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THE BACKWARDS MCSHANE INTEGRAL: AN ALTERNATIVE APPROACH USING FULL DIVISIONS

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Abstract. We introduce the backwards McShane integral on the interval $[0, T]$, constructed via (δ, η)-fine divisions that combine backwards δ-fine and McShane δ-fine components. We develop several properties of the backwards McShane integral and establish its equivalence to the Lebesgue and backwards Henstock integrals. Within this framework, the absolute continuity of the primitive follows by refinement, eliminating the need for Vitali coverings, and the passage from Lebesgue integrability to backwards McShane integrability is achieved through a concise, single-step construction. The equivalence proofs remain substantive but provide an alternative characterization of backwards integration where Cousin's Lemma applied.

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1. Introduction

In the 1950s, Henstock [6] and Kurzweil [7] independently defined an integral by replacing a positive constant δ in the Riemann integral with a positive function δ. This idea is known as the Henstock approach. This small modification is the cornerstone of what is now known as the Henstock–Kurzweil integral, significantly extending the class of integrable functions beyond those permitted by the Riemann integral. Subsequently, McShane modified the division in the Henstock–Kurzweil integral by allowing each tag to be chosen outside its associated subinterval [11]. This gave rise to the McShane integral, which is a special case of the Henstock-Kurzweil integral.

The backwards Henstock integral was introduced by Arcede and Cabral [1] as a deterministic counterpart to the backwards Itô integral in stochastic calculus. They established that the backwards Henstock integral, defined using backwards δ-fine partial divisions, is equivalent to the Lebesgue integral. A main technical feature is partial divisions: the tag ξ always be the right endpoint of each

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interval $(u, \xi]$, and the requirement that $(u, \xi] \subseteq (\xi - \delta(\xi), \xi]$ means that full backwards δ-fine divisions may not exist for some gauge functions δ. For instance, if $\delta(\xi) = \frac{\xi}{2}$, the interval $(0, T]$ cannot be covered by any finite collection of backwards δ-fine intervals. To handle this, Arcede and Cabral allowed partial divisions that "fail to cover $[0, T]$ by at most Lebesgue measure η," proving integrability results using Vitali covering arguments to manage the uncovered intervals.

While the backwards Henstock integral was motivated by applications to backwards stochastic integration [2], the equivalence to the Lebesgue integral establishes it as a legitimate object of study in deterministic integration theory independent of its stochastic origins. Subsequent work by Sasam and Labendia [13] extended this framework to set-valued stochastic processes, relying on the foundational equivalence results from [1].

In [2], Arcede and Cabral introduced hybrid divisions combining backwards and McShane components to define the stochastic integral with full coverage, which are (δ, η)-fine divisions. In this paper, we systematically apply the same divisions to the (deterministic) backwards Henstock integral, which was previously constructed using partial divisions (backwards divisions) rather than full ones. As well known that backwards divisions may fail to achieve full coverage due to the restriction that tags always be the right endpoints. A natural question arises: Can we modify the backwards Henstock integration framework to permit (δ, η)-fine divisions while preserving the class of integrable functions? The McShane integral [4] offers a precedent for such modifications. By allowing tags to lie outside their associated intervals, McShane δ-fine divisions satisfy Cousin's Lemma, guaranteeing the existence of full divisions. Moreover, McShane integrability is equivalent to Lebesgue integrability. This pattern—proving that alternative characterizations yield equivalent integrals—is well-established in gauge integration theory, where Henstock, Kurzweil, and Denjoy integrals have been shown to coincide [4,8,14]. Although the division structure resembles that of Arcede and Cabral, the resulting integral is nontrivial when compared to their stochastic construction: in Arcede and Cabral, the integrator is a stochastic process and the integral is defined in the L^2 sense, yielding a random variable even when the integrand is deterministic, whereas the present integral operates on deterministic functions with respect to the identity function as integrator and is defined in the usual real setting. This shift from a stochastic to a deterministic framework alters both the underlying convergence and the nature of the integral, leading to a nontrivial extension of the (δ, η)-fine divisions approach.

In this work, we answer the research question: Can we take a backwards partial division definition and hybridize it with McShane divisions to get full divisions, recovering the exact same integral? The answer is affirmative. We introduce the backwards McShane integral using (δ, η)-fine divisions, establish some properties, and prove it is equivalence to the Lebesgue integral and backwards Henstock integral. The (δ, η)-fine division framework provides two concrete advantages: First, the absolute continuity of the primitive admits an easier proof. Our proof (Theorem 2.9) uses simple refinement arguments rather than the Vitali covering technique required in the partial division approach. Second, when considering the direction from Lebesgue to backwards McShane, the proof can be significantly simplified, as it relies only on a direct observation (Theorem 2.12) without requiring several complicated steps. Finally, the equivalence proof of the backwards McShane integral to the Lebesgue integral and the backwards Henstock integral (Corollary 2.13 and Corollary 2.14) remains substantive but provides an alternative characterization of backwards integration. While this paper focuses on the deterministic theory, the framework has potential applications in stochastic settings, which remain an avenue for future research.

Throughout the course of this paper, we consider functions and integrals defined on a closed interval $[0, T]$.

A finite collection $P = \{[u_i, v_i]\}_{i=1}^n$ of non-overlapping subintervals of $[0, T]$ is called a partial division of $[0, T]$. In addition, P is called a full division of $[0, T]$ if $\cup_{i=1}^n [u_i, v_i] = [0, T]$.

