ON REDUCED WEIGHTED BOMBIERI l_2-NORM

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Abstract. In this article, P is a given polynomial with real coefficients. Our aim is to study, with various normalisations on the polynomial Q with real coefficients, the quantity $\inf\,[PQ]_2$. When Q is monic or $Q(0)=1$, this quantity is known as the reduced weighted Bombieri l_2-norm of P and denoted by $[P]$. On the other hand if Q is monic and $Q(0)\geq 1$, it is denoted by $[P]_0$. More precisely we study some properties of these norms and give bounds on the ratio $\frac{[P']_0}{[P]_0}$ under certain conditions on the roots of P.

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1. Introduction

The study of polynomial complexity, often quantified through various heights or norms, is a fundamental topic in number theory and analysis. For a real polynomial of degree d,

$$P = a_dX^d + a_{d-1}X^{d-1} + \ldots + a_1X + a_0 = a_d(X-\alpha_1)(X-\alpha_2)\ldots(X-\alpha_d),$$

classical measures include its height $\mathrm{ht}(P)=\max_{0\leq j\leq d}|a_j|$, its length $L(P)=\sum_{j=0}^{d}|a_j|$, its Euclidean norm $\|P\|=\left(\sum_{j=0}^{d}|a_j|^2\right)^{1/2}$, and its Mahler measure $M(P)=|a_d|\prod_{j=1}^{d}\max\{1,|\alpha_j|\}$.

Another important measure is the weighted l_2-norm of Bombieri, defined by

$$[P]_2=\left(\sum_{j=0}^{d}\frac{|a_j|^2}{\binom{d}{j}}\right)^{1/2}. \qquad (1)$$

This norm accounts for the distribution of coefficients across the polynomial's degree and possesses several desirable algebraic and analytic properties.

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A natural line of inquiry involves studying a polynomial's "minimal" size relative to multiplicative adjustments. For a given norm $N(\cdot)$, this leads to the investigation of quantities like

$$\inf\{N(PQ)\},$$

where the infimum is taken over all monic polynomials $Q \in \mathbb{R}[X]$. Such problems have been studied extensively for the norms $\|\cdot\|$, $L(\cdot)$, and $\mathrm{ht}(\cdot)$ (see, e.g., [8], [2], [1], [9]).

In this work, we focus on the Bombieri norm. Let $\widehat{\Gamma}$ be the set of all real polynomials with either constant term 1 or leading coefficient 1. We define the reduced weighted Bombieri norm of P by

$$\widehat{[P]} = \inf_{Q\in\widehat{\Gamma}} [PQ]_2. \tag{2}$$

Letting $\Gamma \subset \widehat{\Gamma}$ denote the subset of monic polynomials, it is straightforward to show that $\widehat{[P]} = \min\{[P], [P^*]\}$, where

$$[P] = \inf_{Q\in\Gamma} [PQ]_2 \tag{3}$$

and $P^*(X) = X^d P(1/X)$ is the reverse polynomial of P. Thus, the study of $\widehat{[P]}$ reduces to that of $[P]$, which we will henceforth refer to as the reduced Bombieri norm.

We also introduce a related quantity, the special reduced Bombieri norm, defined by

$$[P]_0 = \inf_{Q\in\Gamma_0} [PQ]_2, \tag{4}$$

where Γ_0 is the set of monic polynomials Q satisfying $|Q(0)| \geq 1$.

This paper has two primary aims. Firstly, we establish elementary properties of the norms $[P]$ and $[P]_0$. Secondly, we derive estimates for the ratio

$$\frac{[P']_0}{[P]_0} \tag{5}$$

under specific assumptions on the roots of P. This continues a tradition of bounding ratios between the heights of a polynomial and its derivative [6,7], but within the framework of these novel reduced norms.

The paper is organized as follows. Section 2 presents our main results, and Section 3 is devoted to their proofs.

2. Mains results

We begin with the following basic properties of the reduced weighted l_2-norm Bombieri $[\cdot]$.

