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ALGEBRAS OF TERMS DEFINED BY TRANSFORMATIONS PRESERVING A LENGTH

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Abstract. This paper discusses the concept of algebras of terms defined by transformations that preserve a length on a finite set. The study introduces various types and forms of term algebras and explores transformations that adhere to specific length preservation criteria. The results extend existing theories on algebraic structures and hypersubstitutions.

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1. Introduction

Let $X := \{x_1, x_2, \ldots\}$ be a countably infinite set of symbols called *variables* and let $X_n := \{x_1, x_2, \ldots, x_n\}$. Let $(f_i)_{i\in I}$ be an indexed set which is disjoint from X. Each f_i is called an n_i*-ary operation symbol*, where $1 \leq n_i \leq n$ is a natural number. Let τ be a function which assigns to every f_i the number n_i as its arity. The sequence of the values of function τ, written as $(n_i)_{i\in I}$, is called a *type*. An n*-ary term of type* τ is defined inductively as follows: (i) Every variable $x_j \in X_n$ is an n-ary term of type τ. (ii) The composition $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ where $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol. The set of all n-ary terms of type τ, closed under finite number of applications of (ii), is denoted by $W_\tau(X_n)$. The symbol $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ stands for the set of all terms of type τ.

The set of all terms of type τ can be used as the universe of an algebra of type τ. For every $i \in I$, an n_i-ary operation $\bar{f}_i : W_\tau(X)^{n_i} \longrightarrow W_\tau(X)$ is defined by

$$\bar{f}_i\left(t_1, \ldots, t_{n_i}\right) := f_i\left(t_1, \ldots, t_{n_i}\right).$$

The algebra $\mathcal{F}_\tau(X) := (W_\tau(X); (\bar{f}_i)_{i\in I})$ is called the *absolutely free algebra* of type τ over the set X.

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There is another way to consider the operation on the set of terms. Now, we recall the concept of superposition operation of terms. For each natural numbers $m, n \geq 1$, the superposition operation is a many-sorted mapping

$$S^n_m : W_\tau(X_n) \times (W_\tau(X_m))^n \to W_\tau(X_m)$$

defined by

(i) $S^n_m(x_j, t_1, \ldots, t_n) := t_j$, if $x_j \in X_n$,

(ii) $S^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) := f_i(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n))$.

Then the many-sorted algebra can be defined by

$$\text{clone } \tau := ((W_\tau(X_n))_{n\in\mathbb{N}^+}; (S^n_m)_{n,m\in\mathbb{N}^+}, (x_i)_{i\leq n\in\mathbb{N}^+}),$$

which is called *the clone of all terms of type* τ.

Let $\tau_n = (n, n, \ldots, n)$ be a type consisting of the same values equal to n, i.e. $\tau_n = (n_i)$ with $n_i = n$ for all $i \in I$. In 2004, Denecke and Jampachon [2] inductively defined n-ary full terms of type τ_n, based on the full transformations (mappings) instead of the permutations, as follows:

(i) $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an n-ary full term of type τ_n if f_i is an n-ary operation symbol and $\alpha \in T_n$ where T_n is the set of all full transformation on $\{1, 2, \ldots, n\}$;

(ii) $f_i\,(t_1, \ldots, t_n)$ is an n-ary full term of type τ_n if f_i is an n-ary operation symbol and $t_1, \ldots, t_n$ are n-ary full terms of type τ_n.

The set of all n-ary full terms of type τ_n, closed under finite application of (ii), is denoted by $W^F_{\tau_n}(X_n)$. Refer to [3,5–7,9,10] for more details about terms, full terms and thier properties. This paper aims to apply the notion of transformations preserving a length to define the set of terms and construct the algebra satisfying some laws. Moreover, we apply our results to the theory of hyperidentities and solid varieties.

2. THE ALGEBRA OF $T_n(l)$-FULL TERMS

Let X be a nonempty set and let $T(X)$ denote the semigroup of the full transformations from X into itself under composition of mappings. Throughout the paper, we let $\bar{n} := \{1, 2, \ldots, n\}$ $(n \geq 2)$ and l an integer such that $1 \leq l \leq n-1$. We define

$$T_n(l) = \{\alpha \in T_n : \forall x, y \in \bar{n}, |x-y| = l \Rightarrow |x\alpha - y\alpha| = l\}$$

where $|x - y|$ is the absolute difference of numbers x and y.

Note that if $\alpha \in T_n(l)$, then we say that α *preserves the length* l. For any $\alpha, \beta \in T_n(l)$ and $x, y \in \bar{n}$ such that $|x - y| = l$. Then $|x\alpha - y\alpha| = l$. Because $x\alpha, y\alpha \in \bar{n}$ and β preserves the length l, so $|(x\alpha)\beta - (y\alpha)\beta| = l$. Thus $T_n(l)$ is a subsemigroup of T_n. We call $T_n(l)$ the *semigroup of transformations preserving a length* l.

