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ON (n, m)-D-QUASI OPERATOR ON HILBERT SPACE

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Abstract. This study seeks to present a novel category of operators, termed (n, m)-D-quasi operator on Hilbert space, defined via the Drazin inverse.This framework examines various fundamental operations, including addition, multiplication, tensor product, direct sum, and the Drazin inverse.These operations are demonstrated to be valid, although certain instances necessitate supplementary conditions.

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1. Introduction

In recent decades, numerous scholars have concentrated on the Drazin inverse, introduced by Michael P. Drazin in 1958 [1], Within the context of bounded linear operators on complex Banach spaces, the Drazin inverse was examined by Caradus [2] and King [3]. The Drazin inverse has demonstrated its use across various domains, including differential and difference equations, Markov chains, and iterative approaches (see [4,5]). In 2018, M. Dana and R. Youssfi presented a series of classes derived from the D-normal operator and examined their fundamental features and interconnections [6]. In 2019, Sid Ahmed Ould Ahmed Mahmoud and Ould Beinane Sid Ahmed introduced two novel operator classes on Hilbert spaces related to the Drazin inverse: one pertaining to the (n, m)-power form of D-normality, and the other to its quasi-normal equivalent were examined for their principal features and the inclusion relations that link them [7]. In 2020, Eiman and Samira investigated further aspects of the D-operator, proving that it preserves unitary equivalence and the scalar product, along with some other results. [8]. In 2021, Wanjala and Nyongesa presented a novel operator class known as the N_{op} quasi-D-operator.

We say that operator T_{op} is a N_{op} quasi-D-operator if

$$T_{op}({T_{op}}^{*2}({T_{op}}^{D})^2) = N_{op}({T_{op}}^{*}T_{op}^{D})^2 T_{op} \tag{1}$$

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where N_{op} is a bounded operator on Hilbert space [9].In 2024 [10], S. D. Mohsen introduced the class of (n, D)-quasi operators on Hilbert spaces, based on the Drazin inverse, and studied their main properties along with operations such as addition, multiplication, direct sum, and tensor product.
We say that operator T_{op} is a (n, D)-quasi operators if

$$T_{op}({T_{op}}^{*2}({T_{op}}^{D})^{2n}) = ({T_{op}}^{*}({T_{op}}^{D})^{n})^{2}T_{op} \tag{2}$$

Throughout the paper, $\mathscr{H}$ denotes a Hilbert space and $\mathscr{B}(\mathscr{H})$ denotes the algebra of all bounded linear operators on $\mathscr{H}$. Moreover, the subscript "op" is used to indicate operators, for example A_{op}, T_{op} denote bounded linear operators on $\mathscr{H}$. This convention will be used consistently throughout the paper.

[11]We say that T_{op} is a normal operator if

$$T_{op}{T_{op}}^{*} = {T_{op}}^{*}T_{op} \tag{3}$$

And self-adjoint if

$$T_{op} = {T_{op}}^{*} \tag{4}$$

There are four different sections in the paper.The first section addresses the introduction of the book, and the second section discusses the fundamental ideas. The first two deals with the fundamental ideas, section three we study most of properties of (n, m)-D-quasi operator, and lastly we draw up a conclusion.

2. Basic Concepts

Definition 2.1. [11] The vector space V over $\mathscr{F}$ is said to be an inner product space if there is defined for any two vectors $v, u \in$ V an element $<v, u>$ in $\mathscr{F}$ such that:

(1) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0 \Leftrightarrow u = 0$
(2) $\langle u, v\rangle = \overline{\langle v, u\rangle}$
(3) $\langle \gamma u + \delta v, c\rangle = \gamma\langle u, c\rangle + \delta\langle v, c\rangle$, for any $v, u, c \in$ V and $\gamma, \delta \in \mathscr{F}$

Definition 2.2. [4,6,12] Let A_{op} belong to $\mathscr{B}(\mathscr{H})$.The Darzin inverse of A_{op} is $A_{op}^{D} \in B(H)$ that satisfies the following condition:

(1) $A_{op}A_{op}^{D} = A_{op}^{D}A_{op}$.
(2) $A_{op}^{D}A_{op}A_{op}^{D} = A_{op}^{D}$.
(3) $A_{op}^{k+1}A_{op}^{D} = A_{op}^{k}$, for at least one integer $k \geq 0$.