Theorem 1. *Suppose that $\psi, \varphi \in \mathbb{R}$, $\mu \in \mathbb{C}$, $|\varphi| < 1$, $|\mu| < 1$, $\psi \neq 0$ and $k \in \mathbb{N}$. Then for all polynomials with real coefficients P, we have*

(1) $[\psi P] = |\psi|\,[P]$;

(2) $[(X-\varphi)P] = [P]$;

(3) $[(X-\mu)(X-\overline{\mu})P]=[P]$;

(4) $[P(-X)]=[P]$

(5) $\left[P(X^k)\right]=[P]$.

Remark 1. *The properties of the theorem remain valid even if we replace the reduced weighted Bombieri l_2-norm by the special one. It seems difficult to determine $[X-\omega]$ with ω a given real. The following proposition allows us to calculate $[X-\omega]_0$.*

Proposition 1. *Suppose that $\omega \in \mathbb{R}$, then*

(1) $1 \leq [X-\omega] \leq \left(1+|\omega|^2\right)^{1/2}$

(2) $[X+\omega]_0 = \left(1+|\omega|^2\right)^{1/2}$.

Proposition 2 (A. DURAND [2])**.** *For any polynomial $P \neq 0$, we have*

$$M(P) = \inf_{Q\in\mathbb{C}[X];Q(0)=1} \|PQ\|_\mu$$

where $\|PQ\|_\mu = \left(\int_0^1 \left|PQ(e^{2i\pi\theta})\right|^\mu d\theta\right)^{1/\mu}$ *for every* $\mu>0$.

We deduce from Proposition 2, the identity $M(P)=\inf_{Q\in\widehat{\Gamma}}\|PQ\|$. Then we can prove the following.

Proposition 3. *For all polynomials with real coefficients P and Q, we have*

(1) $\max\{[P],[Q]\} \leq [PQ] \leq [P]\,[Q]$.

(2) $[P] \leq M(P)$.

The previous proposition gives an answer to the question : for which polynomials P the reduced l_2-wieghted norm of Bombieri is smaller than its Mahler's measure? A similar question was asked in [4]. The authors gave an answer in terms of the localization of the roots of the polynomial P.

Proposition 4. *If a polynomial $P \neq 0$ has no root outside the unit disc then $[P]=|a_d|$. If α is the unique root of P outside the unit disc then $[P]=|a_d|\,|\alpha|\left[\frac{X}{\alpha}-1\right]$.*

Corollary 1. *If α is the unique root of a polynomial $P \neq 0$ out side the unit disc then $[P]_0 = |a_d|\left(1+|\alpha|^2\right)^{1/2}$.*

Determining the reduced weighted Bombieri l_2-norm of a quadratic polynomial is not easy, if all its roots are outside the unit disk. On the other hand, the norm $[\cdot]_0$ of such a polynomial is easily obtained using Corollary 1. In addition, the corollary below allows the calculation of the special reduced weighted Bombieri l_2-norm of any polynomial P.

Proposition 5. *For all polynomials with real coefficients P, $|P(0)| \leq [P] \leq \|P\|$.*

Corollary 2. *For all polynomials with real coefficients P, $[P]_0 = \left(|P(0)|^2+|P^*(0)|^2\right)^{1/2}$.*

Now, we state the results relating to the problem of finding bounds for the ratio $\dfrac{\left[P'\right]_0}{[P]_0}$ under assumptions on the roots of the polynomial P. We will exploit the following proposition, the proof of which is done by simple calculations.

Proposition 6. *For all polynomials P of degree d, we have*

$$\left[P'\right]_0^2 + \left[(P^*)'\right]_0^2 = d^2\left[P\right]_0^2 + \left[P - P^*(0)X^d - P(0) + P'(0)\right]_0^2 \tag{6}$$

where P^* denotes the reverse of the polynomial P.

