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REPRESENTATION OF $\frac{1}{2}(L_n - 1)(L_{n+1} - 1)$ AND $\frac{1}{2}(L_n - 1)(L_{n+2} - 1)$

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ABSTRACT. Let L_n be the Lucas sequence: $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$ for $n \in \mathbb{Z}$. We obtain new identities involving Lucas numbers from the Diophantine equations $ax + by = (a-1)(b-1)/2$ and $ax + by + 1 = (a-1)(b-1)/2$ when $(a,b) = (L_n, L_{n+1})$ and (L_n, L_{n+2}) for $n \geq 0$.

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1. Introduction

The Lucas numbers L_n and the Fibonacci numbers F_n are defined by the recurrence relations $L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$ and $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for all $n \in \mathbb{Z}$, respectively. The two equations

$$ax + by = \frac{(a-1)(b-1)}{2} \quad \text{and} \quad ax + by + 1 = \frac{(a-1)(b-1)}{2}, \tag{1}$$

where a and b are relatively prime positive integers, arose from the study of cyclotomic polynomials [2]. Chu [4] used these two equations with $(a,b) = (F_n, F_{n+1})$ and (F_n, F_{n+2}) to obtain new identities involving Fibonacci numbers. Later, Davala proved new identities for balancing numbers [5]. Subsequently, Arachchi et al. established a criterion to check which equation is used for a given pair (a,b) and applied it to consecutive terms of various sequences [1]. More recently, Chen et al. [3] studied the cases $(a,b) = (F_n^2, F_{n+1}^2)$ and (F_n^3, F_{n+1}^3) and gave new identities involving squared and cubed Fibonacci numbers.

Continuing this line of work, we let $(a,b) = (L_n, L_{n+1})$ and (L_n, L_{n+2}) and prove new identities for Lucas numbers.

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2. Preliminaries

This section covers the necessary concepts including theorems and lemmata used in this study. The proofs can be found in [4,6,7]. The following theorem is due to Chu [4].

Theorem 2.1. *Let $a, b \in \mathbb{N}$ be relatively prime. Consider the following two equations*

$$ax + by = \frac{(a-1)(b-1)}{2}, \tag{2}$$

$$ax + by + 1 = \frac{(a-1)(b-1)}{2}. \tag{3}$$

Exactly one of the two equations has nonnegative integer solutions, and the solution is unique.

We also note the following properties of Lucas and Fibonacci numbers:

Property 2.2. *Let $n \in \mathbb{N}_0$. The following properties hold:*

(i) $\gcd(L_n, L_{n+1}) = 1$;

(ii) $\gcd(L_n, L_{n+2}) = 1$;

(iii) $L_{-n} = (-1)^n L_n$ *and* $F_{-n} = (-1)^{n+1} F_n$;

(iv) $F_{n+1} + F_{n-1} = L_n$;

(v) $L_{n+1} + L_{n-1} = 5F_n$;

(vi) $F_{m+n} + (-1)^n F_{m-n} = F_m L_n$;

(vii) $L_n L_{n+1} - (-1)^n = L_{2n+1}$;

$(viii)$ $L_n L_{n+2} - 3(-1)^n = L_{2n+2}$;

(ix) $2 \mid F_n$ *iff* $n \equiv 0 \pmod 3$, *and* $2 \mid L_n$ *iff* $n \equiv 0 \pmod 3$.

3. Main results

In this part, we investigate the nonnegative integer solutions to:

$$L_n x_n + L_{n+1} y_n = (L_n - 1)(L_{n+1} - 1)/2, \tag{4}$$

$$1 + L_n x_n + L_{n+1} y_n = (L_n - 1)(L_{n+1} - 1)/2, \tag{5}$$

where L_n are nth Lucas number. Furthermore, $\gcd(L_n, L_{n+1}) = 1$, and Theorem 2.1 guarantees that only one of these equations has a unique nonnegative integer solution. The first several cares are illustrated in the following table.

