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LAYER-SHIFT EQUITABLE COLORING OF PAN GRAPHS AND THEIR CARTESIAN PRODUCTS WITH PATHS: THEORY AND VISUALIZATION

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ABSTRACT. We determine the equitable chromatic number of the pan graph $C_n^{(+1)}$ and its Cartesian product with a path P_m. We prove that $\chi_=\left(C_n^{(+1)}\right) = 2$ when n is even and $\chi_=\left(C_n^{(+1)}\right) = 3$ when n is odd, and show that the same values hold for the Cartesian product $C_n^{(+1)}\square P_m$ for all $m \geq 2$. The proof uses a layer-shift construction that extends an equitable coloring along the path without increasing the number of colors. A second contribution of this work is an interactive visualization framework implemented in Python. The tool generates pan and pan–path graphs, applies the equitable coloring algorithm step by step, and produces both static snapshots and animations. This visualization supports verification of our results and enhances conceptual understanding of independence, equitability, and graph-product structure. The framework strengthens mathematics education through accessible computational exploration and directly supports *Sustainable Development Goal (SDG) 4: Quality Education*, particularly in advancing technology-enhanced learning in higher mathematics.

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Key words and phrases. equitable coloring; pan graph; path graph; Cartesian product; equitable chromatic number; graph visualization.

1. Introduction

Equitable graph coloring captures the notion of fairness in allocation: resources, workloads, or schedules should be shared almost equally while respecting independence constraints. This balance between structural restrictions and equitable distribution makes equitable coloring both mathematically rich and pedagogically relevant, connecting abstract combinatorial principles with applications in scheduling, load balancing, and visualization-enhanced mathematics education. Such connections also support broader educational development goals, particularly *Sustainable Development Goal (SDG) 4: Quality Education*, by promoting accessible and technology-supported learning tools in advanced mathematics.

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Equitable colorings have been fully characterized for several classical families of graphs—including paths, cycles, trees, and complete bipartite graphs—and for various graph products (see Lih [7] and Furmańczyk [5]). However, two important gaps remain. First, for the pendant–cycle or *pan graph* $C_n^{(+1)}$ (formed by attaching a single pendant vertex to a cycle C_n), no explicit closed formula for its equitable chromatic number $\chi_=(C_n^{(+1)})$ appears in the literature, even though the pan graph naturally lies between fans and wheels. Second, for Cartesian products with paths, existing results typically provide only upper bounds or require the path length m to be sufficiently large relative to the maximum degree $\Delta(G)$; a simple invariant formula that holds for all $m \geq 2$ has not yet been published.

This paper addresses both gaps. First, we establish a closed-form expression for the equitable chromatic number of the pan graph $C_n^{(+1)}$, showing that it equals 2 when n is even and 3 when n is odd, the latter demonstrated through a concise three-case $n \bmod 3$ argument. Second, we introduce a constructive *layer-shift coloring method* that extends an equitable coloring of the pan graph along a path and prove that this coloring remains equitable on the full Cartesian product $C_n^{(+1)} \square P_m$ for every $m \geq 2$. The result is strikingly simple: the product graph always requires the same number of colors as the base pan graph—two for even n and three for odd n—regardless of the path length.

To complement these theoretical contributions, we also develop an interactive Python-based visualization framework that dynamically illustrates equitable colorings on pan and pan–path graphs. This tool enhances verification and comprehension of equitable coloring behavior and aligns with SDG 4 by integrating computational visualization into the study of modern graph theory.

2. Preliminaries

We follow standard graph-theoretic notation and terminology throughout this paper. All graphs considered are simple, finite, and undirected. For a graph G, we denote its vertex set by $V(G)$, its edge set by $E(G)$, and the number of vertices by $|V(G)|$.

2.1. **Path, Pan, and Cartesian Product.**

We recall the definitions of paths, pan graphs, and Cartesian products of graphs following standard references such as Bondy and Murty [1], Diestel [4], Harary [6], and Varghese and Babu [10].