It is well known that if P is $\pm$-palindromic its roots are all on the unit circle. So when P is palindromic of degree d, Proposition 6 shows that

$$\frac{[P']_0}{[P]_0} \le d \tag{7}$$

if and only if $|a_1| \le d|a_0|$. Of course, this bound is optimal since equality holds for $P = (X+1)^d$. In general, the quotient can be arbitrarily large. In addition, we have that

$$\frac{[P']_0}{[P]_0} \ge \frac{d}{\sqrt{2}} \tag{8}$$

with equality when $P = X^d + 1$.

If P is any polynomial of degree d with roots in the disc $\{|z| < R\}$, we can also deduce from Proposition 6 that

$$\frac{[P']_0}{[P]_0} \le d\sqrt{1 + R^2 + \left(1 - \frac{1}{d}\right)^2 R^{2(d-1)}} \quad \text{where} \quad R \le 1 \tag{9}$$

and

$$\frac{[P']_0}{[P]_0} \le d\sqrt{1 + \frac{1}{R^{2(d-1)}} + \left(1 - \frac{1}{d}\right)^2 R^2} \quad \text{where} \quad R \ge 1 \tag{10}$$

Hence the ratio is close to d if the roots of P are small. The theorem below gives lower bound of the ratio $\dfrac{\left[P'\right]_0}{[P]_0}$ according to the values of the roots modulus of the polynomial P.

Theorem 2. *Let P be a polynomial with degree d. If P has all roots in the disc $\{|z| \le R\}$ then*

$$\frac{\left[P'\right]_0}{[P]_0} \ge \frac{d}{\sqrt{1+R^2}} \quad \text{if} \quad R \le 1 \tag{11}$$

and

$$\frac{\left[P'\right]_0}{[P]_0} \ge \frac{d}{\sqrt{1+R^{2d}}} \quad \text{if} \quad R \ge 1 \tag{12}$$

3. Proof of mains results

Proof of Theorem 1. 1. The proof of this point is deduced from the homogeneity of the weighted l_2-norm of Bombieri.
2. Set $Q_0 = \varphi^{m-1}+\varphi^{m-2}X+\varphi^{m-2}X^2+\ldots+\varphi X^{m-2}+X^{m-1}$. Then, for arbitrary $Q \in \Gamma$, $(X-\varphi)PQQ_0 = PQ(X^m - \varphi^m)$, giving $[(X-\varphi)PQQ_0]_2 \leq [PQ]_2 \left(1+|\varphi|^{2m}\right)^{1/2}$ using submultiplicative property of the norm $[\cdot]_2$. Since $QQ_0 \in \Gamma$ and $|\omega| < 1$, this yields $[(X-\varphi)P] \leq [P]$ since we can take m arbitrarily large.
On the other hand, we have $(X-\varphi)Q \in \Gamma$ when $Q \in \Gamma$ and from $[P] \leq [(X-\varphi)PQ]_2$ we deduce that $[P] \leq [(X-\varphi)P]$. Hence $[(X-\varphi)P] = [P]$.
3. We prove this point by following the previous reasoning, since $(X-\mu)(X-\overline{\mu}) \in \Gamma$, so setting

$$Q_0 = \left(\mu^{m-1} + \mu^{m-2}X + \ldots + \mu X^{m-2} + X^{m-1}\right)\left(\overline{\mu}^m + \overline{\mu}^{m-1}X + \ldots + \overline{\mu}X^{m-1} + X^m\right) \in \Gamma$$

we can argue as above.
4. For any polynomial Q in the set Γ, the proof of this point follows from

$$[P(-X)] \leq [P(-X)Q(-X)(-1)^m]_2 = [P(X)Q(X)]_2 \,,$$

then

$$[P(-X)] \leq \inf_{Q\in\Gamma} [PQ]_2 = [P]\,. \tag{13}$$

5. Similarly, for this property, we have

$$\left[P(X^k)\right] \leq \left[P(X^k)Q(X^k)\right]_2 = [PQ]_2$$

hence

$$\left[P(X^k)\right] \leq \inf_{Q\in\Gamma} [PQ]_2 = [P]\,.$$

In the other hand, we can put

$$P(X^k)Q = \sum_{i=0}^{m} X^i A_i(X^k), \qquad \text{where} \qquad A_i \in \mathbb{R}[X]. \tag{14}$$