In Table 1, three patterns are observed. First, Equations (4) and (5) are used alternatively in period 3. Second, when $n \equiv 0, 1 \pmod 3$, then $x_n = y_{n+1}$. Lastly, when $n \equiv 2 \pmod 3$, then $x_n = x_{n+2} = y_{n+3}$. These patterns are summarized by the following theorem.

TABLE 1. Some Nonnegative Solutions of Equations (4) and (5)

n	L_n	L_{n+2}	Which equation	x_n	y_n
0	2	1	(4)	0	0
1	1	3	(4)	0	0
2	3	4	(4)	1	0
3	4	7	(5)	2	0
4	7	11	(5)	1	2
5	11	18	(5)	6	1
6	18	29	(4)	10	2
7	29	47	(4)	6	10
8	47	76	(4)	27	6
9	76	123	(5)	44	10
10	123	199	(5)	27	44
11	199	322	(5)	116	27
12	322	521	(4)	188	44
13	521	843	(4)	116	188
14	843	1364	(4)	493	116

Theorem 3.1. *For $k \geq 0$, we have the following identities:*

$$\left(\frac{L_{6k+3}+L_{6k+1}-5}{10}\right)L_{6k}+\left(\frac{L_{6k}+L_{6k-2}-5}{10}\right)L_{6k+1}=\frac{(L_{6k}-1)(L_{6k+1}-1)}{2} \tag{6}$$

$$\left(\frac{L_{6k+2}+L_{6k}-5}{10}\right)L_{6k+1}+\left(\frac{L_{6k+3}+L_{6k+1}-5}{10}\right)L_{6k+2}=\frac{(L_{6k+1}-1)(L_{6k+2}-1)}{2} \tag{7}$$

$$\left(\frac{L_{6k+5}+L_{6k+3}-5}{10}\right)L_{6k+2}+\left(\frac{L_{6k+2}+L_{6k}-5}{10}\right)L_{6k+3}=\frac{(L_{6k+2}-1)(L_{6k+3}-1)}{2} \tag{8}$$

$$1+\left(\frac{L_{6k+6}+L_{6k+4}-5}{10}\right)L_{6k+3}+\left(\frac{L_{6k+3}+L_{6k+1}-5}{10}\right)L_{6k+4}=\frac{(L_{6k+3}-1)(L_{6k+4}-1)}{2} \tag{9}$$

$$1+\left(\frac{L_{6k+5}+L_{6k+3}-5}{10}\right)L_{6k+4}+\left(\frac{L_{6k+6}+L_{6k+4}-5}{10}\right)L_{6k+5}=\frac{(L_{6k+4}-1)(L_{6k+5}-1)}{2} \tag{10}$$

$$1+\left(\frac{L_{6k+8}+L_{6k+6}-5}{10}\right)L_{6k+5}+\left(\frac{L_{6k+5}+L_{6k+3}-5}{10}\right)L_{6k+6}=\frac{(L_{6k+5}-1)(L_{6k+6}-1)}{2} \tag{11}$$

Proof. The proofs of Equations (6), (7), (8), (9), (10) and (11) are similar. We only prove Equations (6) and (9). We first prove Equation (6). We write