Definition 1. A *path graph* P_m is a graph with vertex set

$$V(P_m) = \{v_1, v_2, \ldots, v_m\}$$

and edge set

$$E(P_m) = \{v_i v_{i+1} \mid 1 \leq i \leq m-1\}.$$

Definition 2. A *pan graph* $C_n^{(+1)}$ is obtained from a cycle C_n by attaching exactly one pendant vertex to a single vertex of the cycle. Equivalently, if u is a vertex of C_n and x is a new vertex, then

$$V(C_n^{(+1)}) = V(C_n) \cup \{x\}, \quad E(C_n^{(+1)}) = E(C_n) \cup \{ux\}.$$

Definition 3. The *Cartesian product* of two graphs G and H, denoted by $G\Box H$, is the graph with vertex set

$$V(G\Box H) = V(G) \times V(H),$$

where two vertices (g, h) and (g', h') are adjacent if and only if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$.

2.2. **Equitable Coloring.** We follow the standard definitions of equitable coloring as presented in Chartrand and Zhang [2], Lih [7], and Meyer [8].

Definition 4. A *proper k-coloring* of a graph G is an assignment of colors (labels) from the set $\{1, 2, \ldots, k\}$ to the vertices of G such that no two adjacent vertices receive the same color.

Definition 5. A *color class* is the subset of vertices assigned the same color.

Definition 6. Let $G = (V, E)$ be a graph. A partition

$$\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$$

of $V(G)$ is called an *equitable chromatic set* if it satisfies the following conditions:

(1) Each subset $A_p \subseteq V(G)$ is an independent set; that is, no two vertices in A_p are adjacent (independence).
(2) The sizes of any two color classes differ by at most one (equitability), that is,

$$\big|\,|A_p| - |A_q|\,\big| \leq 1, \quad \text{for all } p, q \in \{1, 2, \ldots, k\}.$$

Definition 7. A graph G is said to be *equitably k-colorable* if it admits an equitable chromatic set with exactly k color classes. The smallest integer k for which such a coloring exists is called the *equitable chromatic number* of G, denoted by $\chi_=(G)$.

Remark 2.1. If a graph G contains at least one edge, then $\chi_=(G) \geq 2$. Indeed, the two endpoints of an edge must receive different colors in any proper coloring, and an equitable coloring is, in particular, proper.

Remark 2.2. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ be an equitable chromatic set of a graph G. Since $\mathcal{A}$ is a partition of $V(G)$, we have

$$\sum_{i=1}^{k} |A_i| = |V(G)|.$$

Consequently,

(1) $k \leq |V(G)|$, and

(2) the average size of a color class is $\frac{|V(G)|}{k}$.

Example 2.1. Consider the graph G of order 6 shown in Figure 1. Define two color classes by

$$A_1 = \{u_1, u_3, u_5\}, \qquad A_2 = \{u_2, u_4, u_6\}.$$

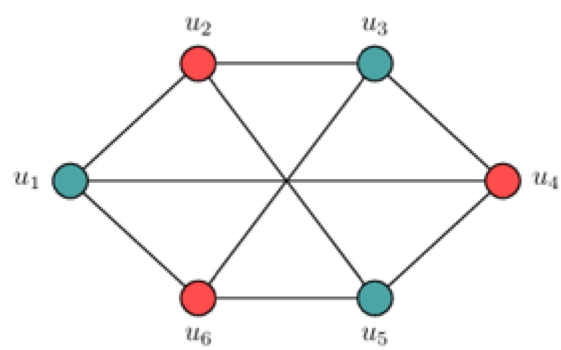


FIGURE 1. An equitable 2-coloring of the graph G.

We verify that the partition $\mathcal{A} = \{A_1, A_2\}$ is an equitable chromatic set.

Independence. Every edge of G joins a vertex in A_1 to a vertex in A_2. Hence, both A_1 and A_2 are independent sets.

Equitability. We have $|A_1| = |A_2| = 3$, so the sizes of the two color classes differ by 0 (and hence by at most 1).