A finite collection of interval–point pairs $D = \{([u_i, \xi_i], \xi_i)\}_{i=1}^n$ is called a backwards partial division of $[0, T]$ if $\{([u_i, \xi_i]\}_{i=1}^n$ is a finite collection of non-overlapping subintervals of $[0, T]$. Let δ be a positive function on $[0, T]$. An interval–point pair $([u, \xi], \xi)$ is said to be backwards δ-fine if $[u, \xi] \subseteq (\xi - \delta(\xi), \xi]$, whenever $[u, \xi] \subseteq [0, T]$. We call $D = \{([u_i, \xi_i], \xi_i)\}_{i=1}^n$ a backwards δ-fine partial division of $[0, T]$ if D is a backwards partial division of $[0, T]$ and for each i, an interval-point pair $([u_i, \xi_i], \xi_i)$ is backwards δ-fine. A backwards δ-fine partial division might fail to covers the interval $[0, T]$. For example, let $\delta : [0, T] \to \mathbb{R}^+$ be defined by $\delta(\xi) = \frac{\xi}{3}$ for $\xi \in (0, T]$ and $\delta(0) = \frac{1}{2}$.

Let δ be a positive function on $[0, T]$ and $\eta > 0$ be given. We call $D = \{([u_i, \xi_i], \xi_i)\}_{i=1}^n$ a backwards (δ, η)- fine partial division of $[0, T]$ if it fails to cover an interval $[0, T]$ by at most Lebesgue measure η, that is,

$$T - (D) \sum_{i=1}^{n} (\xi_i - u_i) \leq \eta.$$

We emphasize that a backwards (δ, η)-fine partial division of $[0, T]$ always exists by the Vitali covering theorem. However, as stated above, it might fail to cover the interval $[0, T]$. If we want to have a full cover on $[0, T]$, then we need the concept of McShane δ-fine as define below.

Let δ be a positive function on $[0, T]$, $[u, v] \subseteq [0, T]$, and $\xi \in [0, T]$. An interval-point pair $([u, v], \xi)$ is said to be McShane δ-fine if $[u, v] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))$. Note that we do not require the tag ξ to belong in the subinterval $[u, v]$. A finite collection of interval-point pairs $D' = \{([w_j, z_j], \xi_j)\}_{j=1}^m$ is called a McShane δ-fine partial division of $[0, T]$ if $\{[w_j, z_j]\}_{j=1}^m$ is a partial division of $[0, T]$ and for each j, an interval-point pair $([w_j, z_j], \xi_j)$ is McShane δ-fine. In addition, we call D' a McShane δ-fine division of $[0, T]$ if $\cup_{j=1}^m [w_j, z_j] = [0, T]$.

We will combine the two concepts, i.e., backwards McShane δ-fine and McShane δ-fine to create a hybrid full division on $[0, T]$. Let δ be a positive function on $[0, T]$ and $\eta > 0$ be given. A full division

D'' is called a (δ, η)-fine division of $[0, T]$ if $D'' = D_1 \cup D_2$, with $D_1 = \{([u_i, \xi_i], \xi_i)\}_{i=1}^n$ is a backwards (δ, η)-fine partial division of $[0, T]$ and $D_2 = \{([x_k, y_k], \tau_k)\}_{k=1}^r$ is a McShane δ-fine partial division of $[0, T]$ where $(D_2) \sum_{k=1}^{r} (y_k - x_k) \leq \eta$. Note that the existence of D'' as a (full) division is now immediate. Indeed, whenever D_1 fails to cover the interval $[0, T]$, we can always find a full McShane δ-fine division D_2 of $\overline{[0, T] \setminus \bigcup_{[u_i, \xi_i] \in D_1} [u_i, \xi_i]}$.

Let δ be a positive function on $[0, T]$ and $\eta > 0$ be given. A division Q is called a (δ, η)-fine partial division of $[0, T]$ if Q is a subset of some (δ, η)-fine division Q' of $[0, T]$.

Subsequently, we present definitions of the McShane integral and the backwards Henstock integral on $[0, T]$ together with some facts.

Definition 1.1. [4] *A function $f : [0, T] \to \mathbb{R}$ is said to be McShane integrable to $A \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists a positive function δ on $[0, T]$ such that for any McShane δ-fine division $P = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[0, T]$, we have*

$$\left| (P) \sum_{i=1}^{n} f(\xi_i)(v_i - u_i) - A \right| < \varepsilon.$$

In [4], it is a fact that f is McShane integrable on $[0, T]$ if and only if f it is Lebesgue integrable on $[0, T]$ and the values of both integrals are equal.

Definition 1.2. [1] *A function $f : [0, T] \to \mathbb{R}$ is said to be backwards Henstock integrable to $A \in \mathbb{R}$ if for every $\varepsilon > 0$, there exist a positive function δ on $[0, T]$ and a positive number η such that for any backwards (δ, η)-fine partial division $D = \{([u_i, \xi_i], \xi_i)\}_{i=1}^n$ of $[0, T]$, we have*

$$\left| (D) \sum_{i=1}^{n} f(\xi_i)(\xi_i - u_i) - A \right| < \varepsilon.$$

Theorem 1.3. [1] *A function f is backwards Henstock integrable on $[0, T]$ if and only if f is Lebesgue integrable on $[0, T]$ and the values of both integrals are equal.*

2. Main Results

In this section, we define the backwards McShane integral for real-valued function defined on $[0, T]$. We establish some properties of the backwards McShane integral and prove its equivalence to both the Lebesgue integral and the backwards Henstock integral.