Let $A_i = Q_iP + R_i$, where $Q_i, R_i \in \mathbb{R}[X]$ and deg$R_i <$ deg$P$. It follows that

$$P(X^k) \mid \sum_{i=0}^{m} X^i R_i(X^k)$$

and since deg $\sum_{i=0}^{m} X^i R_i(X^k) <$ deg $P(X^k)$, we deduce that $R_i = 0$ for $i = 0, 1, \ldots, m$. It's easy to see that deg $X^i A_i(X^k) <$ deg $X^m A_m(X^k)$ for $i = 0, 1, \ldots, m-1$. It follows from equality (14) that Q_m is in Γ. Hence, by (14)

$$\left[P(X^k)Q\right]_2 \geq [A_m]_2 = [PQ_m]_2 \geq [P]\,,$$

thus $[P] \leq \left[P(X^k)\right]$, which together with (13) implies the sixth point. □

Proof of Proposition 1. For a real ω and a polynomial $Q = b_0 + b_1 X + \ldots + b_n X^m$ with real coefficients, the weighted l_2-norm of Bombieri of the polynomial

$$(\omega + X)(b_0 + b_1 X + \ldots + b_m X^m)$$

is equal to

$$\left(\frac{|b_0 \omega|^2}{\binom{0}{m+1}} + \frac{|b_1 \omega + b_0|^2}{\binom{1}{m+1}} + \frac{|b_2 \omega + b_1|^2}{\binom{2}{m+1}} + \ldots + \frac{|b_m \omega + b_{m-1}|^2}{\binom{m}{m+1}} + \frac{|b_m|}{\binom{m+1}{m+1}} \right)^{1/2}.$$

we deduce that

$$\frac{|b_1 \omega + b_0|^2}{\binom{1}{m+1}} + \frac{|b_2 \omega + b_1|^2}{\binom{2}{m+1}} + \ldots + \frac{|\omega + b_{m-1}|^2}{\binom{m}{m+1}} \geq \frac{|b_1 \omega + b_0|^2 + |b_2 \omega + b_1|^2 + \ldots + |b_m \omega + b_{m-1}|^2}{\binom{\lfloor \frac{m+1}{2} \rfloor}{m+1}}.$$

Hence $[(\omega + X)Q]_2 \geq \left(|b_0|^2 |\omega|^2 + 1 \right)^{1/2}$ for any $Q \in \Gamma$, giving $[\omega + X] \geq 1$. If we consider $Q \in \Gamma_0$, then

$$[\omega + X]_0 \geq \left(|\omega|^2 + 1 \right)^{1/2}.$$

Combining these inequalities with the trivial inequality

$$[\omega + X]_0 \leq [\omega + X]_2 = \left(|\omega|^2 + 1 \right)^{1/2}$$

we obtain the results in Proposition 1. □

Proof of Proposition 2. See [2]. □

Proof of Proposition 3. For all polynomials G and H with real coefficients, we have by definition

$$[P] \leq [PQG]_2 \quad \text{and} \quad [PQ] \leq [PQGH]_2 \leq [PG]_2 [QH]_2 .$$

Hence

$$[P] \leq \inf_{G \in \Gamma} [PQG]_2 = [PQ] \quad \text{and} \quad [PQ] \leq \inf_{G \in \Gamma} [PG]_2 \inf_{H \in \Gamma} [QH]_2 = [P] [Q] .$$

Trivially we have $[P] \leq \|P\|$ since $\frac{|a_j|^2}{\binom{d}{j}} \leq |a_j|^2$ for $j = 0, 1, \ldots, d$. Then for any Q in the set Γ, we have $[PQ] \leq \|PQ\|$. Thus $\inf_{Q \in \Gamma} [P] \leq \inf_{Q \in \Gamma} \|PQ\|$ and the inquality of the proposition follows from Proposition 2. □