The smallest nonnegative integer k is referred to as the index of A_{op}, represented as $ind(A_{op})$ and $ind(A_{op}) = 0$ if and only if $A_{op}^{D} = A_{op}^{-1}$.

The following lemma states several fundamental properties of Drazin operators established in [4,12].

Lemma 2.3. *Let A_{op}, T_{op} be any Drazin invertible in $\mathscr{B}(\mathscr{H})$:*

(1) $(A^*_{op})^D = (A^D_{op})^*$
(2) $(A^D_{op})^\lambda = (A^\lambda_{op})^D$, $\lambda \geq 1$.
(3) *If* $A_{op}T_{op} = T_{op}A_{op}$ *then* $(A_{op}T_{op})^D = T^D_{op}A^D_{op} = A^D_{op}T^D_{op}$, $A_{op}T^D_{op} = T^D_{op}A_{op}$.
(4) *If* $A_{op}T_{op} = T_{op}A_{op} = 0$ *then* $A^D_{op} + T^D_{op} = (A_{op} + T_{op})^D$.
(5) $(T^{-1}_{op}A_{op}T_{op})^D = T^{-1}_{op}A^D_{op}T_{op}$

3. Main Results

In this section, we define (n, m)-D-quasi operators on $\mathscr{H}$ and examine certain characteristics.

Definition 3.1. Assume that $A_{op} \in \mathscr{B}(\mathscr{H})$ be a Drazin operator then, A_{op} is called an (n, m)-D-quasi operator if

$$A^m_{op}(A^{*2}_{op}(A^D_{op})^{2n}) = (A^*_{op}(A^D_{op})^n)^2 A^m_{op} \tag{5}$$

where the integers $m, n \geq 1$.
The collection of all (n, m)-D-quasi operators is represented by $[D_{n,m}]$.

Proposition 3.2. *Let $A_{op} \in \mathscr{B}(\mathscr{H})$ be a Drazin operator then*

(1) *Every $A_{op} \in [D_{n,1}]$ is (n, D)-quasi operator.*
(2) *If A_{op} is (n, D)-quasi operator and normal then $A_{op} \in [D_{n,m}]$.*

Proof.

(1) Putting $m = 1$ in (5), we obtain (2).
(2) Since A_{op} is normal, we have $A_{op}A^*_{op} = A^*_{op}A_{op}$. Moreover, by Lemma 2.3 (3), $A^D_{op}A^*_{op} = A^*_{op}A^D_{op}$. It follows that

$$A^{*2}_{op}(A^D_{op})^{2n} = \left(A^*_{op}(A^D_{op})^n\right)^2.$$

Thus,

$$\begin{aligned}
A^m_{op}\left(A^{*2}_{op}(A^D_{op})^{2n}\right) &= A^{m-1}_{op}A_{op}\left(A^{*2}_{op}(A^D_{op})^{2n}\right) \\
&= A^{m-1}_{op}\left(A^*_{op}(A^D_{op})^n\right)^2 A_{op} \\
&= A^{m-2}_{op}A_{op}\left(A^{*2}_{op}(A^D_{op})^{2n}\right) A_{op} \\
&= A^{m-2}_{op}\left(A^*_{op}(A^D_{op})^n\right)^2 A^2_{op} \\
&\vdots \\
&= \left(A^*_{op}(A^D_{op})^n\right)^2 A^m_{op}.
\end{aligned}$$

Hence $A_{op} \in [D_{n,m}]$.

□

Proposition 3.3. *Suppose that $A_{op} \in [D_{n,m}]$ then*

$$[A_{op}^m(A_{op}^{*2}(A_{op}^D)^{2n})]^r = [(A_{op}^*(A_{op}^D)^n)^2 A_{op}^m]^r, \quad r \in \mathbb{N}. \tag{6}$$

Proof. Let

$$X = A_{op}^m(A_{op}^{*2}(A_{op}^D)^{2n}), Y = (A_{op}^*(A_{op}^D)^n)^2 A_{op}^m$$

By using the principle of mathematical induction, When $r = 1$ then

$$X = Y$$

Suppose that the statement holds at k, so become

$$X^k = Y^k$$

Now, prove that when $r = k + 1$,

$$\begin{aligned} X^{k+1} &= X^k X \\ &= Y^k Y \\ &= Y^{k+1} \end{aligned}$$

Consequently,

$$[A_{op}^m(A_{op}^{*2}(A_{op}^D)^{2n})]^r = [(A_{op}^*(A_{op}^D)^n)^2 A_{op}^m]^r, \quad r \in \mathbb{N}.$$