$$\begin{aligned}&\left(\frac{L_{6k+3}+L_{6k+1}-5}{10}\right)L_{6k}+\left(\frac{L_{6k}+L_{6k-2}-5}{10}\right)L_{6k+1}\\ =\ &\frac{1}{2}\left(F_{6k+2}-1\right)L_{6k}+\frac{1}{2}\left(F_{6k-1}-1\right)L_{6k+1} \qquad \text{due to Property 2.2}(v)\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\left(F_{6k+2}L_{6k} - L_{6k}\right) + \frac{1}{2}\left(F_{6k-1}L_{6k+1} - L_{6k+1}\right) \\
&= \frac{1}{2}\left(F_{6k+2}L_{6k} + F_{6k-1}L_{6k+1}\right) - \frac{1}{2}\left(L_{6k} + L_{6k+1}\right) \\
&= \frac{1}{2}\left(F_{12k+2} + F_2 + F_{12k} - F_{-2}\right) - \frac{1}{2}\left(L_{6k} + L_{6k+1}\right) \qquad \text{due to Property 2.2}(vi) \\
&= \frac{1}{2}\left(F_{12k+2} + F_{12k} + 2\right) - \frac{1}{2}\left(L_{6k} + L_{6k+1}\right) \qquad \text{due to Property 2.2}(iii) \\
&= \frac{1}{2}\left(L_{12k+1} + 2\right) - \frac{1}{2}\left(L_{6k} + L_{6k+1}\right) \qquad \text{due to Property 2.2}(iv) \\
&= \frac{1}{2}\left(L_{6k}L_{6k+1} + 1\right) - \frac{1}{2}\left(L_{6k} + L_{6k+1}\right) \qquad \text{due to Property 2.2}(vii) \\
&= \frac{1}{2}\left(L_{6k} - 1\right)\left(L_{6k+1} - 1\right)
\end{aligned}$$

This proves Equation (6).

Now, we proceed with Equation (9). The left side is

$$\begin{aligned}
& 1 + \left(\frac{L_{6k+6} + L_{6k+4} - 5}{10}\right) L_{6k+3} + \left(\frac{L_{6k+3} + L_{6k+1} - 5}{10}\right) L_{6k+4} \\
&= 1 + \frac{1}{2}\left(F_{6k+5} - 1\right) L_{6k+3} + \frac{1}{2}\left(F_{6k+2} - 1\right) L_{6k+4} \qquad \text{due to Property 2.2}(v) \\
&= 1 + \frac{1}{2}\left(F_{6k+5}L_{6k+3} - L_{6k+3}\right) + \frac{1}{2}\left(F_{6k+2}L_{6k+4} - L_{6k+4}\right) \\
&= 1 + \frac{1}{2}\left(F_{6k+5}L_{6k+3} + F_{6k+2}L_{6k+4}\right) - \frac{1}{2}\left(L_{6k+3} + L_{6k+4}\right) \\
&= 1 + \frac{1}{2}\left(F_{12k+8} - F_2 + F_{12k+6} + F_{-2}\right) - \frac{1}{2}\left(L_{6k+3} + L_{6k+4}\right) \qquad \text{due to Property 2.2}(vi) \\
&= 1 + \frac{1}{2}\left(F_{12k+8} + F_{12k+6} - 2\right) - \frac{1}{2}\left(L_{6k+3} + L_{6k+4}\right) \qquad \text{due to Property 2.2}(iii) \\
&= 1 + \frac{1}{2}\left(L_{12k+7} - 2\right) - \frac{1}{2}\left(L_{6k+3} + L_{6k+4}\right) \qquad \text{due to Property 2.2}(iv) \\
&= 1 + \frac{1}{2}\left(L_{6k+3}L_{6k+4} - 1\right) - \frac{1}{2}\left(L_{6k+3} + L_{6k+4}\right) \qquad \text{due to Property 2.2}(vii) \\
&= \frac{1}{2}\left(L_{6k+3} - 1\right)\left(L_{6k+4} - 1\right)
\end{aligned}$$

This proves Equation (9).

□

Remark 3.2. *By Property 2.2(v), we have the identity* $\frac{1}{10}(L_{n+1} + L_{n-1} - 5) = \frac{1}{2}(F_n - 1)$. *Using this, we can rewrite Theorem 3.1 as follows:*

$$\begin{aligned}
\frac{1}{2}\left(F_{6k+2} - 1\right) L_{6k} + \frac{1}{2}\left(F_{6k-1} - 1\right) L_{6k+1} &= \frac{1}{2}(L_{6k} - 1)(L_{6k+1} - 1); \\
\frac{1}{2}\left(F_{6k+1} - 1\right) L_{6k+1} + \frac{1}{2}\left(F_{6k+2} - 1\right) L_{6k+2} &= \frac{1}{2}(L_{6k+1} - 1)(L_{6k+2} - 1); \\
\frac{1}{2}\left(F_{6k+4} - 1\right) L_{6k+2} + \frac{1}{2}\left(F_{6k+1} - 1\right) L_{6k+3} &= \frac{1}{2}(L_{6k+2} - 1)(L_{6k+3} - 1);
\end{aligned}$$

