

EIGENVALUES OF AN INHOMOGENEOUS ELLIPTIC EQUATION INVOLVING THE $p(x)$ -LAPLACIAN OPERATOR AND LOGARITHMIC NONLINEARITY, PART I

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ABSTRACT. The main aim of this paper is to examine the eigenvalues of doubly inhomogeneous elliptic equations involving the $p(x)$ -Laplacian operator and logarithmic nonlinearity. We employ Ekeland's variational principle via sublinearity to analyse the spectrum structure and obtain results concerning multiplicity by establishing sufficient conditions. Our results extend those of a recent paper that appears in the literature.

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1. INTRODUCTION

One of the most significant topics in elliptic differential equations is the study of eigenvalue problems. This area of research dates back to the work of Euler and Lagrange on mechanical vibrations. The formalism of the calculus of variations developed by Lagrange led to the Euler-Lagrange equation, which characterizes eigenvalues as solutions to a variational problem.

Recently, many studies have focused on semi-linear eigenvalue elliptic equations, where the Laplacian is treated as the linear operator. This type of equation has been widely studied (see, for example, [4]). And these studies later evolved into quasilinear eigenvalue equations that include the p -Laplacian operator (see [11–13, 22, 26]).

On the other hand, we observe that numerous prior studies have transitioned to the spaces of variable exponents, and numerous mathematicians have concentrated on investigating differential equations that exhibit nonstandard $p(x)$ -growth conditions. This behavior arises from complexities associated

with nonlinear elasticity theory, electrorheological fluids, image processing or calculus of variations, and similar fields (see [2,3,6,23,31–34]).

Without the logarithmic term, problem (1.1), when $p(\cdot) = q(\cdot)$, has been examined from various perspectives. The existence of eigenvalues and their qualitative characteristics, including simplicity and multiplicity, have been explored (see, for instance, [8,9,16,19,20]).

In [8,9] the authors investigated the Steklov and Robin eigenvalue problems. They proved that there exist infinitely many eigenvalue sequences, established that the principal eigenvalue does not exist, and indicated that, under certain assumptions, the set of all eigenvalues is not closed.

Fan, Zhang, and Zhao [19] demonstrated the existence of an endless sequence of eigenvalues and established that $\sup \sigma_\lambda = +\infty$, where σ_λ is the set of all nonnegative eigenvalues. They also identified specific conditions that determine whether $\inf \sigma_\lambda$ equals 0 or is strictly positive.

M. Mihăilescu, and V. Rădulescu in [28] studied the problem

$$\begin{cases} -\Delta_{p(x)}u(x) = \lambda|u(x)|^{q(x)-2}u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

They establish that any sufficiently small $\lambda > 0$ is an eigenvalue of the above nonhomogeneous quasilinear problem.

N. Benouhiba [5] addresses the problem

$$-\Delta_{p(x)}u(x) = \lambda V(x)|u(x)|^{q(x)-2}u(x), \quad x \in \mathbb{R}^n, \quad (\text{P})$$

where V is an indefinite weight function that can change sign in \mathbb{R}^n . The author expanded the investigation to the entire space \mathbb{R}^n , demonstrating that the spectrum includes a continuous family of eigenvalues.

The same problem (P) has been considered by M. Aboubacar and S. Janvier in [27], in a bounded domain Ω of \mathbb{R}^n . According to the authors, the spectrum comprises a continuous set of eigenvalues, which in certain cases is the whole set \mathbb{R}_+^* .

In this paper, we investigate eigenvalues for doubly inhomogeneous elliptic problems with the $p(x)$ -Laplacian operator and logarithmic nonlinearity, namely

$$\begin{cases} -\Delta_{p(x)}u(x) = \lambda|u(x)|^{q(x)-2}u(x) \log(|u(x)|), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, Ω is a bounded domain in \mathbb{R}^n ($n \geq 2$) with smooth boundary $\partial\Omega$ and λ is a real parameter. The functions $p(\cdot)$ and $q(\cdot)$ are in $C(\bar{\Omega}, \mathbb{R}_+)$ and satisfy $p(\cdot) \neq q(\cdot)$ with

$$2 < p_- \leq p(x) \leq p_+ < +\infty, \quad (1.2)$$

and

$$1 < q_- \leq q(x) \leq q_+ < p^*(x), \quad (1.3)$$

$p(\cdot)$ and $q(\cdot)$ also, satisfy the following assumption

$$1 < q(x) < p(x), \quad (\text{H1})$$

where

$$p_- = \operatorname{ess\,inf}_{x \in \Omega} p(x), \quad p_+ = \operatorname{ess\,sup}_{x \in \Omega} p(x), \quad q_- = \operatorname{ess\,inf}_{x \in \Omega} q(x), \quad q_+ = \operatorname{ess\,sup}_{x \in \Omega} q(x),$$

and

$$p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & \text{if } p_+ < n, \\ +\infty, & \text{if } p_+ \geq n. \end{cases}$$

The aim of this paper is to investigate the spectrum structure of problem (1.1) through several nonlinear analytic approaches. In this first part of the paper, which focuses on the sublinear case, we demonstrate, utilizing Ekeland's variational principle, that under assumptions (H1)–(H3) problem (1.1) admits a continuous family of eigenvalues. We also highlight that this family of eigenvalues corresponds precisely to the whole \mathbb{R}_+^* . Furthermore, we establish sufficient conditions that enable each continuous spectrum eigenvalue to support an infinite countable (possibly unbounded) family of eigenfunctions.

We highlight several technical challenges arising from the nonlinear logarithmic term in the study of the eigenvalue problem, particularly due to its failure to maintain a consistent sign. Given the importance of this problem, the main difference between the problems studied in eigenvalue contexts is that in this case, the functions with changing signs are related to the spatial variable (see [5,27]). In contrast, our problem involves a nonlinear logarithm related to the unknown function, which complicates its study. Therefore, we define the following sets:

$$\mathcal{W} = \left\{ v \in W_0^{1,p(x)}(\Omega) \setminus \{0\} \mid g(|v|) \in L^1(\Omega) \right\}, \quad (1.4)$$

$$\mathcal{W}^+ = \{ v \in \mathcal{W} \mid F(|v|) > 0 \}, \quad (1.5)$$

where

$$g(s) = s^{q(x)} \log(s), \quad F(s) = \int_{\Omega} g(s) dx. \quad (1.6)$$

In this study, we only consider the case where the logarithm is positive; a negative sign case is omitted.

The paper is structured as follows: Section 2, outlines several classical properties of the variable exponent Lebesgue and Sobolev spaces. Section 3, is devoted to prove some auxiliary Lemmas required to deal with the next results. In Section 4, we analyze the sublinear problem, and provide proof of the main result, along with several corollaries.

2. PRELIMINARIES

We recall some needed properties of variable exponent Lebesgue and Sobolev spaces, which will be required later. Regarding the basic properties of these spaces, we advise the reader to browse the references [10, 24, 30]. Later we will use the notation: $\|v\|_{p(\cdot)} = \|v\|_{L^{p(\cdot)}(\Omega)}$.

Let Ω be a bounded domain in \mathbb{R}^n , we denote by $\mathcal{P}(\Omega)$ the set of all Lebesgue measurable functions $p(\cdot) : \Omega \rightarrow [1, \infty]$.