Definition 2.1. *A function $f : [0, T] \to \mathbb{R}$ is said to be backwards McShane integrable to $A \in \mathbb{R}$ on $[0, T]$ if for every $\varepsilon > 0$, there exist a positive function δ on $[0, T]$ and a positive number η such that for any (δ, η)-fine division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[0, T]$, we have*

$$\left| (D) \sum_{i=1}^{n} f(\xi_i)(v_i - u_i) - A \right| < \varepsilon.$$

Note that A is unique; we denote it by $\int_0^T f(t)\,dt$.

We provide Theorem 2.2, which shows the linearity of the backwards McShane integral on $[0,T]$.

Theorem 2.2 (Linearity)**.** *Let $\alpha \in \mathbb{R}$. If f and g are backwards McShane integrable on $[0,T]$ then $f+g$ and αf are backwards McShane integrable on $[0,T]$ and*

i. $\int_0^T (f(t)+g(t))\,dt = \int_0^T f(t)\,dt + \int_0^T g(t)\,dt$;
ii. $\int_0^T \alpha f(t)\,dt = \alpha \int_0^T f(t)\,dt$.

In some cases, it is not necessary to know the integral value. Therefore, the Cauchy criterion in Theorem 2.3 can be applied to verify integrability.

Theorem 2.3 (Cauchy Criterion)**.** *A function f is backwards McShane integrable on $[0,T]$ if and only if for every $\varepsilon > 0$, there exist a positive function δ on $[0,T]$ and a positive number η such that for any two (δ,η)-fine divisions $D_1 = \{([u_i,v_i],\xi_i)\}_{i=1}^n$ and $D_2 = \{([w_j,z_j],\tau_j)\}_{j=1}^m$ on $[0,T]$, we have*

$$\left|(D_1)\sum_{i=1}^{n} f(\xi_i)(v_i-u_i) - (D_2)\sum_{j=1}^{m} f(\tau_j)(z_j-w_j)\right| < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given and let $\int_0^T f(t)\,dt = A$. Since f is backward McShane integrable on $[0,T]$ to $A \in \mathbb{R}$, then there exist a positive function δ on $[0,T]$ and a positive number η such that for any two (δ,η)-fine divisions $D_1 = \{([u_i,v_i],\xi_i)\}_{i=1}^n$ and $D_2 = \{([z_j,w_j],\tau_j)\}_{j=1}^m$ of $[0,T]$, we have

$$\left|(D_1)\sum_{i=1}^{n} f(\xi_i)(v_i-u_i) - A\right| < \frac{\varepsilon}{2}$$

and

$$\left|(D_2)\sum_{j=1}^{m} f(\tau_j)(z_j-w_j) - A\right| < \frac{\varepsilon}{2}.$$

Therefore,

$$\left|(D_1)\sum_{i=1}^{n} f(\xi_i)(v_i-u_i) - (D_2)\sum_{j=1}^{m} f(\tau_j)(z_j-w_j)\right| < \varepsilon.$$

Conversely, let $k \in \mathbb{N}$ be given. It follows that there exist a positive function δ_k on $[0,T]$ and a positive number η_k such that for any two (δ_k,η_k)-fine divisions $D' = \{([\vartheta_r,\beta_r],\lambda_r)\}_{r=1}^s$ and $D'' = \{([w_h,\theta_h],\rho_h)\}_{h=1}^q$ of $[0,T]$, we have

$$\left|(D')\sum_{r=1}^{s} f(\lambda_r)(\beta_r-\vartheta_r) - (D'')\sum_{h=1}^{q} f(\rho_h)(\theta_h-w_h)\right| < \frac{1}{k}.$$

Choose $\{\delta_p\}_{p\in\mathbb{N}}$ and $\{\eta_p\}_{p\in\mathbb{N}}$, which are decreasing sequences. Let D_p be a (δ_p,η_p)-fine division of $[0,T]$. For each $m \geq p$, $D_m = \{([b_{r^{(m)}},\psi_{r^{(m)}}],\varphi_{r^{(m)}})\}_{r^{(m)}=1}^{s^{(m)}}$ is also a (δ_p,η_p)-fine division of $[0,T]$. It follows that for any $m', m'' \geq p$ where $D_{m'} = \{([x_\varrho, \zeta_\varrho], \mu_\varrho)\}_{\varrho=1}^{c}$ and $D_{m''} = \{([y_g, \phi_g], \gamma_g)\}_{g=1}^{d}$ which are (δ_p, η_p)-fine divisions of $[0, T]$, we have

$$\left|(D_{m'})\sum_{\varrho=1}^{c} f(\mu_\varrho)(\zeta_\varrho - x_\varrho) - (D_{m''})\sum_{g=1}^{d} f(\gamma_g)(\phi_g - y_g)\right| < \frac{1}{p}.$$

Thus, a sequence

$$\left\{\sum_{r^{(m)}=1}^{s^{(m)}} f(\varphi_{r^{(m)}})(\psi_{r^{(m)}} - b_{r^{(m)}})\right\}_{m\in\mathbb{N}}$$

is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete then it converges to $A \in \mathbb{R}$. It means, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ with $N > \frac{2}{\varepsilon}$ such that for any $m \geq N$, we have

$$\left|\sum_{r^{(m)}=1}^{s^{(m)}} f(\varphi_{r^{(m)}})(\psi_{r^{(m)}} - b_{r^{(m)}}) - A\right| < \frac{\varepsilon}{2}.$$

