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MONOTONOUS SOLUTIONS AND MULTIPLE SOLUTIONS FOR SOME SECOND ORDER EQUATIONS WITH NEUMANN-STEKLOV BOUNDARY CONDITIONS AND $\varphi-$LAPLACIAN OPERATOR

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Abstract. We study the existence of solutions of equation

$$(\varphi(u'(s)))' = f(s, u(s), u'(s)), \quad a.e. \quad s \in [0, e]$$

submitted to nonlinear Neumann-Steklov boundary conditions on $[0, e]$ where $f : [0, e] \times \mathbb{R}^2 \to \mathbb{R}$ is a L^1-Carathéodory function. $\varphi : \mathbb{R} \to \mathbb{R}$, is an increasing homeomorphism such that $\varphi(0) = 0$. We show the existence of multiple solutions using some sign conditions.

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1. Introduction

The aim of this paper is to study the existence of solutions for differential equation

$$\begin{cases} (\varphi(u'(s)))' = f(s, u(s), u'(s)), \quad a.e. \quad s \in [0, e]; \\ \varphi(u'(0)) = g_0(u(0)), \\ \varphi(u'(e)) = g_1(u(e)), \end{cases} \tag{1}$$

where $0 < e$, $g_0 : \mathbb{R} \to \mathbb{R}$ and $g_1 : \mathbb{R} \to \mathbb{R}$ are two continuous functions, $f : [0, e] \times \mathbb{R}^2 \to \mathbb{R}$ is assumed to be L^1-Carathéodory function and $\varphi : \mathbb{R} \to \mathbb{R}$, is an increasing homeomorphism such that $\varphi(0) = 0$. The study of equation $(\varphi(u'(s)))' = f(s, u(s), u'(s)), \quad s \in [0, e]$ is a classical subject that has drawn the attention of numerous researchers due to its relevance in applications. Typically, a $\varphi-$Laplacian operator is called singular when the domain of φ is bounded (i.e., $\varphi :]-c, c[\to \mathbb{R}$ with $0 < c < +\infty$); conversely, it is termed regular. More recently, operators that are either singular or regular have also

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been investigated.

In this area, C. Bereanu and J. Mawhin in [1] and [2], have obtained existence and multiplicity results for equation (1) with different boundary conditions where φ is an increasing homeomorphism on a bounded interval $]-a, a[$ to $\mathbb{R}$ for $a > 0$.

In 2008, Cristian Bereanu and Jean Mawhin studied in [3] problem (1) where $\varphi :]-c, c[\to \mathbb{R}$, $(c \in]0, +\infty])$, is an increasing homeomorphism such that $\varphi(0) = 0$ and $f : [0, e] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous function.

In 2015 and 2017, Goli and Adjé in [4] and [6] extend Cristian Bereanu and Jean Mawhin results to $f : [0, e] \times \mathbb{R}^2 \to \mathbb{R}$, L^1-Carathéodory.

After introducing notations and preliminaries in section 2, using two sign conditions [7], we prove in section 3 that the problem (1) admits at least one solution whose derivative function has its image contained in a given segment $[a, b] \subset \mathbb{R}$.

In section 4, we apply the results of sections 3 to obtain existence of strictly increasing solution, and existence of strictly decreasing solution. The method allows us to find a monotonic solution when f and g_0 have the same sign or not.

In section 5 we apply the results of sections 3 to prove the existence of multiple solutions of the problem (1).

Generally, to prove existence of multiple solutions of a problem, one shows the existence of certain solutions and then compares the position of the image set of these solutions in $\mathbb{R}$ to assert that these solutions are different. In our method, after showing the existence of certain solutions, we compare the position of the image set of the derivatives of these solutions in $\mathbb{R}$ to assert that these solutions are different.

In our method, we only use sign conditions.

2. Notations and preliminaries

We denote:

∘ $C = C([0, e])$, the Banach space of continuous functions on $[0, e]$;

∘ $||u||_C = ||u||_\infty = \max\{|u(t)|; t \in [0, e]\}$, the norm of C;

∘ $C^1 = C^1([0, e])$, the Banach space of continuous functions on $[0, e]$ having continuous first derivative on $[0, e]$;

∘ $||u||_{C^1} = ||u||_C + ||u'||_C$, the norm of C^1;

∘ $AC = AC([0, e])$, the set of absolutely continuous functions on $[0, e]$;

∘ $L^1 = L^1(0, e)$, the Banach space of functions Lebesgue integrable on $[0, e]$;

∘ $||x||_{L^1} = \int_0^e |x(t)|dt$, the norm of L^1.

Definition.

$f:[0,e]\times\mathbb{R}^2\to\mathbb{R}$ is L^1-Carathéodory if:

(i): $f(.,x,y):[0,e]\to\mathbb{R}$ is measurable for all $(x,y)\in\mathbb{R}^2$;

(ii): $f(s,.,.):\mathbb{R}^2\to\mathbb{R}$ is continuous for a.e. $s\in[0,e]$;

(iii): For each compact set $A\subset\mathbb{R}^2$ there is a function $\mu_A\in L^1$ such that $|f(s,x,y)|\le\mu_A(s)$ for a.e. $s\in[0,e]$ and all $(x,y)\in A$.