Therefore, $\mathcal{A}$ is an equitable chromatic set, and G is equitably 2-colorable. Consequently,

$$\chi_{=}(G) = 2.$$

2.3. Layer-shift coloring on the Cartesian product $G\square P_m$.

Definition 8. Let G be any graph and let $\gamma : V(G) \to \{1, 2, \ldots, k\}$ be an equitable k-coloring of G. Let P_m be the path with ordered vertices $v_1, v_2, \ldots, v_m$. The *layer-shift coloring* of the Cartesian product $G\square P_m$ induced by γ is the map

$$\Gamma(u, v_j) = \big((\gamma(u) + j - 1) \bmod k\big) + 1, \qquad (u, v_j) \in V(G\square P_m). \tag{1}$$

Thus, each layer $G \times \{v_j\}$ uses the colors of G shifted cyclically by $j - 1$.

Remark 2.3. (Layer-shift preserves coloring properties). Let γ be a proper (respectively, equitable) k-coloring of a graph G. When the layer-shift rule (1) is applied to the Cartesian product $G\square P_m$, the resulting coloring Γ is again proper (respectively, equitable).

Indeed, within each layer $G \times \{v_j\}$ the shift is uniform, so adjacencies inside a layer inherit the propriety (or equitability) of γ. Between consecutive layers, corresponding vertices differ in color index by exactly 1 modulo k, so no monochromatic edge is created. Moreover, each color class in $G \square P_m$ receives the sizes of the base color classes in a cyclic pattern, so the global class sizes remain balanced up to ± 1.

3. Main Results

In this section, we present the main theoretical results of the paper. We first determine the equitable chromatic number of the pan graph $C_n^{(+1)}$, establishing an exact formula that depends only on the parity of n. We then extend this result to the Cartesian product $C_n^{(+1)} \square P_m$ by applying the layer-shift coloring introduced in Section 2.3. Remarkably, the equitable chromatic number of the product graph coincides with that of the base pan graph and is independent of the path length m.

3.1. Equitable Coloring of Pan Graphs $C_n^{(+1)}$.

Theorem 3.1. *Let $C_n^{(+1)}$ denote the pan graph obtained by attaching a single pendant vertex to the cycle C_n, where $n \geq 3$. Then*

$$\chi_{=}\left(C_n^{(+1)}\right) = \begin{cases} 2, & \text{if } n \text{ is even}, \\ 3, & \text{if } n \text{ is odd}. \end{cases}$$

Proof. Let $C_n^{(+1)}$ have vertex set

$$V = \{u_0, u_1, u_2, \dots, u_n\},$$

where u_0 is the pendant vertex adjacent to u_1, and $u_1 u_2 \cdots u_n u_1$ forms the cycle C_n.

Case 1. n is even. Define two color classes by

$$A_1 = \{u_i \mid i \text{ is odd}\}, \qquad A_2 = \{u_0\} \cup \{u_i \mid i \text{ is even}\}.$$

Independence. Any two vertices in A_1, as well as in A_2, are at distance at least 2 along the cycle, and u_0 is adjacent only to $u_1 \notin A_2$. Hence, no edge lies entirely within either class.

Equitability. We have

$$|A_1| = \frac{n}{2}, \qquad |A_2| = \frac{n}{2} + 1,$$

so the sizes of the two color classes differ by exactly 1. Thus, $\{A_1, A_2\}$ is an equitable 2-coloring, and

$$\chi_{=}\left(C_n^{(+1)}\right) \leq 2.$$

Since $C_n^{(+1)}$ contains at least one edge, $\chi_{=}\left(C_n^{(+1)}\right) \geq 2$, and therefore

$$\chi_{=}\left(C_n^{(+1)}\right) = 2.$$

Case 2. n **is odd.** Write $n = 3q + r$, where $r \in \{0, 1, 2\}$. Color the cycle vertices periodically by

$$u_{3i-2} \in A_1, \quad u_{3i-1} \in A_2, \quad u_{3i} \in A_3, \qquad 1 \le i \le q,$$

and assign the pendant vertex u_0 to A_2 or A_3 when $r = 0$, and A_3 when $r = 1, 2$.