We say that the function $p(\cdot)$ fulfills the log-Hölder continuity condition in Ω if

$$\forall x, y \in \Omega \text{ with } |x - y| \leq \frac{1}{2}, \quad |p(x) - p(y)| \leq \frac{C_0}{-\log(|x - y|)}, \quad (2.1)$$

where $C_0 > 0$ is a constant.

We call that $p(\cdot)$ satisfies the log-Hölder decay condition in Ω if

$$\forall x \in \Omega, \quad |p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}, \quad (2.2)$$

where $p_\infty = \lim_{|x| \rightarrow \infty} p(x)$ and $C_\infty > 0$ are constants.

The class of variable exponents is defined as follows:

$$\mathcal{P}^{\log}(\Omega) = \{p(\cdot) \in \mathcal{P}(\Omega) \mid 1/p(\cdot) \text{ is globally log-Hölder continuous}\}.$$

Note that $p(\cdot) \in \mathcal{P}(\Omega)$ is globally log-Hölder continuous in Ω , if $p(\cdot)$ satisfies both (2.1)–(2.2) conditions.

Proposition 2.1 (see [7]). *Given a domain Ω*

- (1) *If $p(\cdot)$ fulfills (2.1), then it is uniformly continuous and fulfills (2.2) on every bounded subset $E \subset \Omega$.*
- (2) *If $p(\cdot) \in \mathcal{P}(\Omega)$ and $p_+ < +\infty$, then $1/p(\cdot)$ satisfies either conditions (2.1), (2.2) or both if and only if $p(\cdot)$ is also.*

Remark 2.1. *From Proposition 2.1 we deduce that if Ω is bounded, $p(\cdot) \in C(\bar{\Omega})$ and satisfies the conditions (1.2), (2.1) then $p(\cdot), 1/p(\cdot) \in \mathcal{P}^{\log}(\Omega)$.*

Now we define the $p(\cdot)$ modular of a measurable function $u : \Omega \rightarrow \mathbb{R}$ as follows:

$$\rho_{p(x)}(u) = \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} dx + \text{ess sup}_{x \in \Omega_\infty} |u(x)|,$$

where

$$\Omega_\infty = \{x \in \Omega \mid p(x) = \infty\}.$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of those measurable functions u defined on Ω as follows:

$$L^{p(x)}(\Omega) = \left\{ u \in \mathcal{P}(\Omega) \mid \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

we define the Luxembourg norm of the space $L^{p(x)}(\Omega)$ by

$$\|u\|_{p(x)} = \inf \{ \kappa > 0 \mid \rho_{p(x)}(u/\kappa) \leq 1 \}.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with this norm, is a Banach space.

Now we present results related to variable exponent Lebesgue spaces

Proposition 2.2 (generalized Hölder inequality, see [24,30]). Let $p(\cdot), p'(\cdot) \in \mathcal{P}(\Omega)$ such that $p_- > 1$ and

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1, \quad \text{a.e. } x \in \Omega.$$

Then the inequality

$$\int_{\Omega} |u(x)v(x)| dx \leq \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) \|u\|_{p(x)} \|v\|_{p'(x)},$$

holds for every $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.

Proposition 2.3 (see [17,18]). Let $u \in L^{p(x)}(\Omega)$, $(u_m)_{m \in \mathbb{N}} \subset L^{p(x)}(\Omega)$, then

$$(1) \|u\|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1 (= 1; > 1),$$

$$(2) \text{ If } \|u\|_{p(x)} > 1, \text{ then } \|u\|_{p(x)}^{p_-} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p_+},$$

$$(3) \text{ If } \|u\|_{p(x)} < 1, \text{ then } \|u\|_{p(x)}^{p_+} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p_-},$$

$$(4) \|u_m\|_{p(x)} \xrightarrow{m \rightarrow \infty} 0 \Leftrightarrow \rho_{p(x)}(u_m) \xrightarrow{m \rightarrow \infty} 0,$$

$$(5) \|u_m\|_{p(x)} \xrightarrow{m \rightarrow \infty} \infty \Leftrightarrow \rho_{p(x)}(u_m) \xrightarrow{m \rightarrow \infty} \infty.$$

Proposition 2.4 (see [21]). Assume that Ω has finite measure, $p_1(x), p_2(x) \in \mathcal{P}(\Omega)$. If $p_1(x) \leq p_2(x)$ for almost all $x \in \Omega$ and $1 \leq p_{i-} \leq p_{i+} < +\infty$, $(i = 1, 2)$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2.5 (see [21]). Let $u, u_m \in L^{p(x)}(\Omega)$, $m = 1, 2, \dots$. Then the following statements are equivalent to each other

$$(1) \lim_{m \rightarrow \infty} \|u_m - u\|_{p(x)} = 0,$$

$$(2) \lim_{m \rightarrow \infty} \rho_{p(x)}(u_m - u) = 0,$$

$$(3) u_m \text{ converges to } u \text{ in } \Omega \text{ in measure and } \lim_{m \rightarrow \infty} \rho_{p(x)}(u_m) = \rho_{p(x)}(u).$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

This space endowed with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)},$$

is a Banach space.

The space $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)} = \|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)}$. And $W^{-1,p'(x)}(\Omega)$ is the dual space of $W_0^{1,p(x)}(\Omega)$ where $p'(x)$ is the conjugate exponent function of $p(x)$ such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

We denote $\mathcal{X}_0 = W_0^{1,p(x)}(\Omega) \setminus \{0\}$, with the norm $\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}$.

Proposition 2.6 (see [10, Theorem 8.1.13]). Let $p \in \mathcal{P}(\mathbb{R}^n)$. The space $W_0^{1,p(\cdot)}(\Omega)$ is a Banach space, which is separable if $p(\cdot)$ is bounded, and reflexive and uniformly convex if $1 < p_- \leq p_+ < +\infty$.

Proposition 2.7. Assume that $1 \leq \text{ess inf}_{x \in \Omega} p_i(x) \leq p_i(x) \leq \text{ess sup}_{x \in \Omega} p_i(x) < +\infty$, ($i = 1, 2$). If $p_1(x) \leq p_2(x)$, then $W^{1,p_2(x)}(\Omega) \hookrightarrow W^{1,p_1(x)}(\Omega)$.

Proposition 2.8 (see [7, Theorem 6.29] and [10, Theorem 8.3.1]).

(1) Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p_+ < n$ suppose that the maximal operator is bounded on $L^{(p^*(\cdot)/n)'}(\Omega)$. Then $W_0^{1,p(\cdot)}(\Omega) \subset L^{p^*(\cdot)}(\Omega)$, and

$$\|u\|_{p^*(\cdot)} \leq C \|\nabla u\|_{p(\cdot)}.$$

(2) Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 \leq p_- \leq p_+ < n$. Then for every $u \in W_0^{1,p(\cdot)}(\Omega)$, the inequality

$$\|u\|_{p^*(\cdot)} \leq c \|\nabla u\|_{p(\cdot)},$$

holds with a constant c depending only on the dimension n , $c_{\log}(p)$, and p_+ .

Proposition 2.9 (see [29]). If Ω is bounded, $p(x) \in C(\bar{\Omega})$ such that $p_+ < n$ and $q(x)$ defined in Ω with $q_- \geq 1$ and

$$\text{ess inf}_{x \in \Omega} (p^*(x) - q(x)) > 0,$$

then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.

