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FORECASTING COMMODITY PRICES WITH MEMORY: AN APPLICATION OF APPROXIMATE ARITHMETIC FRACTIONAL BROWNIAN MOTION TO PET MARKETS

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Abstract. This study explores the use of fractional Brownian motion (fBm) and its approximation in modeling polyethylene terephthalate (PET) prices. We compare the performance of the Arithmetic Brownian Motion (ABM) model with the approximate arithmetic fractional Brownian motion (AAFBM) model, which accounts for long-range dependencies in price data. Using PET price data from 2018 to 2021, our results show that the AAFBM model outperforms the ABM model, offering improved forecasting accuracy. This approach provides valuable insights for commodity price forecasting and risk management.

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Key words and phrases. fractional brownian motion; price modeling; commodity forecasting; PET.

1. Introduction

Commodity pricing plays a critical role in financial analysis, industrial procurement, and supply chain management. In particular, the price dynamics of polyethylene terephthalate (PET)—a widely used thermoplastic polymer—are of significant interest due to their implications in packaging, textiles, and recycling markets. Traditional models such as the Arithmetic Brownian Motion (ABM) have long been employed to simulate commodity price movements. However, these models assume that price increments are independent and identically distributed, failing to account for long-range dependencies and temporal correlations often observed in real-world price series. Recent work in commodities confirms persistent or scale-dependent behavior in returns and especially volatility, where long-memory and wavelet/EVT-augmented frameworks materially improve forecasting and VaR/ES estimation [1,16]. Related evidence from spot–futures systems shows that allowing fractional dynamics can raise predictive power over non-fractional VAR/CVAR benchmarks [7].

To address these limitations, fractional Brownian motion (fBm), introduced by Mandelbrot and Van Ness [13], provides a more flexible framework by incorporating memory effects through the Hurst exponent H. Nevertheless, fBm is not a semimartingale when $H \neq \frac{1}{2}$, which prevents the use of classical

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stochastic calculus and complicates applications in financial modeling. For background on fractional integration and calculus with fractal functions—and their connections to finance—see [15,17], and for a concise overview of properties specific to fBm, see [14]. To overcome tractability challenges, Alòs, Mazet, and Nualart [2] proposed using an alternative process derived from fBm, which retains long-memory features while facilitating mathematical analysis. Building upon this insight, Dung and Thao [9] introduced the approximate arithmetic fractional Brownian motion (AAFBM), a semimartingale-based model that closely approximates fBm via a smoothing parameter $\varepsilon > 0$ and permits Itô-type analysis. Taken together, these methodological contributions—semimartingale/Volterra approximations and transaction-cost frameworks—align with our strategy to preserve memory under a no-arbitrage, tractable setup [3,5,6,8]. In parallel, the rough-volatility literature documents volatility dynamics consistent with fractional noise (with H typically below $\frac{1}{2}$) and provides fast simulation and approximation schemes that make non-Markovian models practical [3,4,11].

In this paper, we apply the AAFBM framework to model and forecast quarterly PET prices. We construct binomial pricing trees under both the ABM and AAFBM models using historical price data from 2018 to 2021, then assess their predictive accuracy over the 2022–2023 period. Our results show that AAFBM significantly outperforms ABM in terms of forecast precision, as measured by the sum of squared errors (SSE). Furthermore, we conduct a sensitivity analysis over a grid of AAFBM parameters to demonstrate the model's robustness and its potential application in commodity pricing. We note that recent estimation technology—time-varying and wavelet-based Hurst procedures—improves inference robustness versus early plug-ins and supports the practical value of memory-aware modeling for forecasting and regime identification [10,12].

2. Preliminaries

2.1. **Fractional Brownian Motion and Its Approximation.**

In 1968, Mandelbrot and Van Ness [13] introduced a generalization of Brownian motion known as *fractional Brownian motion* (*fBm*). A fractional Brownian motion B_t^H with Hurst index $H \in (0,1)$ is a zero-mean Gaussian process with covariance function

$$R(t,s) = \mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\left(|t|^\gamma + |s|^\gamma - |t-s|^\gamma\right),$$

where $\gamma = 2H$.