Choose $\delta(\xi) = \delta_N(\xi)$ for all $\xi \in [0, T]$ and $\eta = \eta_N$. This implies for any (δ, η)-fine division $D = \{([z_r, y_r], p_r)\}_{r=1}^{s}$ on $[0, T]$, we have

$$\begin{aligned}\left|\sum_{r=1}^{s} f(p_r)(y_r - z_r) - A\right| &\leq \left|\sum_{r=1}^{s} f(p_r)(y_r - z_r) - \sum_{r^{(N)}=1}^{s^{(N)}} f(\varphi_{r^{(N)}})(\psi_{r^{(N)}} - b_{r^{(N)}})\right| \\ &\quad + \left|\sum_{r^{(N)}=1}^{s^{(N)}} f(\varphi_{r^{(N)}})(\psi_{r^{(N)}} - b_{r^{(N)}}) - A\right| \\ &< \frac{1}{N} + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

□

Theorem 2.4. *Let $0 \leq a \leq T$. If f is backwards McShane integrable on $[0, a]$ and $[a, T]$ then f is backwards McShane integrable on $[0, T]$ and*

$$\int_0^a f(t)\,dt + \int_a^T f(t)\,dt = \int_0^T f(t)\,dt.$$

Proof. Let $\varepsilon > 0$ be given. Let $\int_0^a f(t)\,dt = A_1$ and $\int_a^T f(t)\,dt = A_2$. Since f is backwards McShane integrable on $[0, a]$, it follows that there exist a positive function δ_1 on $[0, T]$ and a positive number η_1 such that for any (δ_1, η_1)-fine division $D_1 = \{([u_i, y_i], \varphi_i)\}_{i=1}^{n}$ of $[0, T]$, we have

$$\left|(D_1)\sum_{i=1}^{n} f(\varphi_i)(y_i - u_i) - A_1\right| < \frac{\varepsilon}{2}.$$

Since f is backwards McShane integrable on $[a, T]$ then there exist a positive function δ_2 on $[0, T]$ and a positive number η_2 such that for any (δ_2, η_2)-fine division $D_2 = \{([v_j, z_j], \tau_j)\}_{j=1}^{m}$ of $[a, T]$, we have

$$\left|(D_2)\sum_{j=1}^{m} f(\tau_j)(z_j - v_j) - A_2\right| < \frac{\varepsilon}{2}.$$

We define a positive function δ on $[0, T]$, as follows :

$$\delta(\xi) = \begin{cases} \min\{\delta_1(\xi), a - \xi\}, & \text{if } \xi \in [0, a), \\ \min\{\delta_1(\xi), \delta_2(\xi)\}, & \text{if } \xi = a, \\ \min\{\delta_2(\xi), \xi - a\}, & \text{if } \xi \in (a, T]. \end{cases}$$

and $\eta = \min\{\eta_1, \eta_2\}$.

Therefore, for any (δ, η)-fine division $D = \{([w_k, x_k], \xi_k)\}_{k=1}^{r}$ of $[0, T]$, we have

$$\left| (D) \sum_{k=1}^{r} f(\xi_k)(x_k - w_k) - (A_1 + A_2) \right| < \varepsilon.$$

□

Theorem 2.5. *If f is backwards McShane integrable on $[0, T]$ then f is backwards McShane integrable on any subinterval $[a, b]$ of $[0, T]$.*

Proof. Let $\varepsilon > 0$ and $a \in (0, T)$ be given. By Theorem 2.3, there exist a positive function δ on $[0, T]$ and a positive number η such that for any two (δ, η)-fine divisions $D' = \{([u_i, z_i], \xi_i)\}_{i=1}^{n}$ and $D'' = \{([w_j, x_j], \tau_j)\}_{j=1}^{m}$ of $[0, T]$, we have

$$\left| \sum_{i=1}^{n} f(\xi_i)(z_i - u_i) - \sum_{j=1}^{m} f(\tau_j)(x_j - w_j) \right| < \varepsilon.$$

If no confusion arises, we may write $(D) \sum$ instead of $(D) \sum_{i=1}^{n}$ when summing over a finite collection D.

Choose $\eta' = \frac{\eta}{2}$. Let δ_1 and δ_2 be the restriction of δ to $[0, a]$ and $[a, T]$, respectively. Let $D^{*}_{[0,a]}$ is (δ_1, η')-fine division of $[0, a]$. Let $D^{(**)}_{[a,T]}$ and $D^{(***)}_{[a,T]}$ be arbitrary two (δ_2, η')-fine divisions of $[a, T]$. Since $D^{*}_{[0,a]}$ is (δ_1, η')-fine division of $[0, a]$, then there exists a McShane δ_1-fine partial division $P^{*}_{[0,a]}$ of $[0, a]$ such that $(P^{*}_{[0,a]}) \sum (v - u) \leq \eta'$. Since $D^{(**)}_{[a,T]}$ and $D^{(***)}_{[a,T]}$ are (δ_2, η')-fine division of $[a, T]$, then there exist McShane δ_2-fine partial divisions $P^{(**)}_{[a,T]}$ and $P^{(***)}_{[a,T]}$ of $[a, T]$ such that $(P^{(**)}_{[a,T]}) \sum (v - u) \leq \eta'$ and $(P^{(***)}_{[a,T]}) \sum (v - u) \leq \eta'$. Hence,

$$(P^{*}_{[0,a]}) \sum (v - u) + (P^{(**)}_{[a,T]}) \sum (v - u) \leq 2\eta' = \eta,$$

and similarly,

$$(P^{*}_{[0,a]}) \sum (v - u) + (P^{(***)}_{[a,T]}) \sum (v - u) \leq 2\eta' = \eta.$$