Proposition 2.10 (see [5, 9, 15, 18, 20]). The mapping \mathcal{J}_1 is coercive, convex and sequentially weakly lower semi-continuous; $u_m \rightharpoonup u$ in $\mathcal{X}_0 \Rightarrow \mathcal{J}_1(u) \leq \liminf_{m \rightarrow \infty} \mathcal{J}_1(u_m)$.

Lemma 2.1. Let $\xi(\cdot) \in C(\bar{\Omega}, \mathbb{R}_+^*)$ such that $0 < \text{ess inf}_{x \in \Omega} \xi(x) = \xi_- \leq \xi(x) \leq \xi_+ = \text{ess sup}_{x \in \Omega} \xi(x) < +\infty$. Then the following inequalities hold

$$\log(s) \leq \frac{e^{-1}}{\xi(x)} s^{\xi(x)}, \quad \forall s \in [1, +\infty) \quad \text{and} \quad -\frac{e^{-1}}{\xi(x)} \leq s^{\xi(x)} \log(s) \leq 0, \quad \forall s \in (0, 1].$$

Remark 2.2. It follows from Lemma 2.1 that

$$s^{q(x)} \log(s) \leq \frac{e^{-1}}{\xi(x)} s^{q(x)+\xi(x)}, \quad \forall s \in [1, +\infty).$$

Definition 2.1 (Weak solution). Let $\lambda \in \mathbb{R}$, $u \in \mathcal{X}_0$. We call that (λ, u) is a weak solution of problem (1.1) on Ω if

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla w \, dx = \lambda \int_{\Omega} |u|^{q(x)-2} u \log(|u|) w \, dx, \quad (2.3)$$

for all $w \in W_0^{1,p(x)}(\Omega)$, with λ as an eigenvalue and u as its corresponding eigenfunction.

3. AUXILIARY RESULTS

We first define the functionals \mathcal{J}_λ and \mathcal{I}_λ on \mathcal{X}_0 by

$$\mathcal{J}_\lambda(u) = \mathcal{J}_1(u) - \lambda \mathcal{J}_2(u) \quad \text{and} \quad \mathcal{I}_\lambda(u) = \mathcal{I}_1(u) - \lambda \mathcal{I}_2(u), \quad (3.1)$$

where

$$\mathcal{J}_1(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx, \quad \mathcal{J}_2(u) = \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} \log(|u|) dx - \int_{\Omega} \frac{1}{q^2(x)} |u|^{q(x)} dx, \quad (3.2)$$

and

$$\mathcal{I}_1(u) = \int_{\Omega} |\nabla u|^{p(x)} dx, \quad \mathcal{I}_2(u) = \int_{\Omega} |u|^{q(x)} \log(|u|) dx. \quad (3.3)$$

The functionals \mathcal{J}_λ and \mathcal{I}_λ are well defined in \mathcal{X}_0 . Moreover $\mathcal{J}_\lambda \in C^1(\mathcal{X}_0, \mathbb{R})$, (see [25, Remark 2.4]), we also have for all $w \in W_0^{1,p(x)}(\Omega)$ the following identities

$$\langle \mathcal{J}'_1(u), w \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla w dx, \quad \langle \mathcal{J}'_2(u), w \rangle = \int_{\Omega} |u|^{q(x)-2} u \log(|u|) w dx. \quad (3.4)$$

It is commonly recognized that u is a weak solution of problem (1.1) on \mathcal{X}_0 if and only if u is a critical point of the energy functional $\mathcal{J}_\lambda(u) = \mathcal{J}_1(u) - \lambda \mathcal{J}_2(u)$. In other words

$$\mathcal{J}_\lambda(u) = \mathcal{J}_1(u) - \lambda \mathcal{J}_2(u) = 0. \quad (3.5)$$

The restricted variational method is typically used to solve the eigenvalue problem (3.5) (see [1,27]). Here, we consider \mathcal{J}_1 to be a restricted functional and \mathcal{J}_2 to be an objective functional. For such a strictly non-negative real number α , we define the level sets

$$\mathcal{M}_\alpha^+ = \{u \in \mathcal{W}^+ \mid \mathcal{J}_1(u) \leq \alpha\}, \quad (3.6)$$

with boundary

$$\partial \mathcal{M}_\alpha^+ = \{u \in \mathcal{W}^+ \mid \mathcal{J}_1(u) = \alpha\}. \quad (3.7)$$

In what follows, we will use the following hypothesis

$$q(x) + \xi(x) < p^*(x), \quad (\text{H2})$$

where $\xi(x)$ is the function defined in Lemma 2.1.

Lemma 3.1. *Let $(u_m)_m \subset \mathcal{X}_0$ such that $u_m \rightharpoonup u$. Then*

$$\lim_{m \rightarrow \infty} \int_{\Omega} |u_m|^{q(x)-2} u_m \log(|u_m|) (u_m - u) dx = 0.$$

Proof. By Hölder inequality we have

$$\begin{aligned} \left| \int_{\Omega} |u_m|^{q(x)-2} u_m \log(|u_m|) (u_m - u) dx \right| &\leq \int_{\Omega} \left| |u_m|^{q(x)-1} \log(|u_m|) (u_m - u) \right| dx \\ &\leq \left(\frac{1}{q_-} + \frac{1}{q'_-} \right) \left\| |u_m|^{q(x)-1} \log(|u_m|) \right\|_{q'(x)} \|u_m - u\|_{q(x)}. \end{aligned}$$

Moreover, from Remark 2.2 where we choose $\xi(x) \ll \frac{(q(x)-1)(p^*(x)-q(x))}{q(x)}$ since by (1.3), we get

$$\begin{aligned} & \left\| |u_m|^{q(x)-1} \log(|u_m|) \right\|_{q'(x)} = \int_{\Omega} \left| |u_m|^{q(x)-1} \log(|u_m|) \right|^{q'(x)} dx \\ & \leq \int_{\Omega_-} \left| |u_m|^{q(x)-1} \log(|u_m|) \right|^{q'(x)} dx + \int_{\Omega_+} \left| |u_m|^{q(x)-1} \log(|u_m|) \right|^{q'(x)} dx \\ & \leq \left(\frac{e^{-1}}{q_- - 1} \right)^{q'_+} |\Omega| + \left(\frac{e^{-1}}{\xi_-} \right)^{q'_+} \int_{\Omega} |u_m|^{r(x)} dx \\ & \leq \left(\frac{e^{-1}}{q_- - 1} \right)^{q'_+} |\Omega| + \left(\frac{e^{-1}}{\xi_-} \right)^{q'_+} \|u_m\|_{r(x)}^{r_+}, \end{aligned} \quad (3.8)$$

since by Proposition 2.3 where

$$q(x) < r_- \leq r(x) = \frac{(q(x) - 1 + \xi(x))q(x)}{q(x) - 1} \leq r_+ \ll p^*(x), \quad (3.9)$$

and

$$\Omega_- = \{x \in \Omega \mid |u| \leq 1\}, \quad \Omega_+ = \{x \in \Omega \mid |u| > 1\}. \quad (3.10)$$

So, we obtain

$$\begin{aligned} & \left| \int_{\Omega} |u_m|^{q(x)-2} u_m \log(|u_m|) (u_m - u) dx \right| \\ & \leq \left(\frac{1}{q_-} + \frac{1}{q'_-} \right) \left(\left(\frac{e^{-1}}{q_- - 1} \right)^{q'_+} |\Omega| + \left(\frac{e^{-1}}{\xi_-} \right)^{q'_+} \|u_m\|_{r(x)}^{r_+} \right) \|u_m - u\|_{q(x)}. \end{aligned}$$