$$1+\frac{1}{2}\left(F_{6k+5}-1\right)L_{6k+3}+\frac{1}{2}\left(F_{6k+2}-1\right)L_{6k+4}=\frac{1}{2}(L_{6k+3}-1)(L_{6k+4}-1);$$
$$1+\frac{1}{2}\left(F_{6k+4}-1\right)L_{6k+4}+\frac{1}{2}\left(F_{6k+5}-1\right)L_{6k+5}=\frac{1}{2}(L_{6k+4}-1)(L_{6k+5}-1);$$
$$1+\frac{1}{2}\left(F_{6k+7}-1\right)L_{6k+5}+\frac{1}{2}\left(F_{6k+4}-1\right)L_{6k+6}=\frac{1}{2}(L_{6k+5}-1)(L_{6k+6}-1).$$

Based on Property 2.2(ix), if $n \equiv 0 \pmod 3$, then F_n is even. However, we observe from the left-hand sides of the equations that F_n are odd. Thus, the expressions $(F_n - 1)$ are always divisible by 2. Furthermore, the right-hand sides are also divisible by 2, since $\gcd(L_n, L_{n+1}) = 1$, which implies that at least one of $(L_n - 1)$ or $(L_{n+1} - 1)$ is even.

In a similar manner, we now investigate the nonnegative integer solutions to the following equations:

$$L_n x_n + L_{n+2} y_n = (L_n - 1)(L_{n+2} - 1)/2, \tag{12}$$

$$1 + L_n x_n + L_{n+2} y_n = (L_n - 1)(L_{n+2} - 1)/2, \tag{13}$$

where L_n is the nth Lucas number. We observe that $\gcd(L_n, L_{n+2}) = 1$. Consequently, by Theorem 2.1, only one of these equations has a unique nonnegative integer solution. The following table provides the first several cases.

TABLE 2. Some Nonnegative Solutions of Equations (12) and (13)

n	L_n	L_{n+2}	Which equation	x_n	y_n
1	1	4	(12)	0	0
2	3	7	(12)	2	0
3	4	11	(12)	1	1
4	7	2	(13)	2	2
5	11	29	(13)	10	1
6	18	47	(13)	6	6
7	29	76	(12)	10	10
8	47	123	(12)	44	6
9	76	199	(12)	27	27
10	123	322	(13)	44	44
11	199	521	(13)	188	27
12	322	843	(13)	116	116
13	521	1364	(12)	188	188
14	843	2207	(12)	798	116
15	1364	3571	(12)	493	493

Three patterns are observed in Table 2. First, Equation (12) and Equation (13) are used alternatively in period 3. Second, the first and third rows of each period give $x_n = y_n$. Lastly, when $n \equiv 2 \pmod 3$, then $x_n = x_{n+2}$, and $y_n = y_{n-2}$ for $n \geq 4$. These patterns are summarized by the following theorem.

Theorem 3.3. *For $k \geq 0$, we have the following identities:*

$$\left(\frac{L_{6k+3}+L_{6k+1}-5}{10}\right)L_{6k+1}+\left(\frac{L_{6k+3}+L_{6k+1}-5}{10}\right)L_{6k+3}=\frac{(L_{6k+1}-1)(L_{6k+3}-1)}{2}; \quad (14)$$

$$\left(\frac{L_{6k+6}+L_{6k+4}-5}{10}\right)L_{6k+2}+\left(\frac{L_{6k+2}+L_{6k}-5}{10}\right)L_{6k+4}=\frac{(L_{6k+2}-1)(L_{6k+4}-1)}{2}; \quad (15)$$