Let us consider the problem

$$\begin{cases} (\phi(u'(s)))'=f(s,u(s),u'(s)), & a.e.\ s\in[0,e],\\ \phi(u'(0))=g_0(u(0)), & \\ \phi(u'(e))=g_1(u(e)) & \end{cases} \tag{2}$$

with $f:[0,e]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ a L^1-Carathéodory function, $g_0:\mathbb{R}\to\mathbb{R}$ and $g_1:\mathbb{R}\to\mathbb{R}$ two continuous functions and $\phi:]-\delta,\delta[\to\mathbb{R}$, $(\delta\in]0;+\infty[)$, an increasing homeomorphism such that $\varphi(0)=0$.

Definition.

A solution of problem (2) is a function $u\in C^1$ such that
$\phi(u')\in AC$, $\|u'\|_\infty<\delta$ and satisfies (2).

Theorem 2.1.

Assume that there exist $\rho>0$ and $\varepsilon\in\{-1,1\}$ such that

$$u_L\ge\rho \text{ and } \|u'\|_\infty<\delta\ \Rightarrow\ \varepsilon\Big\{\int_0^e f(s,u(s),u'(s))ds-(g_1(u(e))-g_0(u(0)))\Big\}>0 \tag{3}$$

and

$$u_M\le-\rho \text{ and } \|u'\|_\infty<\delta\ \Rightarrow\ \varepsilon\Big\{\int_0^e f(s,u(s),u'(s))ds-(g_1(u(e))-g_0(u(0)))\Big\}<0, \tag{4}$$

Then the problem (2) admits at least one solution u such that $\|u\|_\infty<\rho+\delta e$.

Proof. See [5]. □

3. Existence result

We will use Theorem 2.1 to prove a existence results when $\varphi:\mathbb{R}\to\mathbb{R}$.

Definition.

A solution of problem (1) is a function $u\in C^1$ such that $\varphi(u')\in AC$, and satisfies (1).

Let $\theta:\mathbb{R}\to\mathbb{R}$ given by $\theta(x)=\begin{cases} a & if \quad x<a\\ x & if \quad a\le x\le b\\ b & if \quad b<x.\end{cases}$

Theorem 3.1.

Let $a' > \max\{|a|, |b|\}$. Assume that:

(1) *There exist $\rho > 0$ and $\varepsilon \in \{-1, 1\}$ such that*

$$u_L \geq \rho \text{ and } \|u'\|_\infty < a' \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s, u(s), \theta(u'(s)))ds - (g_1(u(e)) - g_0(u(0)))\} > 0 \tag{5}$$

and

$$u_M \leq -\rho \text{ and } \|u'\|_\infty < a' \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s, u(s), \theta(u'(s)))ds - (g_1(u(e)) - g_0(u(0)))\} < 0. \tag{6}$$

(2) *There exists $h \in L^1$ such that, for a.e. $s \in [0, e]$, and all (u, v) with*

$$(s, u, v) \in \{(s, u, v) \in [0, e] \times \mathbb{R}^2, \quad -\rho - a'e < u < \rho + a'e, \quad a \leq v \leq b\},$$

$$f(s, u, v) \leq h(s);$$

(3) $\max_{\Delta} g_0 + \|h\|_{L^1} \leq \varphi(b)$ *and* $\min_{\Delta} g_1 - \|h\|_{L^1} \geq \varphi(a)$,
where $\Delta = [-\rho - a'e, \rho + a'e]$.

Then, the problem (1) admits at least one solution U, with

$$-\rho - a'e < U(s) < \rho + a'e \quad \text{and} \quad a \leq U'(s) \leq b, \ \forall s \in [0, e].$$

Proof.

We have three possible cases: $0 \in [a, b]$, $a > 0$ and $b < 0$.

Let $c_0 = \max_{[-\rho - a'e, \rho + a'e]} g_0$ and $c_T = \min_{[-\rho - a'e, \rho + a'e]} g_1$.

Case 1: $0 \in [a, b]$.

Let $k \in \mathbb{R}$ be such that $\varphi(0) + k = 0$. Let $\Lambda :]-a', a'[\rightarrow \mathbb{R}$ given by

$$\Lambda(x) = \begin{cases} \varphi(a) - \frac{1}{\sqrt{a'+x}} + \frac{1}{\sqrt{a'+a}} + k & if \quad -a' < x < a \\ \varphi(x) + k & if \quad a \leq x \leq b \\ \varphi(b) + \frac{1}{\sqrt{a'-x}} - \frac{1}{\sqrt{a'-b}} + k & if \quad b < x < a'. \end{cases}$$

Λ is an increasing homeomorphism such that $\Lambda(0) = 0$. Consider the functions $G_0 : \mathbb{R} \longrightarrow \mathbb{R}$ and $G_1 : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$G_0(x) = g_0(x) + k \qquad \text{and} \qquad G_1(x) = g_1(x) + k.$$

We introduce the problem

$$\begin{cases} (\Lambda(u'(s)))' = f_1(s, u(s), u'(s)), \ a.e. \ s \in [0, e], \\ \Lambda(u'(0)) = G_0(u(0)), \qquad \Lambda(u'(s)) = G_1(u(e)). \end{cases} \tag{7}$$

where $f_1(s,x,y) = f(s,x,\theta(y))$, for a.e. $s \in [0,e]$, and all $(u,v) \in \mathbb{R}\times\mathbb{R}$.
For $u \in C^1$, we have:

$$\int_0^e f_1(s,u(s),u'(s))ds - (G_1(u(e)) - G_0(u(0))) = \int_0^e f(s,u(s),\theta(u'(s)))ds - (g_1(u(e)) - g_0(u(0))).$$