The compact embeddings $W_0^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and $W_0^{p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ given in Proposition 2.9, along with the holdness of (1.3) and (3.9) and in view of $u_m \rightharpoonup u$ yield that $(u_m)_m$ is bounded in $L^{q(x)}(\Omega) \cap L^{r(x)}(\Omega)$ with $\|u_m\|_{r(x)} \rightarrow \|u\|_{r(x)}$ and $\|u_m - u\|_{q(x)} \rightarrow 0$. The proof is complete. \square

Proposition 3.1. *The functional \mathcal{J}_2 exhibits weak-strong continuity; i.e., if $u_m \rightharpoonup u$ in \mathcal{X}_0 then $\mathcal{J}_2(u_m) \rightarrow \mathcal{J}_2(u)$.*

Proof. Let $(u_m)_m \subset \mathcal{X}_0$ be a sequence and $u \in \mathcal{X}_0$ such that $u_m \rightharpoonup u$, we have

$$\begin{aligned} |\mathcal{J}_2(u_m) - \mathcal{J}_2(u)| & \leq \frac{1}{q_-} \left| \int_{\Omega} \left(|u_m|^{q(x)} \log(|u_m|) - |u|^{q(x)} \log(|u|) \right) dx \right| \\ & \quad + \frac{1}{q_-^2} \left| \int_{\Omega} \left(|u_m|^{q(x)} - |u|^{q(x)} \right) dx \right|. \end{aligned} \quad (3.11)$$

The first term on the right-hand side of (3.11) can be estimated as

$$\begin{aligned} & \left| \int_{\Omega} \left(|u_m|^{q(x)} \log(|u_m|) - |u|^{q(x)} \log(|u|) \right) dx \right| \\ & \leq \left| \int_{\Omega} |u_m|^{q(x)-2} u_m \log(|u_m|) (u_m - u) dx \right| \\ & \quad + \left| \int_{\Omega} u \left(|u_m|^{q(x)-2} u_m \log(|u_m|) - |u|^{q(x)-2} u \log(|u|) \right) dx \right|. \end{aligned}$$

Lemma 3.1 and (3.8) due to the continuity of the function $u \mapsto |u|^{q(x)-2}u \log|u|$ with $u_m \rightharpoonup u$ and the compact embedding $W_0^{p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ (Proposition 2.9), where $r(\cdot)$ is the function given in (3.9), shows that

$$\left| \int_{\Omega} \left(|u_m|^{q(x)} \log(|u_m|) - |u|^{q(x)} \log(|u|) \right) dx \right| \rightarrow 0.$$

For the second term on the right-hand side of (3.11) we may apply the renowned inequality

$$||a|^r - |b|^r| \leq c|a - b|(|a| + |b|)^{r-1}, \quad \forall r > 1 \text{ and } a, b \in \mathbb{R},$$

where c is a non-negative constant, we get

$$\left| \int_{\Omega} \left(|u_m|^{q(x)} - |u|^{q(x)} \right) dx \right| \leq c \int_{\Omega} |u_m - u| (|u_m| + |u|)^{q(x)-1} dx,$$

Hölder inequality yields

$$\left| \int_{\Omega} \left(|u_m|^{q(x)} - |u|^{q(x)} \right) dx \right| \leq c \|u_m - u\|_{q(x)} \| |u_m| + |u| \|_{q(x)},$$

from (1.3) the compact embedding $W_0^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ gives us $\|u_m - u\|_{q(x)} \rightarrow 0$ and $\| |u_m| + |u| \|_{q(x)} \rightarrow 2^{q_{\pm}} \|u\|_{q(x)}$, which complet the proof. \square

Lemma 3.2. For all $u \in \mathcal{X}_0$, we have

$$\mathcal{J}_2(u) \leq \left(\frac{1}{q_{-}\xi_{-}} - \frac{\bar{C}}{q_{+}^2} \right) \max \left\{ \|u\|_{q(x)+\xi(x)}^{q_{-}+\xi_{-}}, \|u\|_{q(x)+\xi(x)}^{q_{+}+\xi_{+}} \right\},$$

where \mathcal{J}_2 is the functional defined in (3.2) and \bar{C} is positive constant.

Proof. We have

$$\begin{aligned} \mathcal{J}_2(u) &= \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} \log(|u|) dx - \int_{\Omega} \frac{1}{q^2(x)} |u|^{q(x)} dx \\ &\leq \frac{1}{q_{-}} \int_{\Omega} |u|^{q(x)} \log(|u|) dx - \frac{1}{q_{+}^2} \int_{\Omega} |u|^{q(x)} dx, \end{aligned} \quad (3.12)$$

thanks to Remark 2.2 and Proposition 2.3 we may write

$$\begin{aligned} \int_{\Omega} |u|^{q(x)} \log(|u|) dx &\leq \int_{\Omega_{-}} |u|^{q_{+}} \log(|u|) dx + \int_{\Omega_{+}} |u|^{q(x)} \log(|u|) dx \\ &\leq \int_{\Omega_{+}} |u|^{q(x)} \log(|u|) dx \\ &\leq \frac{1}{\xi_{-}} \int_{\Omega} |u|^{q(x)+\xi(x)} dx \\ &\leq \frac{1}{\xi_{-}} \max \left\{ \|u\|_{q(x)+\xi(x)}^{q_{-}+\xi_{-}}, \|u\|_{q(x)+\xi(x)}^{q_{+}+\xi_{+}} \right\}, \end{aligned} \quad (3.13)$$

where Ω_{-} and Ω_{+} are as defined in (3.10).

On the other hand, using Proposition 2.3, we go through the continuous embedding given in Proposition 2.4, since $q(x) \leq q(x) + \xi(x)$, we conclude that

$$\begin{aligned} \int_{\Omega} |u|^{q(x)} dx &\geq \min \left\{ \|u\|_{q(x)}^{q_-}, \|u\|_{q(x)}^{q_+} \right\} \\ &\geq C_1 \max \left\{ \|u\|_{q(x)}^{q_-}, \|u\|_{q(x)}^{q_+} \right\} \\ &\geq C_1 C_2 \max \left\{ \|u\|_{q(x)+\xi(x)}^{q_-+\xi_-}, \|u\|_{q(x)+\xi(x)}^{q_++\xi_+} \right\}, \end{aligned} \quad (3.14)$$

where

$$C_1 = \frac{\min \left\{ \|u\|_{q(x)}^{q_-}, \|u\|_{q(x)}^{q_+} \right\}}{\max \left\{ \|u\|_{q(x)}^{q_-}, \|u\|_{q(x)}^{q_+} \right\}}, \quad C_2 = \frac{\max \left\{ \|u\|_{q(x)}^{q_-}, \|u\|_{q(x)}^{q_+} \right\}}{\max \left\{ \|u\|_{q(x)+\xi(x)}^{q_-+\xi_-}, \|u\|_{q(x)+\xi(x)}^{q_++\xi_+} \right\}}.$$

By setting $\bar{C} = C_1 C_2$, we obtain the result from (3.12), (3.13) and (3.14). \square