Then we present the definition of n-ary $T_n(l)$-full term of type τ_n.

Definition 2.1. *Let f_i be an n-ary operation symbol and $\alpha \in T_n(l)$. An n-ary $T_n(l)$-full term of type τ_n is defined in the following way:*

(i) *$f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an n-ary $T_n(l)$-full term of type τ_n;*

(ii) *if $t_1, \ldots, t_n$ are n-ary $T_n(l)$-terms of type τ_n, then $f_i\left(t_1, \ldots, t_n\right)$ is an n-ary $T_n(l)$-full term of type τ_n.*

Let $W_{\tau_n}^{T_n(l)}(X_n)$ be the set of all n-ary $T_n(l)$-full terms of type τ_n.

Now we give an example of Definition 2.1. For any $c_1, c_2, \ldots, c_n \in \bar{n}$ (not necessarily distinct) we shall use the notation

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix}$$

to mean $\alpha \in T_n$ defined by $i\alpha = c_i$ for all $i = 1, 2, \ldots, n$.

Example 2.2. *Let $\tau_n = (n)$ be a type with one operation symbol f. Put $n = 3$, and $l = 1$. We have*

$$T_3(1) = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix} \right\}.$$

Then $f(x_1, x_2, x_1), f(x_1, x_2, x_3), f(x_2, x_1, x_2), f(x_2, x_3, x_2), f(x_3, x_2, x_1), f(x_3, x_2, x_3)$ are elements in $W_{\tau_3}^{T_3(1)}(X_3)$.

Normally, terms have many measures of their complexity, see [4,8]. As a result, there is a possibility to measure a complexity of $T_n(l)$-full terms. The depth of a $T_n(l)$-full term t, denoted by $Depth(t)$, is the longest distance from a first operation symbol that appears in a term (from the left) to variables. It can be inductively defined by

(i) $Depth(t) = 1$ if $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ and $\alpha \in T_n(l)$;

(ii) $Depth(t) = 1 + max\{Depth(t_j) \mid 1 \leq j \leq n\}$ if $t = f_i(t_1, \ldots, t_n)$.

On the set $W_{\tau_n}^{T_n(l)}(X_n)$, we define an $(n+1)$-ary operation S^n,

$$S^n : \left(W_{\tau_n}^{T_n(l)}(X_n)\right)^{n+1} \longrightarrow W_{\tau_n}^{T_n(l)}(X_n)$$

for all $t_1, \ldots, t_n, s_1, \ldots, s_n \in W_{\tau_n}^{T_n(l)}(X_n)$ by

(i) $S^n\left(f_i\left(x_{\alpha(1)}, \ldots, x_{\alpha(n)}\right), t_1, \ldots, t_n\right) := f_i\left(t_{\alpha(1)}, \ldots, t_{\alpha(n)}\right)$;

(ii) $S^n\left(f_i\left(t_1, \ldots, t_n\right), s_1, \ldots, s_n\right) :=$
$f_i\left(S^n\left(t_1, s_1, \ldots, s_n\right), \ldots, S^n\left(t_n, s_1, \ldots, s_n\right)\right)$.

Then we form the algebra

$$clone_{T_n(l)}(\tau_n) := \left(W_{\tau_n}^{T_n(l)}(X_n), S^n\right)$$

which is called *the clone of all $T_n(l)$-full terms of type τ_n*. Theorem 2.3, presented below, shows that the algebra $\left(W_{\tau_n}^{T_n(l)}(X_n), S^n\right)$ satisfies the superassociative law (SASS):

$$S^n(X_0, S^n(Y_1, Z_1, \dots, Z_n), \dots, S^n(Y_n, Z_1, \dots, Z_n))$$

$$\approx S^n(S^n(X_0, Y_1, \dots, Y_n), Z_1, \dots, Z_n) \tag{2.1}$$

where S^n is an $(n+1)$-ary operation symbol and X_0, Y_j, Z_j are variables for all $1 \leq j \leq n$.

Next, we shall show that the superassociative law is satisfied in the clone of all $T_n(l)$-full terms.