If $r = 1$, assign the remaining vertex u_n to A_2. If $r = 2$, assign u_{n-1} to A_1 and u_n to A_2. In all cases, each color class is independent, and the sizes of the color classes differ by at most 1. Hence, an equitable 3-coloring exists, so

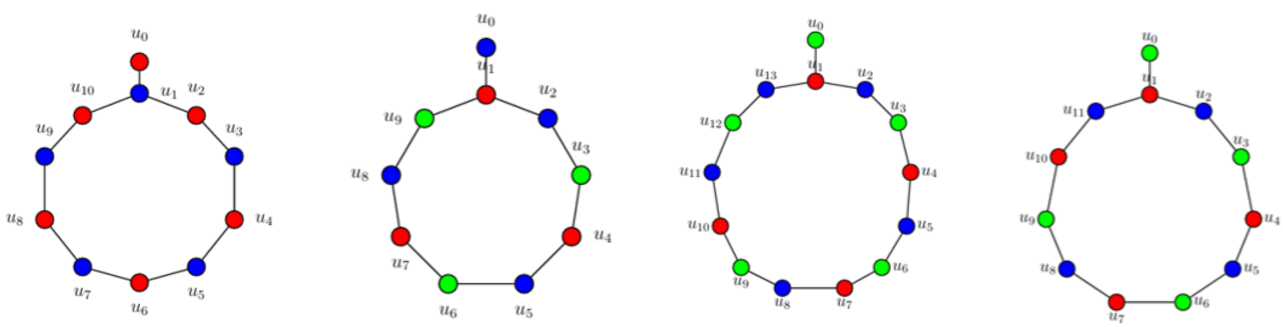
$$\chi_{=}\left(C_n^{(+1)}\right) \le 3.$$

To show that 2 colors are insufficient, suppose that an equitable 2-coloring exists. The pendant vertex u_0 must belong to one color class, say A_1, forcing its neighbor u_1 to belong to A_2. Proceeding around the odd cycle alternately assigns all even-indexed vertices to A_1 and all odd-indexed vertices to A_2. Consequently, both u_1 and u_n lie in A_2, contradicting independence since u_1u_n is an edge of the cycle. Thus, no equitable 2-coloring exists, and

$$\chi_{=}\left(C_n^{(+1)}\right) = 3.$$

Combining both cases completes the proof. □

Below are examples of an equitable 2-coloring of the pan graph $C_{10}^{(+1)}$, and equitable 3-colorings of the pan graphs $C_9^{(+1)}$, $C_{13}^{(+1)}$, and $C_{11}^{(+1)}$.



(A) $C_{10}^{(+1)}$ (B) $C_9^{(+1)}$ (C) $C_{13}^{(+1)}$ (D) $C_{11}^{(+1)}$

FIGURE 2. Equitable colorings of pan graphs: (a) an equitable 2-coloring of $C_{10}^{(+1)}$, and equitable 3-colorings of (b) $C_9^{(+1)}$, (c) $C_{13}^{(+1)}$, and (d) $C_{11}^{(+1)}$.

3.2. Equitable Coloring of the Cartesian Product $C_n^{(+1)} \square P_m$.

Theorem 3.2. *Let $C_n^{(+1)}$ be the pan graph with $n \ge 3$ and let P_m be the path graph with $m \ge 2$. Then*

$$\chi_{=}\left(C_n^{(+1)} \square P_m\right) = \begin{cases} 2, & \text{if } n \text{ is even}, \\ 3, & \text{if } n \text{ is odd}. \end{cases}$$

Proof. We establish matching lower and upper bounds.

Lower bound. Since $C_n^{(+1)}$ is an induced subgraph of $C_n^{(+1)} \square P_m$ (corresponding to any fixed layer), we have

$$\chi_=\left(C_n^{(+1)} \square P_m\right) \geq \chi_=\left(C_n^{(+1)}\right).$$

By Theorem 3.1, this gives a lower bound of 2 when n is even and 3 when n is odd.