Let $\alpha = H - \frac{1}{2}$ and b_0 be a real constant. The fBm B_t^H was originally defined in [13] as $B_0^H = b_0$, and for $t > 0$,

$$B_t^H = B_0^H + \frac{1}{\Gamma(1+\alpha)}\left[Z_t + \int_0^t (t-s)^\alpha \, dW_s\right],$$

where W_s is a standard Brownian motion and

$$Z_t = \int_{-\infty}^0 \left[(t-s)^\alpha - (-s)^\alpha\right] dW_s.$$

Note that when $H = \frac{1}{2}$, B_t^H reduces to standard Brownian motion, which is a memoryless process. For $H \neq \frac{1}{2}$, B_t^H is not a semimartingale, so classical Itô calculus cannot be applied. Since Z_t has absolutely continuous paths, Alòs et al. [2] proposed using the process

$$B_t = \int_0^t (t-s)^\alpha \, dW_s$$

instead of B_t^H in fractional stochastic calculus. Note also that only the term B_t exhibits long-range dependence.

For every $\varepsilon > 0$, define

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha \, dW_s,$$

where $-\frac{1}{2} < \alpha < \frac{1}{2}$ and $\alpha \neq 0$ (i.e., $H \neq \frac{1}{2}$).

In [15], Thao proved that the process $(B_t^\varepsilon)_{t \geq 0}$ is a semimartingale and approximates B_t in the sense that

$$\|B_t^\varepsilon - B_t\|^2 \to 0 \quad \text{as } \varepsilon \to 0,$$

where $\| \cdot \|$ denotes the $L^2(\Omega)$ norm. That is, $B_t^\varepsilon \to B_t$ uniformly on $[0, T]$ in L^2.

Theorem 1. *The process B_t^ε is normally distributed with mean zero and variance*

$$\int_0^t (t-s+\varepsilon)^{2\alpha} \, ds.$$

Proof. It is well known that if f is a deterministic square-integrable function, then

$$\int_0^t f(s) \, dW_s \sim \mathcal{N}\left(0, \int_0^t f^2(s) \, ds\right).$$

Therefore,

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha \, dW_s \sim \mathcal{N}\left(0, \int_0^t (t-s+\varepsilon)^{2\alpha} \, ds\right),$$

as required. □

2.2. **An Approximate Approach to Arithmetic Fractional Brownian Motion.** The classical arithmetic Brownian motion (ABM) with drift μ and volatility σ is given by

$$dp_t = \mu \, dt + \sigma \, dW_t,$$

where W_t is standard Brownian motion. Since W_t has no memory, this model is not suitable for systems with long memory.

To account for long memory, we consider the fractional extension

$$dp_t = \mu \, dt + \sigma \, dB_t, \tag{1}$$

where $B_t = \int_0^t (t-s)^\alpha \, dW_s$ and $\alpha = H - \frac{1}{2}$. This model is known as *arithmetic fractional Brownian motion* (AFBM). The solution is

$$p_t = p_0 + \mu t + \sigma \int_0^t dB_s, \quad p_0 = p_t|_{t=0}. \tag{2}$$

Here, the integral is defined pathwise in the sense of Zähle [17]. However, as B_t is not a semimartingale, this model is not ideal for applications. To resolve this, Dung and Thao [9] proposed the approximation

$$dp_t = \mu \, dt + \sigma \, dB_t^\varepsilon, \tag{3}$$

called the *approximate AFBM* (AAFBM). Its solution is

$$p_t^\varepsilon = p_0 + \mu t + \sigma B_t^\varepsilon. \tag{4}$$

For $\frac{1}{2} < H < 1$, it was shown in [9] that $p_t^\varepsilon \to p_t$ in $L^1(\Omega)$ uniformly in $t \in [0, T]$ as $\varepsilon \to 0$. From Theorem 1, we have

$$\mathbb{E}[p_t^\varepsilon] = p_0 + \mu t \quad \text{and} \quad \text{Var}[p_t^\varepsilon] = \sigma^2 \int_0^t (t - s + \varepsilon)^{2\alpha} \, ds.$$