Theorem 2.3. *The algebra* $\left(W_{\tau_n}^{T_n(l)}(X_n), S^n\right)$ *satisfies the superassociative law.*

Proof. We give a proof by induction on the depth of an n-ary $T_n(l)$-full term t which is substituted for X_0 from (2.1). If we substitute for X_0 from (2.1) by a $T_n(l)$-full term $t = f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$ where $\alpha \in T_n(l)$, and $Depth(t) = 1$, then we have

$$\begin{aligned}
&S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n))\\
&\quad= f_i(S^n(x_{\alpha(1)}, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)), \dots,\\
&\qquad S^n(x_{\alpha(n)}, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)))\\
&\quad= f_i(S^n(t_{\alpha(1)}, s_1, \dots, s_n), \dots, S^n(t_{\alpha(n)}, s_1, \dots, s_n))\\
&\quad= S^n(f_i(t_{\alpha(1)}, \dots, t_{\alpha(n)}), s_1, \dots, s_n)\\
&\quad= S^n(S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), t_1, \dots, t_n), s_1, \dots, s_n).
\end{aligned}$$

If we substitute for X_0 from (2.1) by a $T_n(l)$-full term $t = f_i(r_1, \dots, r_n)$ where $r_1, \dots, r_n \in W_{\tau_n}^{T_n(l)}(X_n)$ and assume that

$$S^n(r_k, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)) = S^n\left(S^n(r_k, t_1, \dots, t_n), s_1, \dots, s_n\right)$$

for all $1 \leq k \leq n$, and $max_{1 \leq k \leq n} Depth(r_k) = m$, then $Depth(t) = m + 1$ and we have

$$\begin{aligned}
&S^n(f_i(r_1, \dots, r_n), S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n))\\
&\quad= f_i(S^n(r_1, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)), \dots,\\
&\qquad S^n(r_n, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)))\\
&\quad= f_i\left(S^n\left(S^n(r_1, t_1, \dots, t_n), s_1, \dots, s_n\right), \dots, \left(S^n(r_n, t_1, \dots, t_n), s_1, \dots, s_n\right)\right)\\
&\quad= S^n(f_i(S^n(r_1, t_1, \dots, t_n), \dots, S^{n_i}(r_n, t_1, \dots, t_n)), s_1, \dots, s_n)\\
&\quad= S^n(S^n(f_i(r_1, \dots, r_n), t_1, \dots, t_n), s_1, \dots, s_n).
\end{aligned}$$

□

An algebra $\mathcal{M} := (M, S^n)$ of type $\tau = (n+1)$ is called a *Menger algebra* of rank n if $\mathcal{M}$ satisfies the condition (SASS) [1]. It follows immediately from Theorem 2.3 that $clone_{T_n(l)}(\tau_n)$ is a Menger algebra of rank n.

It is clear that $clone_{T_n(l)}(\tau_n)$ is generated by

$$F_{W_{\tau_n}^{T_n(l)}(X_n)} := \left\{ f_i \left(x_{\alpha(1)}, \ldots, x_{\alpha(n)} \right) \mid i \in I, \alpha \in T_n(l) \right\}.$$

Let $V^{T_n(l)}$ be the variety of type $\tau = (n+1)$ generated by the superassociative law (SASS). Now let $\mathcal{F}_{V^{T_n(l)}}(\{Y_k \mid k \in J\})$ be the free algebra with respect to $V^{T_n(l)}$, freely generated by an alphabet $\{Y_k \mid k \in J\}$ where $J = \{(i, \alpha) \mid i \in I\ , \alpha \in T_n(l)\}$. The operation of $\mathcal{F}_{V^{T_n(l)}}(\{Y_k \mid k \in J\})$ is denoted by $\tilde{S}^n$. Next, we are going to prove that the clone of all $T_n(l)$-full terms is a free algebra with respect to the variety $V^{T_n(l)}$.

Theorem 2.4. *The algebra $clone_{T_n(l)}(\tau_n)$ is isomorphic to $\mathcal{F}_{V^{T_n(l)}}(\{Y_k \mid k \in J\})$ and therefore it is free with respect to the variety $V^{T_n(l)}$, and freely generated by the set*

$$\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}) \mid i \in I, \alpha \in T_n(l)\}.$$

Proof. We define the mapping $\varphi : W_{\tau_n}^{T_n(l)}(X_n) \longrightarrow \mathcal{F}_{V^{T_n(l)}}(\{Y_k \mid k \in J\})$ inductively as follows:

(i) $\varphi(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}) = y_{(i,\alpha)}$;

(ii) $\varphi(f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)})) = \tilde{S}^n(y_{(i,\alpha)}, \varphi(t_1), \ldots, \varphi(t_n))$.