Upper bound. Let $\gamma : V(C_n^{(+1)}) \to \{1, \dots, k\}$ be an equitable coloring of $C_n^{(+1)}$, where $k = \chi_=(C_n^{(+1)})$. Apply the layer-shift coloring defined in Section 2.3 to obtain a coloring

$$\Gamma(u, v_j) = \big((\gamma(u) + j - 1) \bmod k\big) + 1, \qquad (u, v_j) \in V(C_n^{(+1)} \square P_m).$$

By Remark 2.3, the coloring Γ is proper and equitable on $C_n^{(+1)} \square P_m$ and uses exactly k colors. Hence,

$$\chi_=\left(C_n^{(+1)} \square P_m\right) \leq k = \chi_=\left(C_n^{(+1)}\right).$$

Conclusion. Combining the lower and upper bounds yields

$$\chi_=\left(C_n^{(+1)} \square P_m\right) = \chi_=\left(C_n^{(+1)}\right),$$

which equals 2 when n is even and 3 when n is odd.

□

We now illustrate Theorem 3.2 with representative examples of Cartesian products of pan graphs and paths constructed using the layer-shift coloring.

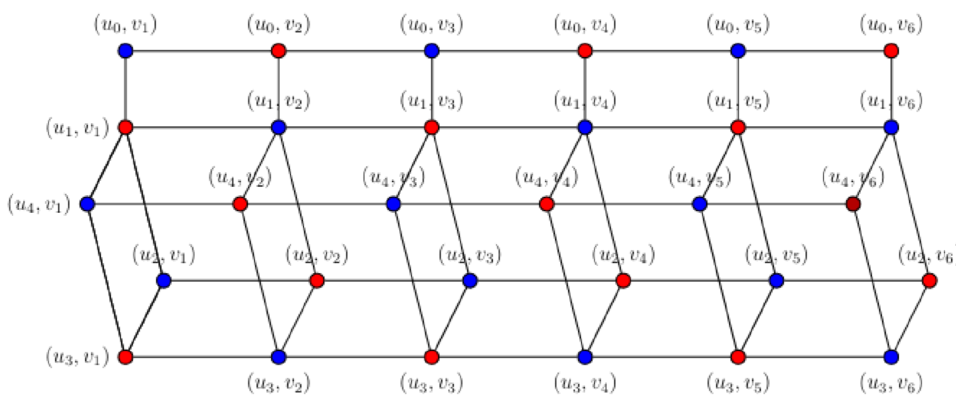


FIGURE 3. An equitable 2-coloring of the Cartesian product $C_4^{(+1)} \square P_6$ obtained via the layer-shift construction.

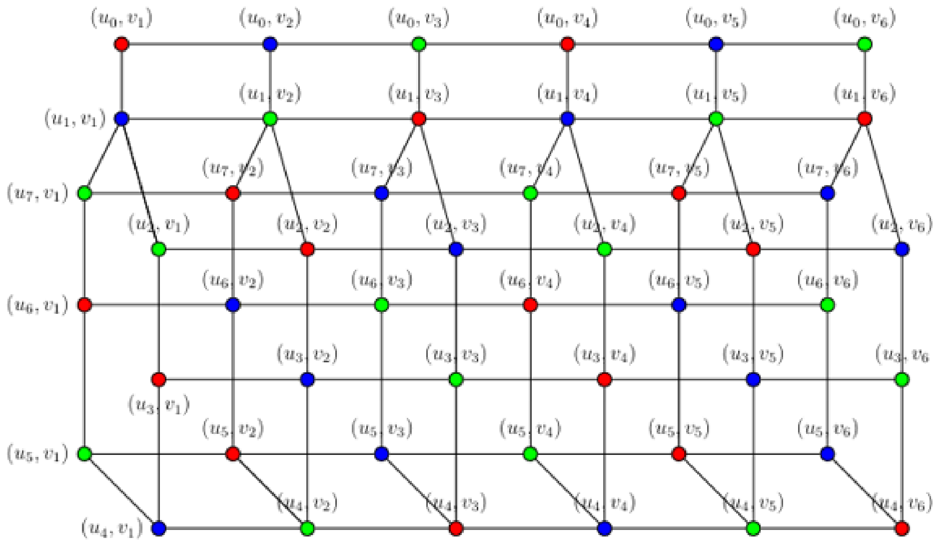


FIGURE 4. An equitable 3-coloring of the Cartesian product $C_7^{(+1)} \square P_6$ obtained via the layer-shift construction.