□

Proposition 3.4. *Suppose that $A_{op} \in [D_{n,m}]$ then $A_{op|N} \in [D_{n,m}]$, N is closed subspace of $\mathscr{H}$.*

Proof.

$$\begin{aligned} (A_{op|N})^m((A_{op|N})^{*2}((A_{op|N})^D)^{2n}) &= (A_{op|N}^m)((A_{op|N})^*((A_{op|N})^*((A_{op|N})^D)^n((A_{op|N})^D)^n) \\ &= (A_{op|N}^m)((A_{op|N}^{*2})(((A_{op}^D)_{|N}^{2n})) \\ &= ((A_{op}{}^m)(A_{op}{}^{*2}((A_{op}^D)^{2n}))_{|N} \\ &= ((A_{op}^*(A_{op}^D)^n)^2 A_{op}{}^m)_{|N} \\ &= ((A_{op}^*(A_{op}^D)^n)^2)_{|N}(A_{op}{}^m)_{|N} \\ &= ((A_{op}^*(A_{op}^D)^n))_{|N}((A_{op}^*(A_{op}^D)^n))_{|N}(A_{op}{}^m)_{|N} \\ &= ((A_{op|N})^*(A_{op|N})^D)^n))^2(A_{op|N})^m \end{aligned} \tag{7}$$

Hence $A_{op|N} \in [D_{n,m}]$. □

Proposition 3.5. *Suppose that $A_{op} \in \mathscr{B}(\mathscr{H})$ and for any $T_{op} \in \mathscr{B}(\mathscr{H})$ which is unitary equivalent to A_{op} then, $A_{op} \in [D_{n,m}] \Leftrightarrow T_{op} \in [D_{n,m}]$*

Proof. Let $A_{op} \in [D_{n,m}]$ and T_{op} is unitary equivalent to A_{op} then $T_{op} = B_{op}A_{op}B_{op}^*$, such that B_{op} is unitary operator.
Therefore,

$$\begin{aligned}
{T_{op}}^m({T_{op}}^{*2}{(T_{op}^D)}^{2n}) &= (B_{op}A_{op}B_{op}^*)^m((B_{op}A_{op}B_{op}^*)^{*2}{(B_{op}A_{op}B_{op}^*)^D)}^{2n})\\
&= B_{op}({A_{op}}^m({A_{op}}^{*2}{(A_{op}^D)}^{2n}))B_{op}^*\\
&= B_{op}(A_{op}^*{(A_{op}^D)}^n)^2{A_{op}}^m B_{op}^*\\
&= (B_{op}A_{op}^*B_{op}^*)(B_{op}{(A_{op}^D)}^n B_{op}^*)(B_{op}A_{op}^*B_{op}^*)(B_{op}{(A_{op}^D)}^n B_{op}^*)(B_{op}{A_{op}}^m B_{op}^*)\\
&= (B_{op}A_{op}B_{op}^*)^*(B_{op}(A_{op}^D)B_{op}^*)^n(B_{op}A_{op}B_{op}^*)^*(B_{op}(A_{op}^D)B_{op}^*)^n(B_{op}A_{op}B_{op}^*)^m\\
&= (T_{op}^*{(T_{op}^D)}^n)^2{T_{op}}^m \qquad (8)
\end{aligned}$$

Suppose that $T_{op} \in [D_{n,m}]$ then

$$\begin{aligned}
&{T_{op}}^m({T_{op}}^{*2}{(T_{op}^D)}^{2n}) = (T_{op}^*{(T_{op}^D)}^n)^2{T_{op}}^m\\
&\Rightarrow (B_{op}A_{op}B_{op}^*)^m\Big((B_{op}A_{op}B_{op}^*)^{*2}\big((B_{op}A_{op}B_{op}^*)^D\big)^{2n}\Big)\\
&= (B_{op}A_{op}B_{op}^*)^*(B_{op}A_{op}^DB_{op}^*)^n(B_{op}A_{op}B_{op}^*)^*(B_{op}A_{op}^DB_{op}^*)^n(B_{op}A_{op}B_{op}^*)^m.\\
&\Rightarrow B_{op}({A_{op}}^m({A_{op}}^{*2}{(A_{op}^D)}^{2n}))B_{op}^* = B_{op}(A_{op}^*{(A_{op}^D)}^n)^2{A_{op}}^m B_{op}^*\\
&\Rightarrow B_{op}^*B_{op}({A_{op}}^m({A_{op}}^{*2}{(A_{op}^D)}^{2n}))B_{op}^*B_{op} = B_{op}^*B_{op}(A_{op}^*{(A_{op}^D)}^n)^2{A_{op}}^m B_{op}^*B_{op}\\
&\Rightarrow {A_{op}}^m({A_{op}}^{*2}{(A_{op}^D)}^{2n}) = (A_{op}^*{(A_{op}^D)}^n)^2{A_{op}}^m \qquad (9)
\end{aligned}$$