The following assumption is also needed, assume that

$$\min_{\Omega} \int_{\Omega} |u|^{q(x)} \log(|u|) dx > 0. \quad (\text{H3})$$

Lemma 3.3. *Under hypothesis (H2) and (H3), the following statements hold*

- (1) *The sets \mathscr{W} , \mathscr{W}^+ and \mathcal{M}_{α}^+ defined in (1.4), (1.5) and (3.6) respectively, are not empty.*
- (2) *The closed subsets \mathcal{M}_{α}^+ endowed with the norm of \mathcal{X}_0 is a Banach subspace of \mathcal{X}_0 .*

Proof. (1) Define the functional $\mathcal{G} : \mathcal{X}_0 \rightarrow \mathbb{R}$ by

$$\mathcal{G}(u) = \frac{1}{p_-} \max \left\{ \|\nabla u\|_{p(x)}^{p_-}, \|\nabla u\|_{p(x)}^{p_+} \right\}.$$

It is clear by construction that \mathcal{G} is bounded from \mathcal{X}_0 to \mathbb{R} . Let α , α_0 , and $\bar{\alpha}$ be non-negative real numbers, and let u_0 be a fixed element in \mathcal{X}_0 , then from Lemma 2.1 and (3.13), using the embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega) \hookrightarrow L^{q(x)+\xi(x)}(\Omega)$ (Proposition 2.4 and Proposition 2.8) in satisfying (H2), we get

$$\begin{aligned} \left\| |u_0|^{q(x)} \log(|u_0|) \right\|_{L^1(\Omega)} &\leq \int_{\Omega_-} \left| |u_0|^{q(x)} \log(|u_0|) \right| dx + \int_{\Omega_+} \left| |u_0|^{q(x)} \log(|u_0|) \right| dx \\ &\leq \frac{e^{-1}}{q_-} |\Omega| + \frac{(CS)^{q_{\pm}+\xi_{\pm}}}{\xi_-} \max \left\{ \|\nabla u_0\|_{p(x)}^{q_-+\xi_-}, \|\nabla u_0\|_{p(x)}^{q_++\xi_+} \right\}, \end{aligned}$$

where C is the constants of the Lebesgue embedding and S is the best constant of the Sobolev embedding. This indicates that u_0 belongs to \mathscr{W} which implies that it is not empty. Furthermore, assume that $|u_0| > 1$ a.e. in Ω , the strictly increasing of the function $g : s \mapsto s^{q(x)} \log(s)$ on $(1, +\infty)$ yields

$$0 < \int_{\Omega} |u_0|^{q(x)} \log(|u_0|) dx \leq \frac{(CS)^{q_{\pm}+\xi_{\pm}}}{\xi_-} \max \left\{ \|\nabla v_0\|_{p(x)}^{q_-+\xi_-}, \|\nabla v_0\|_{p(x)}^{q_++\xi_+} \right\},$$

then $u_0 \in \mathscr{W}^+$ and \mathscr{W}^+ is not empty.

On the other hand, choose $\alpha_0 = \mathcal{G}(u_0)$; then from (1.2) and Proposition 2.3 (with u is replaced by ∇u_0) we obtain

$$\mathcal{J}_1(u_0) = \int_{\Omega} \frac{1}{p(x)} |\nabla u_0|^{p(x)} dx \leq \mathcal{G}(u_0) = \alpha_0.$$

Therefore $u_0 \in \mathcal{M}_{\alpha_0}^+$ and $\mathcal{M}_{\alpha_0}^+ \neq \emptyset$.

(2) Let $(u_m)_{m \geq 1}$ be a Cauchy sequences in \mathcal{W} and let $u \in \mathcal{X}_0$ such that $u_m \rightarrow u$ in \mathcal{X}_0 . Due to (1.2), Proposition 2.5 and Proposition 2.6 ensure that the sequences $(u_m)_{m \geq 1}$ is bounded in \mathcal{X}_0 , there exist a subsequence of $(u_m)_{m \geq 1}$ still denoted by $(u_m)_{m \geq 1}$ such that $u_m \rightarrow u$ a.e. in Ω . This together with the continuity of the function $g : s \mapsto s^{q(x)} \log(s)$ on $(0, +\infty)$ gives us

$$\int_{\Omega} |u_m|^{q(x)} \log(|u_m|) dx \rightarrow \int_{\Omega} |u|^{q(x)} \log(|u|) dx, \quad \text{a.e. in } \Omega, \quad (3.15)$$

and

$$\int_{\Omega} \left| |u_m|^{q(x)} \log(|u_m|) \right| dx \rightarrow \int_{\Omega} \left| |u|^{q(x)} \log(|u|) \right| dx, \quad \text{a.e. in } \Omega.$$

Thus $u \in \mathcal{W}$.

Moreover, suppose that $(u_m)_{m \geq 1} \subset \mathcal{W}^+$, that is to say by the definition of \mathcal{W}^+ we have

$$\int_{\Omega} |u_m|^{q(x)} \log(|u_m|) dx > 0, \quad \forall m \geq 1. \quad (3.16)$$

Using the continuity of the function $g : s \mapsto s^{q(x)} \log(s)$ we get relation as (3.15).

It remains to show that $\int_{\Omega} |u|^{q(x)} \log(|u|) dx > 0$. Indeed, assume the opposite, meaning that

$$\int_{\Omega} |u|^{q(x)} \log(|u|) dx \leq 0, \quad (3.17)$$

there exists m_0 large enough such that

$$\int_{\Omega} |u_m|^{q(x)} \log(|u_m|) dx \leq 0, \quad \forall m > m_0.$$

This contradict (3.16) and (H3). So, u is in \mathcal{W}^+ .

Next, the boundedness and weakly lower semicontinuous of the functional \mathcal{G} allows us to choose $\bar{\alpha} \geq \liminf_{n \rightarrow \infty} \mathcal{G}(u_m)$. Also the fact that the functional \mathcal{J}_1 is weakly lower semicontinuous (see Proposition 2.10), along with (1.2) and Proposition 2.3 (where u is replaced by ∇u_m), leads to

$$\mathcal{J}_1(u) \leq \liminf_{m \rightarrow \infty} \mathcal{J}_1(u_m) \leq \liminf_{m \rightarrow \infty} \mathcal{G}(u_m) \leq \bar{\alpha}.$$

Finally, by choosing $\alpha = \max \{ \alpha_0, \bar{\alpha} \}$, we get that u_0 and u are both in \mathcal{M}_{α}^+ , which means that it is not empty and every Cauchy sequence in \mathcal{M}_{α}^+ converges in \mathcal{M}_{α}^+ , thus the result. \square

4. MAIN RESULTS

This section begins by proving that, in the case where the ranges of $p(\cdot)$ and $q(\cdot)$ intersect, there exists a continuous family of eigenvalues. To this end, we present the subsequent eigenvalue sets

$$\sigma_\lambda^+ = \{\lambda \in \mathbb{R} : \exists u \in \mathcal{W}^+ \text{ such that } (\lambda, u) \text{ is an eigenpair of (1.1)}\},$$

and for any $\alpha > 0$,

$$\sigma_{\alpha, \lambda}^+ = \{\lambda \in \mathbb{R} : \exists u \in \mathcal{M}_\alpha^+ \text{ such that } (\lambda, u) \text{ is an eigenpair of (1.1)}\}.$$

