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GENERALIZED BUZANO-TYPE INEQUALITIES FOR A-NUMERICAL RADIUS INEQUALITIES: AN IMPROVEMENT

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Abstract. This paper introduces enhanced upper bounds for the A-numerical radius of bounded operators through the development of a novel class of Buzano-type inequalities in semi-Hilbertian spaces. In particular, for any operator $\mathfrak{L} \in \mathbb{B}_A(\mathcal{H})$ and any function $\chi : [0,1] \to \left[\frac{1}{4}, 1\right]$, we obtain the estimate

$$\omega_A^4(\mathfrak{L}) \leq \frac{\chi(\mu)}{8} \left\| \left(\mathfrak{L}^{\sharp_A}\mathfrak{L}\right)^2 + \left(\mathfrak{L}\mathfrak{L}^{\sharp_A}\right)^2 \right\|_A + \frac{1-\chi(\mu)}{4} \left\| \mathfrak{L}^{\sharp_A}\mathfrak{L} + \mathfrak{L}\mathfrak{L}^{\sharp_A} \right\|_A \omega_A\left(\mathfrak{L}^2\right).$$

The results contribute to the theoretical framework of operator theory by providing refined analytical tools that improve the estimation of the A-numerical radius and deepen the understanding of Buzano-type inequalities in semi-Hilbertian settings.

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1. Introduction

Let $\mathbb{B}(\mathcal{H})$ be a complex Hilbert space furnished with the inner product $\langle \cdot, \cdot \rangle$. Let represent the C^*-algebra of all linear operators on limited spaces. The semi-inner product $\langle s, t \rangle_A = \langle As, t \rangle$ induces the seminorm $\|s\|_A = \sqrt{\langle s, s \rangle_A}$ for $A \in \mathbb{B}(\mathcal{H})^+$ a positive operator is defined by

$$\omega_A(\mathfrak{L}) = \sup \left\{ |\langle \mathfrak{L}s, s \rangle_A| : \|s\|_A = 1 \right\}.$$

Buzano-type extensions [1, 2, 3] have been employed in recent work to refine A-numerical radius inequalities in [7].

These are useful in a variety of fields, including probability theory, analysis, and linear algebra. New Buzano-type inequalities take on an especially beautiful shape when applied to Hilbert spaces, which is why functional analysis and related fields frequently employ them. The idea of Hilbert spaces, which are generally explored in analysis and linear algebra, is extended by pseudo-Hilbert spaces. Unlike Hilbert spaces, pseudo-Hilbert spaces allow for more flexible inner products that do not need to satisfy all the usual requirements. Because of their adaptability, semi-Hilbert spaces can be used

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to examine a wide range of mathematical and physical issues involving singularities, unbounded operators, and nonlocal interactions. Semi-Hilbert spaces have garnered significant interest in the realm of operator theory. One way to design them is to begin with a positive semi-definite semi-linear model and then build a space around it. This field has benefited greatly from recent studies. The A-numeric radius's exact boundaries are then obtained by applying these inequalities. [26] [29]. for any operator $\mathfrak{L} \in \mathbb{B}_A(\mathcal{H})$ and a mapping $\chi : [0,1] \to \left[\frac{1}{2}, 1\right]$, the following inequality holds:

$$\omega_A^2(\mathfrak{L}) \leq \frac{\chi(\mu)}{8} \left\| (\mathfrak{L}^* A \mathfrak{L})^2 + (\mathfrak{L}\mathfrak{L}^* A)^2 \right\|_A + \frac{(1-\chi(\mu))}{4} \left\| \mathfrak{L}^* A \mathfrak{L} + \mathfrak{L}\mathfrak{L}^* A \right\|_A \omega_A\left(\mathfrak{L}^2\right) [1].$$

These results expand and unify earlier inequalities put forth by AI-Dolat, Kittaneh, and Bhunia. Applications to 2×2 operator matrices are also included in the study, along with numerical examples that demonstrate the superiority of the derived bounds [6]. The *A-numerical radius* of an operator $\mathfrak{L}$, denoted by $\omega_A(\mathfrak{L})$, is given by

$$\omega_A(\mathfrak{L}) = \sup_{\|s\|_A = 1} \left| (\mathfrak{L}s,\, s)_A \right|.$$

The semi-norm $\|s\|_A$ is given by the following formula: The semi-inner product related to the operator A is given by

$$(s, t)_A = (As,\, t).$$

The operator A is presumed to be positive and is an element of $\mathbb{B}(\mathcal{H})_+$, where $\mathcal{H}$ represents a complex Hilbert space equipped with the standard inner product $\langle \cdot, \cdot \rangle$ [2]. This framework broadens the traditional concept of the numerical radius, facilitating a more comprehensive examination of operator behavior in relation to the geometry generated by the operator A. Recent work has refined these A-numerical radius inequalities using Buzano-type extensions [15]. In conclusion, the introduction establishes the scene by outlining the primary goal of the paper, which is to build upon and expand on earlier research in the field by introducing novel Buzano-type inequalities and using them to enhance the bounds for the A-numerical radius of operators.

2. Preliminaries

Lemma 2.1. [16, p.20] Let $A \in \mathbb{B}(\mathcal{H})$ be a positive operator, and let $s \in \mathcal{H}$ be such that $\|s\| = 1$. Then, for all real numbers $r \geq 1$, the following inequality holds:

$$\langle As, s \rangle^r \leq \langle A^r s, s \rangle.$$

Lemma 2.2. [[8], Th. 5] $A \in \mathbb{B}(\mathcal{H})$, for example. Let f and g be continuous functions on $[0, \infty)$ that, for any $\mu \in [0, \infty)$, satisfy the relation $f(\mu)g(\mu) = \mu$. Next,

$$|\langle As, t \rangle| \leq \|f(|A|)s\| \left\| g\left(|A^*|\right) t \right\|,$$

for all $s, t \in \mathcal{H}$.

Lemma 2.3. [[23], p. 44] If $A=[a_{ij}]$ is an $n\times n$ complex matrix such that $a_{ij}\geq 0$ for all $i,j=1,2,\ldots,n$, then $w(A)=w\left(\frac{A+A^*}{2}\right)=r\left(\frac{A+A^*}{2}\right)$, where $r(\cdot)$ denotes the spectral radius.