4. VISUALIZATION AND VERIFICATION

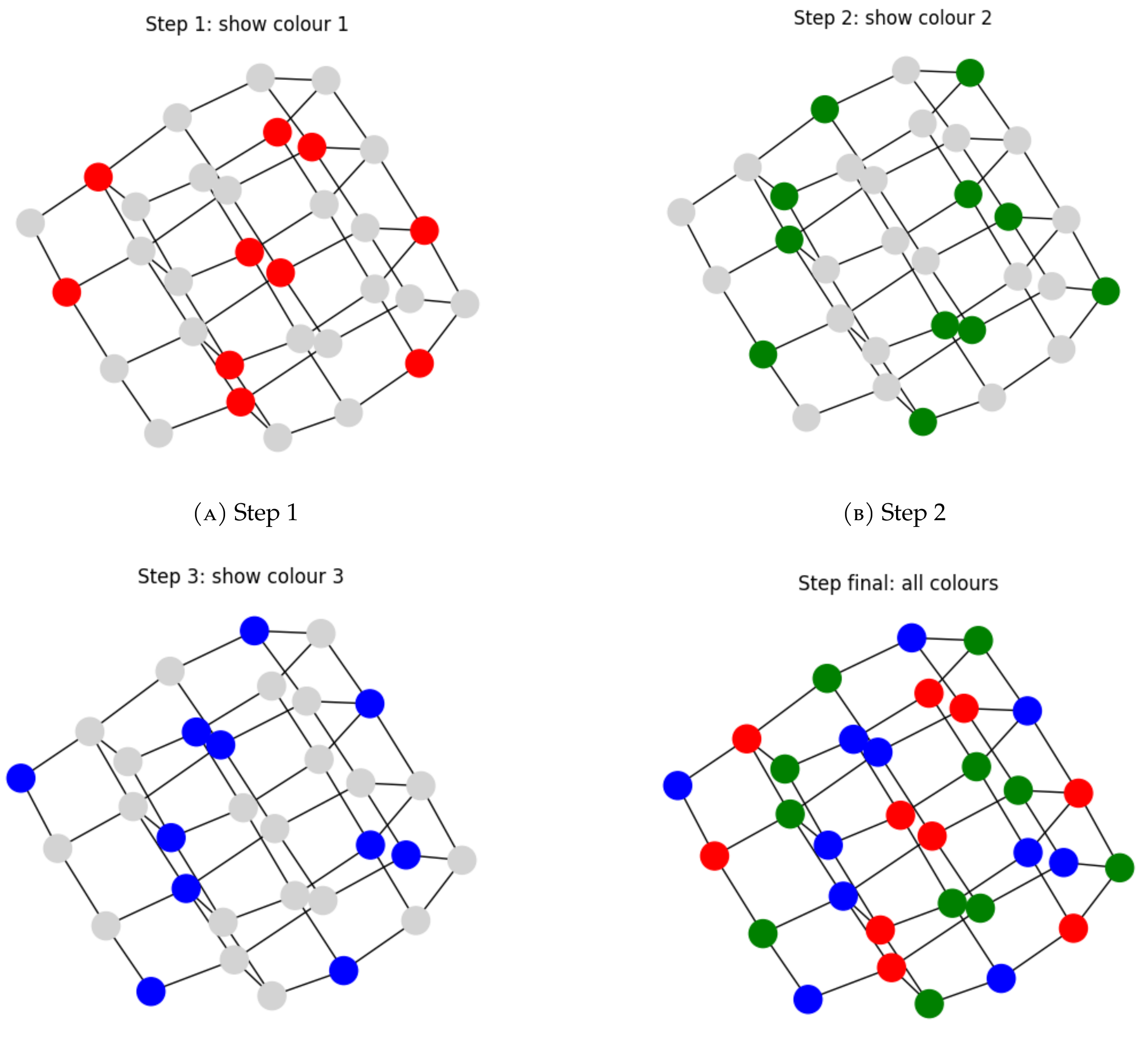
To illustrate and support the theoretical results established in Sections 3.1 and 3.2, we developed an open-source Jupyter notebook, `Equitable Visual.ipynb`, which provides a computational and visual implementation of equitable colorings for pan graphs and their Cartesian products with paths.