Now, we denote by: η_* , λ_* the quotients of Rayleigh as follows:

$$\eta_* = \inf_{u \in \mathcal{W}^+} \frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{q(x)} \log(|u|) dx} = \inf_{u \in \mathcal{W}^+} \frac{\mathcal{I}_1(u)}{\mathcal{I}_2(u)}, \quad (4.1)$$

$$\lambda_* = \inf_{u \in \mathcal{M}_\alpha^+} \frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{q(x)} \log(|u|) dx} = \inf_{u \in \mathcal{M}_\alpha^+} \frac{\mathcal{I}_1(u)}{\mathcal{I}_2(u)}. \quad (4.2)$$

Lemma 4.1. *Let α be a strictly non-negative real number, assume that (H2) holds, then there exists $\lambda_\alpha > 0$ such that for any $\lambda \in (0, \lambda_\alpha)$ we have*

$$\mathcal{I}_\lambda(u) \geq \frac{\alpha}{2} > 0 \text{ for any } u \in \partial \mathcal{M}_\alpha^+. \quad (4.3)$$

Proof. Suppose that $u \in \partial \mathcal{M}_\alpha^+$ for such real number α strictly non-negative, then from (3.1), Lemma 3.2 and the definition of $\partial \mathcal{M}_\alpha^+$ we have

$$\begin{aligned} \mathcal{I}_\lambda(u) &= \alpha - \lambda \mathcal{I}_2(u) \\ &\geq \alpha - \lambda \left(\frac{1}{q - \xi_-} - \frac{\bar{C}}{q_+^2} \right) \max \left\{ \|u\|_{q(x) + \xi(x)}^{q_- + \xi_-}, \|u\|_{q(x) + \xi(x)}^{q_+ + \xi_+} \right\}. \end{aligned}$$

Choose $\xi_- < q_+^2 / \bar{C} q_-$. From Proposition 2.1, Proposition 2.4 and Proposition 2.8 in view of (H2), the embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega) \hookrightarrow L^{q(x) + \xi(x)}(\Omega)$ give us

$$\mathcal{I}_\lambda(u) \geq \alpha - \lambda \left(\frac{1}{q - \xi_-} - \frac{\bar{C}}{q_+^2} \right) (CS)^{q_\pm + \xi_\pm} \max \left\{ \|\nabla u\|_{p(x)}^{q_- + \xi_-}, \|\nabla u\|_{p(x)}^{q_+ + \xi_+} \right\},$$

where C is the constant of the embedding $L^{p^*(x)}(\Omega) \hookrightarrow L^{q(x) + \xi(x)}(\Omega)$ and S is the best constant of the Sobolev embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$. Moreover, since $u \in \partial \mathcal{M}_\alpha^+$, so by Proposition 2.3 (with u is replaced by ∇u) we may write

$$\mathcal{I}_\lambda(u) \geq \alpha - \lambda \left(\frac{1}{q - \xi_-} - \frac{\bar{C}}{q_+^2} \right) (CS)^{q_\pm + \xi_\pm} \max \left\{ \beta^{\frac{q_- + \xi_-}{p_-}}, \beta^{\frac{q_- + \xi_-}{p_+}}, \beta^{\frac{q_+ + \xi_+}{p_-}}, \beta^{\frac{q_+ + \xi_+}{p_+}} \right\}, \quad (4.4)$$

where $\beta = \alpha p_+$.

Finally, by defining

$$\lambda_\alpha = \frac{\alpha q_+^2 q_- \xi_-}{2 \left(q_+^2 - \bar{C} q_- \xi_- \right) (CS)^{q_\pm + \xi_\pm} \max \left\{ \beta^{\frac{q_- + \xi_-}{p_-}}, \beta^{\frac{q_- + \xi_-}{p_+}}, \beta^{\frac{q_+ + \xi_+}{p_-}}, \beta^{\frac{q_+ + \xi_+}{p_+}} \right\}}, \quad (4.5)$$

then, for any $\lambda \in (0, \lambda_\alpha)$ and any $u \in \partial \mathcal{M}_\alpha^+$, we get the result. \square

Now, let us define by \mathcal{Z} the set of all zeros of \mathcal{I}_λ , i.e.,

$$\mathcal{Z} = \{u \in \mathcal{X}_0 \mid \mathcal{I}_\lambda(u) = 0\}.$$

Proposition 4.1. *Let $\alpha > 0$; the set \mathcal{Z} is contained within $\mathcal{M}_{\alpha p_+}^+$, meaning that $\mathcal{Z} \subset \mathcal{M}_{\alpha p_+}^+$.*

Proof. Let $\alpha > 0$ and let $u \in \mathcal{Z}$, then by the definition of \mathcal{Z} we have $\mathcal{I}_\lambda(u) = 0$ or even from (1.2), (3.2) and the definition of $\mathcal{M}_{\alpha p_+}^+$, we get

$$\lambda \int_{\Omega} |u|^{q(x)} \log(|u|) dx = \int_{\Omega} |\nabla u|^{p(x)} dx \leq p_+ \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \leq \alpha p_+.$$

This means that $u \in \mathcal{M}_{\alpha p_+}^+$ and the proof is achieved. \square

Lemma 4.2.

- (1) *Suppose that $p_- < q_+ + \xi_+ < p_-^*$, then \mathcal{I}_λ is coercive on \mathcal{Z} .*
- (2) *The functionals \mathcal{J}_λ and \mathcal{I}_λ are weakly lower semicontinuous.*

Proof. To prove Lemma 4.2, we can use the same method as in the proof of [25, Lemma 2.6]. Therefore, the proof is omitted. \square

At this point, we present the main result of this paper.

Theorem 4.1. *Suppose that (1.2), (1.3) and (H1)–(H3) are satisfied*

- (1) *If $\lambda \in (0, \lambda_*)$ then $\lambda \notin \sigma_{\alpha, \lambda}^+$.*
- (2) *If $q_- < p_-$ then for every $\alpha > 0$, there exists a $\lambda_\alpha > 0$ such that eigenvalues of problem (1.1) are all real numbers $\lambda \in (0, \lambda_\alpha)$, with corresponding eigenfunctions belonging to \mathcal{M}_α^+ . Additionally, $\lambda_* = 0$.*

Proof. (1) By contradiction, assume that there exists $\lambda \in \sigma_{\alpha, \lambda}^+$ and $\lambda \in (0, \lambda_*)$. Then there exists $u \in \mathcal{M}_\alpha^+$ such that by the Definition 2.1 (replacing $w = u$) and (3.3) we have

$$\mathcal{I}_1(u) = \lambda \mathcal{I}_2(u). \quad (4.6)$$

That is to say $\lambda \in (0, \lambda_*)$, (4.2) yields

$$\lambda < \lambda_* = \inf_{u \in \mathcal{M}_\alpha^+} \frac{\mathcal{I}_1(u)}{\mathcal{I}_2(u)} \leq \frac{\mathcal{I}_1(u)}{\mathcal{I}_2(u)},$$

this gives us

$$\lambda \mathcal{I}_2(u) < \mathcal{I}_1(u),$$

which is a contradiction with (4.6), and (1) follows.