Lemma 2.4. [[12], Cor. 2.5] Let $s_1,s_2,m\in\mathcal{H}$ with $\|m\|=1$ and $\beta\in\mathbb{C}\backslash\{0\}$. Then

$$|\langle s_1,m\rangle\langle m,s_2\rangle|\leq\frac{1}{|\beta|}\left(\max\{1,|\beta-1|\}\|s_1\|\,\|s_2\|+|\langle s_1,s_2\rangle|\right)$$

Specifically, the aforementioned inequality becomes Buzano's inequality for $\beta=2$ [18]in [1] [2]

$$|\langle s_1,e\rangle\langle e,s_2\rangle|\leq\frac{1}{2}\left(\|s_1\|\,\|s_2\|+|\langle s_1,s_2\rangle|\right)$$

The following is satisfied by the Moore-Penrose inverse $A^\dagger$ of $A\in\mathcal{L}(\mathcal{H})^+$:

$$AA^\dagger A=A,\quad A^\dagger AA^\dagger=A^\dagger.$$

The A-adjoint of $\mathfrak{L}$ is $\mathfrak{L}^{\sharp A}=A^\dagger\mathfrak{L}^*A$.

Lemma 2.5 (Buzano-type inequality). For $s,t,m\in\mathcal{H}$ with $\|m\|_A=1$ and $\chi:[0,1]\to\left[\frac{1}{3},1\right]$:

$$|\langle s,m\rangle_A\langle m,t\rangle_A|^2\leq\chi(\mu)\|s\|_A^2\|t\|_A^2+(1-\chi(\mu))\,|\langle s,t\rangle_A|^2.$$

Lemma 2.6. [6]. Let $s,t,m\in\mathcal{H}$ with $\|z\|_A=1$, let J be a set such that $(0,1)\subset J\subset\mathbb{R}$, and let χ be a mapping $\chi:J\to\left[\frac{1}{2},1\right]$ .. If the function $\varphi:[0,\infty)\to[0,\infty)$ is growing and convex, then

$$\varphi\left(|\langle s,m\rangle_A\langle m,t\rangle_A|\right)\leq\chi(\beta)\varphi\left(\|s\|_A\|t\|_A\right)+(1-\chi(\beta))\varphi\left(|\langle s,t\rangle_A|\right).$$

In particular, for $\kappa\geq 1$, we have

$$|\langle s,m\rangle_A\langle m,t\rangle_A|^\kappa\leq\chi(\beta)\|s\|_A^\kappa\|t\|_A^\kappa+(1-\chi(\beta))\,|\langle s,t\rangle_A|^\kappa.$$

Proof. Given that φ is a convex function that increases monotonically and the well-known Cauchy-Schwarz inequality, we have

$$\begin{aligned}\varphi\left(|\langle s,t\rangle_A|\right)&=\varphi\left((2\chi(\beta)-1)\,|\langle s,t\rangle_A|+(2-2\chi(\beta))\,|\langle s,t\rangle_A|\right)\\&\leq\varphi\left((2\chi(\beta)-1)\left(\|s\|_A\|t\|_A\right)+(2-2\chi(\beta))\,|\langle s,t\rangle_A|\right)\\&\leq(2\chi(\beta)-1)\varphi\left(\|s\|_A\|t\|_A\right)+(2-2\chi(\beta))\varphi\left(|\langle s,t\rangle_A|\right).\end{aligned}\tag{2.1}$$

The convexity of the function φ can now be integrated with Lemma 2.5 to derive that

$$\begin{aligned}\varphi\left(|\langle s,m\rangle_A\langle m,t\rangle_A|\right)&\leqslant\varphi\left(\frac{\|s\|_A\|t\|_A+|\langle s,t\rangle_A|}{2}\right)\\&\leqslant\frac{1}{2}\left[\varphi\left(\|s\|_A\|t\|_A\right)+\varphi\left(|\langle s,t\rangle_A|\right)\right]\end{aligned}\tag{2.2}$$

So, based on inequalities (2.1) and (2.2), one obtains

$$\varphi\left(|\langle s,m\rangle_A\langle m,t\rangle_A|\right)\leqslant\frac{1}{2}\left(\varphi\left(\|s\|_A\|t\|_A\right)+\varphi\left(|\langle s,t\rangle_A|\right)\right)$$
$$\leqslant\frac{1}{2}\left[\varphi\left(\|s\|_A\|t\|_A\right)+(2\chi(\beta)-1)\varphi\left(\|s\|_A\|t\|_A\right)+(2-2\chi(\beta))\varphi\left(|\langle s,t\rangle_A|\right)\right]$$
$$=\chi(\beta)\varphi\left(\|s\|_A\|t\|_A\right)+(1-\chi(\beta))\varphi\left(|\langle s,t\rangle_A|\right)$$

This concludes the proof.

Lemma 2.7 ([1]). For $\mathfrak{L}\in\mathcal{L}_A(\mathcal{H})$:

$$\frac{1}{4}\left\|\mathfrak{L}^{\sharp_A}\mathfrak{L}+\mathfrak{L}\mathfrak{L}^{\sharp_A}\right\|_A\leq\omega_A^2(\mathfrak{L})\leq\frac{1}{2}\left\|\mathfrak{L}^{\sharp_A}\mathfrak{L}+\mathfrak{L}\mathfrak{L}^{\sharp_A}\right\|_A.$$

3. Generalized Buzano Inequalities

We start this section by using the framework of rank-one operators to provide a fresh and simplified demonstration of an extended form of Buzano's inequality [11]. In addition to providing a succinct derivation, this method offers a better understanding of the structural characteristics that underlie the inequality, opening the door for more generalizations in operator theory and inner product spaces.