□

Proposition 3.6. *Let $A_{op} \in \mathscr{B}(\mathscr{H})$ be self-adjoint and bijective operator then*

(1) $A_{op} \in [D_{n,m}]$.
(2) *If $A_{op} \in [D_{n,m}]$ then $(A_{op})^{-1} \in [D_{n,m}]$.*

Proof.

(1) The proof is easy to understand.
(2) Since A_{op} is self-adjoint then $A_{op} = A_{op}^*$ Now, $({A_{op}}^{-1})^* = (A_{op}^*)^{-1}$. So ${A_{op}}^{-1}$ is self-adjoint, by 1.

Hence, $(A_{op})^{-1} \in [D_{n,m}]$

□

This example illustrates that the opposite of proposition 3.5 (1) is not universally valid.

Example 3.7. Let $A_{op} = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}$ be an operator on $\mathbb{C}^2$

Then $A^*_{op} = \begin{pmatrix} 0 & 0 \\ 0 & -i \end{pmatrix}$ and $A^D_{op} = \begin{pmatrix} 0 & 0 \\ 0 & -i \end{pmatrix}$, A_{op} is evidently not self-adjoint but, where $m = 2, \quad n = 1$

then ${A_{op}}^m({A_{op}}^{*2}(A^D_{op})^{2n}) = (A^*_{op}(A^D_{op})^n)^2 {A_{op}}^m = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ thus $A_{op} \in [D_{n,m}]$.

Theorem 3.8. *Let $T_{op} \in \mathscr{B}(\mathscr{H})$ and for every non-zero $\lambda \in \mathscr{F}$ then,$T_{op} \in [D_{n,m}]$ if and only if $\lambda T_{op} \in [D_{n,m}]$.*

Proof. Suppose that $T_{op} \in [D_{n,m}]$

$$\begin{aligned}
(\lambda T_{op})^m((\lambda T_{op})^{*2}((\lambda T_{op})^D)^{2n}) &= \lambda^m {T_{op}}^m((\overline{\lambda})^2 {T_{op}}^{*2} \lambda^{2n} (T^D_{op})^{2n}) \\
&= \lambda^m (\overline{\lambda})^2 \lambda^{2n} T^m_{op}({T_{op}}^{*2}(T^D_{op})^{2n}) \\
&= \lambda^m (\overline{\lambda})^2 \lambda^{2n} (T^*_{op}(T^D_{op})^n)^2 {T_{op}}^m \\
&= (\lambda T_{op})^*(\lambda T^D_{op})^n)^2(\lambda {T_{op}}^m)
\end{aligned} \tag{10}$$

Suppose that $\lambda T_{op} \in [D_{n,m}]$ and $\lambda \neq 0$,

Therefore,

$$\begin{aligned}
(\lambda T_{op})^m((\lambda T_{op})^{*2}((\lambda T_{op})^D)^{2n}) &= (\lambda T_{op})^*(\lambda T_{op})^D)^n)^2(\lambda {T_{op}}^m) \\
\Rightarrow \lambda^m {T_{op}}^m((\overline{\lambda})^2 {T_{op}}^{*2} \lambda^{2n}(T^D_{op})^{2n}) &= (\lambda T_{op})^*(\lambda T_{op})^D)^n)(\lambda T_{op})^*(\lambda T_{op})^D)^n)\lambda^m T^m_{op} \\
\Rightarrow \lambda^m (\overline{\lambda})^2 \lambda^{2n} T^m_{op}({T_{op}}^{*2}(T^D_{op})^{2n}) &= \lambda^m (\overline{\lambda})^2 \lambda^{2n} (T^*_{op}(T^D_{op})^n)^2 T^m_{op}
\end{aligned}$$

From some basic rules, then

$$T^m_{op}({T_{op}}^{*2}(T^D_{op})^{2n}) = (T^*_{op}(T^D_{op})^n)^2 T^m_{op} \tag{11}$$

□

Remark 3.9. In Theorem 3.8, if $\lambda = 0$ then $\lambda T_{op} \in [D_{n,m}]$ but T_{op} not necessarily belong to $[D_{n,m}]$.