- (2) Let $\alpha > 0$, λ_α be given as in (4.5) with $\lambda \in (0, \lambda_\alpha)$ and let \mathcal{M}_α^+ given as in (3.6). We consider \mathcal{U}_α^+

the interior of the closed set \mathcal{M}_α^+ which when it equipped with the norm of \mathcal{X}_0 is a Banach subspace of \mathcal{X}_0 (Lemma 3.3). Since \mathcal{J}_λ is a coercive on $\mathcal{L} \subset \mathcal{M}_{\alpha p^+}^+$ with

$$\lim_{u \rightarrow 0} \mathcal{J}_\lambda(u) = 0, \quad (4.7)$$

we deduce from Lemma 4.1 for some α large enough that \mathcal{J}_λ attained its infimum within \mathcal{U}_α^+ the interior of \mathcal{M}_α^+ . Moreover, from (4.4) we get

$$\mathcal{J}_\lambda(u) \geq -\lambda \left(\frac{1}{q-\xi_-} - \frac{\bar{C}}{q_+^2} \right) (CS)^{q_\pm + \xi_\pm} \max \left\{ \beta^{\frac{q_- + \xi_-}{p^-}}, \beta^{\frac{q_- + \xi_-}{p^+}}, \beta^{\frac{q_+ + \xi_+}{p^-}}, \beta^{\frac{q_+ + \xi_+}{p^+}} \right\}, \quad (4.8)$$

meaning that \mathcal{J}_λ is bounded below and then

$$\inf_{\mathcal{M}_\alpha^+} \mathcal{J}_\lambda = \inf_{\mathcal{U}_\alpha^+} \mathcal{J}_\lambda > -\infty,$$

resulting by the definition of infimum, for any $\varepsilon > 0$, there exists $u_\varepsilon \in \mathcal{U}_\alpha^+$ such that

$$\inf_{\mathcal{U}_\alpha^+} \mathcal{J}_\lambda + \varepsilon = \inf_{\mathcal{M}_\alpha^+} \mathcal{J}_\lambda + \varepsilon > \mathcal{J}_\lambda(u_\varepsilon).$$

Next, applying Ekeland's variational principle (see [14]) to $\mathcal{J}_\lambda : \mathcal{M}_\alpha^+ \rightarrow \mathbb{R}$, for any $u \in \mathcal{M}_\alpha^+$ with $u \neq u_\varepsilon$, we obtain

$$\mathcal{J}_\lambda(u) > \mathcal{J}_\lambda(u_\varepsilon) - \varepsilon \|\nabla u - \nabla u_\varepsilon\|_{p(x)}. \quad (4.9)$$

Choosing $u = u_\varepsilon \pm tv$ in (4.9) for any $v \in \mathcal{M}_\alpha^+$ and $t > 0$ small enough (resp $t < 0$), it then follows that

$$\left| \frac{\mathcal{J}_\lambda(u_\varepsilon \pm tv) - \mathcal{J}_\lambda(u_\varepsilon)}{tv} v \right| \leq \varepsilon \|\nabla v\|_{p(x)},$$

Hölder inequality gives us

$$\left\| \frac{\mathcal{J}_\lambda(u_\varepsilon \pm tv) - \mathcal{J}_\lambda(u_\varepsilon)}{tv} \right\|_{\mathcal{X}_0^*} \|\nabla v\|_{p(x)} \leq \varepsilon \|\nabla v\|_{p(x)},$$

Letting $t \rightarrow 0$, we find

$$\|\mathcal{J}'_\lambda(u_\varepsilon)\|_{\mathcal{X}_0^*} \leq \varepsilon.$$

Therefore, we conclude that there exists a sequence $(u_m)_m \subset \mathcal{U}_\alpha^+$ such that

$$-\infty < \mathcal{J}_\lambda(u_m) \rightarrow \inf_{\mathcal{U}_\alpha^+} \mathcal{J}_\lambda = a \quad \text{and} \quad \mathcal{J}'_\lambda(u_m) \rightarrow 0 \text{ in } \mathcal{X}_0^*. \quad (4.10)$$

Clearly, $(u_m)_m$ is bounded and there exists $u \in \mathcal{X}_0$ such that $u_m \rightharpoonup u$. So sequentially weakly lower semi-continuity of \mathcal{J}_1 (see Proposition 2.10) leads to $\mathcal{J}_1(u) \leq \alpha$; that is to say, with the help of Lemma 3.3 that $u \in \mathcal{M}_\alpha^+$. Moreover, we have

$$\langle \mathcal{J}'_1(u_m), u_m - u \rangle = \langle \mathcal{J}'_\lambda(u_m), u_m - u \rangle + \lambda \langle \mathcal{J}'_2(u_m t), u_m - u \rangle.$$

Then (4.10) and Lemma 3.1 with (3.4) leads to

$$\langle \mathcal{J}'_1(u_m), u_m - u \rangle \rightarrow 0, \quad (4.11)$$

we also have by $u_m \rightharpoonup u$ and (4.11) that

$$\begin{aligned} & \overline{\lim}_{m \rightarrow +\infty} \langle \mathcal{J}'_1(u_m) - \mathcal{J}'_1(u), u_m - u \rangle \\ &= \overline{\lim}_{m \rightarrow +\infty} \langle \mathcal{J}'_1(u_m), u_m - u \rangle - \overline{\lim}_{m \rightarrow +\infty} \langle \mathcal{J}'_1(u), u_m - u \rangle = 0. \end{aligned}$$

Since \mathcal{J}'_1 is of type (S_+) (see [20, Theorem 3.1]) the sequence $(u_m)_m$ converges strongly to u in \mathcal{X}_0 .

From (3.2) and (3.4) (with w is replaced by u) the functional \mathcal{J}_λ is in $C^1(\mathcal{X}_0, \mathbb{R})$, we then deduce that

$$\mathcal{J}'_\lambda(u_m) \rightarrow \mathcal{J}'_\lambda(u) \text{ as } m \rightarrow \infty. \quad (4.12)$$

Thus (4.10) and (4.12) yield that

$$\mathcal{J}_\lambda(u) = a \quad \text{and} \quad \mathcal{J}'_\lambda(u) = 0, \quad (u \in \mathcal{L}).$$

To complete the proof, looking to (4.7), it remains to prove that u is not in the neighbourhood of 0. To this end, it suffices to show that $a < 0$. Indeed, since (H1) holds, we can choose $\varepsilon > 0$ such that $q_- + \varepsilon < p_-$. By the continuity of the function $q(\cdot)$, we deduce the existence of an open subset $\Omega_0 \subset \Omega$ such that $q(x) \leq q_- + \varepsilon < p_-$ for all $x \in \Omega_0$. Let $v \in \mathcal{M}_\alpha^+$. It is obvious that $tv \in \mathcal{U}_\alpha$ for any $t \in (0, 1)$. So, thanks to Lemma 2.1 (choosing $\xi_- < \frac{q_+ e^{-1} |\Omega|}{\max\{\|v\|_{q(x)}^{q_-}, \|v\|_{q(x)}^{q_+}\}}$) and Proposition 2.3 we get

$$\begin{aligned} \mathcal{J}_\lambda(tv) &= \int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla v|^{p(x)} dx - \lambda \left(\int_\Omega \frac{t^{q(x)}}{q(x)} |v|^{q(x)} \log(|v|) dx - \int_\Omega \frac{t^{q(x)}}{q^2(x)} |v|^{q(x)} dx \right) \\ &\leq t^{p_-} \alpha - \frac{\lambda t^{(q_- + \varepsilon)}}{q_+} \left(\frac{e^{-1}}{\xi_-} |\Omega_0| - \frac{1}{q_+} \int_{\Omega_0} |v|^{q(x)} dx \right) \\ &\leq t^{p_-} \alpha - \frac{\lambda t^{(q_- + \varepsilon)}}{q_+} \left(\frac{e^{-1}}{\xi_-} |\Omega| - \frac{1}{q_+} \max\{\|v\|_{q(x)}^{q_-}, \|v\|_{q(x)}^{q_+}\} \right). \end{aligned}$$