Proposition 3.1. Assuming that $s,t,m\in\mathcal{H}$, $\|m\|=1$, and $\beta\in\mathbb{C}$

$$|\beta\langle s,m\rangle\langle m,t\rangle-\langle s,t\rangle|\leq\max\{1,|\beta-1|\}\|s\|\|t\|. \tag{3.1}$$

Proof. If $s\in\mathcal{H}$ with $\|s\|=1$, then $\mathfrak{L}=\beta(z\otimes z)$.

$$\|\mathfrak{L}s-s\|^2=\left(|\beta-1|^2-1\right)\leq\max\left\{1,|\beta-1|^2\right\}\|s\|^2|\langle m,s\rangle|^2+\|s\|^2.$$

Thus, $\|\mathfrak{L}-H\|\leq\max\{1,|\beta-1|\}$ and
$|\beta\langle s,m\rangle\langle m,t\rangle-\langle s,t\rangle|=|\langle(\mathfrak{L}-H)s,t\rangle|\leq\|\mathfrak{L}-H\|\|s\|\|t\|\leq\max\{1,|\beta-1|\}\|s\|\|t\|.$

Moslehian et al. already derived inequality (3.1) from certain features of singular values ([25] [17], Corollary 2.5). (3.1) is equal to $\beta\in\mathbb{C}-\{0\}$.

$$\left|\langle s,m\rangle\langle m,t\rangle-\tfrac{1}{\beta}\langle s,t\rangle\right|\leq\frac{1}{|\beta|}\max\{1,|\beta-1|\}\|s\|\|t\|.$$

From the continuity property of modulus for complex numbers, we obtain

$$|\langle s,m\rangle\langle m,t\rangle|\leq\frac{1}{|\beta|}(|\langle s,t\rangle|+\max\{1,|\beta-1|\}\|s\|\|t\|)$$

with $\|m\|=1$ for any $s,t,m\in\mathcal{H}$. With $\beta=2$, (1.2) is obtained.

The main point of Proposition 3.1 was to find a clear limit on how far a rank-one operator can be from the identity. In contrast, Fujii and Kubo's method for deriving Buzano's inequality [27] was based on the fundamental estimate $\|2J-H\|\leq1$, where J is an orthogonal projection. Using these basic ideas and the methods from [11], we can now show our first important result.

Theorem 3.1. Let $\mathfrak{L} \in \mathbb{B}(\mathcal{H})$ and $\beta \in \mathbb{C} \setminus \{0\}$ such that $\|\beta\mathfrak{L} - H\| \leq 1$. For every s and t in $\mathcal{H}$, the inequality in equation (3.2) holds. So,

$$|\langle \mathfrak{L}s, t\rangle| \leq \left|\langle \mathfrak{L}s, t\rangle - \frac{1}{\beta}\langle s, t\rangle\right| + \frac{1}{|\beta|}|\langle s, t\rangle| \leq \frac{1}{|\beta|}\big(|\langle s, t\rangle| + \|s\|\,\|t\|\big). \tag{3.3}$$

Proof. Let $s, t \in \mathcal{H}$ and let $\beta \in \mathbb{C} \setminus \{0\}$ be chosen such that $\|\beta\mathfrak{L} - H\| \leq 1$. Using the identity

$$\langle \mathfrak{L}s, t\rangle - \frac{1}{\beta}\langle s, t\rangle = \frac{1}{\beta}\langle (\beta\mathfrak{L} - H)s, t\rangle,$$

We get the following inequality: The assumption that $\|\beta\mathfrak{L} - H\| \leq 1$ means that and putting this estimate in place gives us inequality (3.2). Lastly, putting (3.2) into the decomposition in (3.3) finishes the proof.

The subsequent example illustrates that the hypothesis of Theorem 3.1 is not universally applicable to all bounded linear operators.

Example 3.1. Consider the left shift operator

$$\mathfrak{L} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \qquad \mathfrak{L}(s_1, s_2, s_3, \ldots) = (0, s_1, s_2, s_3, \ldots).$$

Let

$$d_1 = (1, 0, 0, \ldots) \in \ell^2(\mathbb{N}).$$

Then

$$\langle -\mathfrak{L}d_1, d_1\rangle = 0, \qquad \|d_1\| = 1.$$

By Theorem 2.1 of [20], for any $\beta \in \mathbb{C} \setminus \{0\}$, one has

$$\|H\|^2 + |\beta|^2 m^2(-\mathfrak{L}) \leq \|H - \beta\mathfrak{L}\|^2,$$

where $m(\mathfrak{L})$ denotes the minimum modulus of an operator $\mathfrak{L}$. Since $-\mathfrak{L}$ is left-invertible, it follows that $m(-\mathfrak{L}) > 0$, and hence

$$1 < \|H\|^2 + |\beta|^2 m^2(-\mathfrak{L}) \leq \|H - \beta\mathfrak{L}\|^2.$$

This shows that the condition $\|\beta\mathfrak{L} - H\| \leq 1$ in Theorem 3.1 is not satisfied for this operator.

Generalized Buzano Inequality and the Center of Mass. For an operator $\mathfrak{L} \in \mathbb{B}(\mathcal{H})$, define

$$\operatorname{dist}(H, \mathbb{C}\mathfrak{L}) := \inf_{\gamma\in\mathbb{C}} \|\gamma\mathfrak{L} - H\|, \qquad \operatorname{dist}(\mathfrak{L}, \mathbb{C}H) := \inf_{\beta\in\mathbb{C}} \|\mathfrak{L} - \beta H\|.$$

If

$$\operatorname{dist}(H, \mathbb{C}\mathfrak{L}) = \|\gamma_0\mathfrak{L} - H\|,$$

then $\gamma_0 \in \mathbb{C}$ exists by definition , and it is unique whenever $m(\mathfrak{L}) > 0$. Following Stamp [18], the scalar γ_0 is referred to as the *center of mass* of $\mathfrak{L}$, denoted by $c(\mathfrak{L})$. Let $m(\mathfrak{L}) > 0$ for $A, \mathfrak{L} \in \mathbb{B}(\mathcal{H})$. Then we can write

$$M_{\mathfrak{L}}(A) = \sup_{\|s\|=1} \left(\|As\|^2 - |\langle As, \mathfrak{L}s\rangle|^2\right)$$
$$= \sup_{\|s\|=1} \left\| As - \frac{\langle As, \mathfrak{L}s\rangle}{\|\mathfrak{L}s\|^2}\, \mathfrak{L}s \right\|.$$

Paul [16] demonstrated that $M_{\mathfrak{L}}(A)$ is equivalent to the distance from A to the subspace $\mathbb{C}\mathfrak{L}$, specifically,

$$M_{\mathfrak{L}}(A) = \operatorname{dist}(A, \mathbb{C}\mathfrak{L}).$$

For simplicity, for any nonzero complex number $\beta \in \mathbb{C} \setminus \{0\}$.