Using (5) and (6), there exist $\rho > 0$ and $\varepsilon \in \{-1,1\}$ such that

$$u_L \geq \rho \text{ and } \|u'\|_\infty < a' \;\Rightarrow\; \varepsilon\Big\{\int_0^e f_1(s,u(s),u'(s))ds - (G_1(u(e)) - G_0(u(0)))\Big\} > 0$$

and

$$u_M \leq -\rho \text{ and } \|u'\|_\infty < a' \;\Rightarrow\; \varepsilon\Big\{\int_0^e f_1(s,u(s),u'(s))ds - (G_1(u(e)) - G_0(u(0)))\Big\} < 0.$$

By Theorem (2.1), the problem (7) admits at least one solution U, with

$$-\rho - a'e < U(s) < \rho + a'e, \qquad \forall s \in [0,e].$$

Therefore we have, $\forall s \in [0,e]$,

$$\begin{aligned}\Lambda(U'(s)) &= G_0(u(0)) + \int_0^s f(t,U(t),\theta(U'(t)))dt \\ &\leq c_0 + k + \int_0^s h(t)dt \\ &\leq c_0 + k + \|h\|_{L^1} \\ &\leq \varphi(b) + k,\end{aligned}$$

and

$$\begin{aligned}\Lambda(U'(s)) &= G_1(u(e)) - \int_s^e f(t,U(t),\theta(U'(t)))dt \\ &\geq c_T + k - \int_s^e h(t)dt \\ &\geq c_T + k - \|h\|_{L^1} \\ &\geq \varphi(a) + k.\end{aligned}$$

Hence,

$$\forall s \in [0,e], \qquad \Lambda(a) \leq \Lambda(U'(s)) \leq \Lambda(b).$$

Moreover

$$\forall s \in [0,e], \qquad a \leq U'(s) \leq b.$$

It follows that

$$\forall s \in [0,e], \qquad \Lambda(U'(s)) = \varphi(U'(s)) + k \qquad \text{and} \qquad \theta(U'(s)) = U'(s),$$

hence U is also a solution of problem (1).

Case 2: $a > 0$.

Let $k \in \mathbb{R}$ be such that $\varphi(a) + k > 0$.

Let $\Gamma :]-a', a'[\to \mathbb{R}$ given by

$$\Gamma(x) = \begin{cases} -\frac{1}{\sqrt{a'+x}} + \frac{1}{\sqrt{a'-b}} - \frac{b(\varphi(a)+k)}{a} & if \quad -a' < x < -b \\ \frac{(\varphi(a)+k)x}{a} & if \quad -b \leq x < a \\ \varphi(x) + k & if \quad a \leq x \leq b \\ \varphi(b) + \frac{1}{\sqrt{a'-x}} - \frac{1}{\sqrt{a'-b}} + k & if \quad b < x < a'. \end{cases}$$

Γ is an increasing homeomorphism such that $\Gamma(0) = 0$. Consider the functions $G_0 : \mathbb{R} \longrightarrow \mathbb{R}$ and $G_1 : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$G_0(x) = g_0(x) + k \qquad \text{and} \qquad G_1(x) = g_1(x) + k.$$

Consider the problem

$$\begin{aligned} &(\Gamma(u'(s)))' = f_1(s, u(s), u'(s)), \ a.e. \ s \in [0, e], \\ &\Gamma(u'(0)) = G_0(u(0)), \qquad \Gamma(u'(s)) = G_1(u(e)). \end{aligned} \tag{8}$$

where $f_1(s, x, y) = f(s, x, \theta(y))$, for a.e. $s \in [0, e]$, and all $(u, v) \in \mathbb{R} \times \mathbb{R}$.

For $u \in C^1$, we have:

$$\int_0^e f_1(s, u(s), u'(s))ds - (G_1(u(e)) - G_0(u(0))) = \int_0^e f(s, u(s), \theta(u'(s)))ds - (g_1(u(e)) - g_0(u(0))).$$

Using (5) and (6), there exist $\rho > 0$ and $\varepsilon \in \{-1, 1\}$ such that

$$u_L \geq \rho \text{ and } \|u'\|_\infty < a' \ \Rightarrow \ \varepsilon\Big\{\int_0^e f_1(s, u(s), u'(s))ds - (G_1(u(e)) - G_0(u(0)))\Big\} > 0$$

and

$$u_M \leq -\rho \text{ and } \|u'\|_\infty < a' \ \Rightarrow \ \varepsilon\Big\{\int_0^e f_1(s, u(s), u'(s))ds - (G_1(u(e)) - G_0(u(0)))\Big\} < 0.$$

By Theorem (2.1), the problem (8) admits at least one solution V, with

$$-\rho - a'e < V(s) < \rho + a'e, \qquad \forall s \in [0, e].$$

We have, $\forall s \in [0, e]$,

$$\begin{aligned} \Gamma(V'(s)) &= G_0(V(0)) + \int_0^s f(t, V(t), \theta(V'(t)))dt \\ &\leq c_0 + k + \int_0^s h(t)dt \\ &\leq c_0 + k + \|h\|_{L^1} \\ &\leq \varphi(b) + k, \end{aligned}$$

and

$$\begin{aligned}\Gamma(V'(s)) &= G_1(V(s)) - \int_s^e f(t, V(t), \theta(V'(t)))dt \\ &\geq c_T + k - \int_s^e h(t)dt \\ &\geq c_T + k - \|h\|_{L^1} \\ &\geq \varphi(a) + k.\end{aligned}$$

Hence,

$$\forall s \in [0, e], \qquad \Gamma(a) \leq \Gamma(V'(s)) \leq \Gamma(b).$$

Moreover

$$\forall s \in [0, e], \qquad a \leq V'(s) \leq b.$$

It follows that

$$\forall s \in [0, e], \quad \Gamma(V'(s)) = \varphi(V'(s)) + k \qquad \text{and} \qquad \theta(V'(s)) = V'(s),$$

hence V is also a solution of problem (1).

