infty_+\left(0,T; L^2(Q_{A,S})\right), u \longrightarrow \Phi(u)$

where

$$\Phi(u)(x,t,a;s) = \begin{cases} T(t)u(0,a-t,G^{-1}(G(s)-t)) + \int_0^t T(t-\sigma)F(u)(\sigma) & \text{a.e. } a > t \\ T(a)u(t-a,0,G^{-1}(G(s)-a)) + \int_0^a T(a-\sigma)F(u)(\sigma) & \text{a.e. } a < t \\ T(G(s))u(t-G(s),a-G(s),0) + \int_0^{G(s)} T(G(s)-\sigma)F(u)(\sigma(\sigma) & \text{a.e. } a > G(s),\ t > G(s). \end{cases}$$

Let $p_1, p_2 \in L^2(Q)$, by linearity, $\Phi(p_1) - \Phi(p_2)$ satisfies (8) with $u_0 \equiv 0$ and $f \equiv 0$. Hence by (11), we have

$$\|\Phi(p_1)(.,t,.,.) - \Phi(p_2)(.,t,.,.)\|^2_{L^2(Q_{A,S})} \leqslant C_T \int_0^t \|p_1(.,\sigma,.,.) - p_2(.,\sigma,.,.)\|^2_{L^2(Q_{A,S})} d\sigma$$

where C_T is a positve constante. Thus,,

$$\begin{aligned} \|\Phi(p_1) - \Phi(p_2)\|_\gamma &\leqslant e^{-2\gamma t} \|\Phi(p_1)(\cdot,t,.,.) - \Phi(p_1)(\cdot,t,.,.)\|^2_{L^2(Q_{A,S})} \\ &\leq C_T e^{-2\gamma t} \int_0^t e^{2\gamma\sigma} e^{-2\gamma\sigma} \|p_1(\cdot,\sigma) - p_2(\cdot,\sigma)\|^2_{L^2(Q)} dt \leq \frac{C_T}{2\gamma} \|p_1 - p_2\|^2_\gamma \end{aligned}$$

Therefore, for sufficiently large γ, Φ is a contraction in $L^\infty_+\left(0,T;L^2(Q_{A,S})\right)$.

The comparison result in Garroni et al [11] and in Langlais [12] implies that the solution u^v satisfies

$$0 \leqslant u^v(x,t,a,) \leqslant \overline{u}(x,t,a,s) \text{ a.e in } Q, \tag{13}$$

where $\overline{u} \in L^\infty_+(Q)$ is the solution of (4) corresponding to $\mu \equiv 0$ and $\beta \equiv \|\beta\|_{L^\infty(Q)}$
and $\lambda \equiv \|\lambda\|_{L^\infty(Q)}$ □

Corollary 2.1. *The model* (1) *has a unique positive solution in* $L^\infty\left(0,T;L^2(Q_{A,S})\right)$

Proof of corollary 2.1

It suffit to apply the theorem (2.1) taking into account (4) the mortality $\mu = \mu + v$ □

In the rest of our work, we will take $g(s) = 1,$ a.e $s \in (0,S)$.

3. Existence of an optimal control

The following proposition guarantees that the solution of the model (1) depends continuously on the control v, i.e., a small change in the control v does not cause a large change in the solution

Proposition 3.1. *The mapping* $v \longrightarrow u(v)$ *is lipschitzian of* $\mathcal{U}$ *in* $L^2(0,T;L^2(Q_{A,S}))$

Proof of proposition 3.1

Let for any v_1 and v_2 in $\mathcal{U}$, let u_i and u_2 be the corresponding solution of (1). Let $y = y_1 - y_2$ with $y_i = e^{-\lambda_0 t}u_i$ and $\tilde{v}_i = e^{-\lambda_0 t}v_i$ for $i \in \{1,2\}$, where λ_0 is positive parameter that will be choose later. Then y satisfies the following system

$$\begin{cases} \partial_t y + \partial_a y + \partial_s y - \Delta y + (\mu+\lambda_0)y = e^{\lambda_0 t}\left[\tilde{v}_1 y + y_2(\tilde{v}_1 - \tilde{v}_2)\right] & \text{in } Q, \\ y(x,t,0,s) = \int_0^A \int_0^S \beta(a,p,s)y(x,t,a,s)dpda & \text{in } Q_{T,S}, \\ y(x,t,a,0) = \int_0^S \int_0^A \lambda(a,p,s)y(x,t,a,s)dpds & \text{in } Q_{T,A}, \\ y(x,0,a,s) = 0 & \text{in } Q_{A,S} \\ y(x,t,a,s) = 0 & \text{on } \Sigma \end{cases} \tag{14}$$