It follows that $D^*_{[0,a]} \cup D^{(**)}_{[a,T]}$ and $D^*_{[0,a]} \cup D^{(***)}_{[a,T]}$ are (δ, η)-fine divisions of $[0, T]$. Therefore,

$$\begin{aligned}&\left|(D^{(**)}_{[a,T]}) \sum f(\xi)(v-u) - (D^{(***)}_{[a,T]}) \sum f(\xi)(v-u))\right| \\ &= \left|(D^*_{[0,a]} \cup D^{(**)}_{[a,T]}) \sum f(\xi)(v-u) - (D^*_{[0,a]} \cup D^{(***)}_{[a,T]}) \sum f(\xi)(v-u))\right| \\ &< \varepsilon.\end{aligned}$$

By Theorem 2.3, f is backwards McShane integrable on $[a, T]$. Using the same argument, one obtains that f is backwards McShane integrable on $[a, b]$ for all $b \in [a, T]$. □

We present a sequential definition of the backwards McShane integral on $[0, T]$, useful for future work on the stochastic integral.

Theorem 2.6 (Sequential Definition). *A function $f : [0, T] \to \mathbb{R}$ is backwards McShane integrable on $[0, T]$ if and only if there exist $A \in \mathbb{R}$, a decreasing sequence $\{\delta_n\}_{n\in\mathbb{N}}$ of positive functions on $[0, T]$, and a decreasing sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of positive numbers such that for any (δ_n, η_n)-fine division $D_n = \{([u_{i^{(n)}}, v_{i^{(n)}}], \xi_{i^{(n)}}\}_{i^{(n)}=1}^{m^{(n)}}$, we have*

$$\lim_{n\to\infty} \left|(D_n) \sum_{i^{(n)}=1}^{m^{(n)}} f(\xi_{i^{(n)}})(v_{i^{(n)}} - u_{i^{(n)}}) - A\right| = 0.$$

Proof. Let $\varepsilon > 0$ be given and let $\int_0^T f(t)\,dt = A$. Since f is backwards McShane integrable to $A \in \mathbb{R}$ then for any $n \in \mathbb{N}$, there exist a positive function δ_n on $[0, T]$ and a positive number η_n such that for any (δ_n, η_n)-fine division $D_n = \{([u_{i^{(n)}}, v_{i^{(n)}}], \xi_{i^{(n)}}\}_{i^{(n)}=1}^{m^{(n)}}$ of $[0, T]$, we have

$$\left|(D_n) \sum_{i^{(n)}=1}^{m^{(n)}} f(\xi_{i^{(n)}})(v_{i^{(n)}} - u_{i^{(n)}})\right| < \frac{1}{n}.$$

Choose a decreasing sequence $\{\delta_n(\xi)\}_{n\in\mathbb{N}}$ of positive functions on $[0, T]$ and a decreasing sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of positive numbers. It implies that

$$\lim_{n\to\infty} \left|(D_n) \sum_{i^{(n)}=1}^{m^{(n)}} f(\xi_{i^{(n)}})(v_{i^{(n)}} - u_{i^{(n)}}) - A\right| = 0.$$

Conversely, we assume that there exist $A \in \mathbb{R}$, a decreasing sequence $\{\delta_n\}_{n\in\mathbb{N}}$ of positive functions on $[0, T]$, and a decreasing sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of positive numbers such that for any (δ_n, η_n)-fine division $D_n = \{([u_{i^{(n)}}, v_{i^{(n)}}], \xi_{i^{(n)}}\}_{i^{(n)}=1}^{m^{(n)}}$ of $[0, T]$, we have

$$\lim_{n\to\infty} \left|(D_n) \sum_{i^{(n)}=1}^{m^{(n)}} f(\xi_{i^{(n)}})(v_{i^{(n)}} - u_{i^{(n)}}) - A\right| = 0.$$

Suppose f is not backwards McShane integrable on $[0,T]$. It means, there exists $\varepsilon > 0$ such that for any positive function δ on $[0,T]$ and positive number η, we have

$$\left|(D)\sum_{j=1}^{r} f(\tau_j)\big(z_j - w_j\big) - A\right| \geq \varepsilon,$$

for some (δ,η)-fine division $D = \{([w_j,z_j],\tau_j)\}_{j=1}^{r}$ of $[0,T]$. It implies that for any decreasing sequences $\{\delta_n(\xi)\}_{n\in\mathbb{N}}$ of positive functions on $[0,T]$ and decreasing sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of positive numbers, we have

$$\left|(D_n)\sum_{i^{(n)}=1}^{m^{(n)}} f(\xi_{i^{(n)}})(v_{i^{(n)}} - u_{i^{(n)}}) - A\right| \geq \varepsilon,$$

for some (δ_n,η_n)-fine division D_n of $[0,T]$. This contradicts the assumption. □

Let $f:[0,T]\to\mathbb{R}$ be a backwards McShane integrable function on $[0,T]$. By Theorem 2.5, f is backwards McShane integrable on $[t,T]$ for all $t\in[0,T]$. Hence, we can define a function $F:[0,T]\to\mathbb{R}$ by $F(t) = \int_t^T f(s)\,ds$ for $0\leq t\leq T$. The function F is called a primitive of f on $[0,T]$. This definition relies on a backwards integration structure, defined in a time-reversed manner— proceeding from a fixed terminal time T backward to the current time t, and its structure is reflected in the use of backwards δ-fine division. Although this reversed orientation is not yet essential in the deterministic setting, it anticipates the structure required in the stochastic framework, where time naturally flows backward from a terminal point. To maintain consistency with that formulation, the primitive is defined accordingly.