<u>*Case 3:* $b < 0$</u>.

Let $k \in \mathbb{R}$ be such that $\varphi(b) + k < 0$.

Let $\Psi :]-a', a'[\to \mathbb{R}$ given by

$$\Psi(x) = \begin{cases} \varphi(a) - \frac{1}{\sqrt{a'+x}} + \frac{1}{\sqrt{a'+a}} + k & if \quad -a' < x < a \\ \varphi(x) + k & if \quad a \leq x \leq b \\ \frac{(\varphi(b)+k)x}{b} & if \quad b < x \leq -a \\ \frac{1}{\sqrt{a'-x}} - \frac{1}{\sqrt{a'+a}} - \frac{(\varphi(b)+k)a}{b} & if \quad -a < x < a'. \end{cases}$$

Ψ is an increasing homeomorphism such that $\Psi(0) = 0$. Consider the functions $G_0 : \mathbb{R} \longrightarrow \mathbb{R}$ and $G_1 : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$G_0(x) = g_0(x) + k \qquad \text{and} \qquad G_1(x) = g_1(x) + k.$$

We introduce the problem

$$\begin{aligned} &(\Psi(u'(s)))' = f_1(s, u(s), u'(s)), \ a.e. \ s \in [0, e], \\ &\Psi(u'(0)) = G_0(u(0)), \qquad \Psi(u'(s)) = G_1(u(s)). \end{aligned} \tag{9}$$

where $f_1(s, x, y) = f(s, x, \theta(y))$, for a.e. $s \in [0, e]$, and all $(u, v) \in \mathbb{R} \times \mathbb{R}$.

For $u \in C^1$, we have:

$$\int_0^e f_1(s, u(s), u'(s))ds - (G_1(u(e)) - G_0(u(0))) = \int_0^e f(s, u(s), \theta(u'(s)))ds - (g_1(u(e)) - g_0(u(0))).$$

Using (5) and (6), there exist $\rho > 0$ and $\varepsilon \in \{-1, 1\}$ such that

$$u_L \geq \rho \text{ and } \|u'\|_\infty < a' \;\Rightarrow\; \varepsilon\Big\{ \int_0^e f_1(s, u(s), u'(s))ds - (G_1(u(e)) - G_0(u(0)))\Big\} > 0$$

and

$$u_M \leq -\rho \text{ and } \|u'\|_\infty < a' \;\Rightarrow\; \varepsilon\Big\{ \int_0^e f_1(s, u(s), u'(s))ds - (G_1(u(e)) - G_0(u(0)))\Big\} < 0.$$

By Theorem (2.1), the problem (7) admits at least one solution W, with

$$-\rho - a'e < W(s) < \rho + a'e, \qquad \forall s \in [0, e].$$

We have, $\forall s \in [0, e]$,

$$\begin{aligned} \Psi(W'(s)) &= G_0(W(0)) + \int_0^t f(t, W(t), \theta(W'(t)))dt \\ &\leq c_0 + k + \int_0^t h(t)dt \\ &\leq c_0 + k + \|h\|_{L^1} \\ &\leq \varphi(b) + k, \end{aligned}$$

and

$$\begin{aligned} \Psi(W'(s)) &= G_1(W(s)) - \int_t^e f(t, W(t), \theta(W'(t)))dt \\ &\geq c_T + k - \int_t^e h(t)dt \\ &\geq c_T + k - \|h\|_{L^1} \\ &\geq \varphi(a) + k. \end{aligned}$$

Hence,

$$\forall s \in [0, e], \qquad \Psi(a) \leq \Psi(W'(s)) \leq \Psi(b).$$

Moreover

$$\forall s \in [0, e], \qquad a \leq W'(s) \leq b.$$

It follows that

$$\forall s \in [0, e], \qquad \Psi(W'(s)) = \varphi(W'(s)) + k \qquad \text{and} \qquad \theta(W'(s)) = W'(s),$$

hence W is also a solution of problem (1). □

Corollary 3.2.

Let $a' > \max\{|a|, |b|\}$. Assume that:

(1) *f is continuous;*

(2) *f satisfying the condition*

$$\lim_{u\to-\infty} f(t,u,v) = -\infty \quad \textit{and} \quad \lim_{u\to+\infty} f(t,u,v) = +\infty \tag{10}$$

uniformly in $(t,v) \in [0,e] \times [-a',a']$;

(3) g_0, g_1 *bounded on* $\mathbb{R}$

(4) $\max g_0 + \max_{\Delta_1}|f| \le \varphi(b)$ *and* $\min g_1 - \max_{\Delta_1}|f| \ge \varphi(a)$,
where $\Delta_1 = [0,e] \times [-\rho - a'e, \rho + a'e] \times [-a',a']$ *and* ρ *such that*

$$f(t,u,v) - \frac{1}{e}(g_1(u) - g_0(u)) > 0 \qquad \forall (t,u,v) \in [0,e] \times [\rho; +\infty[\times[-a',a'], \textit{ and}$$

$$f(t,u,v) - \frac{1}{e}(g_1(u) - g_0(u)) < 0 \qquad \forall (t,u,v) \in [0,e]\times]-\infty,-\rho] \times [-a',a'].$$