Multiplying the (14) by y and integrating over Q and using the Cauchy-Schwarz's inequality,we obtain:

$$\begin{aligned} &\frac{1}{2}\|y(T)\|^2_{L^2(Q_{A,S})} + \frac{1}{2}\|y(A)\|^2_{L^2(Q_{T,S})} + \frac{1}{2}\|y(S)\|^2_{L^2(Q_{T,A})} - \frac{1}{2}\|y(.,.,0,.)\|^2_{L^2(Q_{T,S})} \\ &\quad - \frac{1}{2}\|y(.,.,.,0)\|^2_{L^2(Q_{T,A})} + \|\nabla y\|^2_{L^2(Q)} + \|\sqrt{\mu+\lambda_0}\,y\|^2_{L^2(Q)} \\ &\quad \leqslant \|\zeta_2\|_{L^\infty(Q)}\|y\|^2_{L^2(Q)} + \|y\|^2_{L^2(Q)} + \|\overline{u}\|^2_{L^\infty(Q)}\|\tilde{v}_1 - \tilde{v}_2\|^2_{L^2(Q)} \end{aligned} \tag{15}$$

On the one hand, we have the estimates

$$\|y(.,.,0,.)\|^2_{L^2(Q_{T,S})} \leqslant AS^2\|\beta\|^2_{L^\infty(Q)}\|y\|^2_{L^2(Q)} \tag{16}$$

and

$$\|y(.,.,.,0)\|^2_{L^2(Q_{T,A})} \leqslant SA^2\|\lambda\|^2_{L^\infty(Q)}\|y\|^2_{L^2(Q)}. \tag{17}$$

Combining (15), (16 and (17) and choose λ_0 such that
$\lambda_0 = \frac{1}{2}AS^2\|\beta\|^2_{L^\infty(Q)} + \frac{1}{2}SA^2\|\lambda\|^2_{L^\infty(Q)} + \|\zeta_2\|_{L^\infty(Q)} + 1$, we have

$$\|y_1 - y_2\|^2_{L^2(Q)} \leqslant \|\overline{u}\|^2_{L^\infty(Q)}\|\tilde{v}_1 - \tilde{v}_2\|^2_{L^2(Q)} \tag{18}$$

□

Theorem 3.1. *The problem* (2) *admits at least one optimal pair* (u^*, v^*).

Proof of theorem 3.1

Define $\Psi : \mathcal{U} \to \mathbb{R}_+$ by

$$\Psi(v) = \int_Q w(x,t,a,s)v(x,t,a,s)u^v(x,t,a,s)dxdtdads$$

and put $d = \sup_{v\in\mathcal{U}} \Psi(v)$. By (5), we have that

$$\|u^v(\cdot,t\cdot,\cdot)\|_{L^2(Q)} \leq C \tag{3.1}$$

for some constant $C > 0$ independent of $v \in \mathcal{U}$. Hence we have

$$0 \leq d \leqslant \|w\|_{L^\infty(Q)}\|\zeta_2\|_{L^\infty(Q)}\|u^v\|_{L^2(Q)} < \infty.$$

Let $\{v_N\} \subset \mathcal{U}$ be a sequence satisfying $d - \frac{1}{N} < \Psi(v_N) \leq d$. By (3.1), $\|u^v\|_{L^2(\mathcal{Q}_T)}$ is bounded in v and hence there exists a subsequence denoted again by $\{v_N\}$ such that u^{v_N} converges weakly to some u^* in $L^2(Q)$. Using Mazur's theorem ([1]), there exists a sequence $\{\widetilde{u}_N\}$ in $L^2(Q)$ satisfying

$$\widetilde{u}_N(x,t,a,s) = \sum_{i=N+1}^{k_N} \lambda_i^N u^{v_i}(x,t,a,s), \lambda_i^N \geq 0, \sum_{i=N+1}^{k_N} \lambda_i^N = 1\,(k_N \geq N+1),$$
$$\widetilde{u}_N \to u^* \quad \text{strongly in } L^2(Q).$$