Now, from this point function F, one can construct an additive interval function, which is defined by $F(u,v) = F(u) - F(v)$.

Theorem 2.7 (Henstock's Lemma)**.** *If f is backwards McShane integrable on $[0,T]$ and $F(u,v) = \int_u^v f(t)\,dt$ for all $[u,v]\subseteq[0,T]$, then for every $\varepsilon>0$, there exist a positive function δ on $[0,T]$ and a positive number η such that for any (δ,η)-fine partial division $D = \{([u_i,v_i],\xi_i)\}_{i=1}^{n}$ of $[0,T]$, we have*

$$\left|(D)\sum_{i=1}^{n} f(\xi_i)(v_i - u_i) - (D)\sum_{i=1}^{n} F(u_i,v_i)\right| < \varepsilon.$$

Proof. Let $\varepsilon>0$ be given. Since f is backwards McShane integrable on $[0,T]$, then there exist a positive function δ on $[0,T]$ and a positive number η' such that for any (δ,η')-fine division $Q = \{([w_r,z_r],\xi_r)\}_{r=1}^{c}$ of $[0,T]$, we have

$$\left|(Q)\sum_{r=1}^{c} f(\xi_r)(z_r - w_r) - F(0,T)\right| < \frac{\varepsilon}{2}.$$

Choose $\eta = \frac{\eta'}{2}$. Let $D = \{([u_i,v_i],\xi_i)\}_{i=1}^{n}$ be an arbitrary (δ,η)-fine partial division of $[0,T]$, then there exists a (δ,η)-fine division D' of $[0,T]$ such that $D\subseteq D'$ and $D' = D_{\text{back}}\cup D_{\text{McS}}$ where $D_{\text{back}} = \{([x_l,\tau_l],\tau_l)\}_{l=1}^{q}$ is a backwards (δ,η)-fine partial division of $[0,T]$ and D_{McS} is a McShane δ-fine partial division of $[0,T]$ such that $T-(D_{\text{back}})\sum_{l=1}^{q}(\tau_l - x_l)\leq \eta$. We know that the set $\overline{[0,T]\setminus \cup_{i=1}^{n}[u_i,v_i]}$ consists of finitely many non-overlapping subintervals $[a_k,b_k]$ with $k=1,2,\ldots,m$. Since $[a_k,b_k]\subseteq [0,T]$, then by Theorem 2.5, f is backwards McShane integrable on $[a_k,b_k]$ for all k. It implies that for each k, there exist a positive function δ_k on $[a_k,b_k]$ and a positive number η_k such that for any (δ_k,η_k)-fine division $D_k=\{([u_{j^{(k)}},v_{j^{(k)}}],\xi_{j^{(k)}})\}_{j^{(k)}=1}^{r^{(k)}}$ of $[a_k,b_k]$ where $k=1,2,\ldots,m$, we have

$$\left|(D_k)\sum_{j^{(k)}=1}^{r^{(k)}} f(\xi_{j^{(k)}})(v_{j^{(k)}}-u_{j^{(k)}})-F(a_k,b_k)\right|<\frac{\varepsilon}{2m}.$$

Furthermore, choose $\{\delta_k(\xi)\}_{k=1}^{m}$ and $\{\eta_k\}_{k=1}^{m}$ such that $\delta_k(\xi)\leq\delta(\xi)$ for all $\xi\in[0,T]$ and all k, and $\sum_{k=1}^{m}\eta_k\leq\eta$. Since D_k is a (δ_k,η_k)-fine division of $[a_k,b_k]$ for all k, there exists a backwards (δ_k,η_k)-fine partial division $D_k^{\text{back}}=\{([w_{s^{(k)}},z_{s^{(k)}}],z_{s^{(k)}})\}_{s^{(k)}=1}^{p^{(k)}}$ of $[a_k,b_k]$. Let $P=D\cup D_1\cup D_2\cup\ldots\cup D_m=\{([u_{i'},v_{i'}],\xi_{i'})\}_{i'=1}^{n'}$ and $P_{\text{back}}=D''\cup D_1^{\text{back}}\cup D_2^{\text{back}}\cup\ldots\cup D_m^{\text{back}}=\{([y_h,\rho_h],\rho_h)\}_{h=1}^{e}$ where $D''=D\cap D_{\text{back}}$. It follows that P_{back} is a backwards δ-fine partial division of $[0,T]$. Moreover, since