Then, the problem (1) *admits at least one solution* U, *with*

$$-\rho - a'e < U(s) < \rho + a'e \quad \textit{and} \quad a \le U'(s) \le b, \ \forall s \in [0,e].$$

Corollary 3.3.
Let $a' > \max\{|a|,|b|\}$. *Assume that:*

(1) *f is continuous;*

(2) *f satisfying the condition*

$$\lim_{u\to-\infty} f(t,u,v) = +\infty \quad \textit{and} \quad \lim_{u\to+\infty} f(t,u,v) = -\infty. \tag{11}$$

uniformly in $(t,v) \in [0,e] \times [-a',a']$;

(3) g_0, g_1 *bounded on* $\mathbb{R}$

(4) $\max g_0 + \max_{\Delta_1}|f| \le \varphi(b)$ *and* $\min g_1 - \max_{\Delta_1}|f| \ge \varphi(a)$,
where $\Delta_1 = [0,e] \times [-\rho - a'e, \rho + a'e] \times [-a',a']$ *and* ρ *such that*

$$f(t,u,v) - \frac{1}{e}(g_1(u) - g_0(u)) < 0 \qquad \forall (t,u,v) \in [0,e] \times [\rho; +\infty[\times[-a',a'], \textit{ and}$$

$$f(t,u,v) - \frac{1}{e}(g_1(u) - g_0(u)) > 0 \qquad \forall (t,u,v) \in [0,e]\times]-\infty,-\rho] \times [-a',a'].$$

Then, the problem (1) *admits at least one solution* U, *with*

$$-\rho - a'e < U(s) < \rho + a'e \quad \textit{and} \quad a \le U'(s) \le b, \ \forall s \in [0,e].$$

Theorem 3.4.
Let $a' > \max\{|a|,|b|\}$. *Assume that:*

(1) *There exist* $\rho > 0$ *and* $\varepsilon \in \{-1,1\}$ *such that*

$$u_L \ge \rho \textit{ and } \|u'\|_\infty < a' \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s,u(s),\theta(u'(s)))ds - (g_1(u(e)) - g_0(u(0)))\} > 0 \tag{12}$$

and

$$u_M \leq -\rho \text{ and } \|u'\|_\infty < a' \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s,u(s),\theta(u'(s)))ds - (g_1(u(e)) - g_0(u(0)))\} < 0. \tag{13}$$

(2) *There exists* $h \in L^1$ *such that, for a.e.* $s \in [0,e]$*, and all* (u,v) *with*

$$(s,u,v) \in \{(s,u,v) \in [0,e] \times \mathbb{R}^2, \quad -\rho - a'e \leq u \leq \rho + a'e, \quad a \leq v \leq b\},$$

$$f(s,u,v) \geq h(s);$$

(3) $\max_\Delta g_1 + \|h\|_{L^1} \leq \varphi(b)$ *and* $\min_\Delta g_0 - \|h\|_{L^1} \geq \varphi(a)$,
where $\Delta = [-\rho - a'e, \rho + a'e]$.

Then, the problem (1) admits at least one solution U*, with*

$$-\rho - a'e \leq U(s) \leq \rho + a'e \quad \text{and} \quad a \leq U'(s) \leq b, \ \forall s \in [0,e].$$

Proof. The proof is similar to the proof Theorem 3.1. □

Corollary 3.5.

Let $a' > \max\{|a|,|b|\}$*. Assume that:*

(1) f *is continuous;*

(2) f *satisfying the condition*

$$\lim_{u\to-\infty} f(t,u,v) = -\infty \quad \text{and} \quad \lim_{u\to+\infty} f(t,u,v) = +\infty \tag{14}$$

uniformly in $(t,v) \in [0,e] \times [-a',a']$;

(3) g_0, g_1 *bounded on* $\mathbb{R}$

(4) $\max g_1 + \max_{\Delta_1}|f| \leq \varphi(b)$ *and* $\min g_0 - \max_{\Delta_1}|f| \geq \varphi(a)$,
where $\Delta_1 = [0,e] \times [-\rho - a'e, \rho + a'e] \times [-a',a']$ *and* ρ *such that*

$$f(t,u,v) - \frac{1}{e}(g_1(u) - g_0(u)) > 0 \qquad \forall (t,u,v) \in [0,e] \times [\rho;+\infty[\times[-a',a'], \text{ and}$$

$$f(t,u,v) - \frac{1}{e}(g_1(u) - g_0(u)) < 0 \qquad \forall (t,u,v) \in [0,e]\times]-\infty,-\rho] \times [-a',a'].$$

Then, the problem (1) admits at least one solution U*, with*

$$-\rho - a'e < U(s) < \rho + a'e \quad \text{and} \quad a \leq U'(s) \leq b, \ \forall s \in [0,e].$$

Corollary 3.6.