Define now the sequence $\{\widetilde{v}_N\}$ by

$$\widetilde{v}_N(x,t,a,s) = \begin{cases} \frac{\sum_{i=N+1}^{k_N} \lambda_i^N v_i(x,t,a,s)u^{v_i}(x,t,a,s)}{\sum_{i=N+1}^{k_N} \lambda_i^N u^{v_i}(x,t,a,s)}, & \text{if } \sum_{i=N+1}^{k_N} \lambda_i^N u^{v_i}(x,t,a,s) \neq 0, \\ \zeta_1(x,t,a,s), & \text{if } \sum_{i=N+1}^{k_N} \lambda_i^N u^{v_i}(x,t,a,s) = 0. \end{cases}$$

Recalling tha $v_i \in \mathcal{U}$, then $\widetilde{v}_N \in \mathcal{U}$. Since $\{\widetilde{v}_N\}$ is bounded in $L^2(Q)$, there exists a subsequence denoted again by $\{\widetilde{v}_N\}$ which converges weakly to some v^* in $L^2(Q)$. We shall show that this v^* gives an optimal control to the optimal harvesting problem (2). Note that by linearity, $\widetilde{u}_N$ is a solution to (1) with $\widetilde{v}_N$ instead of v.

Since u^{v_i} satisfies

$$\begin{cases} \partial_t u^{v_i} + \partial_a u^{v_i} + \partial_s \left(g(s) u^{v_i}\right) - \Delta u^{v_i} + \mu u^{v_i} = f - u^{v_i} v_i & \text{in } Q, \\ u^{v_i}(x,t,0,s) = \int_0^A \int_0^S \beta(a,p,s) u^{v_i}(x,t,a,s) dpda & \text{in } Q_{T,S}, \\ u^{v_i}(x,t,a,0) = \int_0^S \int_0^A \lambda(a,p,s) u^{v_i}(x,t,a,s) dpds & \text{in } Q_{T,A}, \\ u^{v_i}(x,0,a,s) = u_0(x,a,s) & \text{in } Q_{A,S} \\ u^{v_i}(x,t,a,s) = 0 & \text{on } \Sigma \end{cases} \tag{19}$$

Thus, by linearity, $\widetilde{u}_N$ satisfies

$$\begin{cases} \partial_t \widetilde{u}_N + \partial_a \widetilde{u}_N + \partial_s \left(g(s) \widetilde{u}_N\right) - \Delta \widetilde{u}_N + \mu \widetilde{u}_N = f - \widetilde{u}_N \widetilde{v}_N & \text{in } Q, \\ \widetilde{u}_N(x,t,0,s) = \int_0^A \int_0^S \beta(a,p,s) \widetilde{u}_N(x,t,a,s) dpda & \text{in } Q_{T,S} \\ \widetilde{u}_N(x,t,a,0) = \int_0^S \int_0^A \lambda(a,p,s) \widetilde{u}_N(x,t,a,s) dpds & \text{in } Q_{T,A}, \\ \widetilde{u}_N(x,0,a,s) = u_0(x,a,s) & \text{in } Q_{A,S} \\ \widetilde{u}_N(x,t,a,s) = 0 & \text{on } \Sigma \end{cases} \tag{20}$$

Since $\widetilde{u}_N$ converges strongly to u^* in $L^2\left(\mathcal{Q}_T\right)$ and $\widetilde{u}_N \in L^\infty\left(0,T;L^2(Q)\right)$, we find that $u^* \in L^\infty\left(0,T;L^2(Q)\right)$.