2.3. **A Price That Follows a Random Walk.** Suppose the log-price of an asset follows a random walk with drift. If p_j denotes the j-th observation, then

$$p_{j+1} - p_j = \nu + u_{j+1}, \quad u_{j+1} \sim \mathcal{N}(0, \phi^2),$$

where ν and ϕ are constants. Therefore,

$$p_{j+1} - p_j \sim \mathcal{N}(\nu, \phi^2). \tag{5}$$

To calibrate a binomial tree, we estimate ν and ϕ from historical data. However, data may be observed at a different frequency than the desired tree step size. For example, one might use monthly data to calibrate a quarterly model. Fortunately, stochastic process theory provides a solution.

The classical ABM generalizes (5) to continuous time. In this paper, we extend this further by using the approximate AFBM model (3):

$$dp_t = \mu \, dt + \sigma \, dB_t^\varepsilon.$$

Over a small interval Δt, the increment satisfies:

$$p_{t+\Delta t} - p_t \sim \mathcal{N}\left(\mu \Delta t, \, \sigma^2 \int_0^{\Delta t} (\Delta t - s + \varepsilon)^{2\alpha} ds\right).$$

Indeed, by Theorem 1:

$$\mathbb{E}[p_{t+\Delta t} - p_t] = \mu \Delta t,$$

$$\mathrm{Var}[p_{t+\Delta t} - p_t] = \sigma^2 \int_0^{\Delta t} (\Delta t - s + \varepsilon)^{2\alpha} ds.$$

Thus, the parameters in (5) relate to those in (3) as follows:

$$\nu = \mu \Delta t, \quad \phi^2 = \sigma^2 \int_0^{\Delta t} (\Delta t - s + \varepsilon)^{2\alpha} ds.$$

These relationships will play a central role in our calibration method, as described in the next section.

3. Methodology

In this section, we first present a novel method for modeling asset price paths using the approximate arithmetic fractional Brownian motion (AAFBM). We then construct binomial trees for PET prices under both the ABM and AAFBM models using price data from January 2018 to December 2021 (see Figure 1). In the final step, we assess the efficiency of the ABM and AAFBM models using PET price data from January 2022 to December 2023.

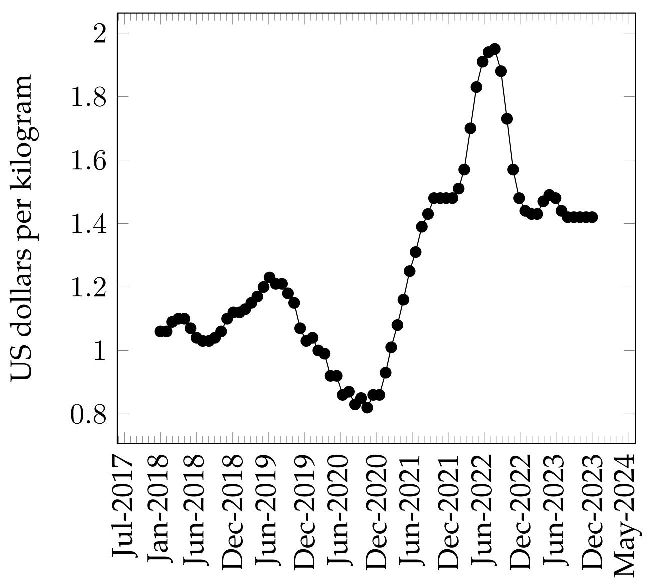

Figure 1. The price of PET in dollars per kilogram from Jan 2018 to Dec 2023
Source : https://businessanalytiq.com/procurementanalytics/index/pet-price-index/