Let $a' > \max\{|a|,|b|\}$*. Assume that:*

(1) f *is continuous;*

(2) f *satisfying the condition*

$$\lim_{u\to-\infty} f(t,u,v) = +\infty \quad \text{and} \quad \lim_{u\to+\infty} f(t,u,v) = -\infty. \tag{15}$$

uniformly in $(t,v) \in [0,e] \times [-a',a']$;

(3) *g_0, g_1 bounded on $\mathbb{R}$*

(4) $\max g_1 + \max_{\Delta_1}|f| \le \varphi(b)$ *and* $\min g_0 - \max_{\Delta_1}|f| \ge \varphi(a)$,
where $\Delta_1 = [0,e] \times [-\rho - a'e, \rho + a'e] \times [-a', a']$ *and* ρ *such that*

$$f(t,u,v) - \frac{1}{e}(g_1(u) - g_0(u)) < 0 \qquad \forall (t,u,v) \in [0,e] \times [\rho; +\infty[\times [-a', a'], \text{ and}$$

$$f(t,u,v) - \frac{1}{e}(g_1(u) - g_0(u)) > 0 \qquad \forall (t,u,v) \in [0,e] \times]-\infty, -\rho] \times [-a', a'].$$

Then, the problem (1) admits at least one solution U, with

$$-\rho - a'e < U(s) < \rho + a'e \quad \text{and} \quad a \le U'(s) \le b, \ \forall s \in [0,e].$$

Example 3.7.

Consider the problem

$$((u'(s))^5)' = \frac{|u'(s)|}{4} + u(s) + \frac{s}{3}, \quad s \in [0,1],$$
$$(u'(0))^5 = \frac{1}{(u(0))^2 + 1} + 10 \quad \text{and} \quad (u'(1))^5 = \sin(u(1)) + 10$$

admits at least one strictly increasing solution U, with $1 \le U'(s) \le 2, \ \forall s \in [0,1]$.

Taking $a = 1$, $b = 2$ and $a' = 3$, we obtain:

$$\varphi(a) = 1^5 = 1, \ \varphi(b) = 2^5 = 32, \ \rho = 3, \ h(s) = \frac{s}{3} - \frac{23}{4}, \ \Delta = [-6, 6],$$

$$\max_{\Delta} g_1 + \|h\|_{L^1} = \frac{197}{12}, \ \min_{\Delta} g_0 - \|h\|_{L^1} = \frac{55}{12}.$$

From Theorem 3.4, we deduce the existence of a solution.

4. Existence of monotonic solutions

4.1. **Existence of strictly increasing solutions.**

Theorem 4.1.

Let $a' > b > 0$. Assume that:

(1) *There exist $\rho > 0$ and $\varepsilon \in \{-1, 1\}$ such that*

$$u_L \ge \rho \text{ and } \|u'\|_\infty < a' \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s,u(s),u'(s))ds - (g_1(u(e)) - g_0(u(0)))\} > 0 \tag{16}$$

and

$$u_M \le -\rho \text{ and } \|u'\|_\infty < a' \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s,u(s),u'(s))ds - (g_1(u(e)) - g_0(u(0)))\} < 0. \tag{17}$$

(2) *There exists $h \in L^1$ such that, for a.e. $s \in [0,e]$, and all (u,v) with*

$$(s,u,v) \in \{(s,u,v) \in [0,e] \times \mathbb{R}^2, \quad -\rho - a'e < u < \rho + a'e, \quad 0 \le v \le b\},$$

$$f(s,u,v) \le h(s);$$

(3) $\max_{\Delta} g_0 + \|h\|_{L^1} \leq \varphi(b)$ *and* $\min_{\Delta} g_1 - \|h\|_{L^1} > 0$,
where $\Delta = [-\rho - a'e, \rho + a'e]$.

Then, the problem (1) admits at least one strictly increasing solution U*, with*

$$-\rho - a'e < U(s) < \rho + a'e \ \forall s \in [0, e].$$

Proof.

We can take a in $\varphi^{-1}(]0, \min_{\Delta} g_1 - \|h\|_{L^1}])$ such that $a < b$ and use Theorem 3.1. □

Theorem 4.2.

Let $a' > b > 0$*. Assume that:*

(1) *There exist* $\rho > 0$ *and* $\varepsilon \in \{-1, 1\}$ *such that*

$$u_L \geq \rho \text{ and } \|u'\|_\infty < a' \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s, u(s), u'(s))ds - (g_1(u(e)) - g_0(u(0)))\} > 0 \tag{18}$$

and

$$u_M \leq -\rho \text{ and } \|u'\|_\infty < a' \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s, u(s), u'(s))ds - (g_1(u(e)) - g_0(u(0)))\} < 0. \tag{19}$$

(2) *There exists* $h \in L^1$ *such that, for a.e.* $s \in [0, e]$*, and all* (u,v) *with*

$$(s, u, v) \in \{(s, u, v) \in [0, e] \times \mathbb{R}^2, \quad -\rho - a'e \leq u \leq \rho + a'e, \quad 0 \leq v \leq b\},$$

$$f(s, u, v) \geq h(s);$$

(3) $\max_{\Delta} g_1 + \|h\|_{L^1} \leq \varphi(b)$ *and* $\min_{\Delta} g_0 - \|h\|_{L^1} > 0$,
where $\Delta = [-\rho - a'e, \rho + a'e]$.

Then, the problem (1) admits at least one strictly increasing solution U*, with*

$$-\rho - a'e \leq U(s) \leq \rho + a'e, \ \forall s \in [0, e].$$

Proof. The proof is similar to the proof Theorem 4.1. □

4.2. Existence of strictly decreasing solutions.

Theorem 4.3.