Since φ maps the generating system of $clone_{T_n(l)}(\tau_n)$ onto the generating system of $\mathcal{F}_{V^{T_n(l)}}(\{Y_k \mid k \in J\})$, it is surjective. We prove the homomorphism property

$$\varphi(S^n(t_0, t_1, \ldots, t_n)) = \tilde{S}^n(\varphi(t_0), \varphi(t_1), \ldots, \varphi(t_n))$$

by induction on the depth of an n-ary $T_n(l)$-full term t_0. If $t_0 = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ where $\alpha \in T_n(l)$, and $Depth(t) = 1$, then we have

$$\begin{aligned}
&\varphi(S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), t_1, \ldots, t_n)) \\
&\qquad = \varphi(f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)})) \\
&\qquad = \tilde{S}^n(y_{(i,\alpha)}, \varphi(t_1), \ldots, \varphi(t_n)) \\
&\qquad = \tilde{S}^n(\varphi(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})), \varphi(t_1), \ldots, \varphi(t_n)).
\end{aligned}$$

If $t_0 = f_i(r_1, \ldots, r_n)$ and assume that

$$\varphi(S^n(r_k, t_1, \ldots, t_n)) = \tilde{S}^n(\varphi(r_k), \varphi(t_1), \ldots, \varphi(t_n))$$

for all $1 \leq k \leq n$ and $max_{1 \leq k \leq n} Depth(r_k) = m$, then $Depth(t) = m + 1$ and we have

$$\begin{aligned}
&\varphi(S^n(f_i(r_1, \ldots, r_n), t_1, \ldots, t_n)) \\
&\quad = \varphi(f_i(S^n(r_1, t_1, \ldots, t_n), \ldots, S^n(r_n, t_1, \ldots, t_n))) \\
&\quad = \tilde{S}^n(y_{(i,1_n)}, \varphi(S^n(r_1, t_1, \ldots, t_n)), \ldots, \varphi(S^n(r_n, t_1, \ldots, t_n)))
\end{aligned}$$

$$\begin{aligned}&= \tilde{S}^n(y_{(i,1_n)}, \tilde{S}^n(\varphi(r_1), \varphi(t_1), \ldots, \varphi(t_n)), \ldots,\\ &\quad \tilde{S}^n(\varphi(r_n), \varphi(t_1), \ldots, \varphi(t_n)))\\ &= \tilde{S}^n(\tilde{S}(y_{(i,1_n)}, \varphi(r_1), \ldots, \varphi(r_n)), \varphi(t_1), \ldots, \varphi(t_n))\\ &= \tilde{S}^n(\varphi(f_i(r_1, \ldots, r_n)), \varphi(t_1), \ldots, \varphi(t_n)).\end{aligned}$$

Thus φ is a homomorphism. The mapping φ is clearly bijective since the set $\{y_{(i,\alpha)} \mid i \in I,\ \alpha \in T_n(l)\}$ is free independent. Therefore we have

$$\begin{aligned}y_{(i,\alpha)} = y_{(j,\beta)} &\Longrightarrow (i, \alpha) = (j, \beta)\\ &\Longrightarrow i = j,\ \alpha = \beta.\end{aligned}$$

So $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}) = f_j(x_{\beta(1)}, \ldots, x_{\beta(n)})$. Thus φ is a bijection between the generating sets of $clone_{T_n(l)}(\tau_n)$ and $\mathcal{F}_{V^{T_n(l)}}(\{Y_k \mid k \in J\})$ and therefore φ is an isomorphism. □

3. $T_n(l)$-FULL HYPERSUBSTITUTIONS

A hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ which maps each operation symbol f_i to an n_i-ary term $\sigma(f_i)$ of type τ. Any hypersubstitution $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ can be uniquely extended to a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ as follows:

(i) $\hat{\sigma}[t] := t$ if $t \in X$; and

(ii) $\hat{\sigma}[t] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if $t = f_i(t_1, \ldots, t_{n_i}) \in W_\tau(X_{n_i})$.

The set $Hyp(\tau)$ of all hypersubstitutions of type τ forms a monoid under the binary operation $\circ_h$, defined by

$$\sigma_1 \circ_h \sigma_2 := \hat{\sigma_1} \circ \sigma_2$$

where $\circ$ denotes the usual composition of mappings. The identity is $\sigma_{id} : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ such that $\sigma_{id}(f_i) = f_i(x_1, ..., x_{n_i})$. Now, we call mapping

$$\sigma : \{f_i \mid i \in I\} \longrightarrow W_{\tau_n}^{T_n(l)}(X_n)$$

$T_n(l)$*-full hypersubstitution* of type τ_n.

For a $T_n(l)$-full term t we need the $T_n(l)$-full term t_β derived from t by replacement a variable $x_{\alpha(j)}$ in t by a variable $x_{\beta(\alpha(j))}$ for a mapping $\beta \in T_n(l)$. This can be defined as follows.

Let $t, t_1, \ldots, t_n \in W_{\tau_n}^{T_n(l)}(X_n)$ and $\alpha, \beta \in T_n(l)$. Then we define the $T_n(l)$-full term t_β in the following steps:

(i) If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$, then $t_\beta := f_i(x_{\beta(\alpha(1))}, \ldots, x_{\beta(\alpha(n))})$.