Proof of Proposition 4. We may assume that for $0 \leq k \leq d$, we have $|\alpha_i| \geq 1$ with $1 \leq i \leq k$ and $|\alpha_i| < 1$ when $i \geq k + 1$. The Mahler measure of P can be written as $M(P) = |a_d| |\alpha_1 \ldots \alpha_k|$ and we have

$$P = a_d (X - \alpha_1) \ldots (X - \alpha_d) = \pm M(P) \left(\frac{X}{\alpha_1} - 1 \right) \ldots \left(\frac{X}{\alpha_k} - 1 \right) (X - \alpha_{k+1}) \ldots (X - \alpha_d).$$

By the properties 1. and 2. of Theorem 1, we get

$$[P] = M(P) \left[\left(\frac{X}{\alpha_1} - 1 \right) \ldots \left(\frac{X}{\alpha_k} - 1 \right) \right].$$

If P has all its roots inside the unit circle, namely $k=0$, then $M(P)=|a_d|$, so $[P]=|a_d|$.

If $\alpha=\alpha_1$ is the unique root of P outside the unit disc, it must be a real number since P is a polynomial with real coefficients. Then $[P]=M(P)\left[\frac{X}{\alpha}-1\right]$. □

Proof of Corollary 1. It follows from Proposition 4 that $[P]_0=|a_d|\left[\frac{X}{\alpha}-1\right]_0$. Since $M(P)=|a_d|\,|\alpha|$ and $\left[\frac{X}{\alpha}-1\right]_0=\left(1+\frac{1}{|\alpha|^2}\right)^{1/2}$, we deduce $[P]_0=|a_d|\left(1+|\alpha|^2\right)^{1/2}$. □

Proof of Proposition 5. The weighted l_2-norm of Bombieri of the product PQ, using the expressions for P and Q given in the introduction, is

$$[PQ]_2=\left((a_db_m)^2+(a_0b_0)^2+\sum_{k=1}^{d+m-1}\frac{|c_k|^2}{\binom{d+m}{k}}\right)^{1/2}$$

where $c_k=\sum_{i+j=k}a_ibj$. Thus $[PQ]_2\geq\left((a_db_m)^2+(a_0b_0)^2\right)^{1/2}$ so that $\inf_{Q\in\Gamma_0}[PQ]_2\geq|a_0|=|P(0)|$. □

Proof of Corollary 2. By Proposition 5 we have

$$\inf_{Q\in\Gamma_0}[PQ]_2\geq\left((a_db_m)^2+(a_0b_0)^2\right)^{1/2}\geq\left((a_d)^2+(a_0)^2\right)^{1/2}$$

when $m\to+\infty$ and since $Q\in\Gamma_0$. In the other hand, we have the inequality $\binom{1}{d+m}\leq\binom{k}{d+m}$ for $k=1,\dots,d+m-1$. That gives

$$\sum_{k=1}^{d+m-1}\frac{|c_k|^2}{\binom{d+m}{k}}\leq\frac{1}{d+m}\sum_{k=1}^{d+m-1}|c_k|^2\,.$$

When $Q\in\Gamma_0$ and $m\to+\infty$ we get

$$\inf_{Q\in\Gamma_0}[PQ]_2\leq\left((a_d)^2+(a_0)^2\right)^{1/2}.$$

We conclude by putting $a_0=P(0)$ et $a_d=P^*(0)$. □

Proof of Proposition 6. Write

$$P=a_0+a_1X+\cdots+a_dX^d,\qquad a_d\neq0,$$

the polynomial,

$$P'=a_1+2a_2X+\cdots+da_dX^{d-1},$$

its derivative and

$$P^*=X^dP(1/X)=a_d+a_{d-1}X+\cdots+a_0X^d$$

his reciprocal. So that $P(0)=a_0$, $P^*(0)=a_d$, $P'(0)=a_1$, and the leading coefficient of P' is da_d. By the definition we have