Theorem 3.10. *Let $T_{op} \in [D_{n,m}]$ and T^D_{op} is normal then $T^D_{op} \in [D_{n,m}]$.*

Proof. Since T^D_{op} is normal then $T^D_{op}(T^D_{op})^* = (T^D_{op})^* T^D_{op}$ and from Definition 2.2, by Fuglede Theorem [13] then $T_{op}(T^D_{op})^* = (T^D_{op})^* T_{op}$. Now, since $T_{op} \in [D_{n,m}]$ then

$$T^m_{op}({T_{op}}^{*2}(T^D_{op})^{2n}) = (T^*_{op}(T^D_{op})^n)^2 T^m_{op}$$

By taking the Drazin of both sides we have

$$T^{2n}_{op}(T^D_{op})^{*2}(T^D_{op})^m = (T^D_{op})^m(T^n_{op}(T^D_{op})^*)^2$$

$$(T_{op}^D)^m T_{op}^{2n} (T_{op}^D)^{*2} = ((T_{op}^D)^* T_{op}^n)^2 (T_{op}^D)^m$$

Then

$$(T_{op}^D)^m ((T_{op}^D)^{*2} ((T_{op}^D)^D)^{2n}) = ((T_{op}^D)^* ((T_{op}^D)^D)^n)^2 (T_{op}^D)^m \quad (12)$$

□

Theorem 3.11. *Let $T_{op} \in [D_{n,m}]$ is normal operator then $T_{op}^* \in [D_{n,m}]$.*

Proof. Since T_{op} is normal and from Definition 2.2, $T_{op} T_{op}^D = T_{op}^D T_{op}$ by Fuglede Theorem [13] then $T_{op}^* T_{op}^D = T_{op}^D T_{op}^*$. Now,

$$T_{op}^m ({T_{op}}^{*2} (T_{op}^D)^{2n}) = (T_{op}^* (T_{op}^D)^n)^2 T_{op}^m$$

By taking the adjoint of both sides we have

$$((T_{op}^D)^*)^{2n} T_{op}^2 (T_{op}^*)^m = (T_{op}^*)^m (((T_{op}^D)^*)^n T_{op})^2$$

$$((T_{op}^D)^*)^{2n} (T_{op}^*)^m T_{op}^2 = ((T_{op}^*)^D)^n T_{op} ((T_{op}^*)^D)^n T_{op} (T_{op}^*)^m$$

Then

$$(T_{op}^*)^m ((T_{op}^*)^{*2} ((T_{op}^*)^D)^{2n}) = ((T_{op}^*)^* ((T_{op}^*)^D)^n)^2 (T_{op}^*)^m \quad (13)$$

□

Theorem 3.12. *Let $T_{op}, Z_{op} \in [D_{n,m}]$ be normal operators such that $T_{op}^m Z_{op}^* (Z_{op}^D)^n = Z_{op}^m T_{op}^* (T_{op}^D)^n = T_{op} Z_{op} = Z_{op} T_{op} = 0$ then $T_{op} + Z_{op} \in [D_{n,m}]$.*

Proof. Since $T_{op} Z_{op} = Z_{op} T_{op} = 0$ then $(T_{op} + Z_{op})^m = T_{op}^m + Z_{op}^m$ and by lemma 2.3(3) we have $T_{op}^D Z_{op} = Z_{op} T_{op}^D = Z_{op}^D T_{op} = T_{op} Z_{op}^D = 0$.
Now, since T_{op} is normal, by Fuglede Theorem [13] we have $T_{op}^* Z_{op}^D = Z_{op}^D T_{op}^* = 0$.
Similarly, $Z_{op}^* T_{op}^D = T_{op}^D Z_{op}^* = 0$, it follows that $T_{op}^{*2} (Z_{op}^D)^{2n} = (Z_{op}^D)^{2n} T_{op}^{*2}$. Thus