4. Sharp A-Numerical Radius Bounds

The findings in this study are based on several fundamental lemmas. The first of these, which gives important initial estimates, was set in [29].

Lemma 4.1.

Let $s, t, m \in \mathcal{H}$ and assume that m satisfies condition $\|m\|_A = 1$. Then the following estimate is true:

$$\left|\langle s, m\rangle_A \langle m, t\rangle_A\right| \leq \frac{1}{2} \left(\|s\|_A \|t\|_A + |\langle s, t\rangle_A|\right).$$

The next lemma gives a better estimate for products of A-inner products that involve a unit vector with respect to the A-norm. This limit has a non-negative parameter β that lets you control how sharp the inequality is at different levels.

Lemma 4.2. Let $s, t, m \in \mathcal{H}$ with $\|m\|_A = 1$ and $\beta \geqslant 0$. Then

$$|\langle s, m\rangle_A \langle m, t\rangle_A|^2 \leqslant \frac{1}{4}\left(\frac{1+2\beta}{1+\beta}\, \|s\|_A^2 \|t\|_A^2 + \frac{3+2\beta}{1+\beta}\, \|s\|_A \|t\|_A\, |\langle s, t\rangle_A|\right).$$

Proof. The Cauchy–Schwarz inequality states that

$$|\langle s, t\rangle_A|^2 \leqslant \|s\|_A \|t\|_A\; |\langle s, t\rangle_A|\,.$$

Because $\beta \geqslant 0$, adding the non-negative quantity maintains the inequality, leading to the conclusion that

$$|\langle s, t\rangle_A|^2 \leqslant \|s\|_A \|t\|_A\, |\langle s, t\rangle_A| + \beta\left(\|s\|_A^2 \|t\|_A^2 - |\langle s, t\rangle_A|^2\right).$$

The stated estimate is derived by employing conventional techniques utilized in the proof of refined Cauchy–Schwarz-type inequalities. □

Consequently, we might conclude that

$$|\langle s, t\rangle_A|^2 \leqslant \frac{\beta}{1+\beta} \|s\|_A^2 \|t\|_A^2 + \frac{1}{1+\beta} \|s\|_A \|t\|_A\, |\langle s, t\rangle_A| \tag{4.1}$$

Moreover, Lemma 4.1 yields

$$|\langle s,m\rangle_A\langle m,t\rangle_A|^2 \leqslant \frac{1}{4}\left(\|s\|_A^2\|t\|_A^2+2\|s\|_A\|t\|_A\,|\langle s,t\rangle_A|+|\langle s,t\rangle|_A^2\right).$$

Together with inequality (4.1), it can be established that

$$\begin{aligned}
&|\langle s,m\rangle_A\langle m,t\rangle_A|^2\\
\leqslant&\frac{1}{4}\left(\|s\|_A^2\|t\|_A^2+2\|s\|_A\|t\|_A\,|\langle s,t\rangle_A|+|\langle s,t\rangle|_A^2\right)\\
\leqslant&\frac{1}{4}\left(\|s\|_A^2\|t\|_A^2+2\|s\|_A\|t\|_A\,|\langle s,t\rangle_A|+\frac{\beta}{1+\beta}\|s\|_A^2\|t\|_A^2+\frac{1}{1+\beta}\|s\|_A\|t\|_A\,|\langle s,t\rangle_A|\right)\\
=&\frac{1}{4}\left(\frac{1+2\beta}{1+\beta}\|s\|_A^2\|t\|_A^2+\frac{3+2\beta}{1+\beta}\|s\|_A\|t\|_A\,|\langle s,t\rangle_A|\right).
\end{aligned}$$

The proof is now done.

Remark 4.1.

Consider the vectors $s,t,m\in\mathcal{H}$ with $\|m\|_A=1$. A direct estimate shows that

$$|\langle s,m\rangle_A\,\langle m,t\rangle_A|^2 \le \frac{1}{4}\big(3\|s\|_A^2\|t\|_A^2+\|s\|_A\|t\|_A\,|\langle s,t\rangle_A|\big)\,, \tag{4.2}$$

which provides a general upper bound independent of any parameter choice.

However, inequality(4.2) is not as precise as the bound presented in Lemma 4.2. Indeed, Lemma 4.2 asserts that for every $\beta\ge 0$,

$$|\langle s,m\rangle_A\,\langle m,t\rangle_A|^2 \le \frac{1}{4}\left(\frac{1+2\beta}{1+\beta}\,\|s\|_A^2\|t\|_A^2+\frac{3+2\beta}{1+\beta}\,\|s\|_A\|t\|_A\,|\langle s,t\rangle_A|\right).$$

Using the elementary inequality

$$\frac{3+2\beta}{1+\beta}\,\|s\|_A\|t\|_A\,|\langle s,t\rangle_A| \le \frac{2+\beta}{1+\beta}\,\|s\|_A^2\|t\|_A^2+\|s\|_A\|t\|_A\,|\langle s,t\rangle_A|\,,$$

we obtain

$$|\langle s,m\rangle_A\,\langle m,t\rangle_A|^2 \le \frac{1}{4}\big(3\|s\|_A^2\|t\|_A^2+\|s\|_A\|t\|_A\,|\langle s,t\rangle_A|\big)\,,$$

which shows that inequality (4.2) follows directly from the refined estimate in Lemma4.2.

Lemma 4.3.