(ii) If $t = f_i(t_1, \ldots, t_n)$, then $t_\beta := f_i((t_1)_\beta, \ldots, (t_n)_\beta)$.

It is observed that if t is an $T_n(l)$-full term of type τ_n, then t_β is an $T_n(l)$-full term of type τ_n for all $\beta \in T_n(l)$. Then an $T_n(l)$-full hypersubstitution $\sigma : \{f_i \mid i \in I\} \longrightarrow W_{\tau_n}^{T_n(l)}(X_n)$ of type τ_n can be extended to a mapping

$$\hat{\sigma} : W_{\tau_n}^{T_n(l)}(X_n) \longrightarrow W_{\tau_n}^{T_n(l)}(X_n)$$

as follows :

(i) $\hat{\sigma}[f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})] := (\sigma(f_i))_{\alpha}$,

(ii) $\hat{\sigma}[f_i(t_1,\ldots,t_n)] := S^n\left(\sigma(f_i),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_{n_i}]\right)$.

The set of all $T_n(l)$-full hypersubstitutions of type τ_n will be denoted by $Hyp^{T_n(l)}(\tau_n)$. It is easy to see that $\left(Hyp^{T_n(l)}(\tau_n);\circ_h,\sigma_{id}\right)$ is a submonoid of $(Hyp(\tau_n);\circ_h,\sigma_{id})$.

The following lemma shows the property of a term t_α and the extension $\hat{\sigma}$.

Lemma 3.5. *Let $t,t_1,...,t_n \in W_{\tau_n}^{T_n(l)}(X_n)$. Then*

$$S^n(t,\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}]) = S^n(t_\alpha,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n])$$

for all $\alpha \in T_n(l)$.

Proof. We begin with the case when $t = f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$, which is the first claim of the first step of the induction $Depth(t) = 1$. In fact, we have

$$\begin{aligned} S^n(f_i(x_1,x_2,\ldots,x_n),\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}]) &= f_i(\hat{\sigma}[t_{\alpha(1)}],\hat{\sigma}[t_{\alpha(2)}],\ldots,\hat{\sigma}[t_{\alpha(n)}]) \\ &= S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n]) \\ &= S^n(f_i(x_1,x_2,\ldots,x_n)_\alpha,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n]). \end{aligned}$$

If $t = f_i(s_1,\ldots,s_n)$ and assume that

$$S^n(s_k,\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}]) = S^n((s_k)_{\alpha_i},\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}])$$

for all $1 \le k \le n$ and $\alpha \in T_n(l)$ then

$$\begin{aligned} &S^n(t,\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}]) \\ &\quad = S^n(f_i(s_1,\ldots,s_n),\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}]) \\ &\quad = f_i(S^n(s_1,\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}]),\ldots,S^n(s_n,\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}])) \\ &\quad = f_i(S^n((s_1)_\alpha,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n]),\ldots,S^n((s_n)_\alpha,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n])) \\ &\quad = S^n(f_i((s_1)_\alpha,\ldots,(s_n)_\alpha),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n]) \\ &\quad = S^n(t_\alpha,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n]). \end{aligned}$$

□

Using Lemma 3.5 we show that the extension $\hat{\sigma}$ of each $T_n(l)$-full hypersubstitution σ preserves the operation S^n on the set $W_{\tau_n}^{T_n(l)}(X_n)$.

Theorem 3.6. *For $\sigma \in Hyp^{T_n(l)}(\tau_n)$, the extension*

$$\hat{\sigma} : W_{\tau_n}^{T_n(l)}(X_n) \longrightarrow W_{\tau_n}^{T_n(l)}(X_n)$$

is an endomorphism on the algebra $clone_{T_n(l)}(\tau_n)$.

Proof. It is clear that $\hat{\sigma} : W_{\tau_n}^{T_n(l)}(X_n) \longrightarrow W_{\tau_n}^{T_n(l)}(X_n)$. Let $t_0,t_1,\ldots,t_n \in W_{\tau_n}^{T_n(l)}(X_n)$. We will show by induction on the depth of t_0 that

$$\hat{\sigma}[S^n(t_0, t_1, \ldots, t_n)] = S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]).$$

If $t_0 = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ where $\alpha \in T_n(l)$, and $Depth(t) = 1$, then we have

$$\begin{aligned}
\hat{\sigma}[S^n(t_0, t_1, \ldots, t_n)] &= \hat{\sigma}[S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), t_1, \ldots, t_n)] \\
&= \hat{\sigma}[f_i(t_{\alpha 1)}, \ldots, t_{\alpha(n)})] \\
&= S^n(\sigma(f_i), \hat{\sigma}[t_{\alpha(1)}], \ldots, \hat{\sigma}[t_{\alpha(n)}]) \\
&= S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]).
\end{aligned}$$