Let $a < 0$ *and* $a' > |a|$*. Assume that:*

(1) *There exist* $\rho > 0$ *and* $\varepsilon \in \{-1, 1\}$ *such that*

$$u_L \geq \rho \text{ and } \|u'\|_\infty < a' \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s, u(s), u'(s))ds - (g_1(u(e)) - g_0(u(0)))\} > 0 \tag{20}$$

and

$$u_M \leq -\rho \text{ and } \|u'\|_\infty < a' \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s, u(s), u'(s))ds - (g_1(u(e)) - g_0(u(0)))\} < 0. \tag{21}$$

(2) *There exists* $h \in L^1$ *such that, for a.e.* $s \in [0,e]$*, and all* (u,v) *with*

$$(s,u,v) \in \{(s,u,v) \in [0,e] \times \mathbb{R}^2, \quad -\rho - a'e < u < \rho + a'e, \quad a \leq v \leq 0\},$$

$$f(s,u,v) \leq h(s);$$

(3) $\max_{\Delta} g_0 + \|h\|_{L^1} < 0$ *and* $\min_{\Delta} g_1 - \|h\|_{L^1} \geq \varphi(a)$,
where $\Delta = [-\rho - a'e, \rho + a'e]$.

Then, the problem (1) *admits at least one strictly decreasing solution* U*, with*

$$-\rho - a'e < U(s) < \rho + a'e, \ \forall s \in [0,e].$$

Proof.
We can take b in $\varphi^{-1}([\max_{\Delta} g_0 + \|h\|_{L^1}, 0[)$ such that $a < b$ and use Theorem 3.1. □

Theorem 4.4.
Let $a < 0$ *and* $a' > |a|$*. Assume that:*

(1) *There exist* $\rho > 0$ *and* $\varepsilon \in \{-1,1\}$ *such that*

$$u_L \geq \rho \text{ and } \|u'\|_\infty < a' \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s,u(s),u'(s))ds - (g_1(u(e)) - g_0(u(0)))\} > 0 \tag{22}$$

and

$$u_M \leq -\rho \text{ and } \|u'\|_\infty < a' \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s,u(s),u'(s))ds - (g_1(u(e)) - g_0(u(0)))\} < 0. \tag{23}$$

(2) *There exists* $h \in L^1$ *such that, for a.e.* $s \in [0,e]$*, and all* (u,v) *with*

$$(s,u,v) \in \{(s,u,v) \in [0,e] \times \mathbb{R}^2, \quad -\rho - a'e \leq u \leq \rho + a'e, \quad a \leq v \leq 0\},$$

$$f(s,u,v) \geq h(s);$$

(3) $\max_{\Delta} g_1 + \|h\|_{L^1} < 0$ *and* $\min_{\Delta} g_0 - \|h\|_{L^1} \geq \varphi(a)$,
where $\Delta = [-\rho - a'e, \rho + a'e]$.

Then, the problem (1) *admits at least one strictly decreasing solution* U*, with*

$$-\rho - a'e \leq U(s) \leq \rho + a'e, \ \forall s \in [0,e].$$

Proof. The proof is similar to the proof Theorem 4.3. □

5. Existence of multiple solutions

Theorem 5.1.

Assume that:

(1) *There exist* $\{[a_i, b_i]\}_{1\leq i\leq n}$ *such that*
 (a) $\forall i \in \{1;\cdots;n\}$, $[a_i, b_i] \subset \mathbb{R}$ *and* $a_i < b_i$.
 (b) $\forall i \in \{1;\cdots;n-1\}$, $b_i < a_{i+1}$.

(2) $\forall i \in \{1;\cdots;n\}$, *there exist* $\rho_i > 0$ *and* $\varepsilon \in \{-1, 1\}$ *such that*

$$u_L \geq \rho_i \text{ and } \|u'\|_\infty < a'_i \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s, u(s), \theta_i(u'(s)))ds - (g_1(u(e)) - g_0(u(0)))\} > 0 \tag{24}$$

and

$$u_M \leq -\rho_i \text{ and } \|u'\|_\infty < a'_i \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s, u(s), \theta_i(u'(s)))ds - (g_1(u(e)) - g_0(u(0)))\} < 0; \tag{25}$$

Where $a'_i > \max\{|a_i|, |b_i|\}$ *and* $\theta_i : \mathbb{R} \to \mathbb{R}$ *given by*

$$\theta_i(x) = \begin{cases} a_i & if \quad x < a_i \\ x & if \quad a_i \leq x \leq b_i \\ b_i & if \quad b < x. \end{cases}$$

(3) $\forall i \in \{1;\cdots;n\}$, *there exists* $h_i \in L^1$ *such that, for a.e.* $s \in [0, e]$, *and all* (u,v) *with*

$$(s, u, v) \in \{(s, u, v) \in [0, e] \times \mathbb{R}^2, \quad -\rho_i - a'_i e < u < \rho_i + a'_i e, \quad a_i \leq v \leq b_i\},$$

$$f(s, u, v) \leq h_i(s);$$

(4) $\forall i \in \{1;\cdots;n\}$, $\max\limits_{\Delta_i} g_0 + \|h_i\|_{L^1} \leq \varphi(b_i)$ *and* $\min\limits_{\Delta_i} g_1 - \|h_i\|_{L^1} \geq \varphi(a_i)$, *where* $\Delta_i = [-\rho_i - a'_i e, \rho_i + a'_i e]$.

Then, the problem (1) *admits at least* n *solutions* $U_1, \cdots, U_n$,

$$\forall i \in \{1;\cdots;n\}, \; -\rho_i - a'_i e < U_i(s) < \rho_i + a'_i e \quad \text{and} \quad a_i \leq U'_i(s) \leq b_i, \; \forall s \in [0, e].$$

Proof.