Thus,

$$\int_0^S \int_0^A \beta(a,p,s) \widetilde{u}_N(x,t,a,p) dadp \rightharpoonup \int_0^S \int_0^A \beta(a,p,s) u^*(x,t,a,p) dadp$$

and

$$\int_0^A \int_0^S \lambda(a,p,s) \widetilde{u}_N(x,t,p,s) dsdp \rightharpoonup \int_0^A \int_0^S \lambda(a,p,s) u^*(x,t,p,s) dsdp$$

.

Therefore, by taking the limit in (20), u^* satisfies

$$\begin{cases} \partial_t u^* + \partial_a u^* + \partial_s \left(g(s) u^*\right) - \Delta u^* + \mu u^* = f - u^* v^* & \text{in } Q, \\ u^*(x,t,0,s) = \int_0^A \int_0^S \beta(a,p,s) u^*(x,t,a,s) dpda & \text{in } Q_{T,S}, \\ u^*(x,t,a,0) = \int_0^S \int_0^A \lambda(a,p,s) u^*(x,t,a,s) dpds & \text{in } Q_{T,A}, \\ u^*(x,0,a,s) = u_0(x,a,s) & \text{in } Q_{A,S} \\ u^*(x,t,a,s) = 0. & \text{on } \Sigma \end{cases} \tag{21}$$

Then u^* is a solution of (1) for $v = v^*$. By uniqueness of solutions, we conclude that $u^* = u^{v^*}$ and

$$\int_Q w(x,t,a,s) \widetilde{v}_N(x,t,a,s) \widetilde{u}_N(x,t,a,s) dxdtdads$$
$$\to \int_Q w(x,t,a,s) v^*(x,t,a,s) u^*(x,t,a,s) dxdtdads = \Psi\left(v^*\right)$$

as $N \to \infty$. On the other hand,

$$\begin{aligned} &\int_Q \ w(x,t,a,s) \widetilde{v}_N(x,t,a,s) \widetilde{u}_N(x,t,a,s) dxdtdads \\ &= \ \sum_{i=N+1}^{k_N} \lambda_i^N \int_Q w(x,t,a,s) v_i(x,t,a,s) u^{v_i}(x,t,a,s) dxdtdads = \sum_{i=N+1}^{k_N} \lambda_i^N \Psi\left(v_i\right) \to d \end{aligned}$$

as $N \to \infty$. Thus we obtain $d = \Psi(v^*)$ and we conclude that (v^*, u^*) is an optimal solution to the problem (2). □

4. Necessary optimal conditions

Let's (u^*, v^*) is an optimal pair for (2) and consider the following system

$$\begin{cases} -\partial_t q - \partial_a q - \partial_s q - \Delta q + \mu q - q(x,t,0,s)\int_0^S \beta(a,p,s)dp & \\ -q(x,t,a,0)\int_0^A \lambda(a,p,s)dp = -v^*(w+q)(x,t,a,s) & \text{in } Q, \\ q(x,t,A,s) = 0 & \text{in } Q_{T,S}, \\ q(x,t,a,S) = 0 & \text{in } Q_{T,A}, \\ q(x,T,a,s) = 0 & \text{in } Q_{A,S} \\ \frac{\partial q}{\partial \eta} = 0. & \text{on } \Sigma \end{cases} \tag{22}$$

Proposition 4.1. *The system (22) admits an unique solution in $L^2(0,T;L^2(Q_{A,S}))$*

Proof of proposition 4.1

Let's make the following variable changes: $t = t - T$, $a = A - a$ and $s = S - s$ and adopting always the same notations, the system (22) can be written

$$\begin{cases} \partial_t q + \partial_a q + \partial_s q - \Delta q + \mu q - q(x,t,A,s)\int_0^S \beta(a,p,s)dp & \\ -q(x,t,a,S)\int_0^A \lambda(a,p,s)dp = -v^*(w+q)(x,t,a,s) & \text{in } Q, \\ q(x,t,0,s) = 0 & \text{in } Q_{T,S}, \\ q(x,t,a,0) = 0 & \text{in } Q_{T,A}, \\ q(x,0,a,s) = 0 & \text{in } Q_{A,S} \\ \frac{\partial q}{\partial \eta} = 0. & \text{on } \Sigma \end{cases} \tag{23}$$