3.1. **Normalized estimates of parameters.**

Suppose that asset data has one observation every Δt_d years, and that changes in the log-price have a sample mean $\hat{\nu}$ and standard deviation $\hat{\phi}$. Furthermore, suppose that the data-generating process for the log-price follows an Arithmetic Fractional Brownian Motion (AFBM) with drift parameter μ and volatility σ. Then, the population mean of changes in log-price is $\mu \Delta t_d$. Now, we aim to find suitable values for the drift parameter $\hat{\mu}$ and volatility $\hat{\sigma}$ in the AAFBM model. A sensible estimate of the drift parameter $\hat{\mu}$ should ensure that the population mean equals the sample mean. That is, it is reasonable to choose $\hat{\mu}$ such that it satisfies the equation $\hat{\nu} = \hat{\mu} \Delta t_d$, i.e.,

$$\hat{\mu} = \frac{\hat{\nu}}{\Delta t_d}. \tag{6}$$

Similarly, the population variance of changes in the log-price is $\sigma^2 \int_0^{\Delta t} (\Delta t - s + \varepsilon)^{2\alpha} ds$. Therefore, a reasonable estimate of the volatility $\hat{\sigma}$ should satisfy the equation

$$\hat{\phi}^2 = \hat{\sigma}^2 \int_0^{\Delta t} (\Delta t - s + \varepsilon)^{2\alpha} ds.$$

That is, we define

$$\hat{\sigma} = \frac{\hat{\phi}}{\sqrt{\int_0^{\Delta t} (\Delta t - s + \varepsilon)^{2\alpha} ds}}. \tag{7}$$

3.2. **Building the binomial options pricing model.** We now construct the binomial trees for the log-price of PET under both the ABM and AAFBM models. In many industries, driven by production planning, inventory management, and supplier agreements, materials are typically ordered on a quarterly basis. This quarterly cycle facilitates more predictable material flows and helps optimize costs. In alignment with this practice, we develop a pricing tree based on quarterly PET prices. Since our goal is to forecast prices on a quarterly basis, each time step in the tree will represent a 3-month period. That is, we set $\Delta t_m = \frac{1}{4}$. At the initial node $(0, 0)$, the log-price is set to the observed market price on the date the real options analysis is conducted. In each subsequent period, the log-price either increases or decreases by $\hat{\sigma}\sqrt{\Delta t_m}$.

Let $X(i, n)$ denote the price at node (i, n), and $x(i, n)$ denote the corresponding log-price. Then the log-price at node (i, n) is given by:

$$x\,(i, n) = \log P_0 + (n - 2i)\,\hat{\sigma}\sqrt{\Delta t_m},$$

where P_0 is the initial price at node $(0, 0)$. Taking exponential of both sides yields the price at that node:

$$X(i, n) = e^{x(i,n)} = P_0 e^{(n-2i)\hat{\sigma}\sqrt{\Delta t_m}}.$$

The size of an up move at any node is:

$$U = \frac{X\,(i, n+1)}{X\,(i, n)} = e^{\hat{\sigma}\sqrt{\Delta t_m}},$$

and similarly, the size of a down move is:

$$D = \frac{X\,(i+1, n+1)}{X\,(i, n)} = e^{-\hat{\sigma}\sqrt{\Delta t_m}}.$$

Estimating Drift and Volatility. The next step is to estimate the mean $\hat{\nu}$ and variance $\hat{\phi}^2$ of changes in the log-price based on historical price data These statistics were calculated from monthly spot prices of PET, expressed in US dollars per kilogram, obtained from BusinessAnalytiq.com for the period January

2018 to December 2021 (see Figure 1), covering a total of 48 months. The cash price of PET expressed in US dollars per kilogram. The mean $\hat{\nu}$ of the difference of the log-price is 0.0071, with the standard deviation $\hat{\phi}$ of 0.0380. The time step used here $\Delta t_d = \frac{1}{12}$, reflecting monthly intervals.