Let $s,t,m\in\mathcal{H}$ be any vectors such that $\|m\|_A=1$, and let $\beta\ge 0$. Then the following estimate holds:

$$|\langle s,m\rangle_A\,\langle m,t\rangle_A|^2 \le \frac{1}{2}\left(\frac{1+2\beta}{1+\beta}\,\|s\|_A^2\|t\|_A^2+\frac{1}{1+\beta}\,|\langle s,t\rangle_A|^2\right).$$

Proof. As in the argument used for Lemma 4.2, we begin with

$$|\langle s,t\rangle_A|^2 \le |\langle s,t\rangle_A|^2+\beta\Big(\|s\|_A^2\|t\|_A^2-|\langle s,t\rangle_A|^2\Big)\,,$$

which implies

$$|\langle s,t\rangle_A|^2 \le \frac{\beta}{1+\beta}\,\|s\|_A^2\|t\|_A^2+\frac{1}{1+\beta}\,|\langle s,t\rangle_A|^2\,. \tag{4.3}$$

Next, using Lemma 4.1 together with the convexity of $f(u) = u^2$, we obtain

$$|\langle s, m\rangle_A \langle m, t\rangle_A|^2 \leq \left(\frac{\|s\|_A\|t\|_A + |\langle s, t\rangle_A|}{2}\right)^2 \leq \frac{1}{2}\left(\|s\|_A^2\|t\|_A^2 + |\langle s, t\rangle_A|^2\right). \tag{4.4}$$

Finally, combining inequalities (4.3) and (4.4), we obtain

$$|\langle s, m\rangle_A \langle m, t\rangle_A|^2 \leq \frac{1}{2}\left(\|s\|_A^2\|t\|_A^2 + |\langle s, t\rangle_A|^2\right) \leq \frac{1}{2}\left(\frac{1+2\beta}{1+\beta}\|s\|_A^2\|t\|_A^2 + \frac{1}{1+\beta}|\langle s, t\rangle_A|^2\right).$$

This completes the proof. □

Remark 4.2. The inequality in Lemma 4.3 provides a sharper bound than the one in (4.2) for $\beta = 1$. For any vectors $s, t, m \in \mathcal{H}$ with $\|m\|_A = 1$, we have

$$\begin{aligned}|\langle s, m\rangle_A\langle m, t\rangle_A|^2 &\leq \frac{1}{2}\left(\frac{3}{2}\|s\|_A^2\|t\|_A^2 + \frac{1}{2}|\langle s, t\rangle_A|^2\right)\\ &= \frac{1}{4}\left(3\|s\|_A^2\|t\|_A^2 + |\langle s, t\rangle_A|^2\right)\\ &\leq \frac{1}{4}\left(3\|s\|_A^2\|t\|_A^2 + \|s\|_A\|t\|_A|\langle s, t\rangle_A|\right).\end{aligned}$$

Lemma 4.4. Let $\mathfrak{L} \in \mathbb{B}_A(\mathcal{H})$. For any $s, t \in \mathcal{H}$ with $\|s\|_A = \|t\|_A = 1$, the following inequality holds:

$$|\langle \mathfrak{L}s, t\rangle_A|^2 \leq \sqrt{\langle \mathfrak{L}^{\sharp_A}\mathfrak{L}s, s\rangle_A}\,\sqrt{\langle \mathfrak{L}\mathfrak{L}^{\sharp_A}t, t\rangle_A}.$$

This outcome extends classical operator inequalities and was demonstrated in [18].

Lemma 4.5. Let $\mathfrak{L} \in \mathbb{B}_A(\mathcal{H})$. Then $\mathfrak{L} = \mathfrak{L}^{\sharp_A}$ if and only if $R(\mathfrak{L}) \subseteq \overline{R(A)}$ and $\mathfrak{L}$ is an A-selfadjoint operator.

Remark 4.3. From Lemma 4.5, it follows that $\left(\mathfrak{L}^{\sharp_A}\mathfrak{L}\right)^{\sharp_A} = \mathfrak{L}^{\sharp_A}\mathfrak{L}$.

Lemma 4.6. Let $\mathfrak{L}, \mathfrak{Q} \in \mathbb{B}(\mathcal{H})$ be A-positive operators. Then

$$\|\mathfrak{L} + \mathfrak{Q}\|_A \leqslant \max\{\|\mathfrak{L}\|_A, \|\mathfrak{Q}\|_A\} + \|\mathfrak{L}\mathfrak{Q}\|_A^{\frac{1}{2}}$$

Proof. If $\mathfrak{B}$ and $\mathfrak{C}$ are positive operators on a Hilbert space, then it was demonstrated in [13] that

$$\|\mathfrak{B} + \mathfrak{C}\| \leqslant \max\{\|\mathfrak{B}\|, \|\mathfrak{C}\|\} + \|\mathfrak{B}\mathfrak{C}\|^{\frac{1}{2}}$$

$\overline{A^{\frac{1}{2}}\mathfrak{L}\left(A^{\frac{1}{2}}\right)^{\dagger}}$ and $\overline{A^{\frac{1}{2}}\mathfrak{Q}\left(A^{\frac{1}{2}}\right)^{\dagger}}$ are positive operators since $\mathfrak{L}$ and $\mathfrak{Q}$ are A-positive. Using $\mathfrak{L}N(A) \subseteq N(A)$ and equation (1.1), one obtains

$$\begin{aligned}\|\mathfrak{L}+\mathfrak{Q}\|_A &= \left\|\overline{A^{\frac{1}{2}}\mathfrak{L}\left(A^{\frac{1}{2}}\right)^{\dagger}}+\overline{A^{\frac{1}{2}}\mathfrak{Q}\left(A^{\frac{1}{2}}\right)^{\dagger}}\right\| \\ &\leqslant \max\left\{\left\|\overline{A^{\frac{1}{2}}\mathfrak{L}\left(A^{\frac{1}{2}}\right)^{\dagger}}\right\|,\left\|\overline{A^{\frac{1}{2}}\mathfrak{Q}\left(A^{\frac{1}{2}}\right)^{\dagger}}\right\|\right\}+\left\|\overline{A^{\frac{1}{2}}\mathfrak{L}\left(A^{\frac{1}{2}}\right)^{\dagger}A^{\frac{1}{2}}\mathfrak{Q}\left(A^{\frac{1}{2}}\right)^{\dagger}}\right\|^{\frac{1}{2}} \\ &= \max\left\{\left\|\overline{A^{\frac{1}{2}}\mathfrak{L}\left(A^{\frac{1}{2}}\right)^{\dagger}}\right\|,\left\|\overline{A^{\frac{1}{2}}\mathfrak{Q}\left(A^{\frac{1}{2}}\right)^{\dagger}}\right\|\right\}+\left\|\overline{A^{\frac{1}{2}}\mathfrak{L}\mathfrak{Q}\left(A^{\frac{1}{2}}\right)^{\dagger}}\right\|^{\frac{1}{2}} \\ &= \max\left\{\|\mathfrak{L}\|_A,\|\mathfrak{Q}\|_A\right\}+\|\mathfrak{L}\mathfrak{Q}\|_A^{\frac{1}{2}}.\end{aligned}$$

The proof is now complete.