$$\begin{aligned}
T-(P_{\text{back}})\sum_{h=1}^{e}(\rho_h-y_h) &= (D)\sum_{i=1}^{n}(v_i-u_i)+\sum_{k=1}^{m}(b_k-a_k)\\
&\quad -\sum_{k=1}^{m}(D_k^{\text{back}})\sum_{s^{(k)}=1}^{p^{(k)}}(z_{s(k)}-w_{s(k)})\\
&\quad -(D'')\sum_{([x_l,\tau_l],\tau_l)\in D_{\text{back}}}(\tau_l-x_l)\\
&\leq \sum_{k=1}^{m}\left((b_k-a_k)-(D_k^{\text{back}})\sum_{s^{(k)}=1}^{p^{(k)}}(z_{s(k)}-w_{s(k)})\right)+\eta\\
&= \eta'.
\end{aligned}$$

then P_{back} is a backwards (δ,η')-fine partial division of $[0,T]$. This implies that P is a (δ,η')-fine division of $[0,T]$ since $(\cup_{i=1}^{n}[u_i,v_i])\cup(\cup_{k=1}^{m}\cup_{j^{(k)}=1}^{r^{(k)}}[u_{j^{(k)}},v_{j^{(k)}}])=[0,T]$. Therefore,

$$\left|(D)\sum_{i=1}^{n}f(\xi_i)(v_i-u_i)-(D)\sum_{i=1}^{n}F(u_i,v_i)\right|<\varepsilon.$$

□

Next, we need to establish the absolute continuity of the primitive of a backwards McShane integrable function on $[0,T]$ in order to prove the equivalence between the backwards McShane integral and the backwards Henstock integral. We begin with the definition of absolute continuity for real-valued functions.

Definition 2.8. [4] *A function $F:[0,T]\to\mathbb{R}$ is said to be absolutely continuous on $[0,T]$ if for every $\varepsilon>0$, there exists a positive number η such that for any finite collection of non-overlapping subintervals*

$P = \{[x_i, y_i]\}_{i=1}^n$ *of* $[0, T]$, *where* $(P)\sum_{i=1}^n (y_i - x_i) \leq \eta$, *we have*

$$\left|(P)\sum_{i=1}^{n} F(x_i, y_i)\right| < \varepsilon.$$

The following result has an easier proof compared to [1] as it requires only refinement techniques which is an incremental methodological contribution. Here, no Vitali covering and careful management of uncovered intervals were applied.

Theorem 2.9. *If f is backwards McShane integrable on $[0, T]$ with the primitive F, then F is absolutely continuous on $[0, T]$.*

Proof. Let $\varepsilon > 0$ be given. By Henstock's Lemma (Theorem 2.7), there exist a positive function δ and a positive number η such that for any (δ, η)-fine partial division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[0, T]$, we have

$$\left|(D)\sum_{i=1}^{n} f(\xi_i)(v_i - u_i) - (D)\sum_{i=1}^{n} F(u_i, v_i)\right| < \frac{\varepsilon}{2}.$$

It implies that

$$\left|(D)\sum_{i=1}^{n} F(u_i, v_i)\right| < \frac{\varepsilon}{2} + \left|(D)\sum_{i=1}^{n} f(\xi_i)(v_i - u_i)\right|.$$

Furthermore, let $D' = \{([w_j, z_j], \tau_j)\}_{j=1}^m$ be a fixed (δ, η)-fine division of interval $[0, T]$. Let $K = \max_{1\leq j\leq m} |f(\tau_j)|$ (finite since the division has finitely many tags). We choose $\eta' = \frac{\varepsilon}{2K+1}$ and $\eta'' = \min\{\eta, \eta'\}$. Let $P = \{[x_k, y_k]\}_{k=1}^r$ be an arbitrary finite collection of non-overlapping subintervals of $[0, T]$ with $(P)\sum_{k=1}^r (y_k - x_k) \leq \eta''$. Let $\{[a_l, b_l]\}_{l=1}^s$ be the common refinement of $\{[w_j, z_j]\}_{j=1}^m$ and $\{[x_k, y_k]\}_{k=1}^r$ on $\cup_{k=1}^r [x_k, y_k]$. Thus, $\sum_{l=1}^s (b_l - a_l) \leq \eta''$. If $[a_l, b_l] \subseteq [w_j, z_j]$ then we select τ_j as an associate tag of $[a_l, b_l]$, denote it by ρ_l. Hence, we have a (δ, η'')-fine partial division $D'' = \{([a_l, b_l], \rho_l)\}_{l=1}^s$ of $[0, T]$ which is also a (δ, η)-fine partial division of $[0, T]$. Therefore,

$$\begin{aligned}
\left|(P)\sum_{k=1}^{r} F(x_k, y_k)\right| &= \left|(D'')\sum_{l=1}^{s} F(a_l, b_l)\right| \\
&< \frac{\varepsilon}{2} + \left|(D'')\sum_{l=1}^{s} f(\rho_l)(b_l - a_l)\right| \\
&\leq \frac{\varepsilon}{2} + K\left(\frac{\varepsilon}{2K+1}\right) \\
&< \varepsilon.
\end{aligned}$$

□

Theorem 2.10. *If f is backwards McShane integrable on $[0, T]$ then f is backwards Henstock integrable on $[0, T]$ and the values of both integrals are equal.*