Let us introduce the following auxiliary system:

$$\begin{cases} \partial_t y + \partial_a y + \partial_s y - \Delta y + (\lambda_0 + \mu)y - y(x,t,A,s)\int_0^S \beta(a,p,s)dp & \\ -y(x,t,a,S)\int_0^A \lambda(a,p,s)dp = -v^*(\tilde{w}+y)(x,t,a,s) & \text{in } Q, \\ y(x,t,0,s) = 0 & \text{in } Q_{T,S}, \\ y(x,t,a,0) = 0 & \text{in } Q_{T,A}, \\ y(x,0,a,s) = 0 & \text{in } Q_{A;S} \\ \frac{\partial q}{\partial \eta} = 0. & \text{on } \Sigma \end{cases} \tag{24}$$

où $\tilde{w} = e^{-\lambda t} w$ and λ_0 a positive parameter.

Let us show that (24) has an unique solution. For that, consider now the following system

$$\begin{cases} \partial_t y + \partial_a y + \partial_s y - \Delta y + (\lambda_0 + \mu) y - \varphi(x,t,s) \int_0^S \beta(a,p,s) dp & \\ -\theta(x,t,a) \int_0^A \lambda(a,p,s) dp = -v^*(\tilde{w} + y)(x,t,a,s) & \text{in } Q, \\ y(x,t,0,s) = 0 & \text{in } Q_{T,S}, \\ y(x,t,a,0) = 0 & \text{in } Q_{T,A}, \\ y(x,0,a,s) = 0 & \text{in } Q_{A,S} \\ \frac{\partial q}{\partial \eta} = 0. & \text{on } \Sigma \end{cases} \tag{25}$$

$\varphi \in L^\infty(0,T;L^2(Q_S))$ et $\theta \in L^\infty(0,T;L^2(Q_A))$

The system (25) is linear and has bounded coefficients, so it is easy to show that the system (25) has a unique solution y in $L^\infty(0,T;L^2(Q_{A,S}))$.

Let us note $\eta : L^\infty(0,T;L^2(Q_S)) \times L^\infty(0,T;L^2(Q_A)) \to L^\infty(0,T;L^2(Q_S)) \times L^\infty(0,T;L^2(Q_A))$, $(\varphi,\theta) \mapsto (y(.,.,A,.), y(.,.,.,S))$ where y is a solution of (25).

We will show that for some values of λ, η it is strictly contracting.

Let $\varphi_1;\varphi_2$ be elements of $L^\infty(0,T;L^2(Q_S))$ and θ_1 ; θ_2 be in $L^\infty(0,T;L^2(Q_A))$ and y_1, y_2 be the solutions of (25) corresponding to (φ_1,θ_1) and (φ_2,θ_2), respectively. Let $Y = y_1 - y_2$, $\Phi = \varphi_1 - \varphi_2$ and $\Theta = \theta_1 - \theta_2$, then Θ and Φ verify

$$\begin{cases} \partial_t Y + \partial_a Y + \partial_s Y - \Delta Y + (\lambda_0 + \mu) Y - \Phi(x,t,s) \int_0^S \beta(a,p,s) dp & \\ -\Theta(x,t,a) \int_0^A \lambda(a,p,s) dp = -v^* Y(x,t,a,s) & \text{in } Q, \\ y(x,t,0,s) = 0 & \text{in } Q_{T,S}, \\ y(x,t,a,0) = 0 & \text{in } Q_{T,A}, \\ y(x,0,a,s) = 0 & \text{in } Q_{A,S} \\ \frac{\partial q}{\partial \eta} = 0. & \text{on } \Sigma \end{cases} \tag{26}$$