$$\left(\frac{L_{6k+5}+L_{6k+3}-5}{10}\right)L_{6k+3}+\left(\frac{L_{6k+5}+L_{6k+3}-5}{10}\right)L_{6k+5}=\frac{(L_{6k+3}-1)(L_{6k+5}-1)}{2}; \quad (16)$$

$$1+\left(\frac{L_{6k+6}+L_{6k+4}-5}{10}\right)L_{6k+4}+\left(\frac{L_{6k+6}+L_{6k+4}-5}{10}\right)L_{6k+6}=\frac{(L_{6k+4}-1)(L_{6k+6}-1)}{2}; \quad (17)$$

$$1+\left(\frac{L_{6k+9}+L_{6k+7}-5}{10}\right)L_{6k+5}+\left(\frac{L_{6k+5}+L_{6k+3}-5}{10}\right)L_{6k+7}=\frac{(L_{6k+5}-1)(L_{6k+7}-1)}{2}; \quad (18)$$

$$1+\left(\frac{L_{6k+2}+L_{6k}-5}{10}\right)L_{6k}+\left(\frac{L_{6k+2}+L_{6k}-5}{10}\right)L_{6k+2}=\frac{(L_{6k}-1)(L_{6k+2}-1)}{2}. \quad (19)$$

Proof. The proofs of Equations (14), (15), (16), (17), (18) and (19) are similar. We only prove Equations (14) and (17). We first prove Equation (14). We write

$$\begin{aligned}
&\left(\frac{L_{6k+3}+L_{6k+1}-5}{10}\right)L_{6k+1}+\left(\frac{L_{6k+3}+L_{6k+1}-5}{10}\right)L_{6k+3}\\
=\;& \frac{1}{2}\left(F_{6k+2}-1\right)L_{6k+1}+\frac{1}{2}\left(F_{6k+2}-1\right)L_{6k+3} \qquad \text{due to Property 2.2}(v)\\
=\;& \frac{1}{2}\left(F_{6k+2}L_{6k+1}-L_{6k+1}\right)+\frac{1}{2}\left(F_{6k+2}L_{6k+3}-L_{6k+3}\right)\\
=\;& \frac{1}{2}\left(F_{6k+2}L_{6k+1}+F_{6k+2}L_{6k+3}\right)-\frac{1}{2}\left(L_{6k+1}+L_{6k+3}\right)\\
=\;& \frac{1}{2}\left(F_{12k+3}-F_{1}+F_{12k+5}-F_{-1}\right)-\frac{1}{2}\left(L_{6k+1}+L_{6k+3}\right) \qquad \text{due to Property 2.2}(vi)\\
=\;& \frac{1}{2}\left(F_{12k+3}+F_{12k+5}-2\right)-\frac{1}{2}\left(L_{6k+1}+L_{6k+3}\right) \qquad \text{due to Property 2.2}(iii)\\
=\;& \frac{1}{2}\left(L_{12k+4}-2\right)-\frac{1}{2}\left(L_{6k+1}+L_{6k+3}\right) \qquad \text{due to Property 2.2}(iv)\\
=\;& \frac{1}{2}\left(L_{6k+1}L_{6k+3}+1\right)-\frac{1}{2}\left(L_{6k+1}+L_{6k+3}\right) \qquad \text{due to Property 2.2}(viii)\\
=\;& \frac{1}{2}\left(L_{6k+1}-1\right)\left(L_{6k+3}-1\right)
\end{aligned}$$