Using these estimates, the normalized drift $\hat{\mu}$ and volatility $\hat{\sigma}$ for the ABM model are computed as follows:

$$\hat{\mu} = \frac{\hat{\nu}}{\Delta t_d} = 0.0852 \quad \text{and} \quad \hat{\sigma} = \frac{\hat{\phi}}{\sqrt{\Delta t_d}} = 0.1317.$$

Starting from an initial price of $\$1.48$, the binomial tree for the PET prices under the ABM model is presented in Table 1.

Table 1. The binomial tree for the price of PET under the ABM model

i \ n	0	1	2	3	4	5	6	7	8
0	1.48	1.58	1.69	1.80	1.93	2.06	2.20	2.35	2.51
1		1.39	1.48	1.58	1.69	1.80	1.93	2.06	2.20
2			1.30	1.39	1.48	1.58	1.69	1.80	1.93
3				1.21	1.30	1.39	1.48	1.58	1.69
4					1.14	1.21	1.30	1.39	1.48
5						1.06	1.14	1.21	1.30
6							1.00	1.06	1.14
7								0.93	1.00
8									0.87

Secondly, we construct the binomial tree of the PET prices by using the AAFBM model. From Equations (6) and (7), the normalized estimates of the drift and volatility parameters are given by:

$$\hat{\mu} = \frac{\hat{\nu}}{\Delta t_d} \quad \text{and} \quad \hat{\sigma} = \frac{\hat{\phi}}{\sqrt{\int_0^{\Delta t_d} \left[(\Delta t_d - s + \varepsilon)^{\alpha}\right]^2 ds}}.$$

Since $\hat{\sigma}$ depends on the choice of α and ε, these parameters must be explicitly specified when applying the formula. Under the assumption that the price of PET exhibits trending behavior, α should lie within the interval $(0, 0.5)$. Additionally, because B_t^{ε} is used as an approximation of B_t, the smoothing parameter ε must be positive and sufficiently small to ensure accuracy. In this analysis, we choose $\alpha = 0.1$ and $\varepsilon = 0.01$.

Substituting these values into the equations yields:

$$\hat{\mu} = 0.0852 \quad \text{and} \quad \hat{\sigma} = 0.1792.$$

Table 2 presents the binomial tree for the price of PET under the AAFBM model, based on the chosen values of $\alpha = 0.1$ and $\varepsilon = 0.01$.

TABLE 2. The binomial tree for the price of PET under the AAFBM model

i \ n	0	1	2	3	4	5	6	7	8
0	1.48	1.62	1.77	1.94	2.12	2.32	2.53	2.77	3.03
1		1.35	1.48	1.62	1.77	1.94	2.12	2.32	2.53
2			1.24	1.35	1.48	1.62	1.77	1.94	2.12
3				1.13	1.24	1.35	1.48	1.62	1.77
4					1.03	1.13	1.24	1.35	1.48
5						0.95	1.03	1.13	1.24
6							0.86	0.95	1.03
7								0.79	0.86
8									0.72

4. RESULTS

At this stage, we proceed to evaluate the accuracy of the PET pricing models constructed using ABM and AAFBM. The **sum of squared errors (SSE)** is selected as the performance metric, and **quarterly PET prices from January 2022 to December 2023** are used to compare the forecast accuracy of each model.