$$\begin{aligned}
&(T_{op} + Z_{op})^m ((T_{op} + Z_{op})^{*2} ((T_{op} + Z_{op})^D)^{2n}) \\
&= (T_{op} + Z_{op})^m (T_{op}^* + Z_{op}^*)(T_{op}^* + Z_{op}^*)(T_{op}^D + Z_{op}^D)^n (T_{op}^D + Z_{op}^D)^n \\
&= (T_{op} + Z_{op})^m (({T_{op}}^{*2} + Z_{op}^{*2})((T_{op}^D)^{2n} + (Z_{op}^D)^{2n}) \\
&= (T_{op} + Z_{op})^m (T_{op}^{*2} (T_{op}^D)^{2n} + T_{op}^{*2} (Z_{op}^D)^{2n} + {Z_{op}}^{*2} (T_{op}^D)^{2n} + {Z_{op}}^{*2} (Z_{op}^D)^{2n}) \\
&= ({T_{op}}^m + {Z_{op}}^m)(T_{op}^{*2} (T_{op}^D)^{2n} + {Z_{op}}^{*2} (Z_{op}^D)^{2n}) \\
&= ({T_{op}}^m T_{op}^{*2} (T_{op}^D)^{2n} + T_{op}^m {Z_{op}}^{*2} (Z_{op}^D)^{2n} + {Z_{op}}^m T_{op}^{*2} (T_{op}^D)^{2n} + T_{op}^m {Z_{op}}^{*2} (Z_{op}^D)^{2n}) \\
&= T_{op}^m T_{op}^{*2} (T_{op}^D)^{2n} + {Z_{op}}^m {Z_{op}}^{*2} (Z_{op}^D)^{2n}, \qquad T_{op}, Z_{op} \in [D_{n,m}]
\end{aligned}$$

$$= (T^*_{op}(T^D_{op})^n)^2 T^m_{op} + (Z^*_{op}(Z^D_{op})^n)^2 {Z_{op}}^m$$

$$= (T^*_{op}(T^D_{op})^n)^2 + (Z^*_{op}(Z^D_{op})^n)^2 ({Z_{op}}^m + {Z_{op}}^m)$$

$$= (T^*_{op} + Z^*_{op})((T^D_{op})^{2n} + (Z^D_{op})^{2n})(T^*_{op} + Z^*_{op})((T^D_{op})^{2n} + (Z^D_{op})^{2n})({T_{op}}^m + \mathscr{Z}^m)$$

$$= ((T_{op} + Z_{op})^*((T_{op} + Z_{op})^D)^n)^2 (T_{op} + Z_{op})^m \tag{14}$$

□

Theorem 3.13. *Let $T_{op}, Z_{op} \in [D_{n,m}]$ be normal operator such that $T^m_{op} Z^*_{op}(Z^D_{op})^n = {Z_{op}}^m T^*_{op}(T^D_{op})^n = T_{op}Z_{op} = Z_{op}T_{op} = 0$ then $T_{op} - Z_{op} \in [D_{n,m}]$.*

Proof. The proof is a direct use of Theorem 3.12. □

Theorem 3.14. *Let $T_{op}, Z_{op} \in [D_{n,m}]$ then $T_{op}Z_{op} \in [D_{n,m}]$.If the following are holds:*

(1) $T_{op}Z_{op} = Z_{op}T_{op}$
(2) ${T_{op}}^{*2} Z^m_{op} = Z^m_{op} {T_{op}}^{*2}$

Proof. Since $T_{op}Z_{op} = Z_{op}T_{op}$ and ${T_{op}}^{*2} Z^m_{op} = Z^m_{op} {T_{op}}^{*2}$ then by lemma 2.3(3) we have $Z^*_{op}T^D_{op} = T^D_{op}Z^*_{op}$ and $T^D_{op}Z_{op} = Z_{op}T^D_{op}$ it following that $(T^D_{op})^{2n} Z^m_{op} = Z^m_{op}(T^D_{op})^{2n}$ and $Z^2_{op}(T^D_{op})^{2n} = (T^D_{op})^{2n} Z^2_{op}$.
Therefore,