Multiplying the first equation of (26) by Y and using Cauchy's inequality with $\epsilon > 0$, we show that there exists a positive constant C independent of ϵ and positive constants $C_1(\epsilon)$ and $C_2(\epsilon)$ such that

$$\begin{aligned} \frac{1}{2}\|Y(.,T,.,.)\|^2_{L^2(Q_{A,S})} + \frac{1}{2}\|Y(.,.,A,.)\|^2_{L^2(0,T;L^2(Q_S))} + \frac{1}{2}\|Y(.,.,.,S)\|^2_{L^2(0,T;L^2(Q_A))} + \lambda\|Y\|^2_{L^2(Q)} \\ \leqslant \frac{C}{\epsilon}\left(\|\Phi\|^2_{L^2(0,T;L^2(Q_S))} + \|\Theta\|^2_{L^2(0,T;L^2(Q_A))}\right) + C(\epsilon)\|Y\|^2_{L^2(Q)} \end{aligned}$$

By choosing $\epsilon = 2C$ and $\lambda > \frac{C(\epsilon)}{2}$, one get:

$$\|\eta\| \leqslant \frac{1}{2}\left(\|\phi_1 - \phi_2\|^2_{L^2(0,T;L^2(Q_S))} + \|\theta_1 - \theta_2\|^2_{L^2(0,T;L^2(Q_A))}\right).$$

And therefore η is a contraction. □

We can now give necessary conditions of optimality:

Theorem 4.1. *If (u^*, v^*) is an optimal pair for (2) and if q is the solution of (22) then we have*

$$v^*(x,t,a,s) = \begin{cases} \zeta_1(x,t,a,s) & \text{if } (w+q)(x,t,a,s) < 0 \\ \zeta_2(x,t,a,s) & \text{if } (w+q)(x,t,a,s) > 0 \end{cases}$$

For the proof theorem 4.1, we need the followings lemmas.

Lemma 4.1. *The following convergence holds*

$$u^{v^*+\delta v}(x,t,a,s) \longrightarrow u^{v^*}(x,t,a,s) \text{ in } L^\infty\left(0,T;L^2(Q_{A,S})\right)$$

as $\delta \longrightarrow 0^+$.

Proof of lemma 4.1

Let $h_\delta = u^{v^*+\delta v} - u^{v^*}$ then h_δ satisfy

$$\begin{cases} \partial_t h_\delta + \partial_a h_\delta + \partial_s h_\delta - \Delta h_\delta + (\mu + v^*)h_\delta = -\delta u^{v^*+\delta v} v & \text{in } Q, \\ h_\delta(x,t,0,s) = \int_0^A \int_0^S \beta(a,p,s) h_\delta(x,t,a,s) dpda & \text{in } Q_{T,S} \\ h_\delta(x,t,a,0) = \int_0^S \int_0^A \delta(a,p,s) h_\delta(x,t,a,s) dpds & \text{in } Q_{T,A}, \\ h_\delta(x,0,a,s) = 0 & \text{in } Q_{A,S} \\ \frac{\partial h_\delta}{\partial \eta} = 0 & \text{on } \Sigma \end{cases} \tag{27}$$

Multiplying (27) by h_δ and integrating over Q_t and using Young's inequality, we obtain :

$$\begin{aligned} &\frac{1}{2}\int_0^A \int_0^S \|h_\delta(.,T,.,.)\|^2_{L^2(\Omega)} dsda + \frac{1}{2}\int_0^T \int_0^S \|h_\delta(.,.,A,.)\|^2_{L^2(\Omega)} dsdt \\ &\quad + \frac{1}{2}\int_0^T \int_0^A \|h_\delta(.,.,.,S)\|^2_{L^2(\Omega)} dadt - \frac{1}{2}\int_0^T \int_0^S \|h_\delta(.,.,0,.)\|^2_{L^2(\Omega)} dsdt \\ &\quad - \frac{1}{2}\int_0^T \int_0^A \|h_\delta(.,.,.,0)\|^2_{L^2(\Omega)} dadt + \|\nabla h_\delta\|^2_{L^2(Q)} + \|\sqrt{\mu + v^*} h_\delta\|^2_{L^2(Q)} \\ &\quad \leqslant \delta \int_0^t \int_0^A \int_0^S \int_\Omega \zeta_2(x,a,t,s)\bar{u}(x,a,l,s) \left| h_\delta((x,a,l,s)) \right| dxdsdadl \end{aligned} \tag{28}$$

Using again Young's inequality, the estimation (28) implies that

$$\|h_\delta(t)\|^2_{L^2(Q_{A,S})} \leq C \int_0^t \|h_\delta(t)\|^2_{L^2((Q_{A,S})\times\Omega)} + \delta^2 \int_{Q_T} \zeta_2^2(a,t,x)\bar{u}^2(a,t,x) dxdadt \tag{29}$$

where C is a positive constant.