TABLE 3. SSE of the forecasted PET prices using the ABM and AAFBM models

Quarter (t)	**Actual PET Price**	**Forecast Price**	
		ABM	AAFBM $(\alpha = 0.1, \varepsilon = 0.01)$
1	1.57	1.58	1.61
2	1.91	1.69	1.77
3	1.88	1.80	1.94
4	1.48	1.48	1.48
5	1.43	1.39	1.35
6	1.48	1.48	1.48
7	1.42	1.39	1.35
8	1.42	1.48	1.48
Sum of Squared Errors (SSE)		**0.0619**	**0.0390**

From Table 3, we observe that the AAFBM model, with parameters $\alpha = 0.1$ and $\varepsilon = 0.01$, achieves a lower SSE (**0.0390**) than the ABM model (**0.0619**), indicating that AAFBM provides more accurate forecasts of PET prices in this case. This improvement can be attributed to AAFBM's ability to capture *memory effects and local trends* that ABM overlooks.

To conduct a more comprehensive evaluation, we extend the analysis across a broader parameter space. Specifically, we vary α from 0.001 to 0.149 in increments of 0.001, and ε from 0.001 to 0.499, also in steps of 0.001. The resulting surface plot in Figure 2 illustrates how the SSE of the AAFBM model varies across this range of parameter values.

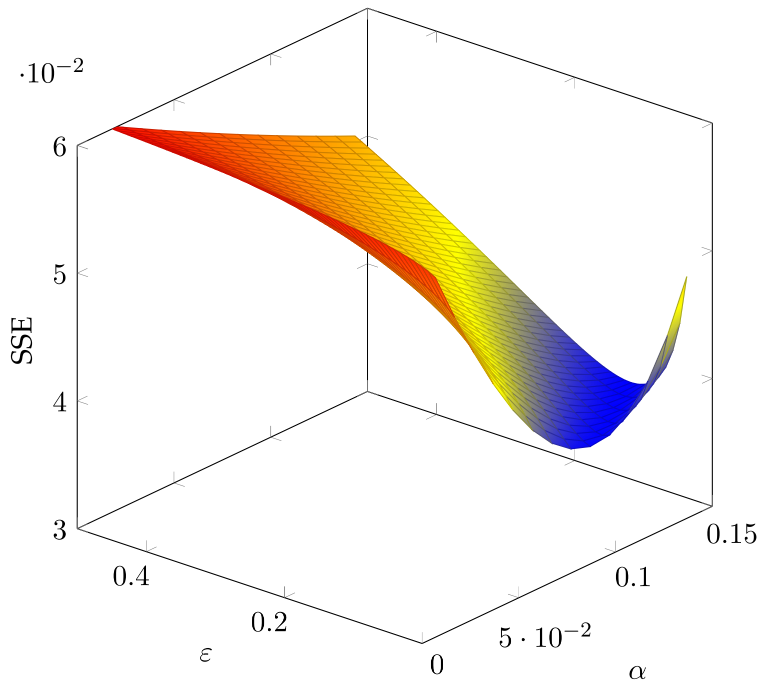

FIGURE 2. SSE of the AAFBM models across the parameter space $0 < \alpha < 0.15$ and $0 < \varepsilon < 0.5$

Within this domain, the maximum SSE of 0.061795 occurs when $\alpha = 0.001$ and $\varepsilon = 0.499$—which is still slightly lower than the SSE of the ABM model. The minimum SSE of 0.038781 is achieved when $\alpha = 0.1$ and $\varepsilon = 0.02$, confirming that the AAFBM model can outperform the ABM model over a wide range of parameter values.

5. Conclusion

The comparative analysis between the ABM and AAFBM models demonstrates that the AAFBM framework offers a more accurate approach to forecasting PET prices. By incorporating memory effects and local trend adjustments, the AAFBM model significantly reduces prediction error, as evidenced by a lower SSE across both fixed and variable parameter evaluations. Even under extensive sensitivity analysis, AAFBM consistently outperforms the ABM model, with its worst-case performance still surpassing that of ABM. These findings confirm the robustness and effectiveness of the AAFBM model, making it a more reliable tool for modeling and forecasting PET price dynamics in real-world scenarios.

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M. Alfeus, C.S. Nikitopoulos, Forecasting Volatility in Commodity Markets with Long-Memory Models, J. Commod. Mark. 28 (2022), 100248. https://doi.org/10.1016/j.jcomm.2022.100248.