If $t_0 = f_i(r_1, \ldots, r_n)$ and we assume that

$$\hat{\sigma}[S^n(r_k, t_1, \ldots, t_n)] = S^n(\hat{\sigma}[r_k], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$$

for all $1 \leq k \leq n$ and $max_{1 \leq k \leq n} Depth(r_k) = m$, then $Depth(t) = m+1$ and we have $\hat{\sigma}[S^n(t_0, t_1, \ldots, t_n)]$

$$\begin{aligned}
&= \hat{\sigma}[S^n(f_i(r_1, \ldots, r_n), t_1, \ldots, t_n)] \\
&= \hat{\sigma}[f_i(S^n(r_1, t_1, \ldots, t_n), \ldots, S^{n_i}(r_n, t_1, \ldots, t_n))] \\
&= S^n(\sigma(f_i), \hat{\sigma}[S^n(r_1, t_1, \ldots, t_n)], \ldots, \hat{\sigma}[S^n(r_n, t_1, \ldots, t_n)]) \\
&= S^n(\sigma(f_i), S^n(\hat{\sigma}[r_1], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]), \ldots, S^n(\hat{\sigma}[r_n], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])) \\
&= S^n(S^n(\sigma(f_i), \hat{\sigma}[r_1], \ldots, \hat{\sigma}[r_n]), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \\
&= S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]).
\end{aligned}$$

□

We complete this section by studying the connection between $T_n(l)$-full terms and the extension of a mapping which maps fundamental term to any $T_n(l)$-full terms.

As mentioned, the algebra $clone_{T_n(l)}(\tau_n)$ is generated by the set

$$F_{W_{\tau_n}^{T_n(l)}(X_n)} := \left\{ f_i\left(x_{\alpha(1)}, \ldots, x_{\alpha(n)}\right) \mid i \in I, \alpha \in T_n(l) \right\}.$$

Thus, any mapping

$$\eta : F_{W_{\tau_n}^{T_n(l)}(X_n)} \longrightarrow W_{\tau_n}^{T_n(l)}(X_n)$$

called $T_n(l)$*-full clone substitution*, can be uniquely extended to endomorphism

$$\bar{\eta} : W_{\tau_n}^{T_n(l)}(X_n) \longrightarrow W_{\tau_n}^{T_n(l)}(X_n).$$

Let $Subst_{T_n(l)}(\tau_n)$ be the set of all $T_n(l)$-full clone substitutions. On the set $Subst_{T_n(l)}(\tau_n)$, a binary operation $\odot$ can be defined by

$$\eta_1 \odot \eta_2 := \bar{\eta_1} \circ \eta_2$$

where $\circ$ denotes the usual composition of mappings. Furthermore, the identity mapping with respect to $\odot$ is denoted by $id_{F_{W_{\tau_n}^{T_n(l)}(X_n)}}$.

Then clearly, $\left(Subst_{T_n(l)}(\tau); \odot, id_{F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)}}\right)$ forms a monoid.

Consider $\sigma \in Hyp^{T_n(l)}(\tau_n)$ and by Theorem 3.6, $\hat{\sigma} : W^{T_n(l)}_{\tau_n}(X_n) \longrightarrow W^{T_n(l)}_{\tau_n}(X_n)$ is an endomorphism. Since $F_{W^{T_n(l)}_{\tau_n}(X_n)}$ generates $clone_{T_n(l)}(\tau_n)$, we have $\hat{\sigma}|_{F_{W^{T_n(l)}_{\tau_n}(X_n)}}$ is an $T_n(l)$-full clone substitution with

$$\overline{\hat{\sigma}|_{F_{W^{T_n(l)}_{\tau_n}(X_n)}}} = \hat{\sigma}.$$

Define a mapping $\psi : Hyp^{T_n(l)}(\tau_n) \longrightarrow Subst_{T_n(l)}(\tau_n)$ by

$$\psi(\sigma) = \hat{\sigma}|_{F_{W^{T_n(l)}_{\tau_n}(X_n)}}.$$

We have that ψ is a homomorphism. In fact: Let $\sigma_1, \sigma_2 \in Hyp^{T_n(l)}(\tau_n)$. Then

$$\begin{aligned}
\psi(\sigma_1 \circ_h \sigma_2) &= (\sigma_1 \circ_h \sigma_2)\hat{}|_{F_{W^{T_n(l)}_{\tau_n}(X_n)}} \\
&= (\hat{\sigma_1} \circ \hat{\sigma_2})|_{F_{W^{T_n(l)}_{\tau_n}(X_n)}} \\
&= \overline{\hat{\sigma_1}|_{F_{W^{T_n(l)}_{\tau_n}(X_n)}}} \circ \hat{\sigma_2}|_{F_{W^{T_n(l)}_{\tau_n}(X_n)}} \\
&= \overline{\psi(\sigma_1)} \circ \psi(\sigma_2) \\
&= \psi(\sigma_1) \odot \psi(\sigma_2).
\end{aligned}$$

Clearly, ψ is an injection. Hence we have proved, the following corollary.