Choose t such that

$$0 < t < \min \left\{ 1, \frac{\lambda}{\alpha q_+} \left(\frac{e^{-1}}{\xi_-} |\Omega| - \frac{1}{q_+} \max\{\|v\|_{q(x)}^{q_-}, \|v\|_{q(x)}^{q_+}\} \right) \right\}^{\frac{1}{p_- - q_- - \varepsilon}},$$

we obtain

$$\mathcal{J}_\lambda(tv) < 0,$$

Consequently, we find

$$\inf_{\mathcal{U}_\alpha^+} \mathcal{J}_\lambda = a < 0.$$

So u is a weak solution for problem (1.1) and thus any $\lambda \in (0, \lambda_\alpha)$ is an eigenvalue of problem (1.1) with corresponding eigenfunction in \mathcal{M}_α^+ . Based on the preceding analysis, there are no eigenvalues inside $(0, \lambda_*)$ therefore $\lambda_* = 0$. \square

Remark 4.1. The expression of λ_α in (4.5) can be analysed to provide further insights into the magnitude of the eigenvalue set in relation to the exponents $p(\cdot)$ and $q(\cdot)$. Indeed, assume that $\alpha p^+ \geq 1$, then

$$\lambda_\alpha = \alpha^{1 - \frac{q_+ + \xi_+}{p_-}} \left(\frac{q_+^2 q_- \xi_- (p_+)^{-\frac{q_+ + \xi_+}{p_-}}}{2(q_+^2 - \overline{C} q_- \xi_-)(CS)^{q_\pm + \xi_\pm}} \right), \quad (4.13)$$

and hence if $q_+ + \xi_+ < p_-$ we have

$$\lim_{\alpha \rightarrow +\infty} \lambda_\alpha = +\infty.$$

But when $q_+ + \xi_+ = p_-$, we may get

$$\lambda_\alpha = \lambda_{p_-} = \frac{q_+^2 q_- \xi_-}{2p_+ (q_+^2 - \overline{C} q_- \xi_-)(CS)^{q_\pm + \xi_\pm}},$$

which no longer depends on α . This case will allow us to present the multiplicity result of the eigenfunctions.

When $\alpha p^+ < 1$

$$\lambda_\alpha = \alpha^{1 - \frac{q_- + \xi_-}{p_+}} \left(\frac{q_+^2 q_- \xi_- (p_+)^{-\frac{q_- + \xi_-}{p_+}}}{2(q_+^2 - \overline{C} q_- \xi_-)(CS)^{q_\pm + \xi_\pm}} \right). \quad (4.14)$$

Then

$$\lim_{\alpha \rightarrow 0} \lambda_\alpha = 0.$$

Based on Remark 4.1, we will examine the eigenvalue set $\sigma_{\alpha, \lambda}^+$ for the case in which problem (1.1) is sublinear. Generally, we assume that $1 < q(x) + \xi(x) < p(x)$.

Corollary 4.1. Suppose that assumptions (H1) and (H2) hold with $q_+ + \xi_+ < p_-$, then $\sigma_\lambda = (0, +\infty)$, that is to say, any $\lambda > 0$ is an eigenvalue of problem (1.1) on \mathcal{X}_0 and hence $\lambda_* = \eta_* = 0$.

Proof. Obviously, $\sigma_\lambda^+ \subset (0, +\infty)$. So let $\lambda \geq 0$. Suppose that $q_+ + \xi_+ < p_-$, then by Theorem 4.1, for any $\alpha > 0$, there is an eigenfunction $u_\lambda \in \mathcal{M}_\alpha^+$ associated to λ . By (4.13), $\lim_{\alpha \rightarrow +\infty} \lambda_\alpha = \infty$. This means for α large enough with $\lambda \in (0, \lambda_\alpha) \subset \sigma_\lambda^+$, that $\sigma_\lambda^+ = (0, +\infty)$ with eigenfunctions in \mathcal{X}_0 .

To show that $\lambda_* = \eta_* = 0$, we observe from their definitions that $\lambda_* \geq \eta_*$. This, together with the fact that $\inf \{\sigma_{\alpha, \lambda}^+\} = 0$, gives the result. \square

Corollary 4.2. Suppose that assumptions (H1) and (H2) hold with $q_- + \xi_- < p_-$ and $q_+ + \xi_+ = p_-$, then each $\eta \in (0, \lambda_{p_-})$, admits at least an infinitely countable family of eigenfunctions in \mathcal{X}_0 .

Proof. Let $\eta \in (0, \lambda_{p_-})$ and consider an increasing sequence of positive real numbers $(\alpha_m)_{m \in \mathbb{N}^*}$ satisfying $\alpha_m p_+ \geq 1$, $\forall m \in \mathbb{N}^*$. By Lemma 4.1, there exists a positive real number λ_{α_m} such that for any $\lambda \in (0, \lambda_{\alpha_m})$, inequality (4.3) holds. But since $q_+ + \xi_+ = p_-$, $\lambda_{\alpha_m} = \lambda_{p_-}$, $\forall m \in \mathbb{N}^*$, then

$\eta \in (0, \lambda_{\alpha_m})$, $\forall m \in \mathbb{N}^*$. Using inequality (4.8) in the proof of Theorem 4.1 along with the fact that $\alpha_m p_+ \geq 1$, $\forall m \in \mathbb{N}^*$ and $q_+ + \xi_+ = p_-$, we get

$$\begin{aligned} \mathcal{J}_\eta(u) &\geq -\lambda_{p_-} \left(\frac{1}{q_- \xi_-} - \frac{\bar{C}}{q_+^2} \right) (CS)^{q_\pm + \xi_\pm} \max \left\{ \beta^{\frac{q_- + \xi_-}{p_-}}, \beta^{\frac{q_- + \xi_-}{p_+}}, \beta^{\frac{q_+ + \xi_+}{p_-}}, \beta^{\frac{q_+ + \xi_+}{p_+}} \right\} \\ &\geq -\lambda_{p_-} \left(\frac{1}{q_- \xi_-} - \frac{\bar{C}}{q_+^2} \right) (CS)^{q_\pm + \xi_\pm} \alpha_m p_+ \\ &\geq -\frac{\alpha_m}{2}. \end{aligned}$$

So $\mathcal{J}_\eta(u)$ is bounded below on $\mathcal{M}_{\alpha_m}^+$ and since \mathcal{J}_η is a coercive on $\mathcal{L} \subset \mathcal{M}_{\alpha_m p_+}^+$ and $\mathcal{J}_\eta(u) \geq \frac{\alpha_m}{2}$ on $\partial \mathcal{M}_{\alpha_m}^+$, it achieves its infimum in $\mathcal{U}_{\alpha_m}^+$ the interior of $\mathcal{M}_{\alpha_m}^+$. Thus, by closely examining the idea developed in the proof of Theorem 4.1, we derive some eigenfunctions associated with η for each $\mathcal{M}_{\alpha_m}^+$ and consequently, as m approaches $+\infty$, we can extract a sequence of eigenfunctions $(u_m)_{m \in \mathbb{N}^*}$ that belong to \mathcal{X}_0 and share the same eigenvalue η . Thus the proof is complete. \square

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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