$$\begin{aligned}
(T_{op}Z_{op})^m((T_{op}Z_{op})^{*2}((T_{op}Z_{op})^D)^{2n}) &= {T_{op}}^m Z^m_{op}({T_{op}}^{*2} Z^{*2}_{op})(T^D_{op}Z^D_{op})^{2n} \\
&= {T_{op}}^m Z^m_{op}({T_{op}}^{*2} Z^{*2}_{op})(T^D_{op})^{2n}(Z^D_{op})^{2n} \\
&= {T_{op}}^m ({T_{op}}^{*2} Z^m_{op})((T^D_{op})^{2n} Z^{*2}_{op})(Z^D_{op})^{2n} \\
&= {T_{op}}^m {T_{op}}^{*2}(Z^m_{op}((T^D_{op})^{2n})(Z^{*2}_{op}(Z^D_{op})^{2n}) \\
&= {T_{op}}^m {T_{op}}^{*2}((T^D_{op})^{2n} {Z_{op}}^m)(Z^{*2}_{op}(Z^D_{op})^{2n}) \\
&= ({T_{op}}^m {T_{op}}^{*2}(T^D_{op})^{2n})(Z^m_{op} Z^{*2}_{op}(Z^D_{op})^{2n}) \\
&= ((T^*_{op}(T^D_{op})^n)^2 {T_{op}}^m)((Z^*_{op}(Z^D_{op})^n)^2 Z^m_{op}) \\
&= {T_{op}}^{*2} Z^{*2}_{op}(T^D_{op})^{2n}(Z^D_{op})^{2n} {T_{op}}^m Z^m_{op} \\
&= (T_{op}Z_{op})^{*2}((T_{op}Z_{op})^D)^{2n}(T_{op}Z_{op})^m
\end{aligned}$$

Then

$$(T_{op}Z_{op})^m((T_{op}Z_{op})^{*2}((T_{op}Z_{op})^D)^{2n}) = ((T_{op}Z_{op})^*((T_{op}Z_{op})^D)^n)^2(T_{op}Z_{op})^m \tag{15}$$

□

The following examples demonstrates that the converse of Theorems (3.12,3.13,3.14) does not hold in general.

Example 3.15. Let $A_{op} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T_{op} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ be an operator on $\mathbb{C}^2$.

Then $A_{op} + T_{op} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in [D_{1,2}]$ but, $A_{op}^2({A_{op}}^{*2}{A_{op}}^{-2}) = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$ and $(A_{op}^* A_{op}^{-1})^2 A_{op}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. So $A_{op} \notin [D_{1,2}]$.

Example 3.16. Let $A_{op} = M_{op} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be an operator on $\mathbb{C}^2$.

Then $A_{op} - M_{op} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in [D_{1,2}]$ but $A_{op} \notin [D_{1,2}]$.

Example 3.17. Let $A_{op} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T_{op} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ be an operator on $\mathbb{C}^2$.

Then $A_{op} T_{op} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in [D_{1,2}]$ but $A_{op} \notin [D_{1,2}]$.

The following examples demonstrates that the theorems (3.12,3.13,3.14) are not necessarily true in general.

Example 3.18. Let $A_{op} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $T_{op} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ be an operator on $\mathbb{C}^2$.

Then $A_{op}^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_{op}^D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $T_{op}^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $T_{op}^D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
We note that $A_{op} T_{op} \neq T_{op} A_{op}$ and $A_{op}, T_{op} \in [D_{1,2}]$.
But it is easy compute that $A_{op} + T_{op} \notin [D_{1,2}]$ and $A_{op} - TT_{op} \notin [D_{1,2}]$.

Example 3.19. Let $Z_{op} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $M_{op} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be an operator on $\mathbb{C}^2$.

Then $Z_{op}^* = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $Z_{op}^D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ and $M_{op}^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $M_{op}^D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Therefore $Z_{op} M_{op} \neq M_{op} Z_{op}$ and $Z_{op}, M_{op} \in [D_{1,2}]$.
But it is easy compute that $Z_{op} M_{op} \notin [D_{1,2}]$

Theorem 3.20. *The collection of all (n, m)-D-quasi operator on $\mathscr{H}$ constitutes a closed subset of $\mathscr{B}(\mathscr{H})$ which is closed under scaler multiplication.*

Proof. Suppose that $(T_{op})_k$ be a sequence of (n,m)-D-quasi operator such that $(T_{op})_k \to T_{op}$. To show that T_{op} is (n,m)-D-quasi operator. Now

Let

$$X = T_{op}^m(T_{op}^{*2}(T_{op}^D)^{2n}), \quad Y = (T_{op}^*(T_{op}^D)^n)^2 T_{op}^m$$

$$\| X - Y \| = \| X - (T_{op}^m)_k((T_{op}{}^{*2})_k((T_{op}^D)^{2n})_k) + (T_{op}^m)_k((T_{op}{}^{*2})_k((T_{op}^D)^{2n})_k - Y \|$$
$$\leq \| X - (T_{op}^m)_k((T_{op}{}^{*2})_k((T_{op}^D)^{2n})_k) \| + \| (T_{op}^m)_k((T_{op}{}^{*2})_k((T_{op}^D)^{2n})_k - Y \| \to 0$$

When $k \to \infty . T_{op} \in [D_{n,m}]$ □

Theorem 3.21. *Let $A_{op_1}, A_{op_2}, ..., A_{op_r} \in [D_{n,m}]$ then*

(1) $A_{op_1} \oplus A_{op_2} \oplus ... \oplus A_{op_r} \in [D_{n,m}]$.
(2) $A_{op_1} \otimes A_{op_2} \otimes ... \otimes A_{op_r} \in [D_{n,m}]$.