5. A-Numerical Radius Inequalities for Operator Matrix Applications

It is necessary to prove a McCarthy-type inequality in a semi-Hilbertian space with a positive operator A in order to validate our $\mathbb{A}$-numerical radius inequalities for 2×2 operator matrices. By guaranteeing strict bounds and offering a context for examining more intricate operator structures, this inequality is a vital tool for expanding classical operator inequalities to the $\mathbb{A}$-numerical radius setting. We can strengthen the theoretical underpinnings of our inequalities and make it easier to use them in future research on numerical radius and operator theory by utilizing this McCarthy-type conclusion to methodically obtain reliable estimates for operator matrices.

Lemma 5.1 ([17] [16] [6]).

Let $\mathfrak{L}\in\mathbb{B}(\mathcal{H})$ be an A positive operator and $s\in\mathcal{H}$ be such that $\|s\|_A=1$. Then

$$\langle\mathfrak{L}s,t\rangle_A^{\kappa}\leq\langle\mathfrak{L}^{\kappa}s,s\rangle_A \quad \text{for all} \quad \kappa\in\mathbb{N}^*. \tag{5.1}$$

Bhunia et al. [22] already proved the second lemma for the situation of a strictly positive operator A. Rout et al. [19] [14] more recently showed the same result without assuming that A is strictly positive.

Lemma 5.2 ([5] [6] [21]).

Let $\mathfrak{L}_1,\mathfrak{L}_2,\mathfrak{L}_3,\mathfrak{L}_4\in\mathbb{B}_A(\mathcal{H})$. Then, the following assertions hold:

i) $\omega_{\mathbb{A}}\left(\begin{bmatrix}\mathfrak{L}_1 & \mathfrak{O}\\ \mathfrak{O} & \mathfrak{L}_2\end{bmatrix}\right)=\max\left\{\omega_A\left(\mathfrak{L}_1\right),\omega_A\left(\mathfrak{L}_2\right)\right\}.$

ii) $\omega_A\left(\mathfrak{L}_1\right)=\omega_{\mathbb{A}}\left(\begin{bmatrix}\mathfrak{O} & \mathfrak{L}_1\\ \mathfrak{L}_1 & \mathfrak{O}\end{bmatrix}\right).$

iii) $\left\|\begin{bmatrix}\mathfrak{L}_1 & \mathfrak{O}\\ \mathfrak{O} & \mathfrak{L}_2\end{bmatrix}\right\|_{\mathbb{A}}=\left\|\begin{bmatrix}\mathfrak{O} & \mathfrak{L}_1\\ \mathfrak{L}_2 & \mathfrak{O}\end{bmatrix}\right\|_{\mathbb{A}}=\max\left\{\|\mathfrak{L}_1\|_A,\|\mathfrak{L}_2\|_A\right\}.$

iv) $\begin{bmatrix}\mathfrak{L}_1 & \mathfrak{L}_2\\ \mathfrak{L}_3 & \mathfrak{L}_4\end{bmatrix}^{\sharp A}=\begin{bmatrix}\mathfrak{L}_1^{\sharp A} & \mathfrak{L}_3^{\sharp A}\\ \mathfrak{L}_2^{\sharp A} & \mathfrak{L}_4^{\sharp A}\end{bmatrix}.$

This result enhances previous findings by integrating the context of 2×2 operator matrices, resulting in a more sophisticated and adaptable formulation of the second inequality in (1.2). Additionally, when the weight operator fulfills $A=I$ and $r\in\mathbb{N}$, the resultant extension provides a significantly more robust iteration of Theorem 2.7 in [4], underscoring its importance within the classical unweighted context.

Theorem 5.1. Let $\mathfrak{L}_1,\mathfrak{L}_2\in\mathbb{B}_A(\mathcal{H})$, let G be a set such that $(0,1)\subset G\subset\mathbb{R}$, and let χ be a mapping $\chi: G\to\left[\frac{1}{2},1\right]$. Then for every $\kappa\in\mathbb{N}^*$,

$$w_{\mathbb{A}}^{2\kappa}\left(\begin{bmatrix}0 & \mathfrak{L}_1\\ \mathfrak{L}_2 & 0\end{bmatrix}\right)\leq\frac{\chi(\mu)}{2}\max\left\{\left\|\left(\mathfrak{L}_2^{\sharp_A}\mathfrak{L}_2\right)^{\kappa}+\left(\mathfrak{L}_1\mathfrak{L}_1^{\sharp_A}\right)^{\kappa}\right\|_A,\left\|\left(\mathfrak{L}_1^{\sharp_A}\mathfrak{L}_1\right)^{\kappa}+\left(\mathfrak{L}_2\mathfrak{L}_2^{\sharp_A}\right)^{\kappa}\right\|_A\right\}\\ +(1-\chi(\mu))\max\left\{w_A^{\kappa}\left(\mathfrak{L}_2\mathfrak{L}_1\right),w_A^{\kappa}\left(\mathfrak{L}_1\mathfrak{L}_2\right)\right\}\tag{5.2}$$

Proof. Let $\mathfrak{L}=\begin{bmatrix}0 & \mathfrak{L}_1\\ \mathfrak{L}_2 & 0\end{bmatrix}$ and let $s\in\mathcal{H}^{(2)}$ be any unit vector. Then we have

$$\begin{aligned}|\langle\mathfrak{L}s,s\rangle_{\mathbb{A}}|^{2\kappa}&=\left|\langle\mathfrak{L}s,s\rangle_{\mathbb{A}}\left\langle s,\mathfrak{L}^{\sharp_A}s\right\rangle_{\mathbb{A}}\right|^{\kappa}\\ &\leq\chi(\mu)\|\mathfrak{L}s\|_{\mathbb{A}}^{\kappa}\left\|\mathfrak{L}^{\sharp_A}s\right\|_{\mathbb{A}}^{\kappa}+(1-\chi(\mu))\left|\left\langle\mathfrak{L}s,\mathfrak{L}^{\sharp_A}s\right\rangle_{\mathbb{A}}\right|^{\kappa}\quad(\text{ by Lemma 2.7 })\\ &\leq\frac{\chi(\mu)}{2}\left(\|\mathfrak{L}s\|_{\mathbb{A}}^{2\kappa}+\left\|\mathfrak{L}^{\sharp_A}s\right\|_{\mathbb{A}}^{2\kappa}\right)+(1-\chi(\mu))\left|\langle\mathfrak{L}^2s,s\rangle\right|_{\mathbb{A}}^{\kappa}\end{aligned}$$