The notebook constructs pan graphs $C_n^{(+1)}$ and pan–path graphs $C_n^{(+1)} \square P_m$ and applies the layer-shift coloring described in Section 2.3. In particular, the visualization framework implements the following features:

- generation of pan and pan–path graph structures;
- step-by-step demonstration of the equitable coloring process;
- grouping of vertices by color class to highlight balanced allocation;
- interactive animations that illustrate the layer-shift mechanism across path layers;
- generation of static snapshots of coloring states for closer inspection and verification.

These visualizations serve a dual purpose. From a theoretical standpoint, they provide independent verification of the correctness and equitability of the proposed colorings. From a pedagogical perspective, they offer an intuitive representation of the layer-shift construction, making the equitable coloring of Cartesian products more accessible for instruction and further exploration.

Figure 5 shows a snapshot from the Jupyter notebook illustrating the equitable coloring of a pan–path graph $C_7^{(+1)} \square P_4$, where each color class has size 3.



(A) Step 1 (B) Step 2

(C) Step 3 (D) Step 4

FIGURE 5. Four-step frame coloring of the pan–path graph $C_7^{(+1)} \square P_4$ obtained via the layer-shift construction.

In addition to animations, the visualization tool produces console-style outputs and static image snapshots of the coloring state. These snapshots allow users to pause the dynamic process, examine the distribution of colors, and verify equitability by directly inspecting the sizes of the color classes. As a result, the visualization is useful not only for illustrating the theoretical results but also for independently confirming correctness across different parameter values.

The Jupyter notebook is organized into four main computational components:

- **Setup.** This component initializes the computational environment by importing the required Python packages, including `networkx`, `matplotlib`, and `ipywidgets`, and by defining consistent plotting styles.
- **Graph builders.** This component constructs pan graphs $C_n^{(+1)}$ and their Cartesian products with paths P_m.
- **Equitable coloring and visualization.** This component implements the equitable coloring of pan graphs, extends it to pan–path graphs using the layer-shift construction, and generates snapshot outputs for inspection and verification.
- **Animations.** This component provides interactive animations that demonstrate the four-step equitable coloring reveal.

Together, these visual and computational components enable learners to observe independence, equitability, and Cartesian product structure dynamically, thereby bridging the gap between theoretical graph coloring results and intuitive understanding.

5. Conclusion and Outlook

In this paper, we determined the equitable chromatic number of the pan graph $C_n^{(+1)}$, showing that it equals 2 when n is even and 3 when n is odd, and extended this result to the Cartesian product $C_n^{(+1)} \square P_m$. These results close gaps in the literature on equitable coloring by providing exact values in cases where only partial results or upper bounds were previously available.

In addition to the theoretical contributions, we developed an interactive visualization tool that illustrates the equitable coloring process and enables independent verification of equitability across different parameter values. By combining rigorous proofs with visualization-based verification, this work demonstrates how constructive methods and computational tools can complement each other in the study of equitable graph coloring.

Future work may include extending the layer-shift approach to other graph families and graph products, as well as enhancing the visualization framework to support broader classes of equitable coloring problems.

Code and Data Availability. The visualization tool and all scripts supporting this study are available as an open-source Jupyter notebook hosted on GitHub at `https://github.com/mflibao/Equitable-Theory-and-Visualization`. The repository includes sample animations (MP4/GIF) generated by the notebook, illustrating the four-step equitable coloring process. A static copy of the notebook and the generated snapshot images can also be provided as supplementary material upon request. No additional datasets were generated or analyzed in this study.

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Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed to this work in the following ways:

- M.F.L. contributed to the conceptualization and investigation, designed the methodology, assisted in the formal analysis, developed the visualization framework, validated the results, wrote the original draft of the manuscript, performed the review and editing of the manuscript.
- R.G.D.A. conceptualized the study, investigated, and performed the formal analysis.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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