Proof.

(1)

$$(A_{op_1} \oplus ... \oplus A_{op_r})^m((A_{op_1} \oplus ... \oplus A_{op_r})^{*2}(A_{op_1} \oplus ... \oplus A_{op_r})^D)^{2n}))$$
$$= (A_{op_1^m} \oplus ... \oplus A_{op_r^m})((A_{op_1^{*2}} \oplus ... \oplus A_{op_r^{*2}})((A_{op_1^D})^{2n} \oplus ... \oplus (A_{op_r^D})^{2n}))$$
$$= A_{op_1^m} A_{op_1^{*2}}(A_{op_1^D})^{2n} \oplus ... \oplus A_{op_r^m} A_{op_r^{*2}}(A_{op_r^D})^{2n})$$
$$= (A_{op_1^*}(A_{op_1^D})^n)^2 A_{op_1^m} \oplus ... \oplus (A_{op_r^*}(A_{op_r^D})^n)^2 A_{op_r^m}$$
$$= (A_{op_1^*}(A_{op_1^D})^n \oplus ... \oplus A_{op_r^*}(A_{op_r^D})^n)^2(A_{op_1} \oplus ... \oplus A_{op_r})^m$$

(2) Let $x_1, x_2, ..., x_r \in \mathscr{H}$, it is following that

$$(A_{op_1} \otimes ... \otimes A_{op_r})^m((A_{op_1} \otimes ... \otimes A_{op_r})^{*2}((A_{op_1} \otimes ... \otimes A_{op_r})^D)^{2n}))(x_1 \otimes x_2 \otimes ... \otimes x_r)$$
$$= (A_{op_1^m} \otimes ... \otimes A_{op_r^m})((A_{op_1^{*2}} \otimes ... \otimes A_{op_r^{*2}})((A_{op_1^D})^{2n} \otimes ... \otimes (A_{op_r^D})^{2n}))(x_1 \otimes x_2 \otimes ... \otimes x_r)$$
$$= A_{op_1^m} A_{op_1^{*2}}(A_{op_1^D})^{2n}(x_1) \otimes ... \otimes A_{op_r^m} A_r^{*2}(A_{op_r^D})^{2n}(x_r)$$
$$= (A_{op_1^*}(A_{op_1^D})^n)^2 A_{op_1^m}(x_1) \otimes ... \otimes (A_{op_r^*}(A_{op_r^D})^n)^2 A_{op_r^m}(x_r)$$
$$= (A_{op_1^*}(A_{op_1^D})^n \otimes ... \otimes A_{op_r^*}(A_{op_r^D})^n)^2(A_{op_1} \otimes ... \otimes A_{op_r})^m(x_1 \otimes x_2 \otimes ... \otimes x_r)$$

□

The following example demonstrates that the converse of Theorem 3.21(2) does not hold in general.

Example 3.22. Let $A_{op} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $M_{op} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ be operators on $\mathbb{C}^2$. Then

$$A_{op} \otimes M_{op} = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (A_{op} \otimes M_{op})^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix},$$

and

$$(A_{op} \otimes M_{op})^D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to compute that $A_{op} \otimes M_{op} \in [D_{1,2}]$, but $A_{op} \notin [D_{1,2}]$

4. Conclusion

The purpose of this study has been to introduce and investigate the class of (n, m)-D-quasi operator, together with several operations naturally associated with them. Throughout the paper, we established a number of structural properties for this class, showing that many of the required conditions are weaker than those found in previously known operator families. This demonstrates that the present work provides a genuine extension of earlier results in the field. Moreover, we observed that the operators studied here maintain an essential connection to the corresponding operators, although certain fundamental features such as those typically satisfied by hyponormal operators may fail to hold for quasi operators without additional assumptions.These limitations can nonetheless be overcome by imposing suitable conditions, particularly in cases involving sums and products of operators.

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