Corollary 3.7. *The monoid* $\left(Hyp^{T_n(l)}(\tau_n); \circ_h, \sigma_{id}\right)$ *can be embedded into* $(Subst_{T_n(l)}(\tau_n); \odot, id_{F_{W^{T_n(l)}_{\tau_n}(X_n)}})$.

4. $T_n(l)$-FULL HYPERIDENTITIES AND CLONE IDENTITIES

Let V be a variety of type τ_n and let IdV be the set of all identities of V. Let $Id^{T_n(l)}V$ be the set of all $s \approx t$ of V such that s and t are both $T_n(l)$-full term of type τ_n; that is

$$Id^{T_n(l)}V := \left(W^{T_n(l)}_{\tau_n}(X_n)\right)^2 \cap IdV.$$

It is well-known that IdV is a congruence on the free algebra $\mathcal{F}_\tau(X)$. However, in general this is not true for $Id^{T_n(l)}V$. The following theorem shows that $Id^{T_n(l)}V$ is a congruence on $clone_{T_n(l)}(\tau_n)$.

Theorem 4.8. *Let V be a variety of type τ_n. Then $Id^{T_n(l)}V$ is a congruence on the algebra $clone_{T_n(l)}(\tau_n)$.*

Proof. We will prove that if $t \approx r$, $t_k \approx r_k \in Id^{T_n(l)}V, k = 1, 2, \ldots, n$, then $S^n(t, t_1, \ldots, t_n) \approx S^n(r, r_1, \ldots, r_n) \in Id^{T_n(l)}V$. Firstly, we give a proof by induction on the depth of a term $t \in W^{T_n(l)}_{\tau_n}(X_n)$ that for every $i \in I$ from $t_k \approx r_k \in Id^{T_n(l)}V, k = 1, 2, \ldots, n$, there follows $S^n(t, t_1, \ldots, t_n) \approx S^n(t, r_1, \ldots, r_n) \in Id^{T_n(l)}V$. If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ where $\alpha \in T_n(l)$, and $Depth(t) = 1$, then we have

$$\begin{aligned} &S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),t_1,\ldots,t_n)\\ &\quad = \quad f_i(t_{\alpha(1)},\ldots,t_{\alpha(n)})\\ &\quad \approx \quad f_i(r_{\alpha(1)},\ldots,r_{\alpha(n)})\\ &\quad = \quad \overline{\psi(\sigma_1)}\circ\psi(\sigma_2)\\ &\quad = \quad S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),r_1,\ldots,r_n)\in Id^{T_n(l)}V. \end{aligned}$$

since IdV is compatible with the operation $\overline{f_i}$ of the absolutely free algebra $\mathcal{F}_\tau(X)$ and by the definition of $T_n(l)$-full terms.

If $t=f_i(l_1,\ldots,l_n)\in W_{\tau_n}^{T_n(l)}(X_n)$ and assume that

$$S^n(l_k,t_1,...,t_n)\approx S^n(l_k,r_1,...,r_n)\in Id^{T_n(l)}V.$$

for all $1\leq k\leq n$ and $max_{1\leq k\leq n}Depth(r_k)=m$, then $Depth(t)=m+1$ and we obtain

$$\begin{aligned} S^n(f_i(l_1,\ldots,l_n),t_1,\ldots,t_n) &= f_i(S^n(l_1,t_1,\ldots,t_n),\ldots,S^n(l_n,t_1,\ldots,t_n))\\ &\approx f_i(S^n(l_1,r_1,\ldots,r_n),\ldots,S^{n_i}(l_n,r_1,\ldots,r_n))\\ &= S^n(f_i(l_1,\ldots,l_n),r_1,\ldots,r_n)\in Id^{T_n(l)}V. \end{aligned}$$

This means

$$S^n(t,t_1,\ldots,t_n)\approx S^n(t,r_1,\ldots,r_n)\in Id^{T_n(l)}V.$$

This is a consequence of the fact that IdV is a fully invariant congruence on the absolutely free algebra $\mathcal{F}_\tau(X)$. Assume now that $t\approx r, t_k\approx r_k\in Id^{T_n(l)}V$. Then

$$S^n(t,t_1,\ldots,t_n)\approx S^n(r,t_1,\ldots,t_n)\approx S^n(r,r_1,\ldots,r_n)\in Id^{T_n(l)}V.$$

□

By using the concepts of $T_n(l)$-full hypersubstitution as we presented in Section 3. We shall define $T_n(l)$-full hyperidentities in a variety of typer τ_n.