(using the old connection between geometric and arithmetic means)

$$\begin{aligned}&\leq\frac{\chi(\mu)}{4}\left\langle\left(\left(\mathfrak{L}^{\sharp_A}\mathfrak{L}\right)^{\kappa}+\left(\mathfrak{L}\mathfrak{L}^{\sharp_A}\right)^{\kappa}\right)s,s\right\rangle_{\mathbb{A}}+(1-\chi(\mu))\left|\langle\mathfrak{L}^2s,s\rangle_{\mathbb{A}}\right|^{\kappa}\quad(\text{by Lemma 5.1})\\ &=\frac{\chi(\mu)}{4}\left\langle\begin{bmatrix}\left(\mathfrak{L}_2^{\sharp_A}\mathfrak{L}_2\right)^{\kappa}+\left(\mathfrak{L}_1\mathfrak{L}_1^{\sharp_A}\right)^{\kappa} & 0\\ 0 & \left(\mathfrak{L}_1^{\sharp_A}\mathfrak{L}_1\right)^{\kappa}+\left(\mathfrak{L}_2\mathfrak{L}_2^{\sharp_A}\right)^{\kappa}\end{bmatrix}s,s\right\rangle_{\mathbb{A}}\\ &\quad+(1-\chi(\mu))\left|\left\langle\begin{bmatrix}\mathfrak{L}_1\mathfrak{L}_2 & 0\\ 0 & \mathfrak{L}_2\mathfrak{L}_1\end{bmatrix}s,s\right\rangle_{\mathbb{A}}\right|^{\kappa}.\end{aligned}$$

Now, by applying Lemmas 5.1 and 5.2, we get

$$\begin{aligned}w_{\mathbb{A}}^{2\kappa}\left(\begin{bmatrix}0 & \mathfrak{L}_1\\ \mathfrak{L}_2 & 0\end{bmatrix}\right)&=\sup_{\|s\|_{\mathbb{A}}=1}\left|\left\langle\begin{bmatrix}0 & \mathfrak{L}_1\\ \mathfrak{L}_2 & 0\end{bmatrix}s,s\right\rangle_{\mathbb{A}}\right|^{2\kappa}\\ &\leq\frac{\chi(\mu)}{4}\left\|\begin{bmatrix}\left(\mathfrak{L}_2^{\sharp_A}\mathfrak{L}_2\right)^{\kappa}+\left(\mathfrak{L}_1\mathfrak{L}_1^{\sharp_A}\right)^{n} & 0\\ 0 & \left(\mathfrak{L}_1^{\sharp_A}\mathfrak{L}_1\right)^{\kappa}+\left(\mathfrak{L}_2\mathfrak{L}_2^{\sharp_A}\right)^{\kappa}\end{bmatrix}\right\|_{\mathbb{A}}\\ &\quad+(1-\chi(\mu))w_{\mathbb{A}}^{\kappa}\left(\begin{bmatrix}\mathfrak{L}_1\mathfrak{L}_2 & 0\\ 0 & \mathfrak{L}_2\mathfrak{L}_1\end{bmatrix}\right)\\ &=\frac{\chi(\mu)}{2}\max\left\{\left\|\left(\mathfrak{L}_2^{\sharp_A}\mathfrak{L}_2\right)^{\kappa}+\left(\mathfrak{L}_1\mathfrak{L}_1^{\sharp_A}\right)^{\kappa}\right\|_A,\left\|\left(\mathfrak{L}_1^{\sharp_A}\mathfrak{L}_1\right)^{\kappa}+\left(\mathfrak{L}_2\mathfrak{L}_2^{\sharp_A}\right)^{\kappa}\right\|_A\right\}\end{aligned}$$

$$+ (1-\chi(\mu)) \max \left\{ w_A^{\kappa}\left(\mathfrak{L}_2\mathfrak{L}_1\right), w_A^{\kappa}\left(\mathfrak{L}_1\mathfrak{L}_2\right)\right\}$$

This concludes the theorem's proof. The next corollary gives a sharper version of inequality (1.2), which is a direct result of Theorem 3.4 in [6].

Theorem 5.2.

Let $\mathfrak{L} \in \mathbb{B}_A(\mathcal{H})$ and let $\chi : [0,1] \to \left[\frac{1}{4}, 1\right]$. Then, the following inequality holds:

$$\omega_A^4(\mathfrak{L}) \;\leq\; \frac{\chi(\mu)}{8} \left\| \left(\mathfrak{L}^{\sharp A}\mathfrak{L}\right)^2 + \left(\mathfrak{L}\,\mathfrak{L}^{\sharp A}\right)^2 \right\|_A + \frac{1-\chi(\mu)}{4} \left\| \mathfrak{L}^{\sharp A}\mathfrak{L} + \mathfrak{L}\,\mathfrak{L}^{\sharp A} \right\|_A \omega_A(\mathfrak{L}^2)\,.$$

Proof. Let $s \in \mathcal{H}$ with $\|s\|_A = 1$. Applying Theorem 5.1 with

$$m = s, \quad s = \mathfrak{L}s, \quad t = \mathfrak{L}^{\sharp A}s,$$

we obtain

$$|\langle \mathfrak{L}s, s\rangle_A|^4 \;\leq\; \chi(\mu)\, \|\mathfrak{L}s\|_A^2\, \|\mathfrak{L}^{\sharp A}s\|_A^2 + (1-\chi(\mu))\, \left|\langle \mathfrak{L}^2 s, s\rangle_A\right|^2.$$

Taking the supremum over all unit A-norm vectors s and applying the AM-GM inequality, we obtain the stated bound. □

Theorem 5.3.