Using Gronwall's inequality, estimate (29) implies

$$\|h_\delta(t)\|^2_{L^2(Q_{A,S})} \leq K\delta^2 \exp\{Ct\}. \tag{30}$$

Passing to the limit in the last inequality we get

$$h_\delta \to 0 \quad \text{in } L^\infty\left(0,T;L^2(Q_{A,S})\right)$$

□

Let

$$z_\delta = \frac{h_\delta}{\delta} \text{ in } Q.$$

Then the function z^δ is a solution of

$$\begin{cases} \partial_t z_\delta + \partial_a z_\delta + \partial_s z_\delta - \Delta z_\delta + \mu z_\delta = -v^* z_\delta - u^{v^*+\delta v} v & \text{in } Q, \\ z_\delta(x,t,0,s) = \int_0^A \int_0^S \beta(a,p,s) z_\delta(x,t,a,s) dpda & \text{in } Q_{T,S}, \\ z_\delta(x,t,a,0) = \int_0^S \int_0^A \lambda(a,p,s) z_\delta(x,t,a,s) dpds & \text{in } Q_{T,A}, \\ z_\delta(x,0,a,s) = 0 & \text{in } Q_{A,S} \\ \frac{\partial z_\delta}{\partial \eta} = 0 & \text{on } \Sigma \end{cases} \tag{31}$$

Lemma 4.2. *The following convergence holds*

$$z^\delta \to z \text{ in } L^\infty(Q) \text{ as } \delta \to 0, \text{ where } z \text{ is the solution of}$$

$$\begin{cases} \partial_t z + \partial_a z + \partial_s z - \Delta z + \mu z = -v^* z - u^{v^*} v & \text{in } Q, \\ z(x,t,0,s) = \int_0^A \int_0^S \beta(a,p,s) z(x,t,a,s) dpda & \text{in } Q_{T,S}; \\ z(x,t,a,0) = \int_0^S \int_0^A \lambda(a,p,s) z(x,t,a,s) dpds & \text{in } Q_{T,A}; \\ z(x,0,a,s) = 0 & \text{in } Q_{A,S}; \\ \frac{\partial z}{\partial \eta} = 0 & \text{on } \Sigma. \end{cases} \tag{32}$$

Proof of lemma 4.2

Let $w_\delta = z_\delta - z$, so w_δ satisfies

$$\begin{cases} \partial_t w_\delta + \partial_a w_\delta + \partial_s w_\delta - \Delta w_\delta + \mu w_\delta = -v^* w_\delta - v h_\delta & \text{in } Q; \\ w_\delta(x,t,0,s) = \int_0^A \int_0^S \beta(a,p,s) w_\delta(x,t,a,s) dpda & \text{in } Q_{T,S}; \\ w_\delta(x,t,a,0) = \int_0^S \int_0^A \lambda(a,p,s) w_\delta(x,t,a,s) dpds & \text{in } Q_{T,A}; \\ w_\delta(x,0,a,s) = 0 & \text{in } Q_{A,S}; \\ \frac{\partial w_\delta}{\partial \eta} = 0 & \text{on } \Sigma. \end{cases} \tag{33}$$

Multiplying (33) by w_δ and integrating over $Q_t = \Omega \times (0,t) \times (0,A) \times (0,S)$, we obtain

$$\begin{aligned} \|w_\delta(t)\|^2_{L^2(Q_{A,S})} \leq (C+1) \int_0^t \|w_\delta(l)\|^2_{L^2(Q_{A,S})} \, dl \\ + \|\zeta\|^2_{L^\infty(Q)} \int_0^t \|h_\delta(l)\|^2_{L^2(Q_{A,S})} \, dl \end{aligned}$$