[2] E. Alòs, O. Mazet, D. Nualart, Stochastic Calculus with Respect to Fractional Brownian Motion with Hurst Parameter Lesser Than $1/2$, Stoch. Process. Appl. 86 (2000), 121–139. https://doi.org/10.1016/s0304-4149(99)00089-7.

[3] C. Bayer, S. Breneis, Markovian Approximations of Stochastic Volterra Equations with the Fractional Kernel, Quant. Financ. 23 (2022), 53–70. https://doi.org/10.1080/14697688.2022.2139193.

[4] C. Bayer, P. Friz, J. Gatheral, Pricing Under Rough Volatility, Quant. Financ. 16 (2015), 887–904. https://doi.org/10.1080/14697688.2015.1099717.

[5] M. Bennedsen, A. Lunde, M.S. Pakkanen, Hybrid Scheme for Brownian Semistationary Processes, Financ. Stochastics 21 (2017), 931–965. https://doi.org/10.1007/s00780-017-0335-5.

[6] C. Czichowsky, W. Schachermayer, Portfolio Optimisation Beyond Semimartingales: Shadow Prices and Fractional Brownian Motion, Ann. Appl. Probab. 27 (2017), . https://doi.org/10.1214/16-aap1234.

[7] S. Dolatabadi, P.K. Narayan, M.Ø. Nielsen, K. Xu, Economic Significance of Commodity Return Forecasts from the Fractionally Cointegrated VAR Model, J. Futur. Mark. 38 (2017), 219–242. https://doi.org/10.1002/fut.21866.

[8] N.T. Dung, Semimartingale Approximation of Fractional Brownian Motion and Its Applications, Comput. Math. Appl. 61 (2011), 1844–1854. https://doi.org/10.1016/j.camwa.2011.02.013.

[9] N.T. Dung, T.H. Thao, An Approximate Approach to Fractional Stochastic Integration and Its Applications, Braz. J. Probab. Stat. 24 (2010), 57–67. https://doi.org/10.1214/08-bjps013.

[10] M. Garcin, Estimation of Time-Dependent Hurst Exponents with Variational Smoothing and Application to Forecasting Foreign Exchange Rates, Physica: Stat. Mech. Appl. 483 (2017), 462–479. https://doi.org/10.1016/j.physa.2017.04.122.

[11] J. Gatheral, T. Jaisson, M. Rosenbaum, Volatility Is Rough, Quant. Financ. 18 (2018), 933–949. https://doi.org/10.1080/14697688.2017.1393551.

[12] A. Gómez-Águila, J.E. Trinidad-Segovia, M.A. Sánchez-Granero, Improvement in Hurst Exponent Estimation and Its Application to Financial Markets, Financ. Innov. 8 (2022), 86. https://doi.org/10.1186/s40854-022-00394-x.

[13] B.B. Mandelbrot, J.W. Van Ness, Fractional Brownian Motions, Fractional Noises and Applications, SIAM Rev. 10 (1968), 422–437. https://doi.org/10.1137/1010093.

[14] T.H. Thao, A Note on Fractional Brownian Motion, Vietnam. J. Math. 31 (2003), 255–260.

[15] T.H. Thao, An Approximate Approach to Fractional Analysis for Finance, Nonlinear Anal.: Real World Appl. 7 (2006), 124–132. https://doi.org/10.1016/j.nonrwa.2004.08.012.

[16] M. Youssef, L. Belkacem, K. Mokni, Value-At-Risk Estimation of Energy Commodities: A Long-Memory Garch–Evt Approach, Energy Econ. 51 (2015), 99–110. https://doi.org/10.1016/j.eneco.2015.06.010.

[17] M. Zähle, Integration with Respect to Fractal Functions and Stochastic Calculus. I, Probab. Theory Relat. Fields 111 (1998), 333–374. https://doi.org/10.1007/s004400050171.