This proves Equation (14). Now, we proceed with Equation (17). The left side is

$$\begin{aligned}
& 1+\left(\frac{L_{6k+6}+L_{6k+4}-5}{10}\right)L_{6k+4}+\left(\frac{L_{6k+6}+L_{6k+4}-5}{10}\right)L_{6k+6}\\
=\;& 1+\frac{1}{2}\left(F_{6k+5}-1\right)L_{6k+4}+\frac{1}{2}\left(F_{6k+5}-1\right)L_{6k+6} \qquad \text{due to Property 2.2}(v)\\
=\;& 1+\frac{1}{2}\left(F_{6k+5}L_{6k+4}-L_{6k+4}\right)+\frac{1}{2}\left(F_{6k+5}L_{6k+6}-L_{6k+6}\right)
\end{aligned}$$

$$
\begin{aligned}
&= 1+\frac{1}{2}\left(F_{6k+5}L_{6k+4}+F_{6k+5}L_{6k+6}\right)-\frac{1}{2}\left(L_{6k+4}+L_{6k+6}\right)\\
&= 1+\frac{1}{2}\left(F_{12k+9}+F_{1}+F_{12k+11}+F_{-1}\right)-\frac{1}{2}\left(L_{6k+4}+L_{6k+6}\right) \qquad \text{due to Property 2.2}(vi)\\
&= 1+\frac{1}{2}\left(F_{12k+9}+F_{12k+11}+2\right)-\frac{1}{2}\left(L_{6k+4}+L_{6k+6}\right) \qquad \text{due to Property 2.2}(iii)\\
&= 1+\frac{1}{2}\left(L_{12k+10}+2\right)-\frac{1}{2}\left(L_{6k+4}+L_{6k+6}\right) \qquad \text{due to Property 2.2}(iv)\\
&= 1+\frac{1}{2}\left(L_{6k+4}L_{6k+6}-1\right)-\frac{1}{2}\left(L_{6k+4}+L_{6k+6}\right) \qquad \text{due to Property 2.2}(viii)\\
&= \frac{1}{2}\left(L_{6k+4}-1\right)\left(L_{6k+6}-1\right)
\end{aligned}
$$

This proves Equation (17). □

Remark 3.4. *Again, by Property 2.2(v), we have the identity $\frac{1}{10}(L_{n+1}+L_{n-1}-5)=\frac{1}{2}(F_n-1)$. Using this, we can rewrite Theorem 3.3 as follows:*

$$
\begin{aligned}
\frac{1}{2}\left(F_{6k+2}-1\right)L_{6k+1}+\frac{1}{2}\left(F_{6k+2}-1\right)L_{6k+3} &= \frac{1}{2}(L_{6k+1}-1)(L_{6k+3}-1);\\
\frac{1}{2}\left(F_{6k+5}-1\right)L_{6k+2}+\frac{1}{2}\left(L_{6k+1}-1\right)L_{6k+4} &= \frac{1}{2}(L_{6k+2}-1)(L_{6k+4}-1);\\
\frac{1}{2}\left(F_{6k+4}-1\right)L_{6k+3}+\frac{1}{2}\left(F_{6k+4}-1\right)L_{6k+5} &= \frac{1}{2}(L_{6k+3}-1)(L_{6k+5}-1);\\
1+\frac{1}{2}\left(F_{6k+5}-1\right)L_{6k+4}+\frac{1}{2}\left(F_{6k+5}-1\right)L_{6k+6} &= \frac{1}{2}(L_{6k+4}-1)(L_{6k+6}-1);\\
1+\frac{1}{2}\left(F_{6k+8}-1\right)L_{6k+5}+\frac{1}{2}\left(F_{6k+4}-1\right)L_{6k+7} &= \frac{1}{2}(L_{6k+5}-1)(L_{6k+7}-1);\\
1+\frac{1}{2}\left(F_{6k+1}-1\right)L_{6k}+\frac{1}{2}\left(F_{6k+1}-1\right)L_{6k+2} &= \frac{1}{2}(L_{6k}-1)(L_{6k+2}-1).
\end{aligned}
$$

Based on Property 2.2(ix), if $n \equiv 0 \pmod 3$, then F_n is even. However, we observe from the left-hand sides of the equations that F_n are odd. Thus, the expressions (F_n-1) are always divisible by 2. Furthermore, the right-hand sides are also divisible by 2, since $\gcd(L_n, L_{n+2})=1$, which implies that at least one of (L_n-1) or $(L_{n+2}-1)$ is even.

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