Using Bellman's inequality, this last inequality implies that

$$\|w_\delta(t)\|^2_{L^2(Q_{A,S})} \leq \|\zeta\|^2_{L^\infty(Q)} \exp\{(C+1)t\} \int_0^t \|h_\delta(l)\|^2_{L^2(Q_{A,S})} \, dl$$

So,

$$\|w_\delta\|^2_{L^\infty(0,T;L^2(Q_{A,S}))} \leq C' \, \|h_\delta\|^2_{L^\infty(0,T;L^2(Q_{A,S}))} \tag{34}$$

where C' is a positive constant that depends on T. Taking the limit in (34) as $\delta \to 0^+$, we obtain that

$$w_\delta \to 0 \qquad \text{in} \quad L^\infty(0,T;L^2(Q_{A,S}))$$

□

Proof of theorem 4.1

Let's (u^*, v^*) is an optimal pair for (2).

For all $v \in L^\infty(Q)$ such that $v^* + \delta v \in \mathcal{U}$ and $\forall \delta > 0$,

$$\frac{1}{\delta}\left(\Psi(u^{v^*+\delta v}) - \Psi(v^*)\right) = \int_Q v^*(x,t,a)w(x,t,a)\frac{u^{v^*+\delta v}(x,t,a) - u^{v^*}(x,t,a)}{\delta}dxdtda$$
$$+ \int_Q v(x,t,a)g(x,t,a)u^{v^*+\delta v}(x,t,a)dxdtda \tag{35}$$

Using lemma 4.1 and lemma 4.2, and passing the limit in (35), we get

$$d\Psi(v^*)(v) = \int_Q v^*(x,t,a)w(x,t,a)z(x,t,a)dxdtda + \int_Q v(x,t,a)w(x,t,a)u^{v^*}(x,t,a)dxdtda \tag{36}$$

Multiplying (22) by z and integrating over Q we get

$$\int_Q (\partial_t z + \partial_a z + \partial_s z - \triangle z + \mu z)\, q dxdadsdt + \int_0^T\int_0^S\int_\Omega q(x,t,0,s)z(x,t,0,s)dxdsdt$$
$$\int_0^T\int_0^A\int_\Omega q(x,t,a,0)z(x,t,a,0)dxdadt - \int_0^T\int_0^S\int_0^A\int_\Omega \left(q(x,t,0,s)\int_0^S \beta(a,p,s)dp\right)dxdadsdt$$
$$-\int_0^T\int_0^S\int_0^A\int_\Omega \left(q(x,t,0,s)\int_0^a \lambda(a,p,s)dp\right)dxdadsdt = -\int_Q v^* z(w+q)(x,t,a,s)dxdadsdt. \tag{37}$$

Recalling that z satisfies the system (32), after some calculation the equality (37) becomes

$$-\int_Q (v^* zq)(x,t,a,s)dxdadsdt - \int_Q (vu^{v^*}q)(x,t,a,s)dxdadsdt = -\int_Q v^* z(w+q)(x,t,a,s)dxdadsdt. \tag{38}$$

We then deduce the following equality from (38):

$$\int_Q (v^* wz)(x,t,a)dxdtda = \int_Q \left(vu^{v^*}q\right)(x,t,a)dxdtda \tag{39}$$

So from (36) and (39) we have ,

$$d\Psi(v^*)(v) = \int_Q vu^{v^*}(w+q)(x,t,a,s)dxdsdadt$$

Recalls that v^* is an opimal solution of problem (2), then

$$d\Psi(v^*)(v) \le 0, \quad \forall v \in L^\infty(Q).$$

According to (3), this implies that $u^{v^*}(g+q) \in N_{\mathcal{U}}(v^*)$, where $N_{\mathcal{U}}(v^*)$ is the normal cone at $\mathcal{U}$ in v^* in $L^2(Q)$.

For any $(x,t,a,s) \in Q$ such that $u^{v^*}(x,t,a,s) \neq 0$, we conclude

$$v^*(x,t,a,s) = \begin{cases} \zeta_1(x,t,a,s) & \text{if } (g+q)(x,t,a,s) < 0 \\ \zeta_2(x,t,a,s) & \text{if } (g+q)(x,t,a,s) > 0 \end{cases}$$

□

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