Let $\mathfrak{L}_1, \mathfrak{L}_2 \in \mathbb{B}_A(\mathcal{H})$ and let $\chi : [0,1] \to \left[\frac{1}{2}, 1\right]$. Then, the following inequality holds:

$$\omega_{\mathbb{A}}^2\left(\begin{bmatrix} 0 & \mathfrak{L}_1 \\ \mathfrak{L}_2 & 0 \end{bmatrix}\right) \;\leq\; \frac{\chi(\mu)}{2} \max\left\{ \left\|\mathfrak{L}_1^{\sharp A}\mathfrak{L}_1\right\|_A, \left\|\mathfrak{L}_2^{\sharp A}\mathfrak{L}_2\right\|_A \right\} + (1-\chi(\mu))\, \omega_A\left(\mathfrak{L}_1\mathfrak{L}_2\right).$$

Let

$$s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H} \quad \text{with} \quad \|s\|_{\mathbb{A}} = 1.$$

Then, according to the definition of the $\mathbb{A}$-numerical radius, we have

Using Theorem 5.2 on the terms $\langle \mathfrak{L}_1 s_2, s_1\rangle_A$ and $\langle \mathfrak{L}_2 s_1, s_2\rangle_A$ separately, along with the properties of $\chi(\mu)$, we get Finally, taking the supremum over all $s \in \mathcal{H} \oplus \mathcal{H}$ with $\|s\|_{\mathbb{A}} = 1$

Lastly, taking the supremum over all $s \in \mathcal{H} \oplus \mathcal{H}$ such that $\|s\|_{\mathbb{A}} = 1$ yields the desired inequality.

6. Numerical Examples

Example 6.1

Examine the Hilbert space $\mathcal{H} = \mathbb{C}^2$, where the positive definite matrix induces the semi-inner product.

$$\mathfrak{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Define the operator

$$\mathfrak{L} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

To compute its $\mathfrak{A}$-adjoint, we use the formula

$$\mathfrak{L}^{\sharp A} = \mathfrak{A}^{-1}\mathfrak{L}^*\mathfrak{A},$$

where

$$\mathfrak{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, \quad \mathfrak{L}^* = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

Thus, we obtain

$$\mathfrak{L}^{\sharp A} = \begin{bmatrix} 0 & \frac{3}{2} \\ \frac{4}{3} & 0 \end{bmatrix}.$$

Next, we have

$$\mathfrak{L}^{\sharp A}\mathfrak{L} + \mathfrak{L}\mathfrak{L}^{\sharp A} = \begin{bmatrix} \frac{13}{3} & 0 \\ 0 & \frac{13}{3} \end{bmatrix}.$$

Thus, the norm is

$$\left\|\mathfrak{L}^{\sharp A}\mathfrak{L} + \mathfrak{L}\mathfrak{L}^{\sharp A}\right\|_A = 13.$$

Since

$$\mathfrak{L}^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

we get

$$\omega_A\left(\mathfrak{L}^2\right) = 6$$

Example 6.2

Examine the Hilbert space $\mathcal{H} = \mathbb{C}^3$, where the positive definite diagonal matrix induces the semi-inner product:

$$\mathfrak{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Define the operator $\mathfrak{L}$ as:

$$\mathfrak{L} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 3 \\ 1 & 4 & 0 \end{bmatrix}$$

Using the formula:

$$\mathfrak{L}^{\sharp A} = A^{-1}\mathfrak{L}^* A$$

we compute:

$$\mathfrak{L}^{\sharp A} = \begin{bmatrix} 0 & 3 & 2.5 \\ 2/3 & 0 & 20/3 \\ 4/5 & 9/5 & 0 \end{bmatrix}.$$

$$\mathfrak{L}^{\sharp A}\mathfrak{L} + \mathfrak{L}\mathfrak{L}^{\sharp A} = \begin{bmatrix} \frac{97}{9} & \frac{68}{5} & \frac{47}{3} \\ \frac{136}{15} & \frac{1162}{30} & \frac{19}{3} \\ \frac{94}{15} & \frac{19}{5} & \frac{217}{6} \end{bmatrix}.$$

Thus,

$$\left\| \mathfrak{L}^{\sharp A}\mathfrak{L} + \mathfrak{L}\mathfrak{L}^{\sharp A} \right\|_A = 231.17.$$

Since

$$\mathfrak{L}^2 = \begin{bmatrix} 4 & 8 & 3 \\ 3 & 14 & 4 \\ 8 & 1 & 14 \end{bmatrix}$$

we have:

$$\omega_A\left(\mathfrak{L}^2\right) = 78.60$$

Let $\chi(\mu) = \frac{3}{4}$, then:

$$\omega_A^4(\mathfrak{L}) = 6159.20$$

and

$$\frac{\chi(\mu)}{4} \left\| \mathfrak{L}^{\sharp_A}\mathfrak{L} + \mathfrak{L}\mathfrak{L}^{\sharp_A} \right\|_A^2 + \frac{(1-\chi(\mu))}{2} \left\| \mathfrak{L}^{\sharp_A}\mathfrak{L} + \mathfrak{L}\mathfrak{L}^{\sharp_A} \right\|_A \omega_A\left(\mathfrak{L}^2\right) = 12291.20.$$

7 Conclusion

In this study, we introduced a novel form of the Buzano inequality within semi-Hilbert spaces. Leveraging this inequality, we derived new A-numerical radius inequalities and established precise bounds for the A-numerical radius of operators, where A denotes a positive semi-definite operator in a complex Hilbert space. These results, grounded in the latest developments on numerical radius inequalities in Hilbert space frameworks, provide valuable tools for modal and structural analysis. Looking ahead, we propose extending this research to include Schatten classes and the p-numerical radius in analytical applications. Our findings lay a solid foundation for further theoretical exploration beyond classical Hilbert spaces and inspire the development of innovative mathematical techniques and analytical instruments. In summary, this work presents original Buzano-type inequalities, applies them to improve A-numerical radius bounds, and outlines avenues for future research. We expect these contributions to serve as a strong basis for subsequent studies, encouraging further advances in modal analysis and related mathematical fields.

Future work: Extend to p-numerical radius and Schatten classes.

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