$$[P']_0^2=P'(0)^2+\left(((P')^*(0))\right)^2=a_1^2+(da_d)^2.$$

Similarly,

$$(P^*)'=a_{d-1}+2a_{d-2}X+\cdots+da_0X^{d-1},$$

so $(P^*)'(0) = a_{d-1}$ and the leading coefficient of $(P^*)'$ is da_0, whence

$$\left[(P^*)'\right]_0^2 = a_{d-1}^2 + (da_0)^2.$$

Adding these equalities yields

$$[P']_0^2 + \left[(P^*)'\right]_0^2 = d^2(a_0^2 + a_d^2) + a_1^2 + a_{d-1}^2 = d^2[P]_0^2 + a_1^2 + a_{d-1}^2. \tag{15}$$

Now set

$$Q := P(X) - P^*(0)X^d - P(0) + P'(0).$$

Since $P(0) = a_0$ and $P^*(0) = a_d$, one has

$$Q = a_1X + a_2X^2 + \cdots + a_{d-1}X^{d-1} + a_1,$$

so in particular $Q(0) = a_1 = P'(0)$ and the constant term of Q^* is a_{d-1}. Therefore

$$[Q]_0^2 = Q(0)^2 + Q^*(0)^2 = a_1^2 + a_{d-1}^2. \tag{16}$$

Combining the equalities 15 and 16 gives the desired identity

$$[P']_0^2 + \left[(P^*)'\right]_0^2 = d^2[P]_0^2 + \left[\,P - P^*(0)X^d - P(0) + P'(0)\,\right]_0^2.$$

□

Proof of Theorem 2. We first proves two results which will be used in the proof.

Proposition 7. *Let P and H be two polynomials with degree d. Assume that P has all roots in the closed unit disc $\{|z| \le 1\}$ and that $|P(z)| \ge |H(z)|$ for every z, $|z| = 1$. Then $[P]_0 \ge [H]_0$.*

Proof. To prove the Proposition, lets us consider the two polynomial $R_1 = \alpha P - H$ and $R_2 = \overline{\alpha}P - H$ where $\alpha \in \mathbb{C}$, $|\alpha| > 1$. Let $f = \dfrac{H}{P}$ and $g(z) = f(1/z)$ so that $g = \dfrac{H^*}{P^*}$ where H^* and P^* are respectively the reverse polynomials of H and P. By the assumptions of the Proposition, the polynomial H^* has no roots in the disc $D = \{|z| \le 1\}$ and hence g is holomorphic in D. On the other hand, we easily get $|g(z)| \le 1$ for each z, $|z| = 1$. By the maximum modulus principle, we deduce

$$|g(z)| \le 1, \text{ for every } z,\ |z| \le 1.$$

Hence

$$|f(z)| \le 1, \text{ for every } z,\ |z| \ge 1$$

.i.e.,

$$|P(z)| \ge |H(z)| \qquad \text{for every } z, |z| \ge 1.$$

If $R_1(z_0) = 0$ for some $|z_0| \ge 1$, then $|\alpha||P(z_0)| = |H(z_0)| \le |P(z_0)|$, which forces $|\alpha| \le 1$, contradiction. Thus R_1 has no zero in $\{|z| \ge 1\}$; similarly R_2 has no zero in $\{|z| \ge 1\}$. Hence all zeros of R_1 and R_2 lie in the open unit disc.