Proof. Let $\varepsilon > 0$ be given and let F be the primitive of f. Since f is backwards McShane integrable on $[0, T]$, then by Henstock's Lemma (Theorem 2.7), there exist a positive function δ' on $[0, T]$ and a positive number η' such that for any (δ', η')-fine partial division $D' = \{([u_i, v_i], \xi_i)\}_{i=1}^{n}$ of $[0, T]$, we have

$$\left|(D')\sum_{i=1}^{n} f(\xi_i)(v_i - u_i) - (D')\sum_{i=1}^{n} F(u_i, v_i)\right| < \frac{\varepsilon}{2}. \tag{1}$$

By Theorem 2.9, F is absolutely continuous on $[0, T]$. It follows that there exists a positive number η'' such that for any finite collection of non-overlapping subintervals $\{([x_l, y_l]\}_{l=1}^{q}$ with $\sum_{l=1}^{q}(y_l - x_l) \leq \eta''$, we have

$$\left|\sum_{l=1}^{q} F(x_l, y_l)\right| < \frac{\varepsilon}{2}. \tag{2}$$

Choose $\delta(\xi) = \delta'(\xi)$ for all $\xi \in [0, T]$ and $\eta = \min\{\eta', \eta''\}$. Let $P = \{([w_j, \tau_j], \tau_j)\}_{j=1}^{m}$ be an arbitrary backwards (δ, η)-fine partial division of $[0, T]$. Hence, P is a (δ, η)-fine partial division of $[0, T]$ which is also (δ', η')-fine partial division of $[0, T]$. By (1), we have

$$\left|(P)\sum_{j=1}^{m} f(\tau_j)(\tau_j - w_j) - (P)\sum_{j=1}^{m} F(w_j, \tau_j)\right| < \frac{\varepsilon}{2}.$$

Let $Q = \{([a_k, b_k])\}_{k=1}^{r}$ be a finite collection of non-overlapping subintervals of $[0, T]$ such that $\bigcup_{k=1}^{r}[a_k, b_k] = \overline{[0, T] \setminus \bigcup_{j=1}^{m}[w_j, \tau_j]}$. We have

$$(Q)\sum_{k=1}^{r}(b_k - a_k) = T - (P)\sum_{j=1}^{m}(\tau_j - w_j) \leq \eta.$$

By (2), we have

$$\left|(Q)\sum_{k=1}^{r} F(a_k, b_k)\right| < \frac{\varepsilon}{2}.$$

Therefore,

$$\begin{aligned}\left|(P)\sum_{j=1}^{m} f(\tau_j)(\tau_j - w_j) - F(0, T)\right| &\leq \left|(P)\sum_{j=1}^{m} f(\tau_j)(\tau_j - w_j) - (P)\sum_{j=1}^{m} F(w_j, \tau_j)\right| \\ &\quad + \left|(Q)\sum_{k=1}^{r} F(a_k, b_k)\right| \\ &< \varepsilon.\end{aligned}$$

□

By combining Theorem 1.3 with Theorem 2.10, we arrive at the following result.

Corollary 2.11. *A function $f : [0, T] \to \mathbb{R}$ is backwards McShane integrable on $[0, T]$ then f is Lebesgue integrable on $[0, T]$ and the values of both integrals are equal.*

In Theorem 2.12, we show that every Lebesgue integrable function is backwards McShane integrable on $[0, T]$.

Theorem 2.12. *If f is Lebesgue integrable on $[0, T]$ then f is backwards McShane integrable on $[0, T]$ and the values of both integrals are equal.*

Proof. Let $\varepsilon > 0$ be given. Since f is Lebesgue integrable on $[0, T]$ then f is McShane integrable on $[0, T]$ (see [4]). Let $\int_0^T f(t)\,dt = A$, that means, there exists a positive function δ on $[0, T]$ such that for any McShane δ-fine division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[0, T]$, we have

$$\left|(D)\sum_{i=1}^{n} f(\xi_i)(v_i - u_i) - A\right| < \varepsilon.$$

Choose $\eta = \varepsilon$. For any (δ, η)-fine division $D' = \{([w_j, z_j], \tau_j)\}_{j=1}^m$ of $[0, T]$ which is also a McShane δ-fine division of $[0, T]$, we have

$$\left|(D')\sum_{j=1}^{m} f(\tau_j)(z_j - w_j) - A\right| < \varepsilon.$$

□

By Corollary 2.11 and Theorem 2.12, we obtain the Corollary 2.13.

Corollary 2.13. *A function $f : [0, T] \to \mathbb{R}$ is backwards McShane integrable on $[0, T]$ if and only if f is Lebesgue integrable on $[0, T]$ and the values of the two integrals are equal.*

Furthermore, combining Corollary 2.13 with Theorem 1.3, we establish Corollary 2.14

Corollary 2.14. *A function $f : [0, T] \to \mathbb{R}$ is backwards McShane integrable on $[0, T]$ if and only if f is backwards Henstock integrable on $[0, T]$ and the values of the two integrals are equal.*

3. Conclusions

In this paper, we have defined the backwards McShane integral for real-valued functions on $[0, T]$, using (δ, η)-fine divisions. We established some properties such as uniqueness, linearity, Cauchy criterion, additivity, integrability over subintervals, sequential definition, Henstock's Lemma, and the absolute continuity of the primitive. We have provided that the backwards McShane integral is equivalent to both the Lebesgue integral and the backwards Henstock integral. While this equivalence confirms that the class of integrable functions remains unchanged, the full division framework provides demonstrable advantages: the absolute continuity of the primitive admits an easier proof via refinement arguments and the direction from Lebesgue integrability to backwards McShane integrability simplifies considerably.

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