For $i \in \{1;\cdots;n\}$, The assumptions of Theorem 3.1 are verified. Therefore $\forall i \in \{1;\cdots;n\}$, the problem (1) admits at least one solution U_i, such that

$$-\rho_i - a'_i e < U_i(s) < \rho_i + a'_i e \quad \text{and} \quad a_i \leq U'_i(s) \leq b_i, \; \forall s \in [0, e].$$

Using the fact that $\forall i \in \{1;\cdots;n\}$, $[a_i, b_i] \subset \mathbb{R}$ and $a_i < b_i$, and $\forall i \in \{1;\cdots;n-1\}$, $b_i < a_{i+1}$, we have

$$\forall (i, j) \in \{1;\cdots;n\}^2, \; U_i \neq U_j, \quad \text{for } i \neq j.$$

Hence, the problem (1) admits at least n solutions $U_1, \cdots, U_n$,

$$\forall i \in \{1;\cdots;n\}, \; -\rho_i - a'_i e < U_i(s) < \rho_i + a'_i e \quad \text{and} \quad a_i \leq U'_i(s) \leq b_i, \; \forall s \in [0, e].$$

□

Theorem 5.2.

Assume that:

(1) *There exist* $\{[a_i, b_i]\}_{1\leq i\leq n}$ *such that*
  (a) $\forall i \in \{1; \cdots ; n\}$, $[a_i, b_i] \subset \mathbb{R}$ *and* $a_i < b_i$.
  (b) $\forall i \in \{1; \cdots ; n-1\}$, $b_i < a_{i+1}$.

(2) $\forall i \in \{1; \cdots ; n\}$, *there exist* $\rho_i > 0$ *and* $\varepsilon \in \{-1, 1\}$ *such that*

$$u_L \geq \rho_i \text{ and } \|u'\|_\infty < a'_i \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s, u(s), \theta_i(u'(s)))ds - (g_1(u(e)) - g_0(u(0)))\} > 0 \tag{26}$$

and

$$u_M \leq -\rho_i \text{ and } \|u'\|_\infty < a'_i \quad \Rightarrow \quad \varepsilon\{\int_0^e f(s, u(s), \theta_i(u'(s)))ds - (g_1(u(e)) - g_0(u(0)))\} < 0; \tag{27}$$

Where $a'_i > \max\{|a_i|, |b_i|\}$ *and* $\theta_i : \mathbb{R} \to \mathbb{R}$ *given by*

$$\theta_i(x) = \begin{cases} a_i & if \quad x < a_i \\ x & if \quad a_i \leq x \leq b_i \\ b_i & if \quad b < x. \end{cases}$$

(3) $\forall i \in \{1; \cdots ; n\}$, *there exists* $h_i \in L^1$ *such that, for a.e.* $s \in [0, e]$, *and all* (*u*,*v*) *with*

$$(s, u, v) \in \{(s, u, v) \in [0, e] \times \mathbb{R}^2, \quad -\rho_i - a'_i e < u < \rho_i + a'_i e, \quad a_i \leq v \leq b_i\},$$

$$f(s, u, v) \geq h_i(s);$$

(4) $\forall i \in \{1; \cdots ; n\}$, $\max\limits_{\Delta_i} g_1 + \|h_i\|_{L^1} \leq \varphi(b_i)$ *and* $\min\limits_{\Delta_i} g_0 - \|h_i\|_{L^1} \geq \varphi(a_i)$, *where* $\Delta_i = [-\rho_i - a'_i e, \rho_i + a'_i e]$.

Then, the problem (1) *admits at least* n *solutions* $U_1, \cdots, U_n$,

$$\forall i \in \{1; \cdots ; n\}, \ -\rho_i - a'_i e < U_i(s) < \rho_i + a'_i e \quad \text{and} \quad a_i \leq U'_i(s) \leq b_i, \ \forall s \in [0, e].$$

Proof.

The proof is similar to the proof Theorem 5.1. □

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References

[1] C. Bereanu, J. Mawhin, Nonlinear Neumann Boundary-Value Problems With $\phi-$Laplacian Operators, An. St. Univ. Ovidius Constanta 12 (2004), 73–92.

[2] C. Bereanu, J. Mawhin, Existence and Multiplicity Results for Some Nonlinear Problems with Singular ϕ-Laplacian, J. Differ. Equ. 243 (2007), 536–557. https://doi.org/10.1016/j.jde.2007.05.014.

[3] C. Bereanu, J. Mawhin, Nonhomogeneous Boundary Value Problems for Some Nonlinear Equations with Singular ϕ-Laplacian, J. Math. Anal. Appl. 352 (2009), 218–233. https://doi.org/10.1016/j.jmaa.2008.04.025.

[4] C.E. Goli, A. Adje, Solvability for Some Boundary Value Problems With ϕ-Laplacian Operators and General Nonlinear Boundary Conditions, Far East J. Math. Sci. (FJMS) 98 (2015), 445–476. https://doi.org/10.17654/fjmsoct2015_445_476.

[5] K.C.E. Goli, A. Adje, New Existence Results for Some Periodic and Neumann-Steklov Boundary Value Problems with ϕ-Laplacian, Bound. Value Probl. 2016 (2016), 170. https://doi.org/10.1186/s13661-016-0676-6.

[6] C.E. Goli, A. Adje, Existence of Solutions of Some Nonlinear ϕ-Laplacian Equations With Neumann-Steklov Nonlinear Boundary Conditions, Afr. Diaspora J. Math. (N.S.) 20 (2017), 16–38.

[7] G. Villari, Soluzioni Periodiche Di Una Classe Di Equazioni Differenziali Del Terz? Ordine Quasi Lineari, Ann. Mat. Pura Appl. 73 (1966), 103–110. https://doi.org/10.1007/bf02415083.