Let V be a variety of type τ_n. An identity $s\approx t\in Id^{T_n(l)}V$ is called a $T_n(l)$-*full hyperidentity* of V if $\hat{\sigma}[s]\approx\hat{\sigma}[t]\in IdV$ for all $\sigma\in Hyp^{T_n(l)}(\tau_n)$. Moreover, the variety V is called $T_n(l)$-*full solid* if the following holds:

$$\forall s\approx t\in Id^{T_n(l)}V,\forall\sigma\in Hyp^{T_n(l)}(\tau_n),\hat{\sigma}[s]\approx\hat{\sigma}[t]\in IdV.$$

Next theorem characterizes the $T_n(l)$-full solid variety.

Theorem 4.9. *Let V be a variety of type τ_n. If $Id^{T_n(l)}V$ is a fully invariant congruence on $clone_{T_n(l)}(\tau_n)$, then V is $T_n(l)$-full solid.*

Proof. Assume that $Id^{T_n(l)}V$ is a fully invariant congruence on $clone_{T_n(l)}(\tau_n)$. Let $s\approx t\in Id^{T_n(l)}V$ and $\sigma\in Hyp^{T_n(l)}(\tau_n)$. By Theorem 3.6, $\hat{\sigma}$ is an endomorphism of $clone_{T_n(l)}(\tau_n)$. Hence $\hat{\sigma}[s]\approx\hat{\sigma}[t]\in Id^{T_n(l)}V$, which shows that V is $T_n(l)$-full solid. □

For a variety V of type τ_n, $Id^{T_n(l)}V$ is a congruence on $clone_{T_n(l)}(\tau_n)$ by Theorem 4.8. We can form the quotient algebra

$$clone_{T_n(l)}(V) := clone_{T_n(l)}(\tau_n)/Id^{T_n(l)}V.$$

This quotient algebra belongs to the class of a Menger algebra of rank n. Note that we have a natural homomorphism

$$nat_{Id^{T_n(l)}V} : clone_{T_n(l)}(\tau_n) \longrightarrow clone_{T_n(l)}(V)$$

such that

$$nat_{Id^{T_n(l)}V}(t) = [t]_{Id^{T_n(l)}V}.$$

Finally, we prove the following connection between $T_n(l)$-full hyperidentities of a variety V and clone identities.

Theorem 4.10. *Let V be a variety of type τ_n. If $s \approx t \in Id^{T_n(l)}V$ is an identity in $clone_{T_n(l)}(V)$, then $s \approx t$ is $T_n(l)$-full hyperidentity of V.*

Proof. Assume that $s \approx t \in Id^{T_n(l)}V$ is an identity in $clone_{T_n(l)}(V)$. Let $\sigma \in Hyp^{T_n(l)}(\tau_n)$. Then $\hat{\sigma} : clone_{T_n(l)}(\tau_n) \longrightarrow clone_{T_n(l)}(\tau_n)$ is an endomorphism by Theorem 3.6. Thus

$$nat_{Id^{T_n(l)}V} \circ \hat{\sigma} : clone_{T_n(l)}(\tau_n) \longrightarrow clone_{T_n(l)}(V)$$

is a homomorphism. By assumption,

$$\left(nat_{Id^{T_n(l)}V} \circ \hat{\sigma}\right)(s) = \left(nat_{Id^{T_n(l)}V} \circ \hat{\sigma}\right)(t).$$

That is

$$nat_{Id^{T_n(l)}V}(\hat{\sigma}[s]) = nat_{Id^{T_n(l)}V}(\hat{\sigma}[t]).$$

Thus

$$[\hat{\sigma}[s]]_{Id^{T_n(l)}V} = [\hat{\sigma}[t]]_{Id^{T_n(l)}V},$$

and hence

$$\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{T_n(l)}V.$$

Therefore, $s \approx t$ is a $T_n(l)$-full hyperidentity of V. □

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5. CONCLUSION

Finally, we give two problems and suggestions for the future research in this area.

(1) Determine the semigroup properties of the monoid $\left(Hyp^{T_n(l)}(\tau_n); \circ_h, \sigma_{id}\right)$ Find the order of its elements for the particular type. Describe the idempo tency and several kinds of regularity of the $T_n(l)$-full hypersubstitutions.

(2) Use some difference definions of transformation semigroup, for instance transformations with invariant subset to define new generalizations of full terms. Study the connection between the different kinds of full terms.

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