Let $[\cdot,\cdot]_0$ denotes the scalar product associated with the special weighted l_2-norm of Bombieri defined by

$$\left[\sum_{i=1}^{d} a_i X^i, \sum_{i=0}^{m} b_i X^i\right]_0 = a_d\overline{b_m} + a_0\overline{b_0}.$$

If we write

$$R_1 = r_d X^d + \ldots + r_0 = r_d \prod_{i=1}^{d}(X - z_i) \text{ and } R_2 = s_d X^d + \ldots + s_0 = s_d \prod_{i=1}^{d}(X - z_i'),$$

then by Viète's formulas we have $r_0/r_d = (-1)^d z_1 \cdots z_d$ and $s_0/s_d = (-1)^d z_1' \cdots z_d'$. Thus $[R_1, R_2]_0 = r_d\overline{s_d}\big(1 + z_1 \cdots z_d\overline{z_1' \cdots z_d'}\big)$. If $[R_1, R_2]_0 = 0$, then $|z_1 \cdots z_d\overline{z_1' \cdots z_d'}| = 1$, which contradicts the fact that $|z_i| < 1$ for all i. Hence $[R_1, R_2]_0 \neq 0$.

Expanding this scalar product gives

$$\alpha^2 [P]_0^2 - 2\alpha\Re([P,H]_0) + [H]_0^2 \neq 0.$$

Since this quadratic equation has no zero α satisfying $|\alpha| > 1$, then all its roots lie in the unit disc $|z| \leq 1$. By the standard relation between the moduli of the roots and the coefficients of a polynomial, its constant term cannot exceed its leading one. Consequently,

$$[P]_0 \geq [H]_0\,.$$

This proves the proposition. □

Lemma 1. [5] *Let P be a polynomial with all roots in the disc $\{|z| \leq R\}$ with $R \leq 1$, then*

$$R\left|P'(z)\right| \geq \left|(P^*)'(z)\right|$$

for every z, $|z| = 1$.

We turn now to the proof of theorem.
Assume that $P = a_d X^d + \ldots + a_1 X + a_0$ is a polynomial of degree d whose roots are all inside the disk $\{|z| \leq R\}$. If $R \leq 1$ by Lemma 1, we have $R\,|P'(z)| \geq |(P^*)'(z)|$ and applying Proposition 7 we get

$$R\left[P'\right]_0 \geq \left[(P^*)'\right]_0.$$

From Proposition 6, we deduce that

$$\left(1 + R^2\right)\left[P'\right]_0^2 \geq \left[P'\right]_0^2 + \left[(P^*)'\right]_0^2 = d^2\,[P]_0^2 + a_1^2 + a_{d-1}^2.$$

Thus

$$\left(1 + R^2\right)\left(\frac{[P']_0}{[P]_0}\right)^2 \geq d^2 + \frac{a_1^2 + a_{d-1}^2}{a_0^2 + a_d^2} \geq d^2. \tag{17}$$

We deduce from inequality (17) the first part of the theorem. For the proof of the second part, we set $H(X) = P(RX)$ which has all its roots in the unity disc since P has all roots in $\{|z| \leq R\}$. So inequality (17) gives

$$\frac{[H']_0}{[H]_0} \geq \frac{d}{\sqrt{2}}$$

and from Proposition 6, we get

$$\left[H'\right]_0 \geq \left[(H^*)'\right]_0 \tag{18}$$

As $H'(X) = RP'(RX)$ and $R \geq 1$, then we have

$$\left[H'\right]_0 \leq R^d \left[P'\right]_0 \tag{19}$$

by using the definition of special weighted Bombieri l_2-norm. In addition, we get by computing reverse and derivative of H that

$$H^*(X) = R^d P^* \left(\frac{X}{R}\right) \quad \text{and} \quad (H^*)'(X) = R^{d-1}(P^*)' \left(\frac{X}{R}\right).$$

Therefore, since $R \geq 1$ we have

$$\left[(H^*)'\right]_0 \geq \left[(P^*)'\right]_0. \tag{20}$$

We deduce from inqualities (19), (18) and (20) the following

$$R^d \left[P'\right]_0 \geq \left[(P^*)'\right]_0. \tag{21}$$

From the Proposition 6, the inequality (21) gives

$$\left(1 + R^{2d}\right) \left[P'\right]_0^2 \geq d^2 \left[P\right]_0^2 + a_1^2 + a_{d-1}^2 \geq d^2 \left[P\right]_0^2.$$

This completes the proof of the theorem □

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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